Canad. J. Math. Vol. **61** (3), 2009 pp. 708–720

Regular Homeomorphisms of Finite Order on Countable Spaces

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Abstract. We present a structure theorem for a broad class of homeomorphisms of finite order on countable zero dimensional spaces. As applications we show the following.

(a) Every countable nondiscrete topological group not containing an open Boolean subgroup can be partitioned into infinitely many dense subsets.

(b) If G is a countably infinite Abelian group with finitely many elements of order 2 and βG is the Stone–Čech compactification of G as a discrete semigroup, then for every idempotent $p \in \beta G \setminus \{0\}$, the subset $\{p, -p\} \subset \beta G$ generates algebraically the free product of one-element semigroups $\{p\}$ and $\{-p\}$.

1 Introduction

Let X be a topological space with a distinguished point $e \in X$ and let $f\colon X \to X$ be a homeomorphism with $f(e) = e$. We say that f is *regular* if for every $x \in X \setminus \{e\}$, there is a homeomorphism g_x of a neighborhood of e onto a neighborhood of x such that $fg_x|_U = g_{f(x)}f|_U$ for some neighborhood U of e. All spaces are assumed to be Hausdorff. Notice that if a space X admits a regular homeomorphism, then for any two points $x, y \in X$, there is a homeomorphism g of a neighborhood of x onto a neighborhood of y with $g(x) = y$, and if in addition X is countable and zero dimensional (*i.e.,* has a base of clopen sets), then g can be chosen to be a homeomorphism of X onto itself. Hence, a countable zero dimensional space admitting a regular homeomorphism is homogeneous. The notion of a regular homeomorphism generalizes that of a local automorphism on a local left group [7].

A space X with a partial binary operation $\cdot$ and a distinguished point $e \in X$ is a *local left group* if

(i) $x \cdot e = x$ for all $x \in X$,

(ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ whenever all the products in the equality are defined, and

(iii) for every $x \in X \setminus \{e\}$, there is a neighborhood U of e such that $x \cdot y$ is defined for all $y \in U$, $x \cdot U$ is a neighborhood of x and $\lambda_x\colon U \ni y \mapsto x \cdot y \in x \cdot U$ is a homeomorphism.

A basic example of a local left group is an open neighborhood of the identity of a left topological group. (A group endowed with a topology is *left topological* if all left

Received by the editors July 4, 2006; revised January 25, 2007.

The author was supported by NRF grant FA2007041200005 and The John Knopfmacher Centre for Applicable Analysis and Number Theory.

AMS subject classification: Primary 22A30; secondary 54H11, 20M15, 54A05.

Keywords: Homeomorphism, homogeneous space, topological group, resolvability, Stone–Čech compactification.

shifts are continuous, or equivalently, homeomorphisms.)

A mapping $f\colon X \to X$ of a local left group X onto itself is a *local automorphism* if f is a homeomorphism with $f(e) = e$ and for every $x \in X\setminus\{e\}$, there is a neighborhood U of e such that $f(xy) = f(x)f(y)$ for all $y \in U$.

To see that a local automorphism $f\colon X \to X$ is regular, for every $x \in X \setminus \{e\}$, choose a neighborhood U_x of e such that xy is defined for all $y \in U_x$, xU_x is a neighborhood of x and $\lambda_x\colon U_x \ni y \mapsto xy \in xU_x$ is a homeomorphism, and put $g_x = \lambda_x$. Clearly $g_x(e) = x$. Choose a neighborhood V_x of e such that $V_x \subseteq U_x$, $f(V_x) \subseteq U_{f(x)}$ and $f(xy) = f(x)f(y)$ for all $y \in V_x$. Then for every $y \in V_x$, $fg_x(y) = f(xy) = f(x)f(y) = g_{f(x)}f(y)$.

