Particular Solutions of the Equation of Conduction of Heat in One Dimension.

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§ 1. The problem of the conduction of heat in one dimension is usually concerned with the propagation of a thermal disturbance along a bar or rod of uniform cross section. The solution of the problem is required for a given initial distribution of temperature, and given boundary values, usually at each end of the rod. In most cases this solution is found by assuming a series solution and then proving that the series satisfies the equation of the disturbance as well as all the assigned conditions. Other methods, for example the contour integral method developed by Carslaw,* also introduce this arbitrary element of choice in choosing the integrand and the contour of integration. The object of the present paper is to develop the application of Heaviside's Operational method to the solution of the problem, and to show that it leads in all cases to solutions equivalent to the known forms, although initially no assumptions are made regarding the nature of the solution.

§ 2. Heaviside's† formula, together with the allied ones of Bromwich ‡ and Carson § may be briefly stated. The equation is obtained in the symbolic form

$$v = \frac{Y(p)}{Z(p)} v_0; \qquad \text{(1)}$$

where Y and Z are functions of the differential operator $p = \dfrac{d}{dt}$, v_0 is constant, and the initial instant is so chosen that at $t = 0$ the disturbance is zero. Then the solution of the equation is

$$v = v_0 \left[\frac{Y(0)}{Z(0)} + \sum_m \frac{Y(p_m)}{p_m Z'(p_m)} e^{p_m t} \right] \qquad \text{(2)}$$

* H. S. CARSLAW : *Mathematical Theory of the Conduction of Heat in Solids.* Ch. XI.

† HEAVISIDE : *Electromagnetic Theory*, Vol. II., Chapter V.

‡ BROMWICH : *Phil. Mag., London,* (Ser. 6), 37, p. 407, 1919.

§ CARSON : *Physical Review*, X., 2, 1917.

where p_m is a simple root of the algebraic equation $Z(p)=0$, and the summation extends over all such roots.

Bromwich's symbolic equation is

$$v=\frac{Y(p)}{Z(p)}\,Gt\,,$$

where G is a constant, and the solution he obtains is

$$v=G\left(N_0 t+N_1+\sum_m \frac{Y(p_m)}{p_m{}^2 Z'(p_m)}\,e^{p_m t}\right)$$

when N_0 N_1 are determined by the equation

$$\frac{Y(p)}{Z(p)}=N_0+N_1 p+N_2 p^2+\ldots\;.$$

Carson gives a formula for the solution of the equation

$$v=E\,\frac{Y(p)}{Z(p)}\,e^{\alpha t},\qquad\qquad(3)$$

namely,

$$v=E\left[\frac{Y(\alpha)}{Z(\alpha)}\,e^{\alpha t}-\sum\frac{Y(p_m)}{(\alpha-p_m)Z'(p_m)}e^{p_m t}\right].\qquad\qquad(4)$$

§ 3. This formula of Carson's admits of a generalisation which will prove useful. Consider the symbolic equation

$$u=\frac{Y(p)}{Z(p)}\;\phi(t),\qquad\qquad(5)$$

where $\phi(t)$ is such that it can be expanded as a Fourier integral

$$\phi(t)=\frac{2}{\pi}\int_0^\infty\int_{-\infty}^\infty \cos u\,(\lambda-t)\,\phi(\lambda)\,d\lambda\,du$$

$$=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty\left\{e^{iu(\lambda-t)}+e^{-iu(\lambda-t)}\right\}\phi(\lambda)\,d\lambda\,du$$

Then $$u=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{iu\lambda}\phi(\lambda)\frac{Y(p)}{Z(p)}e^{-iut}\,d\lambda\,du$$

$$+\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{-iu\lambda}\phi(\lambda)\frac{Y(p)}{Z(p)}e^{iut}\,d\lambda\,du$$

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Write $u_1 = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty e^{iu\lambda}\phi(\lambda)\frac{Y(p)}{Z(p)}e^{-iut}d\lambda\, du.$

Then by Carson's formula,

$$u_1 = \frac{1}{\pi}\int_0^\infty du\int_{-\infty}^\infty d\lambda\, e^{iu\lambda}\phi(\lambda)\left\{\frac{Y(-iu)}{Z(-iu)}e^{-iut} - \sum_m \frac{Y(p_m)}{(-iu-p_m)Z'(p_m)}e^{p_m t}\right\}$$

$$= \frac{1}{\pi}\int_{-\infty}^0 du\int_{-\infty}^\infty d\lambda e^{-iu\lambda}\phi(\lambda)\left\{\frac{Y(iu)}{Z(iu)}e^{iut} - \sum_m \frac{Y(p_m)}{(iu-p_m)Z'(p_m)}e^{p_m t}\right\}$$

and the solution of the equation

$$u_2 = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty e^{-iu\lambda}\phi(\lambda)\frac{Y(p)}{Z(p)}e^{iut}d\lambda\, du$$

