A GENERALIZATION OF THE INEQUALITY OF THE ARITHMETIC-GEOMETRIC MEANS

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§1. *Introduction.* The main result in this paper, contained in Theorem 1, is a generalisation of the inequality of the arithmetic-geometric means. A result of a similar character has been proved by Siegel (2). The present result gives an improvement in the inequality in the case when the variables involved are not all distinct, whereas Siegel's result does not. The theorem is used in § 3 to obtain a result in connection with totally real and positive algebraic integers.

§ 2. *The main result.* Let $x_1, \ldots, x_n$ be real and positive and write

$$A = \sum_{i<j} (x_i - x_j)^2, \; ns = \sum_{i=1}^{n} x_i,$$

and $p = \prod_{i=1}^{n} x_i$. Then $A = 0$ if and only if $x_1 = x_2 = \ldots = x_n$, and then $\frac{s^n}{p} = 1$, since we have assumed each $x_i > 0$. In what follows we exclude the case $A = 0$.

THEOREM 1. *If β denotes the root in the interval $0 < \beta < 1$ of the equation*

$$A = \beta^2 (n-1)(ns)^2, \qquad \text{(1)}$$

then

$$\frac{s^n}{p} \geqslant \frac{1}{\{1 + \beta(n-1)\}(1-\beta)^{n-1}}. \qquad \text{(2)}$$

We note:

1. Since $A = (n-1)(\Sigma x_i)^2 - 2n \Sigma x_i x_j = (n-1)(ns)^2 - 2n\Sigma x_i x_j$, we have

$$0 < \frac{A}{(n-1)(ns)^2} < 1,$$

so that β is uniquely determined in the interval $0 < \beta < 1$.

2. If
$$f(\beta) = \frac{1}{\{1 + \beta(n-1)\}(1-\beta)^{n-1}},$$

then, in $0 < \beta < 1$, for $n \geqslant 2$, $f(\beta)$ is a steadily increasing function of β. For,

$$\frac{df}{d\beta} = \frac{\beta n (n-1)}{\{1 + \beta(n-1)\}^2 (1-\beta)^n} > 0, \quad \text{for} \quad 0 < \beta < 1.$$

Also $f(0) = 1$, so that $f(\beta) > 1$ for $0 < \beta < 1$.

3. For $n = 2$, (1) becomes $\beta^2 = \frac{(x_1 - x_2)^2}{(2s)^2} = \frac{(x_1 - x_2)^2}{(x_1 + x_2)^2}$, and the right-hand side of (2) equals

$$\frac{1}{1 - \beta^2} = \frac{1}{1 - \frac{(x_1 - x_2)^2}{(x_1 + x_2)^2}} = \frac{\left(\frac{x_1 + x_2}{2}\right)^2}{x_1 x_2} = \frac{s^2}{p},$$

so that the result is true with equality for $n = 2$. We can thus assume that $n \geqslant 3$.

For the proof we use the following lemma:

LEMMA 1. *If*

$$A = \sum_{i<j} (x_i - x_j)^2,$$

where
$$\sum_{i=1}^{n} x_i = ns, \quad \text{.........................(3)}$$

$$\prod_{i=1}^{n} x_i = p, \quad \text{.........................(4)}$$

with
$$\frac{s^n}{p} > 1,$$

and
$$x_i > 0 \quad (i = 1, \ldots, n), \quad \text{.........................(5)}$$

then
$$A \leqslant n^2(n-1)\alpha^2 s^2,$$

where
$$\{1 + \alpha(n-1)\}(1-\alpha)^{n-1} = \frac{p}{s^n}, \quad \textit{and} \quad 0 < \alpha < 1.$$

For a given set of positive values of $x_3, \ldots, x_n$, the equations

$$x_1 + x_2 = ns - (\sum_{i=3}^{n} x_i), \quad x_1 x_2 = p(\prod_{i=3}^{n} x_i)^{-1}$$

completely determine x_1, x_2 as the roots of the equation

$$y^2 - \{ns - \sum_{i=3}^{n} x_i\}y + p(\prod_{i=3}^{n} x_i)^{-1} = 0.$$

The values of x_1, x_2 are both positive if and only if the following two conditions are satisfied

$$\sum_{i=3}^{n} x_i < ns, \quad \{ns - \sum_{i=3}^{n} x_i\}^2 \geqslant 4p(\prod_{i=3}^{n} x_i)^{-1};$$

also x_1, x_2 are unequal, except when equality arises in the second condition. The two conditions define a closed domain D in R_{n-2}, its points having positive co-ordinates $x_3, \ldots, x_n$. The boundary of D is given by $x_1 = x_2$.