We shall deal with regular homeomorphisms of finite order. A bijection $f\colon X \to X$ has *finite order* if there is $n > 0$ such that $f^n = \mathrm{id}_X$, and the smallest such n is the *order* of f. Regular homeomorphisms of finite order occur very naturally in a large class of topological groups. For example, by Lemma 3.3, if G is a topological group with no elements of order 2, then the inversion $G \ni x \mapsto x^{-1} \in G$ is a regular homeomorphism.

Let X be a space with a distinguished point $e \in X$ and let $f\colon X \to X$ be a homeomorphism of finite order with $f(e) = e$. Define the *spectrum* of f by

$$\operatorname{spec}(f) = \{|O(x)| : x \in X \setminus \{e\}\},$$

where $O(x)$ is the orbit of x with respect to f, and more generally, for any subset $Y \subseteq X$,

$$\operatorname{spec}(f, Y) = \{|O(x)| : x \in Y \setminus \{e\}\}.$$

We say that f is *spectrally irreducible* if for every neighborhood U of e,

$$\operatorname{spec}(f, U) = \operatorname{spec}(f).$$

Also, a neighborhood U of a point $x \in X$ is *spectrally minimal* if for every neighborhood V of x contained in U,

$$\operatorname{spec}(f, V) = \operatorname{spec}(f, U).$$

It is clear that for every neighborhood U of x, there is a spectrally minimal neighborhood of x contained in U and that if U is spectrally minimal, so is every neighborhood of x contained in U. We now show that there is an arbitrarily small open invariant neighborhood of e. This will allow us to restrict ourselves in the study of homeomorphisms of finite order in a neighborhood of a fixed point to considering spectrally irreducible ones.

We say that a family $\mathcal{F}$ of subsets of X is *invariant* with respect to $f\colon X \to X$ if for every $Y \in \mathcal{F}$, $f(Y) \in \mathcal{F}$. In particular, $Y \subseteq X$ is invariant if $f(Y) = Y$. (Since f has finite order, $f(Y) = Y$ is equivalent to $f(Y) \subseteq Y$.)

Lemma 1.1 *Let X be a space, let $f\colon X \to X$ be a homeomorphism of finite order, and let $x \in X$ with $|O(x)| = s$. Then for every neighborhood U of x, there is an open neighborhood V of x contained in U such that the family $\{f^j(V) : j < s\}$ is disjoint and invariant. If X is zero dimensional, then V can be chosen to be clopen.*

Proof For each $j < s$, choose a neighborhood U_j of $f^j(x)$ such that all of them are pairwise disjoint and $U_0 \subseteq U$. Let n be the order of f. Then $n = sl$ for some integer l. Choose a neighbourhood W of x such that $f^{j+is}(W) \subseteq U_j$ for all $j < s$ and $i < l$. Put $V = \bigcup_{i<l} f^{is}(W)$. Then for each $j < s$, $f^j(V) = \bigcup_{i<l} f^{j+is}(W) \subseteq U_j$, so $\{f^j(V) : j < s\}$ is disjoint. Also $f^s(V) = \bigcup_{i<l} f^{(i+1)s}(W)$ and since $f^{ls} = f^0$, $f^s(V) = \bigcup_{i<l} f^{is}(W) = V$, so $\{f^j(V) : j < s\}$ is invariant. ■

In this paper we examine spectrally irreducible regular homeomorphisms of finite order. The following lemma says that the spectrum of such a homeomorphism is a finite subset of $\mathbb{N}$ closed under taking the least common multiple, lcm, (*i.e.*, $\operatorname{lcm}(s,t) \in \operatorname{spec}(f)$ for all $s, t \in \operatorname{spec}(f)$).

Lemma 1.2 *Let X be a space with a distinguished point e and let $f : X \to X$ be a spectrally irreducible regular homeomorphism of finite order. Let $x_0 \in X \setminus \{e\}$ with $|O(x_0)| = s$ and let U be a spectrally minimal neighbourhood of x_0. Then* $\operatorname{spec}(f, U) = \{\operatorname{lcm}(s,t) : t \in \{1\} \cup \operatorname{spec}(f)\}$.