is
$$u_2 = \frac{1}{\pi}\int_0^\infty du\int_{-\infty}^\infty d\lambda\, e^{-iu\lambda}\phi(\lambda)\left\{\frac{Y(iu)}{Z(iu)}e^{iut} - \sum_m \frac{Y(p_m)}{(iu-p_m)Z'(p_m)}e^{p_m t}\right\}$$

$$\therefore u = u_1 + u_2 = \frac{1}{\pi}\int_{-\infty}^\infty du\int_{-\infty}^\infty d\lambda\, e^{-iu\lambda}\phi(\lambda)\left[\frac{Y(iu)}{Z(iu)}e^{iut} - \sum_m \frac{Y(p_m)}{(iu-p_m)Z'(p_m)}e^{p_m t}\right]$$

$$= \frac{1}{\pi}\int_{-\infty}^\infty du\frac{Y(iu)}{Z(iu)}e^{iut}\int_{-\infty}^\infty d\lambda\, e^{-iu\lambda}\phi(\lambda)$$
$$- \frac{1}{\pi}\sum_m \frac{Y(p_m)}{Z'(p_m)}e^{p_m t}\int_{-\infty}^\infty \frac{du}{iu-p_m}\int_\infty^\infty e^{-iu\lambda}\phi(\lambda)\,d\lambda$$

$$= \frac{1}{\pi}\int_{-\infty}^\infty du\frac{Y(iu)}{Z(iu)}e^{iut}\int_{-\infty}^\infty d\lambda\, e^{-iu\lambda}\phi(\lambda)$$
$$- 2\sum_m \frac{Y(p_m)}{2Z'(p_m)}e^{p_m t}\int_{-\infty}^\infty e^{-\lambda p_m}\phi(\lambda)\,d\lambda \quad\ldots\ldots\ldots\ldots(6)$$

§ 4. We shall consider now the various problems in the conduction of heat in a finite rod as enunciated by Carslaw.* The first is: Finite Rod : ends at zero Temperature. Initial Temperature $f(x)$.

* Carslaw : *Loc. cit.*, Chapter IV.

No Radiation at the Surface. Analytically then the problem is the solution of the equations

$$\frac{\partial v}{\partial t} = \kappa \cdot \frac{\partial^2 v}{\partial x^2} \quad (0 < x < l) \quad \ldots\ldots\ldots\ldots(4a)$$

$$v = 0 \text{ when } x = 0 \text{ and } x = l \quad \ldots\ldots\ldots\ldots(4b)$$

$$v = f(x) \text{ when } t = 0 \quad \ldots\ldots\ldots\ldots(4c)$$

It is assumed that $f(x)$ can be expanded as a Fourier sine series

$$f(x)\, \Sigma\, a_n \sin \frac{n\pi x}{l}.$$

To satisfy the condition that the disturbance is zero at the initial instant write

$$v = u + f(x) = u + \Sigma\, a_n \sin \frac{n\pi x.}{l}.$$

The differential equation becomes

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} - \kappa\, \Sigma\, a_n \frac{n^2 \pi^2}{l^2} \sin \frac{n\pi x}{l},$$

or, symbolically

$$\kappa \frac{\partial^2 u}{\partial x^2} - pu = \kappa\, \Sigma\, a_n \frac{n^2 \pi^2}{l^2} \sin \frac{n\pi x}{l}.$$

The solution of this equation, regarding p as an algebraic constant, is

$$u = Ae^{\sqrt{\frac{p}{\kappa}}\, x} + Be^{-\sqrt{\frac{p}{\kappa}}\, x} - \kappa \frac{\pi^2}{l^2} \sum_{n=1}^{\infty} \frac{n^2 a_n \sin \frac{n\pi x}{l}}{\kappa \frac{n^2 \pi^2}{l^2} + p}.$$

Now apply the conditions (4b) which give $u = 0$ where $x = 0$ and $x = l$.

$$\therefore \quad 0 = A + B.$$

$$\therefore \quad 0 = Ae^{\sqrt{\frac{p}{\kappa}}\, l} + Be^{-\sqrt{\frac{p}{\kappa}}\, l}$$

$$\therefore \quad A = B = 0.$$

$$\therefore \quad u = -\kappa \frac{\pi^2}{l^2} \sum_{n=1}^{\infty} \frac{n^2 a_n \sin \frac{n\pi x}{l}}{\kappa \frac{n^2 \pi^2}{l^2} + p}.$$

The zeros of each denominator are different, and therefore each term can be considered separately.

Consider
$$u_n = -\frac{\kappa\frac{n^2\pi^2}{l^2}a_n \sin\frac{n\pi x}{l}}{\kappa\frac{n^2\pi^2}{l^2}+p}.$$

This is in form 1, with $Y(p)$ constant, $Z(p)=\kappa\frac{n^2\pi^2}{l^2}+p$.