Now A attains its maximum in D, and $A = 0$ if and only if $x_1 = x_2 = \ldots = x_n$, so that at the maximum not all of the x_i are equal. Hence, by symmetry, we can suppose that A attains its maximum in D at an inner point of D. The values of x_i at this maximum must satisfy the equations

$$\frac{\partial A}{\partial x_i} + \lambda + \frac{\mu}{x_i} = 0 \quad (i = 1, \ldots, n),$$

for some λ, μ;

that is
$$2n(x_i - s) + \lambda + \frac{\mu}{x_i} = 0 \quad (i = 1, \ldots, n);$$

that is
$$x_i^2 + \left(\frac{\lambda}{2n} - s\right) x_i + \frac{\mu}{2n} = 0 \quad (i = 1, \ldots, n),$$

and so the x_i satisfy a quadratic equation. Hence at the maximum we have k of the x_i equal in value to x, say, and the remaining $n-k$ of the x_i equal to a second number y, say, where, by symmetry, k is one of the integers $0, 1, \ldots\ldots, \left[\frac{n}{2}\right]$, and where, from (3) and (4),

$$kx + (n-k)y = ns, \quad \text{.........................(6)}$$

and
$$x^k y^{n-k} = p. \quad \text{.........................(7)}$$

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The corresponding value of A is

$$A = k(n-k)(x-y)^2. \qquad \text{.......(8)}$$

For $k=0$, $A=0$ and we require $y = s = p^{\frac{1}{n}}$, i.e. $\frac{s^n}{p} = 1$, which we have excluded. Thus we have to consider only $k \geqslant 1$. From (6),

$$k(x-y) = n(s-y).$$

Thus (8) becomes

$$A = k(n-k)\frac{n^2}{k^2}(s-y)^2$$

$$= n^2 s^2 \left(\frac{n}{k} - 1\right)(1-u)^2, \quad \text{where } y = us,$$

$$= n^2 s^2 \alpha^2 \left(\frac{n}{k} - 1\right), \quad \text{where } \alpha = 1 - u, \qquad \text{.......(9)}$$

so that $y = (1-\alpha)s$.

Now, from (6) and (5), $\qquad 0 < y < \frac{n}{n-k} s.$

Thus $\qquad 0 < 1 - \alpha < \frac{n}{n-k},$

and so $\qquad \frac{-k}{n-k} < \alpha < 1. \qquad \text{.......(10)}$

From (6), $\qquad x = s\left\{\frac{n}{k} - \left(\frac{n}{k} - 1\right)u\right\}.$

Thus, from (7), $\qquad \left\{\frac{n}{k} - \left(\frac{n}{k} - 1\right)u\right\}^k u^{n-k} = \frac{p}{s^n},$

and so $\qquad \left\{1 + \alpha\left(\frac{n}{k} - 1\right)\right\}^k (1-\alpha)^{n-k} = \frac{p}{s^n},$

where, by (10), we have to consider the roots of this polynomial in α in the interval

$$-1 \leqslant \frac{-k}{n-k} < \alpha < 1,$$

the -1 arising when n is even, and $k = \left[\frac{n}{2}\right]$.