Proof For each $x \in O(x_0)$, let g_x be a homeomorphism of a neighborhood U_x of e onto a neighborhood of x such that $fg_x|_{V_x} = g_{f(x)}f|_{V_x}$ for some neighborhood $V_x \subseteq U_x$ of e. Choose a neighborhood V of e such that $V \subseteq U \cap \bigcap_{x\in O(x_0)} V_x$ and the subsets $g_x(V)$, where $x \in O(x_0)$, are pairwise disjoint. Let n be the order of f. Choose a neighborhood W of e such that $f^i(W) \subseteq V$ for all $i < n$. Then clearly this inclusion holds for all $i < \omega$, and furthermore, for every $y \in W$, $f^i g_{x_0}(y) = g_{f^i(x_0)} f^i(y)$. Indeed, it is trivial for $i = 0$, and further, by induction, we obtain that $f^i g_{x_0}(y) = f f^{i-1} g_{x_0}(y) = f g_{f^{i-1}(x_0)} f^{i-1}(y) = g_{f^i(x_0)} f^i(y)$.

Now let $y \in W$, $|O(y)| = t$, and $k = \operatorname{lcm}(s,t)$. We claim that $|O(g_{x_0}(y))| = k$. Indeed, $f^k(g_{x_0}(y)) = g_{f^k(x_0)}(f^k(y)) = g_{x_0}(y)$. On the other hand, suppose that $f^i(g_{x_0}(y)) = g_{x_0}(y)$ for some i. Then $g_{f^i(x_0)}(f^i(y)) = g_{x_0}(y)$. Since the subsets $g_x(V)$, $x \in O(x_0)$, are pairwise disjoint, it follows from this that $f^i(x_0) = x_0$, so $s|i$. But then also $f^i(y) = y$, as g_{x_0} is injective, and so $t|i$. Hence $k|i$. ■

Now, given any finite subset of $\mathbb{N}$ closed under lcm, we produce a spectrally irreducible regular homeomorphism of the corresponding spectrum.

Example 1.3 Let S be a finite subset of $\mathbb{N}$ closed under lcm. Put $m = 1 + \sum_{s\in S} s$ and let $\mathbb{Z}(m) = \{0, 1, \dots, m-1\}$ denote the additive group of integers modulo m. Let $Z = \sum_\omega \mathbb{Z}(m)$ be the direct sum of ω copies of $\mathbb{Z}(m)$. Endow Z with the topology induced by the product topology on $\prod_\omega \mathbb{Z}(m)$, so Z is a topological group with a neighborhood base of the zero consisting of subgroups

$$Z_n = \{(x_i) \in Z : x_i = 0 \text{ for all } i \le n\},$$

where $n < \omega$. Enumerate S as $s_0 < s_1 < \dots < s_{t-1}$ and define the permutation π_0 on $\mathbb{Z}(m)$ by the product of disjoint cycles

$$\pi_0 = (1, \dots, s_0)(s_0 + 1, \dots, s_0 + s_1) \cdots (s_0 + \dots + s_{t-2} + 1, \dots, s_0 + \dots + s_{t-1}).$$

https://doi.org/10.4153/CJM-2009-038-x Published online by Cambridge University Press

Let $\pi\colon Z \to Z$ be the coordinatewise permutation induced by π_0, that is, π is defined by $\pi((x_n)) = (\pi_0(x_n))$. Then π is a homeomorphism with $\pi(0) = 0$, $\operatorname{spec}(\pi, Z_n) = S$, and $\pi(x + y) = \pi(x) + \pi(y)$, whenever $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \varnothing$, where, as usual,

$$\operatorname{supp}(x) = \{n < \omega : x_n \neq 0\}.$$

Hence $\pi\colon Z \to Z$ is a spectrally irreducible local automorphism of spectrum S. We call π the *standard* permutation of spectrum S.

We need one more notation concerning the group $Z = \sum_\omega \mathbb{Z}(m)$.