Only root of $Z(p)=0$ is $p=-\kappa\frac{n^2\pi^2}{l^2}$.

For this root $\frac{Y(p)}{pZ'(p)}=a_n \sin\frac{n\pi x}{l}$.

∴ Solution of the equation is
$$u_n = -a_n \sin\frac{n\pi x}{l}+a_n \sin\frac{n\pi x}{l}\, e^{-\kappa\frac{n^2\pi^2}{l^2}t},$$

with similar expressions for $u_1\, u_2$............

$$\therefore \quad u = -\Sigma a_n \sin\frac{n\pi x}{l}+\Sigma a_n \sin\frac{n\pi x}{l}\, e^{-\kappa\frac{n^2\pi^2}{l^2}t}$$

$$\therefore \quad v = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}\, e^{-\kappa\frac{n^2\pi^2}{l^2}t}.$$

which is the usual solution of the problem.

§ 5. **Finite Rod. Ends at Fixed Temperatures. Initial Temperature $f(x)$. No Radiation.**

The equations to be satisfied are

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}\ (0<x<l) \quad\text{.......................(5a)}$$

$$\left.\begin{array}{l} v=v_1 \text{ when } x=0 \\ v=v_2 \text{ when } x=l \end{array}\right\}\text{...........................(5b)}$$

$$v=f(x)=\Sigma\, a_n \sin\frac{n\pi x}{l} \text{ when } t=0 \text{.........(5c)}$$

Again write $v=u+f(x)$ and consider only the n^{th} term of the series for $f(x)$.

Then $(\kappa D^2-p)\, u_n=\kappa a_n\frac{n^2\pi^2}{l^2}\sin\frac{n\pi x}{l}$

$$\therefore \quad u_n = Ae^{\sqrt{\frac{p}{\kappa}}x}+Be^{-\sqrt{\frac{p}{\kappa}}x}-\frac{\kappa a_n\frac{n^2\pi^2}{l^2}\sin\frac{n\pi x}{l}}{\frac{\kappa n^2\pi^2}{l^2}+p}$$

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Substitute from (5*b*), so that

$$v_1 = A \qquad + B$$

$$v_2 = Ae^{\sqrt{\frac{p}{\kappa}}\,l} + Be^{-\sqrt{\frac{p}{\kappa}}\,l}$$

$$\therefore \quad A = \frac{v_2 - v_1 e^{-\sqrt{\frac{p}{\kappa}}\,l}}{2\sinh\sqrt{\frac{p}{\kappa}}\,l}$$

$$B = \frac{v_1 e^{\sqrt{\frac{p}{\kappa}}\,l} - v_2}{2\sinh\sqrt{\frac{p}{\kappa}}\,l}$$

$$\therefore \quad u_n = \frac{v_2\sinh\sqrt{\frac{p}{\kappa}}\,x + v_1\sinh\sqrt{\frac{p}{\kappa}}\left(l-x\right)}{\sinh\sqrt{\frac{p}{\kappa}}\,l} - \frac{\kappa a_n\frac{n^2\pi^2}{l^2}\sin\frac{n\pi x}{l}}{\kappa\frac{n^2\pi^2}{l^2}+p}$$

$$Z(p) = \sinh\sqrt{\frac{p}{\kappa}}\,l\left(\kappa\frac{n^2\pi^2}{l^2}+p\right)$$

with poles at $\sqrt{\frac{p}{\kappa}}\,l = ir\pi \quad (r = 0, 1, 2, \ldots\ldots)$

$$\therefore \quad p_r = -\frac{\kappa r^2\pi^2}{l^2}$$

$$p_r Z'(p_r) = p_r\left(\kappa\frac{n^2\pi^2}{l^2}+p_r\right)\frac{l}{2\sqrt{p_r\kappa}}\cosh\sqrt{\frac{p_r}{\kappa}}\,l + p_r\sinh\sqrt{\frac{p_r}{\kappa}}\,l$$

$$= (-1)^r\frac{ir\pi}{2}\,\kappa\frac{(n^2-r^2)\pi^2}{l^2}.$$

$$Y(p_r) = \frac{\kappa(n^2-r^2)\pi^2}{l^2}\left(v_2\sinh\frac{ir\pi x}{l} + v_1\sinh\frac{ir\pi(l-x)}{l}\right)$$

$$= i\kappa\frac{(n^2-r^2)\pi^2}{l^2}\left(v_2 + (-1)^{r-1}v_1\right)\sin\frac{r\pi x}{l}.$$

$$\therefore \quad \frac{Y(p_r)}{p_r Z'(p_r)} = \frac{2}{r\pi}\left((-1)^r v_2 - v_1\right)\sin\frac{r\pi x}{l}.$$