Let $g(\alpha) = \left\{1 + \alpha\left(\frac{n}{k} - 1\right)\right\}^k (1-\alpha)^{n-k} - \frac{p}{s^n}$, and consider this function in the interval $\left[\frac{-k}{n-k}, 1\right]$. We have $g(1) = g\left(\frac{-k}{n-k}\right) = \frac{-p}{s^n}$, so that

$$0 > \left\{\begin{matrix} g(1) \\ g\left(\frac{-k}{n-k}\right) \end{matrix}\right\} > -1.$$

Also $g(0) = 1 - \frac{p}{s^n}$, so that $0 < g(0) < 1$. Further

$$g'(\alpha) = (n-k)\left\{1 + \alpha\left(\frac{n}{k} - 1\right)\right\}^{k-1} (1-\alpha)^{n-k-1}\left(-\frac{n}{k}\alpha\right).$$

Thus $g'(\alpha) = 0$ at $\alpha = 0$, $\alpha = 1$ (if $n \geqslant 3$), and $\alpha = -k/(n-k)$ if $k \geqslant 2$. Also

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$$g'(\alpha)\begin{cases} >0 & \text{at} \quad \alpha=\dfrac{-k}{n-k} \quad \text{if } k=1, \\ >0 & \text{for} \quad \dfrac{-k}{n-k}<\alpha<0, \\ <0 & \text{for} \quad 0<\alpha<1, \end{cases}$$

since $k=1,2,\ldots,\left[\frac{n}{2}\right]$. Hence $g(\alpha)$ increases steadily from $g\left(\frac{-k}{n-k}\right)<0$ to $g(0)>0$ as α increases from $\frac{-k}{n-k}$ to 0, and decreases steadily from $g(0)>0$ to $g(1)<0$ as α increases from 0 to 1. Thus $g(\alpha)=0$ has one root $-\alpha_1$, say, where $\alpha_1>0$, in $\frac{-k}{n-k}<\alpha<0$ and one root α_2, say, in $0<\alpha<1$. We show, by the following lemma, that we need consider only α_2 in finding the maximum of A.

Lemma 2. *If $-\alpha_1$ and α_2 are the roots of the equation*

$$\left\{1+\alpha\left(\frac{n}{k}-1\right)\right\}^k(1-\alpha)^{n-k}=\frac{p}{s^n}$$

in the intervals $\frac{-k}{n-k}<\alpha<0$ and $0<\alpha<1$, respectively, where $k=1,2,\ldots,\left[\frac{n}{2}\right]$, then

$$\alpha_2\geqslant\alpha_1.$$

We have

$$\left\{1-\alpha_1\left(\frac{n}{k}-1\right)\right\}^k(1+\alpha_1)^{n-k}=\frac{p}{s^n}=\left\{1+\alpha_2\left(\frac{n}{k}-1\right)\right\}^k(1-\alpha_2)^{n-k}.$$

Since, by above, $g'(\alpha)<0$ for $0<\alpha<1$,

$$\left\{1+\alpha\left(\frac{n}{k}-1\right)\right\}^k(1-\alpha)^{n-k}-\frac{p}{s^n}>0 \quad \text{for} \quad 0<\alpha<\alpha_2.$$

Thus, to show that $\alpha_2\geqslant\alpha_1$, it is sufficient to show that

$$\left\{1+\alpha_1\left(\frac{n}{k}-1\right)\right\}^k(1-\alpha_1)^{n-k}\geqslant\left\{1+\alpha_2\left(\frac{n}{k}-1\right)\right\}^k(1-\alpha_2)^{n-k}=\left\{1-\alpha_1\left(\frac{n}{k}-1\right)\right\}^k(1+\alpha_1)^{n-k}.$$

For simplicity of notation put $m=\frac{n}{k}-1$. Then $k=\frac{n}{m+1}$, $n-k=\frac{nm}{m+1}$, and $0<\alpha_1<\frac{1}{m}$, $1\leqslant m\leqslant n-1$. We have then to show that

$$\{(1+\alpha_1 m)(1-\alpha_1)^m\}^{\frac{n}{m+1}}\geqslant\{(1-\alpha_1 m)(1+\alpha_1)^m\}^{\frac{n}{m+1}},$$

and so that

$$(1+\alpha_1 m)(1-\alpha_1)^m\geqslant(1-\alpha_1 m)(1+\alpha_1)^m,$$

where $-\alpha_1$ is the given root such that $0<\alpha_1<\frac{1}{m}$, and $1\leqslant m\leqslant n-1$.