For every $x \in Z$, let σ_x denote the shift in Z by x, that is $\sigma_x\colon Z \to Z$ is defined by $\sigma_x(y) = x + y$.

Our main result tells us that in a certain sense Example 1.3 is universal.

Theorem 1.4 *Let X be a countable nondiscrete zero dimensional space with a distinguished point $e \in X$, let $f\colon X \to X$ be a spectrally irreducible regular homeomorphism of finite order, and let $\pi\colon Z \to Z$ be the standard permutation of the same spectrum as f. Then there is a continuous bijection $h\colon X \to Z$ with $h(e) = 0$ such that*

(1) *$f = h^{-1}\pi h$, and*

(2) *for every $x \in X$, $\lambda_x = h^{-1}\sigma_{h(x)}h$ is a homeomorphism of X onto itself.*

Furthermore, if X is a local left group and f is a local automorphism, then h can be chosen so that

(3) *$\lambda_x(y) = xy$, whenever* $\max\operatorname{supp}(h(x)) + 1 < \min\operatorname{supp}(h(y))$.

The conclusion of Theorem 1.4 can be rephrased as follows:

One can define the operation of the group Z on X in such a way that $0 = e$, the topology of Z is weaker than that of X and

(1) $f = \pi$, and

(2) for every $x \in X$, $\sigma_x\colon X \ni y \mapsto x + y \in X$ is a homeomorphism.

Furthermore, if X is a local left group and f is a local automorphism, then the operation can be defined so that

(3) $x + y = xy$, whenever $\max\operatorname{supp}(x) + 1 < \min\operatorname{supp}(y)$.

The second part of Theorem 1.4, the case when X is a local left group and f is a local automorphism, is a result from [7, Theorem 3.1]. The first part with $f = \mathrm{id}_X$ is a result from [8, Theorem 2].

Actually, Theorem 1.4 characterizes spectrally irreducible regular homeomorphisms on countable zero dimensional spaces. Suppose $f\colon X \to X$ is a spectrally irreducible homeomorphism of finite order and for some m there is a continuous bijection $h\colon X \to Z = \sum_\omega \mathbb{Z}(m)$ with $h(e) = 0$ such that

(1) hfh^{-1} is a coordinatewise permutation on Z, and

(2) for every $x \in X$, $h^{-1}\sigma_{h(x)}h$ is a homeomorphism of X onto itself.

Then f is regular.

To see this, let $\pi = hfh^{-1}$. For every $x \in X \setminus \{e\}$, let $n(x) = \max\operatorname{supp}(h(x)) + 1$, $U_x = h^{-1}(Z_{n(x)})$ and $g_x = h^{-1}\sigma_{h(x)}h|_{U_x}$. Then for every $y \in U_x$, $f(y) \in U_x = U_{f(x)}$

and

$$\begin{aligned} fg_x(y) &= h^{-1}\pi h h^{-1}\sigma_{h(x)}h(y) = h^{-1}\pi\sigma_{h(x)}h(y) \\ &= h^{-1}\pi(h(x)+h(y)) = h^{-1}(\pi(h(x))+\pi(h(y))) \\ &= h^{-1}(h(f(x))+h(f(y))) = h^{-1}\sigma_{h(f(x))}h(f(y)) = g_{f(x)}f(y). \end{aligned}$$

In Section 2, we prove Theorem 1.4. In Section 3, we consider two applications of Theorem 1.4. The first one is concerned with resolvability of topological spaces. The second one deals with the Stone–Čech compactification of a discrete semigroup.