For $p = -\kappa\frac{n^2\pi^2}{l^2}$ we have the additional term

$$\frac{-\kappa a_n\frac{n^2\pi^2}{l^2}\sin\frac{n\pi x}{l}}{-\kappa\frac{n^2\pi^2}{l^2}} = a_n\sin\frac{n\pi x}{l}$$

We have still to determine $\dfrac{Y(0)}{Z(0)}$.

$$Y(p) = \left(\kappa\frac{n^2\pi^2}{l^2} + p\right)\left\{ v_2\left[\sqrt{\frac{p}{\kappa}}\,x + \frac{1}{3!}\left(\sqrt{\frac{p}{\kappa}}\,x\right)^3 + \dots\right]\right.$$

$$\left. + v_1\left[\sqrt{\frac{p}{\kappa}}\,(l - + x)\frac{1}{3!}\left(\sqrt{\frac{p}{\kappa}}\,\overline{l-x}\right)^3 + \dots\right]\right\}$$

$$-\kappa a_n\frac{n^2\pi^2}{l^2}\sin\frac{n\pi x}{l}\left\{\sqrt{\frac{p}{\kappa}}\,l + \frac{1}{3!}\left(\sqrt{\frac{p}{\kappa}}\,l\right)^3 + \dots\right\}$$

$$= \kappa\frac{n^2\pi^2}{l^2}\left(xv_2 + \overline{l-x}\,v_1 - la_n\sin\frac{n\pi x}{l}\right)\sqrt{\frac{p}{\kappa}} + O(p^{3/2}).$$

$$Z(p) = \left(\kappa\frac{n^2\pi^2}{l^2} + p\right)\left(\sqrt{\frac{p}{\kappa}}\,l + \frac{1}{3!}\left(\sqrt{\frac{p}{\kappa}}\,l\right)^3 + \dots\right)$$

$$= \kappa\frac{n^2\pi^2}{l^2}\,l\sqrt{\frac{p}{\kappa}} + O(p^{\frac{3}{2}}).$$

$$\therefore\quad \frac{Y(0)}{Z(0)} = \frac{xv_2 + (l-x)\,v_1 - la_n\sin\dfrac{n\pi x}{l}}{l}$$

$$= v_1 + \frac{(v_2 - v_1)\,x}{l} - a_n\sin\frac{n\pi x}{l}.$$

Collecting terms, the solution is given by

$$u_n = v_1 + \frac{(v_2 - v_1)\,x}{l} - a_n\sin\frac{n\pi x}{l} + a_n\sin\frac{n\pi x}{l}\,e^{-\kappa\frac{n^2\pi^2}{l^2}t}$$

$$+\frac{2}{\pi}\sum_{r=0}^{\infty}\frac{v_2\cos r\pi - v_1}{r}\sin\frac{r\pi x}{l}\,e^{-\kappa\frac{r^2\pi^2}{l^2}t}.$$

$\therefore$ Finally,

$$v = v_1 + \frac{(v_2 - v_1)\,x}{l} + \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{l}\,e^{-\kappa\frac{n^2\pi^2}{l^2}t}$$

$$+\frac{2}{\pi}\sum_{r=0}^{\infty}\frac{v_2\cos r\pi - v_1}{r}\sin\frac{r\pi x}{l}\,e^{-\kappa\frac{r^2\pi^2}{l^2}t}.$$

§ 6. Finite Rod. Ends at Temperatures $\phi_1(t)$ and $\phi_2(t)$. Initial Temperature $f(x)$. No radiation. The equations to be satisfied are

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}\;\dots\dots\dots\dots(6a)$$

$$\left.\begin{aligned} v &= \phi_1(t) \text{ when } x = 0\\ v &= \phi_2(t) \text{ when } x = l\end{aligned}\right\}\;\dots\dots\dots\dots(6b)$$

$$v = f(x) = \Sigma a_n\sin\frac{n\pi x}{l}\text{ when } t = 0\dots\dots(6c).$$

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Write $v = u + f(x)$ and as before

$$u = \frac{\sinh\sqrt{\frac{p}{\kappa}}\,x}{\sinh\sqrt{\frac{p}{\kappa}}\,l}\,\phi_2(t) + \frac{\sinh\sqrt{\frac{p}{\kappa}}\,(l-x)}{\sinh\sqrt{\frac{p}{\kappa}}\,l}\,\phi_1(t)$$

$$- \sum_n \frac{\kappa a_n \frac{n^2\pi^2}{l^2} \sin\frac{n\pi x}{l}}{\kappa\frac{n^2\pi^2}{l^2} + p}$$

Consider first $u_1 = \dfrac{\sinh\sqrt{\frac{p}{\kappa}}\,x}{\sinh\sqrt{\frac{p}{\kappa}}\,l}\,\phi_2(t)$,

and apply the Formula (6).