Let

$$h(\alpha)=(1+\alpha m)(1-\alpha)^m-(1-\alpha m)(1+\alpha)^m.$$

Then

$$\frac{1}{m(m+1)}h'(\alpha)=\alpha\{(1+\alpha)^{m-1}-(1-\alpha)^{m-1}\}$$

$$>0 \quad \text{for } 0<\alpha\leqslant 1, \quad \text{and so for } 0<\alpha\leqslant\frac{1}{m}, \text{ if } m>1,$$

$$\equiv 0, \quad \text{if } m=1.$$

Also $h(0)=0$. Hence $h(\alpha_1)>0$ if $m>1$, and $h(\alpha_1)=0$ if $m=1$. Thus

$$(1+\alpha_1 m)(1-\alpha_1)^m \geqslant (1-\alpha_1 m)(1+\alpha_1)^m,$$

with equality if and only if $m=1$, and so

$$\left\{1+\alpha_1\left(\frac{n}{k}-1\right)\right\}^k (1-\alpha_1)^{n-k} - \frac{p}{s^n} \geqslant \left\{1+\alpha_2\left(\frac{n}{k}-1\right)\right\}^k (1-\alpha_2)^{n-k} - \frac{p}{s^n} = 0,$$

giving

$$\alpha_2 \geqslant \alpha_1.$$

By this lemma, for each fixed n and k,

$$n^2 s^2 \left(\frac{n}{k}-1\right)\alpha_2{}^2 \geqslant n^2 s^2 \left(\frac{n}{k}-1\right)\alpha_1{}^2.$$

Hence, by (9), in finding the maximum of A under the given conditions, we have for each possible k, for any given n, to consider for α only the unique root in the range $0<\alpha<1$ of the equation

$$\left\{1+\alpha\left(\frac{n}{k}-1\right)\right\}^k (1-\alpha)^{n-k} = \frac{p}{s^n}.$$

We show, by the following lemma, that $k=1$ gives the maximum for each n.

Lemma 3. *If*

$$u_k = \alpha^2\left(\frac{n}{k}-1\right),$$

where α is the root in $0<\alpha<1$ of the equation

$$\left\{1+\alpha\left(\frac{n}{k}-1\right)\right\}^k (1-\alpha)^{n-k} = \frac{p}{s^n},$$

and $k=1, 2, \dots, \left[\frac{n}{2}\right]$, then

$$\max(u_1, u_2, \dots, u_{[n/2]}) = u_1.$$

As in Lemma 2 we put $m=\frac{n}{k}-1$, so that $k=\frac{n}{m+1}$, $n-k=\frac{nm}{m+1}$. Then

$$(1+\alpha m)^{\frac{n}{m+1}}(1-\alpha)^{\frac{m}{m+1}n} = \frac{p}{s^n},$$

and so

$$(1+\alpha m)^{\frac{1}{m+1}}(1-\alpha)^{\frac{m}{m+1}} = \left(\frac{p}{s^n}\right)^{\frac{1}{n}}, \quad \text{.............................(11)}$$

where $0<\alpha<1$, and $1 \leqslant m \leqslant n-1$. We have then to consider the maximum of

$$U = \alpha^2 m,$$

under these conditions.

Now

$$\frac{dU}{dm} = \alpha^2 + 2\alpha m\frac{d\alpha}{dm} = \alpha\left\{\alpha + 2m\frac{d\alpha}{dm}\right\}. \quad \text{.............................(12)}$$

Also, from (11).

$$\frac{1}{m+1}\log(1+\alpha m) + \frac{m}{m+1}\log(1-\alpha) = \frac{1}{n}\log\frac{p}{s^n},$$

Thus

$$\frac{-1}{(m+1)^2}\log(1+\alpha m) + \frac{\alpha}{(m+1)(1+\alpha m)} + \frac{1}{(m+1)^2}\log(1-\alpha) + \left\{\frac{1}{1+\alpha m} - \frac{1}{1-\alpha}\right\}\frac{m}{m+1}\frac{d\alpha}{dm} = 0.$$