2 Proof of Theorem 1.4

Let W be the set of all words over the alphabet $\mathbb{Z}(m)$ including the empty word $\varnothing$. For every $w \in W$, let $|w|$ denote the length of w. A nonempty word w is *basic* if all nonzero letters in w form a final subword. In particular, every nonempty *zero* word (*i.e.*, all the letters are zero) is basic. For every $v, w \in W$ such that $|v|+1 < |w|$ and the first $|v|+1$ letters in w are zero, define $v+w \in W$ to be the result of substituting v for the initial subword of length $|v|$ in w. Each nonempty $w \in W$ has a unique *canonical* decomposition in the form $w = w_0 + \cdots + w_k$, where for each $i \le k$, w_i is basic, and for each $i < k$, w_i is nonzero. From now on, when we write $w = w_0 + \cdots + w_k$, we mean that this is the canonical decomposition.

The permutation π_0 on $\mathbb{Z}(m)$, which induces the standard permutation π on Z, also induces the permutation $\pi_1\colon W \to W$. If $w = \xi_0 \cdots \xi_p$, then $\pi_1(w) = \pi_0(\xi_0)\cdots\pi_0(\xi_p)$. But we will write π instead of π_0 and π_1.

For each $x \in X \setminus \{e\}$, choose a homeomorphism g_x of a neighborhood of e onto a neighborhood of x with $g_x(e) = x$ such that $fg_x = g_{f(x)}f|_U$ for some neighborhood U of e. Also put $g_e = \mathrm{id}_X$. If X is a local left group and f is a local automorphism, choose g_x so that $g_x(y) = xy$. Enumerate X as $\{x_n : n < \omega\}$ with $x_0 = e$.

We shall assign to each $w \in W$ a point $x(w) \in X$ and a clopen spectrally minimal neighborhood $X(w)$ of $x(w)$ such that

(i) $x(0^n) = e$ and $X(\varnothing) = X$,
(ii) $\{X(w^\frown\xi) : \xi \in \mathbb{Z}(m)\}$ is a partition of $X(w)$,
(iii) $x(w) = g_{x(w_0)}\cdots g_{x(w_{k-1})}(x(w_k))$ and $X(w) = g_{x(w_0)}\cdots g_{x(w_{k-1})}(X(w_k))$, where $w = w_0 + \cdots + w_k$,
(iv) $f(x(w)) = x(\pi(w))$ and $f(X(w)) = X(\pi(w))$,
(v) $x_n \in \{x(w) : |w| = n\}$.

For this, we need the following lemma.

Let $\mathcal{P}$ and $\mathcal{Q}$ be families of subsets of a set X. We say that $\mathcal{Q}$ is inscribed into $\mathcal{P}$ if every member of $\mathcal{Q}$ is contained in a member of $\mathcal{P}$.

Lemma 2.1 *Let X be a countable nondiscrete zero dimensional space with a distinguished point $e \in X$ and let $f\colon X \to X$ be a homeomorphism of finite order with $f(e) = e$. Let U be a clopen invariant subset of X, let K be a finite invariant subset of U, and let $\mathcal{P}$ be a clopen invariant partition of U such that for each $C \in \mathcal{P}$,* $\mathrm{spec}(f, K\cap C) = \mathrm{spec}(f, C)$. *Then there is a clopen invariant partition $\{U(x)\colon x \in K\}$*

of U inscribed into $\mathcal{P}$ such that for each $x \in K$, $U(x)$ is a spectrally minimal neighborhood of x.

Proof Enumerate U as $\{x_n : n < \omega\}$ with $x_0 \in K$. For each $x \in K$, we shall construct an increasing sequence $(U_n(x))_{n<\omega}$ of clopen spectrally minimal neighborhoods of x such that for every $n < \omega$, the family $\{U_n(x) : x \in K\}$ is disjoint, inscribed into $\mathcal{P}$ and invariant, and $x_n \in U_n = \bigcup_{x \in K} U_n(x)$. Then the subsets $U(x) = \bigcup_{n<\omega} U_n(x)$, $x \in K$, will be as required. We proceed by induction on n.