$$\therefore\quad u_1 = \frac{1}{\pi}\int_{-\infty}^{\infty} d\alpha \frac{\sinh\sqrt{\frac{i\alpha}{\kappa}}\,x}{\sinh\sqrt{\frac{i\alpha}{\kappa}}\,l}\, e^{i\alpha t}\int_{-\infty}^{\infty} e^{-i\alpha\lambda}\phi_2(\lambda)\,d\lambda$$

$$-2\sum_{n=-\infty}^{\infty} \frac{i\sin\frac{n\pi x}{l}}{l^2\cos n\pi}\, 2in\pi\kappa\, e^{-\frac{n^2\pi^2\kappa}{l^2}t}\int_{-\infty}^{\infty} e^{\frac{n^2\pi^2\kappa}{l^2}\lambda}\,\phi_2(\lambda)\,d\lambda.$$

But $$\frac{\sinh\sqrt{\frac{i\alpha}{\kappa}}\,x}{\sinh\sqrt{\frac{i\alpha}{\kappa}}\,l} = 2\pi\kappa\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{i\alpha l^2 + n^2\pi^2\kappa}\sin\frac{n\pi x}{l}$$

$$\therefore\quad \frac{1}{\pi}\int_{-\infty}^{\infty} d\alpha \frac{\sinh\sqrt{\frac{i\alpha}{\kappa}}\,x}{\sinh\sqrt{\frac{i\alpha}{\kappa}}\,l}\, e^{i\alpha t}\int_{-\infty}^{\infty} e^{-i\alpha\lambda}\phi_2(\lambda)\,d\lambda$$

$$= 2\kappa\sum_{n=0}^{\infty} n(-1)^{n+1}\sin\frac{n\pi x}{l}\int_{-\infty}^{\infty} e^{i\alpha(t-\lambda)}\frac{d\alpha}{i\alpha l^2 + n^2\pi^2\kappa}\int_{-\infty}^{\infty}\phi_2(\lambda)\,d\lambda$$

$$= 2\kappa\sum_{n=0}^{\infty} n(-1)^{n+1}\sin\frac{n\pi x}{l}\int_{-\infty}^{\infty}\frac{2\pi}{l^2}\, e^{-\frac{n^2\pi^2\kappa}{l^2}(t-\lambda)}\,\phi_2(\lambda)\,d\lambda$$

$$\therefore \quad u_1 = \frac{4\pi\kappa}{l^2} \sum_{n=0}^{\infty} n(-1)^{n+1} \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2\kappa}{l^2}t} \int_{-\infty}^{\infty} e^{\frac{n^2\pi^2\kappa}{l^2}\lambda} \phi_2(\lambda)\, d\lambda$$

$$+ \frac{4\pi\kappa}{l^2} \sum_{n=-\infty}^{0} n(-1)^n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2\kappa}{l^2}t} \int_{-\infty}^{\infty} e^{\frac{n^2\pi^2\kappa}{l^2}\lambda} \phi_2(\lambda)\, d\lambda$$

$$= \frac{2\pi\kappa}{l^2} \sum_{n=-\infty}^{\infty} n(-1)^{n+1} e^{-\frac{n^2\pi^2\kappa}{l^2}t} \sin\frac{n\pi x}{l} \int_{-\infty}^{\infty} e^{\frac{n^2\pi^2\kappa}{l^2}\lambda} \phi_2(\lambda)\, d\lambda.$$

Similarly if we write $u_2 = \dfrac{\sinh\sqrt{\frac{p}{\kappa}}(l-x)}{\sinh\sqrt{\frac{p}{\kappa}}\,l}\phi_1(t)$

we have $$u_2 = \frac{2\pi\kappa}{l^2} \sum_{n=-\infty}^{\infty} n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2\kappa}{l^2}t} \int_{-\infty}^{\infty} e^{\frac{n^2\pi^2\kappa}{l^2}\lambda} \phi_1(\lambda)\, d\lambda.$$

Also if $u_3 = -\Sigma \dfrac{\kappa a_n \frac{n^2\pi^2}{l^2} \sin\frac{n\pi x}{l}}{\kappa\frac{n^2\pi^2}{l^2} + p}$

the solution is, from § 4

$$u_3 = -\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} e^{-\frac{\kappa n^2\pi^2}{l^2}t}.$$

Finally $v = u_1 + u_2 + u_3 + f(x)$

$$= \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} e^{-\kappa\frac{n^2\pi^2}{l^2}t}$$

$$+ \frac{2\pi\kappa}{l^2} \sum_{n=-\infty}^{\infty} n \sin\frac{n\pi x}{l} e^{-\kappa\frac{n^2\pi^2}{l^2}t} \int_{-\infty}^{\infty} e^{\frac{n^2\pi^2\kappa}{l^2}\lambda} \{\phi_1(\lambda) - (-1)^n\phi_2(\lambda)\}\, d\lambda.$$

$$= \frac{2}{l} \sum_{n=1}^{\infty} e^{-\kappa\frac{n^2\pi^2 t}{l^2}} \sin\frac{n\pi x}{l} \left[\int_0^l f(x') \sin\frac{n\pi x'}{l} dx' \right.$$

$$\left. + \frac{n\pi\kappa}{l} \int_0^t e^{\frac{n^2\pi^2\kappa}{l^2}\lambda} \{\phi_1(\lambda) - (-1)^n\phi_2(\lambda)\}\, d\lambda \right]$$

since $\phi(t)$ must vanish for $t<0$, and the values of ϕ for times later than the instant t under consideration need not be considered.