Hence

$$2m\frac{d\alpha}{dm} = \frac{2(1+\alpha m)(1-\alpha)}{\alpha}\left\{\frac{\alpha}{(m+1)(1+\alpha m)} + \frac{1}{(m+1)^2}\log\frac{1-\alpha}{1+\alpha m}\right\}.$$

Therefore

$$\alpha + 2m\frac{d\alpha}{dm} = \alpha + \frac{2(1-\alpha)}{m+1} + \frac{2(1+\alpha m)(1-\alpha)}{\alpha(m+1)^2}\log\frac{1-\alpha}{1+\alpha m}$$

$$= \frac{2(1+\alpha m)(1-\alpha)}{\alpha(m+1)^2}\left\{\frac{(1+\alpha m)^2-(1-\alpha)^2}{2(1+\alpha m)(1-\alpha)} + \log\frac{1-\alpha}{1+\alpha m}\right\} \qquad (13)$$

Consider
$$B = \frac{(1+\alpha m)^2-(1-\alpha)^2}{2(1+\alpha m)(1-\alpha)} + \log\frac{1-\alpha}{1+\alpha m}.$$

Put
$$t = \frac{1+\alpha m}{1-\alpha}.$$

Then $t > 1$, since $0 < \alpha < 1$ and $m \geqslant 1$, and

$$B = B(t) = \frac{t^2-1}{2t} - \log t = \tfrac{1}{2}\left(t - \frac{1}{t}\right) - \log t.$$

Now
$$\frac{dB}{dt} = \frac{1}{2t^2}(t-1)^2 > 0 \quad \text{for } t > 1, \text{ and } B(1) = 0.$$

Thus
$$B > 0 \quad \text{for } t > 1.$$

Hence, from (13), $\alpha + 2m\frac{d\alpha}{dm} > 0$, and so, from (12), $\frac{dU}{dm} > 0$, under the given conditions. Hence $\frac{dU}{dk} < 0$ and so U is a maximum when $k = 1$, which completes the lemma.

By this lemma, $n^2s^2\left(\frac{n}{k}-1\right)\alpha^2$ is a maximum when $k=1$, α being the unique root in $0<\alpha<1$ of the equation

$$\left\{1+\alpha\left(\frac{n}{k}-1\right)\right\}^k(1-\alpha)^{n-k} = \frac{p}{s^n};$$

that is
$$\max A = n^2(n-1)s^2\alpha^2,$$

where α is the unique root in $0<\alpha<1$ of the equation

$$\{1+\alpha(n-1)\}(1-\alpha)^{n-1} = \frac{p}{s^n}.$$

Lemma 1 now follows.

We can now prove Theorem 1. Let

$$A_0 = (ns)^2(n-1)\alpha^2,$$

where α is the root in $0<\alpha<1$ of the equation

$$\frac{s^n}{p} = \frac{1}{\{1+\alpha(n-1)\}(1-\alpha)^{n-1}} = f(\alpha).$$

Let $x_1, \dots, x_n$ be an arbitrary set of positive numbers such that $\Sigma x_i = ns$, and $\Pi x_i = p$. Then, by Lemma 1,

$$A = \sum_{i<j}(x_i-x_j)^2 \leqslant A_0.$$

As noted earlier there is a unique number β in $0<\beta<1$ such that

$$\frac{A}{(ns)^2(n-1)} = \beta^2.$$

But $\frac{A_0}{(ns)^2(n-1)} = \alpha^2$. Hence $\beta \leqslant \alpha$.

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Now $f(\alpha)$ is monotonic increasing in $0<\alpha<1$. Therefore

$$\frac{s^n}{p} \geqslant \frac{1}{\{1+\beta(n-1)\}(1-\beta)^{n-1}},$$

where β is the root in $0<\beta<1$ of the equation

$$\frac{A}{(ns)^2(n-1)} = \beta^2,$$

which is the required result.