Let y_i, $i < l$, be representatives of all orbits in K and let $|O(y_i)| = s_i$. For each $i < l$, choose a neighborhood W_i of y_i such that $f^j(W_i)$ is a spectrally minimal neighborhood of $f^j(y_i)$ for all $j < s_i$, and the family $\{f^j(W_i) : i < l, j < s_i\}$ is disjoint and inscribed into $\mathcal{P}$. By Lemma 1.1, for each $i < l$, there is a clopen neighborhood V_i of y_i contained in W_i such that the family $\{f^j(V_i) : j < s_i\}$ is invariant. Put $U_0(f^j(y_i)) = f^j(V_i)$.

Fix $n > 0$ and suppose that we have constructed required $U_{n-1}(x)$, $x \in K$. Without loss of generality one may suppose also that $x_n \notin U_{n-1}$. Let $|O(x_n)| = s$ and let $x_n \in C_n \in \mathcal{P}$. Using Lemma 1.1, choose a clopen neighborhood V_n of x_n such that for each $j < s$, $f^j(V_n)$ is a spectrally minimal neighborhood of $f^j(x_n)$, and the family $\{f^j(V_n) : j < s\} \cup \{U_{n-1}(x) : x \in K\}$ is disjoint, inscribed into $\mathcal{P}$ and invariant. Pick $z_n \in K \cap C_n$ with $|O(z_n)| = s$. For each $j < s$, put $U_n(f^j(z_n)) = U_{n-1}(f^j(z_n)) \cup f^j(V_n)$. For each $x \in K \setminus O(z_n)$, put $U_n(x) = U_{n-1}(x)$. ■

Now, enumerate S as $s_0 < s_1 < \cdots < s_{t-1}$ and for each $i < t$, pick a representative ζ_i of the orbit in $\mathbb{Z}(m) \setminus \{0\}$ of length s_i. Choose a clopen invariant neighborhood U_1 of e such that $x_1 \notin U_1$ and $\operatorname{spec}(f, X \setminus U_1) = \operatorname{spec}(f)$. Put $x(0) = e$ and $X(0) = U_1$. Then choose points $a_i \in X \setminus U_1$, $i < t$, with pairwise disjoint orbits of lengths s_i such that $x_1 \in \bigcup_{i<t} O(a_i)$. For each $i < t$ and $j < s_i$, put $x(\pi^j(\zeta_i)) = f^j(a_i)$. By Lemma 2.1, there is an invariant partition $\{X(\xi) : \xi \in \mathbb{Z}(m) \setminus \{0\}\}$ of $X \setminus U_1$ such that $X(\xi)$ is a clopen spectrally minimal neighborhood of $x(\xi)$.

Fix $n > 1$ and suppose that $X(w)$ and $x(w)$ have been constructed for all $w \in W$ with $|w| < n$ so that conditions (i)–(v) are satisfied.

Notice that the subsets $X(w)$, $|w| = n - 1$, form a partition of X. So one of them, say $X(u)$, contains x_n. Let $u = u_0 + \cdots + u_q$. Then $X(u) = g_{x(u_0)} \cdots g_{x(u_{q-1})}(X(u_q))$ and $x_n = g_{x(u_0)} \cdots g_{x(u_{q-1})}(y_n)$ for some $y_n \in X(u_q)$. Choose a clopen invariant neighborhood U_n of e such that for all basic w with $|w| = n - 1$,

(a) $g_{x(w)}(U_n) \subset X(w)$,
(b) $f g_{x(w)}|_{U_n} = g_{f(x(w))} f|_{U_n}$, and
(c) $\operatorname{spec}(f, X(w) \setminus g_{x(w)}(U_n)) = \operatorname{spec}(X(w))$.

If $y_n \neq x(u_q)$, choose U_n in addition so that

(d) $y_n \notin g_{x(u_q)}(U_n)$.

Put $x(0^n) = e$ and $X(0^n) = U_n$.