§ 7. **Finite Rod. Radiation at Ends into a Medium at Zero Temperature. Initial Temperature $f(x)$. No Radiation at the Surface.**

The differential equations are

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} \qquad \text{(7 } a\text{)}$$

$$\left.\begin{aligned} -\frac{\partial v}{\partial x} + hv = 0 \text{ at } x = 0 \\ \frac{\partial v}{\partial x} + hv = 0 \text{ at } x = l \end{aligned}\right\} \qquad \text{(7 } b\text{)}$$

$$v = f(x) = \Sigma a_n \sin \frac{n\pi x}{l} \text{ when } t = 0 \qquad \text{(7c)}$$

Put $v = u + f(x)$, and we have as before,

$$u = Ae^{\sqrt{\frac{p}{\kappa}}x} + Be^{-\sqrt{\frac{p}{\kappa}}x} - \Sigma \frac{\kappa n^2\pi^2 a_n}{l^2} \frac{\sin \frac{n\pi x}{l}}{\kappa \frac{n^2\pi^2}{l^2} + p}$$

Write $\sqrt{\frac{p}{\kappa}} = s$

$$\therefore \quad u = Ae^{sx} + Be^{-sx} - \Sigma \frac{n^2\pi^2 a_n}{n^2\pi^2 + l^2 s^2} \sin \frac{n\pi x}{l}.$$

Now $\quad -\frac{\partial v}{\partial x} + hv = -\frac{\partial u}{\partial x} + hu - \Sigma \frac{n\pi a_n}{l} \cos \frac{n\pi x}{l} + h\Sigma a_n \sin \frac{n\pi x}{l}.$

At $x = 0$, $\; -\frac{\partial v}{\partial x} + hv = -\frac{\partial u}{\partial x} + hu - \Sigma \frac{n\pi a_n}{l}$

$$\therefore \quad 0 = -As + Bs + \Sigma \frac{n\pi}{l} \frac{n^2\pi^2 a_n}{n^2\pi^2 + l^2 s^2}$$

$$+ hA + hB - \Sigma \frac{n\pi a_n}{l}$$

$$= A(h - s) + B(h + s) - \Sigma \frac{n\pi a_n l s^2}{n^2\pi^2 + l^2 s^2} \qquad (7, b_1)$$

At $x = l$. $\; \frac{\partial v}{\partial x} + hv = \frac{\partial u}{\partial x} + hu + \Sigma \frac{n\pi a_n \cos n\pi}{l}$

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$$\therefore \quad 0 = sAe^{sl} - sBe^{-sl} - \Sigma \frac{n^3\pi^3}{l} \frac{a_n \cos n\pi}{n^2\pi^2 + l^2 s^2}$$

$$+ hAe^{sl} + hBe^{-sl} + \Sigma \frac{n\pi a_n \cos n\pi}{l.}$$

$$= (h+s)e^{sl}A + (h-s)e^{-sl}B$$

$$+ \Sigma \frac{n\pi a_n l s^2}{n^2\pi^2 + l^2 s^2} \cos n\pi \ldots\ldots\ldots (7, b_2)$$

Solving $(7b_1)$ and $(7b_2)$ for A and B we have the equation in u.

$$u = \sum_n \frac{n\pi a_n s^2 l \left\{ \left[(h-s)e^{-sl} + (h+s)\cos n\pi \right] e^{sx} - \left[(h+s)e^{sl} + (h-s)\cos n\pi \right] e^{-sx} \right\}}{(n^2\pi^2 + l^2 s^2)\left\{ (h-s)^2 e^{-sl} - (h+s)^2 e^{sl} \right\}}$$

$$- \sum_n \frac{n^2\pi^2 a_n \sin \frac{n\pi x}{l}}{n^2\pi^2 + l^2 s^2}.$$

Consider $u_1 = -\dfrac{n^2\pi^2 a_n \sin \frac{n\pi x}{l}}{n^2\pi^2 + l^2 s^2}$

with pole at $s^2 = -\dfrac{n^2\pi^2}{l^2}$. $\frac{1}{2} s \dfrac{dZ}{ds} = l^2 s^2 = -n^2\pi^2$.