§3. *Application of Theorem* 1. For the application we now suppose that $x_1, \ldots, x_n$ are the roots of an irreducible polynomial equation $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, with rational integral coefficients $a_1, \ldots, a_n$, so that $x_1, \ldots, x_n$ are the conjugates of a totally real and positive algebraic integer.

Write
$$\sum_{i<j} x_i x_j = \binom{n}{2} s_2.$$

Then

$$A = (n-1)(ns)^2 - 2n\Sigma x_i x_j = (n-1)n^2 s^2 - n^2(n-1)s_2 = n^2(n-1)(s^2 - s_2).$$

Theorem 1 can be written :

$$\frac{s^n}{p} \geqslant \frac{1}{\{1+\beta(n-1)\}(1-\beta)^{n-1}},$$

where β is the root in $0<\beta<1$ of the equation

$$s_2 = (1-\beta^2)s^2.$$

Since $\prod_{i=1}^{n} x_i = (-1)^n a_n = p$ is a positive rational integer, $p \geqslant 1$, and so

$$s^n \geqslant \frac{1}{\{1+\beta(n-1)\}(1-\beta)^{n-1}}.$$

Also
$$s_2 \geqslant \frac{1-\beta^2}{\{1+\beta(n-1)\}^{\frac{2}{n}}(1-\beta)^{2-\frac{2}{n}}} = \frac{1+\beta}{\{1+\beta(n-1)\}^{\frac{2}{n}}(1-\beta)^{1-\frac{2}{n}}} = f_1(\beta),$$

say. Now, if $n>2$ and $0<\beta<1$,

$$\frac{df_1}{d\beta} = \frac{2\beta(n-2)}{\{1+\beta(n-1)\}^{1+\frac{2}{n}}(1-\beta)^{2-\frac{2}{n}}} > 0.$$

Also
$$f_1(0) = 1.$$

Thus $s_2 > 1$ for $0<\beta<1$ if $n>2$; and $f_1(\beta) = 1$ for $n=2$. Hence $s_2 \geqslant 1$, which, of course, can be established in other ways.

From a result for s under the present assumptions, obtained by Siegel ((2), Theorem II), we deduce the corresponding result for s_2. Siegel's result is :

Let θ be the positive root of the equation

$$(1+\theta)\log\left(1+\frac{1}{\theta}\right) - \frac{\log\theta}{1+\theta} = 1,$$

and
$$\lambda_0 = e\left(1+\frac{1}{\theta}\right)^{-\theta}.$$

Then, if λ is a real number satisfying $1<\lambda<\lambda_0=1{\cdot}7336\ldots$, there is a positive integer $N=N(\lambda)$ such that $s>\lambda$, for all $n\geqslant N$.

We prove:

THEOREM 2. *If λ satisfies $1<\lambda<\lambda_0$, there exists a positive integer $N=N(\lambda)$ such that*

$$s_2>\lambda, \quad \textit{for all } n\geqslant N.$$

We note:

1. In each of the above we can take $\lambda>1$. For, $s\geqslant 1$ and $s_2\geqslant 1$, and equality arises (for $n\geqslant 2$ in the case $s=1$ and for $n>2$ in the case $s_2=1$) if and only if $x_1=x_2=\ldots=x_n$, in which case the equation $x^n+a_1x^{n-1}+\ldots+a_n=0$ is reducible.

2. For every odd prime p, $4\cos^2\dfrac{\pi}{p}$ is a totally real and positive algebraic integer of degree $n=\frac{1}{2}(p-1)$, the corresponding equation being

$$x^n-(2n-1)x^{n-1}+(n-1)(2n-3)x^{n-2}-\tfrac{1}{3}(n-2)(2n-3)(2n-5)x^{n-3}+\ldots=0. \tag{14}$$

Thus, for this particular case, $\Sigma x_ix_j=(n-1)(2n-3)=\binom{n}{2}4\left(1-\dfrac{3}{2n}\right)$, so that $s_2=4\left(1-\dfrac{3}{2n}\right)$.