Let $w \in W$ be an arbitrary nonzero basic word with $|w| = n - 1$ and let $O(w) = \{w_j : j < s\}$, where $w_{j+1} = \pi(w_j)$ for $j < s - 1$ and $\pi(w_{s-1}) = w_0$. Put $Y_j = X(w_j) \setminus g_{x(w_j)}(U_n)$. Using Lemma 1.2, choose points $b_i \in Y_0$, $i < t$, with pairwise

https://doi.org/10.4153/CJM-2009-038-x Published online by Cambridge University Press

disjoint orbits of lengths $\mathrm{lcm}(s_i, s)$. If $u_q \in O(w)$, choose b_i in addition so that

$$y_n \in \bigcup_{i<t} O(b_i).$$

For each $i < t$ and $j < s_i$, put $x(\pi^j(w^\frown \xi_i)) = f^j(b_i)$. Then, using Lemma 2.1, inscribe an invariant partition

$$\{X(v^\frown \xi) : v \in O(w), \xi \in \mathbb{Z}(m) \setminus \{0\}\}$$

into the partition $\{Y_j : j < s\}$ such that $X(v^\frown \xi)$ is a clopen spectrally minimal neighborhood of $x(v^\frown \xi)$.

For nonbasic $w \in W$ with $|w| = n$, define $x(w)$ and $X(w)$ by condition (iii).

To check (ii) and (iv), let $|w| = n - 1$ and $w = w_0 + \cdots + w_k$. Then

$$X(w^\frown 0) = g_{x(w_0)} \cdots g_{x(w_k)}(X(0^n)) = g_{x(w_0)} \cdots g_{x(w_{k-1})}(g_{x(w_k)}(X(0^n)))$$

and

$$X(w^\frown \xi) = g_{x(w_0)} \cdots g_{x(w_{k-1})}(X(w_k^\frown \xi)),$$

so (ii) is satisfied. Next,

$$\begin{aligned} f(x(w)) &= f g_{x(w_0)} \cdots g_{x(w_{k-1})}(x(w_k)) \\ &= g_{f(x(w_0))} f g_{x(w_1)} \cdots g_{x(w_{k-1})}(x(w_k)) \\ &\;\vdots \\ &= g_{f(x(w_0))} \cdots g_{f(x(w_{k-1}))} f(x(w_k)) \\ &= g_{x(\pi(w_0))} \cdots g_{x(\pi(w_{k-1}))}(x(\pi(w_k))) \\ &= x(\pi(w_0) \cdots \pi(w_{k-1}) \pi(w_k)) \\ &= x(\pi(w)), \end{aligned}$$

so (iv) is satisfied as well.

To check (v), suppose that $x_n \notin \{x(w) : |w| = n - 1\}$. Then

$$x_n = g_{x(u_0)} \cdots g_{x(u_{q-1})}(y_n) = g_{x(u_0)} \cdots g_{x(u_{q-1})}(u_q^\frown \xi) = x(u^\frown \xi).$$

Now, for every $x \in X$, there is $w \in W$ with nonzero last letter such that $x = x(w)$, so $\{v \in W : x = x(v)\} = \{w^\frown 0^n : n < \omega\}$. Hence, we can define $h\colon X \to Z$ by putting for every $w = \xi_0 \cdots \xi_n \in W$,

$$h(x(w)) = \overline{w} = (\xi_0, \ldots, \xi_n, 0, 0, \ldots).$$

It is clear that h is bijective and $h(e) = 0$. Since for every $z = (\xi_i)_{i<\omega} \in Z$,

$$h^{-1}(z + Z_n) = X(\xi_0 \cdots \xi_n),$$

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h is continuous.