$$\frac{2\,Y(s)}{s\,Z'(s)} = a_n \sin \frac{n\pi x}{l.}$$

$$\therefore \quad u_1 = -a_n \sin \frac{n\pi x}{l} + a_n \sin \frac{n\pi x}{l} e^{-\kappa \frac{n^2\pi^2}{l^2} t}$$

Now let $u_2 =$

$$\frac{n\pi a_n s^2 l \left[\left\{ (h-s)e^{-sl} + (h+s)\cos n\pi \right\} e^{sx} - \left\{ (h+s)e^{sl} + (h-s)\cos n\pi \right\} e^{-sx} \right]}{(n^2\pi^2 + l^2 s^2)\left\{ (h-s)^2 e^{-sl} - (h+s)^2 e^{sl} \right\}}$$

$$Z(s) = (n^2\pi^2 + l^2 s^2)\left\{ (h-s)^2 e^{-sl} - (h+s)^2 e^{sl} \right\}.$$

Zeros of $Z(s)$ are at $s^2 = -\dfrac{n^2\pi^2}{l}$ and $(h-s)^2 e^{-sl} = (h+s)^2 e^{sl}$.

$$i.e. \quad e^{sl} = \frac{h-s}{h+s}$$

$$\tfrac{1}{2} s Z'(s) = l^2 s^2 \left\{ (h-s)^2 e^{-sl} - (h+s)^2 e^{sl} \right\} + \tfrac{1}{2} s (n^2\pi^2 + l^2 s^2)$$
$$\left[\left\{ -2(h-s) - l(h-s)^2 \right\} e^{-sl} - \left\{ 2(h+s) + l(h+s)^2 \right\} e^{sl} \right]$$
$$= l^2 s^2 \left\{ (h-s)^2 e^{-sl} - (h+s)^2 e^{sl} \right\} - \tfrac{1}{2} s (n^2\pi^2 + l^2 s^2)$$
$$\left\{ (2 + lh - ls)(h-s) e^{-sl} + (2 + lh + ls)(h+s) e^{sl} \right\}.$$

$s^2 = -\frac{n^2\pi^2}{l^2}$ Then $\frac{1}{2} s Z'(s) = -n^2\pi^2\{(h-s)^2+(h+s)^2\} \cos n\pi$

$$= \frac{4ihn^3\pi^3}{l} \cos n\pi.$$

$$Y(s) = 4\pi a_n s^2 l \left[\{(h-s)e^{-sl} + (h+s)\cos n\pi\} e^{sx} - \{(h+s)e^{sl} + (h-s)\cos n\pi\} e^{-sx}\right].$$

$$\therefore \quad Y\left(\frac{in\pi}{l}\right) = -\frac{n^2\pi^2 a_n}{l}\left[2h \cos n\pi\, e^{\frac{in\pi x}{l}} - 2h\cos n\pi e^{\frac{-in\pi x}{l}}\right]$$

$$= -\frac{4ihn^2\pi^2 a_n \cos n\pi}{l} \sin\frac{n\pi x}{l}.$$

$$\therefore \quad \frac{2Y\left(\frac{in\pi}{l}\right)}{\frac{in\pi}{l} Z'\left(\frac{in\pi}{l}\right)} = -a_n \sin\frac{n\pi x}{l}.$$

Also $\frac{Y(0)}{Z(0)} = 0$; therefore term contributed to u_2 by this root of $Z(s) = 0$ cancels the corresponding term in u_1. Now consider the roots given by $e^{sl} = \frac{h-s}{h+s}$, or $e^{-sl} = \frac{h+s}{h-s}$.

$$\text{Then } \tfrac{1}{2} s Z'(s) = \tfrac{1}{2} s (n^2\pi^2 + l^2 s^2)\{(2+lh-ls)(h-s)e^{-sl} + (2+lh+ls)(h+s)e^{sl}\}$$

$$= \tfrac{1}{2} s (n^2\pi^2 + l^2 s^2)\{(2+lh-ls)(h+s) + (2+lh+ls)(h-s)\}$$

$$= s(n^2\pi^2 + l^2 s^2)(2h + lh^2 - ls^2).$$

$$Y(s) = n\pi a_n s^2 l\left[(h+s)(1+\cos n\pi)e^{sx} - (h-s)(1+\cos n\pi)e^{-sx}\right]$$

$$= n\pi a_n s^2 l (1+\cos n\pi)(h+s)(e^{sx} - e^{s(l-x)}).$$

$$\therefore \quad v = \sum_{n=1}^{\infty} \sum_{s} n\pi a_n l (1+\cos n\pi) \frac{s(h+s)\{e^{sx} - e^{s(l-x)}\}}{(n^2\pi^2 + l^2 s^2)(2h + lh^2 - ls^2)} e^{\kappa s^2 t} \quad \ldots\ldots(7)$$