Hence, if μ is the best-possible constant in Theorem 2, $\lambda_0\leqslant\mu\leqslant 4$. It is fairly clear that $\mu>\lambda_0$.

Proof of Theorem 2. Choose λ_1 with $\lambda<\lambda_1<\lambda_0$ and ϵ in $0<\epsilon<1$ such that $\lambda=\lambda_1^{1-\epsilon}$. Now

$$s_2=(1-\beta^2)s^{1+\epsilon}s^{1-\epsilon}\geqslant\frac{1+\beta}{\{1+\beta(n-1)\}^{\frac{1+\epsilon}{n}}(1-\beta)^{\epsilon-\frac{1+\epsilon}{n}}}s^{1-\epsilon}$$

$$=f_2(\beta)s^{1-\epsilon}, \tag{15}$$

where

$$f_2(\beta)=\frac{1+\beta}{\{1+\beta(n-1)\}^{\frac{1+\epsilon}{n}}(1-\beta)^{\epsilon-\frac{1+\epsilon}{n}}}.$$

Take N_1 such that $\epsilon-\dfrac{1+\epsilon}{n}>0$, that is $\epsilon>\dfrac{1}{n-1}$, for $n\geqslant N_1$. For such an n consider $f_2(\beta)$ in $0<\beta<1$.

$$\frac{df_2}{d\beta}=\frac{\beta(n-1)(1-\epsilon)}{\{1+\beta(n-1)\}^{1+\frac{1+\epsilon}{n}}(1-\beta)^{1+\epsilon-\frac{1+\epsilon}{n}}}\left\{(1-\beta)+\frac{2}{1-\epsilon}\left(\epsilon-\frac{1}{n-1}\right)\right\}>0$$

in $0<\beta<1$, since $\epsilon-\dfrac{1}{n-1}>0$. Now $f_2(0)=1$. Thus $f_2(\beta)>1$ in $0<\beta<1$. Hence, from (15),

$$s_2>s^{1-\epsilon}, \quad \text{for } n\geqslant N_1.$$

By Siegel's result, $\quad s>\lambda_1, \quad$ for $n\geqslant N_2$, say.

Thus $\quad s_2>\lambda_1^{1-\epsilon} \quad$ for $n\geqslant N=\max(N_1, N_2)$;

that is $\quad s_2>\lambda \quad$ for $n\geqslant N$.

§ 4. *Inequalities for* $\dfrac{A}{n^2}$ *and* $\dfrac{\Sigma x_i^2}{n}$. An inequality for $\dfrac{A}{n^2}=\dfrac{1}{n^2}\sum_{i<j}(x_i-x_j)^2$ can be deduced from a result due to Schur (1). The inequality can be stated in the form:

If $x_1,\ldots,x_n$ are the conjugates of a totally real algebraic integer, and if λ is such that

$$0<\lambda<e^{\frac{1}{2}}=1{\cdot}6487\ldots,$$

then there is an integer $N=N(\lambda)$ such that $\dfrac{A}{n^2}>\lambda$ for $n\geqslant N$.

For completeness we include a proof of this result. The proof is based on the following lemmas:

LEMMA 4. *If* $\Delta(x_1, \dots, x_n) \equiv \prod_{i<j} (x_i - x_j)^2$, *where the* x_i *are real numbers such that* $\sum_{i=1}^{n} x_i^2 \leqslant B$, *then*

$$\max \Delta(x_1, \dots, x_n) = R_n \left(\frac{B}{n^2-n}\right)^{\frac{1}{2}(n^2-n)}, \quad \textit{where } R_n = \prod_{m=1}^{n} m^m.$$

This follows from Schur (**1**, §2).

LEMMA 5.

$$R_n = \left(\frac{n}{e^{\frac{1}{2}}}\right)^{\frac{1}{2}(n^2-n)} n^{n+\frac{1}{12}} \exp\{-\tfrac{1}{4}n + O(1)\}.$$

This can be obtained by applying the Euler summation formula

$$\sum_{m=1}^{n} f(m) = \tfrac{1}{2}\{f(n) - f(0)\} + \int_0^n f(x)\,dx + \int_0^n f'(x)(x - [x] - \tfrac{1}{2})\,dx$$

to the function $f(x) = x \log x$.