where the summation for s is extended over all the roots of the transcendental equation

$$e^{sl} = \frac{h-s}{h+s}.$$

§ 8. The solution of the problem of § 7 is usually given in the form

$$v=\sum_n A_n\left(\cos\alpha_n x+\frac{h}{\alpha_n}\sin\alpha_n x\right)e^{-\kappa\alpha_n^2 t}\ldots\ldots\ldots\ldots\ldots(8)$$

where α_n is a root of the equation

$$\tan\alpha l=\frac{2\alpha h}{\alpha^2-h^2}$$

and the summation extends over all such roots. Further $f(x)$ is expanded as a series

$$f(x)=\sum_n A_n\left(\cos\alpha_n x+\frac{h}{a_n}\sin\alpha_n x\right)$$

$$\text{where } A_n=\frac{2\alpha_n^2}{2h+lh^2+l\alpha_n^2}\int_0^l f(x')\left(\cos\alpha_n x'+\frac{h}{a_n}\sin\alpha_n x'\right)dx'.$$

To show the identity of the two solutions, put $s=i\alpha$ in (7),

$$\text{then } v=\sum_n\sum_\alpha\frac{n\pi a_n l(1+\cos n\pi)\,i\alpha\,(h+i\alpha)\left\{e^{i\alpha x}-e^{i\alpha(l-x)}\right\}}{(n^2\pi^2-l^2\alpha^2)(2h+lh^2+l\alpha^2)}e^{-\kappa\alpha^2 t},$$

and the equation for s becomes

$$e^{i\alpha l}=\frac{h-i\alpha}{h+i\alpha}$$

$$\therefore\quad\tan\alpha l=\frac{2\alpha h}{\alpha^2-h^2}.$$

$$\text{Also } v=\sum_n\sum_\kappa\frac{n\pi a_n l(1+\cos n\pi)\,i\alpha}{(n^2\pi^2-l^2\alpha^2)(2h+lh^2+l\alpha^2)}\left\{(h+i\alpha)e^{i\alpha x}-(h-i\alpha)e^{-i\alpha x}\right\}e^{-\kappa\alpha^2 t}$$

$$=\sum_n\sum_\alpha\frac{n\pi a_n l(1+\cos n\pi)\,i\alpha}{(n^2\pi^2-l^2\alpha^2)(2h+lh^2+l\alpha^2)}(2ih\sin\alpha x+2i\alpha\cos\alpha x)e^{-\kappa\alpha^2 t}$$

$$=\sum_n\sum_\alpha-\frac{2n\pi a_n l(1+\cos n\pi)\alpha^2}{(n^2\pi^2-l^2\alpha^2)(2h+lh^2+l\alpha^2)}\left(\cos\alpha x+\frac{h}{\alpha}\sin\alpha x\right)e^{-\kappa\alpha^2 t}\ldots\ldots\ldots(7')$$

In the equation for A_n substitute the value of $f(x)$ given by equation (7c).

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Then $$A_n = \frac{2\alpha_n^2}{2h + lh^2 + l\alpha_n^2}\int_0^l \left(\Sigma a_r \sin\frac{r\pi x}{l}\right)\left(\cos\alpha_n x + \frac{h}{\alpha_n}\sin\alpha_n x\right)dx$$

$$= \Sigma\frac{\alpha_n^2 a_r}{2h + lh^2 + l\alpha_n^2}\int_0^l\left\{\sin\left(\frac{r\pi}{l} + \alpha_n\right)x + \sin\left(\frac{r\pi}{l} - \alpha_n\right)x + \frac{h}{\alpha_n}\cos\left(\frac{r\pi}{l} - \alpha_n\right)x - \frac{h}{\alpha_n}\cos\left(\frac{r\pi}{l} + \alpha_n\right)x\right\}dx$$

$$= \sum_r \frac{\alpha_n^2 a_r l}{2h + lh^2 + l\alpha_n^2}\,\frac{2r\pi}{r^2\pi^2 - \alpha_n^2\lambda^2}\left\{1 - \cos r\pi\left(\cos\alpha_n l + \frac{h}{\alpha_n}\sin a_n l\right)\right\}$$

$$= \sum_r \frac{2\alpha_n^2 a_r l n\pi\,(1 + \cos n\pi)}{(2h + lh^2 + l\alpha_n^2)(r^2\pi^2 - \alpha_n^2 l^2)}.$$

Substituting this value of A_n in (8) we have

$$v = \sum_r\sum_\alpha \frac{2\alpha^2 a_r r\pi l\,(1 + \cos r\pi)}{(2h + lh^2 + l\alpha^2)(r^2\pi^2 - \alpha^2 l^2)}\left(\cos\alpha x + \frac{h}{\alpha}\sin\alpha x\right)e^{-\kappa\alpha^2 t},$$

which is identical with the form (7′).

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