Now $\frac{A}{n^2} = \frac{1}{n^2} \sum_{i<j} (x_i - x_j)^2 = \frac{1}{n^2}\{n\Sigma x_i^2 - (\Sigma x_i)^2\} = \frac{1}{n} \sum_{i=1}^{n} (x_i - s)^2$. By Lemma 4, $\sum_{i=1}^{n} (x_i - s)^2 \leqslant n\lambda$ implies $\Delta(x_1 - s, \dots, x_n - s) \leqslant R_n \left(\frac{n\lambda}{n^2-n}\right)^{\frac{1}{2}(n^2-n)}$. Now $\Delta(x_1 - s, \dots, x_n - s) = \Delta(x_1, \dots, x_n) \geqslant \left(\frac{n^n}{n!}\right)^2$, by Minkowski's discriminant inequality. Hence

$$1 \leqslant \left(\frac{n!}{n^n}\right)^2 R_n \left(\frac{n\lambda}{n^2-n}\right)^{\frac{1}{2}(n^2-n)}.$$

Thus, by Lemma 5 and Stirling's approximation for $n!$,

$$1 < Ce^{-\frac{7n}{4}} n^{n+\frac{13}{12}} \left(\frac{\lambda}{e^{\frac{1}{2}}}\right)^{\frac{1}{2}(n^2-n)}, \qquad (16)$$

where C is a constant. Since $\frac{\lambda}{e^{\frac{1}{2}}} < 1$, the right-hand side of (16) is less than 1 for sufficiently large n. Hence (16) implies that $n < N = N(\lambda)$, say. Thus

$$\frac{1}{n} \sum_{i=1}^{n} (x_i - s)^2 > \lambda \quad \text{for } n \geqslant N,$$

and so

$$\frac{A}{n^2} > \lambda \quad \text{for } n \geqslant N.$$

In the earlier notation, this result can be expressed in the form

$$(n-1)(s^2 - s_2) > \lambda \quad \text{for } n \geqslant N,$$

and in the form

$$(n-1)\beta^2 s^2 > \lambda \quad \text{for } n \geqslant N.$$

We note that the result holds, in particular, when $x_1, \dots, x_n$ are the conjugates of a totally real and positive algebraic integer. By (14), the best-possible constant in this case is μ, where $1{\cdot}6487 \dots \leqslant \mu \leqslant 2$.

In the totally real and positive case we deduce :

If $x_1, \dots, x_n$ *are the conjugates of a totally real and positive algebraic integer and if* λ *is such that* $1 < \lambda < e^{\frac{1}{2}} + \lambda_0^2 = 4{\cdot}654 \dots$, *then there is an integer* $N = N(\lambda)$ *such that* $\frac{\Sigma x_i^2}{n} > \lambda$ *for* $n \geqslant N$.

Write $\lambda = \lambda_1 + \lambda_2^2$, where $0 < \lambda_1 < e^{\frac{1}{2}}$ and $1 < \lambda_2 < \lambda_0$. We have

$$\frac{\Sigma x_i^2}{n} = \frac{A}{n^2} + \left(\frac{\Sigma x_i}{n}\right)^2.$$

Now $$\frac{A}{n^2} > \lambda_1 \quad \text{for } n \geqslant N_1, \text{ say,}$$

and, by Siegel's result, $$\frac{\Sigma x_i}{n} > \lambda_2 \quad \text{for } n \geqslant N_2, \text{ say.}$$

Hence $$\frac{\Sigma x_i^2}{n} > \lambda \quad \text{for } n \geqslant \max(N_1, N_2).$$

Schur gave this result with $e^{\frac{1}{2}} + e = 4{\cdot}367 \ldots$ in place of $4{\cdot}654 \ldots$. By (14), the best-possible constant in this case is μ, where $4{\cdot}654 \ldots \leqslant \mu \leqslant 6$.

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