To see (1), let $x = x(w)$. Then

$$h(f(x(w))) = h(x(\pi(w))) = \overline{\pi(w)} = \pi(\overline{w}) = \pi(h(x(w))).$$

To see (2), let $x = x(w)$, $w = w_0 + \cdots + w_k$ and $n = \max \operatorname{supp}(h(x)) + 1$. We first show that

$$\lambda_x|_{h^{-1}(Z_n)} = g_{x(w_0)} \cdots g_{x(w_k)}|_{h^{-1}(Z_n)}.$$

Let $y \in h^{-1}(Z_n)$, $y = x(v)$, and $v = v_0 + \cdots + v_l$. Then

$$\begin{aligned} hg_{x(w_0)} \cdots g_{x(w_k)}(y) &= hg_{x(w_0)} \cdots g_{x(w_k)} g_{x(v_0)} \cdots g_{x(v_{l-1})}(x(v_l)) \\ &= h(x(w+v)) = \overline{w+v} = \overline{w} + \overline{v} \\ &= h(x(w)) + h(x(v)) = \sigma_{h(x)} h(y). \end{aligned}$$

It follows from (iii) that $g_{x(w_0)} \cdots g_{x(w_k)}$ homeomorphically maps $X(0^n)$, a neighborhood of e, onto $X(w^\frown 0)$, a neighborhood of x, and so does λ_x.

Now, to see that λ_x homeomorphically maps a neighborhood of an arbitrary point $y \in X$ onto a neighborhood of $z = \lambda_x(y)$, it suffices to check that $\lambda_x = \lambda_z(\lambda_y)^{-1}$.

Indeed, $z = h^{-1}\sigma_{h(x)} h(y) = h^{-1}(h(x) + h(y))$, and then

$$\begin{aligned} \lambda_z(\lambda_y)^{-1} &= h^{-1}\sigma_{h(x)+h(y)} h (h^{-1}\sigma_{h(y)} h)^{-1} = h^{-1}\sigma_{h(x)+h(y)} h h^{-1} (\sigma_{h(y)})^{-1} h \\ &= h^{-1}\sigma_{h(x)+h(y)} \sigma_{-h(y)} h = h^{-1}\sigma_{h(x)} h = \lambda_x. \end{aligned}$$

To see (3), let $x = x(w)$ and $w = w_0 + \cdots + w_k$. If $k = 0$, then $\lambda_x(y) = g_x(y) = xy$. Continuing, by induction on k, we obtain that

$$\begin{aligned} \lambda_x(y) &= g_{x(w_0)} \cdots g_{x(w_k)}(y) = g_{x(w_0)} \cdots g_{x(w_{k-1})}(x(w_k) \cdot y) \\ &= x(w_0 + \cdots + w_{k-1}) \cdot (x(w_k) \cdot y) = (x(w_0 + \cdots + w_{k-1}) \cdot x(w_k)) \cdot y \\ &= (g_{x(w_0)} \cdots g_{x(w_{k-1})}(x(w_k))) \cdot y = x(w) \cdot y. \end{aligned}$$

Finally, if X has a countable base, then $\{X(0^n) : n < \omega\}$ can be chosen to be a neighborhood base of e, and then h will be a homeomorphism.

3 Applications

In this section we consider two applications of Theorem 1.4.

The first application is concerned with resolvability of topological spaces. A space is called *resolvable* (ω-*resolvable*) if it can be partitioned into two (into ω) dense subsets [2]. The study of this notion for topological groups was initiated in [1], where it was proved that every Abelian group not containing an infinite Boolean subgroup is resolvable in any nondiscrete group topology. (Under Martin's Axiom, an Abelian group containing an infinite Boolean subgroup admits nondiscrete irresolvable group topologies [5, 10], but the existence of such topologies cannot be established in ZFC, the system of usual axioms of set theory [6].) The Comfort–van Mill Theorem was later extended in several directions. In particular, in [7] it was shown that every countable nondiscrete topological group not containing an open Boolean subgroup is ω-resolvable. We now give a new proof of this result, shorter and more transparent than the original one. And in fact we prove a more general theorem.

Let X be a space with a distinguished point $e \in X$ and let $f\colon X \to X$ be a homeomorphism with $f(e) = e$. We say that f is *nontrivial* if every neighborhood of e contains a nonfixed point.

Theorem 3.1 *If a countable zero dimensional space admits a nontrivial regular homeomorphism of finite order, then it is ω-resolvable.*

Proof Let X be a countable zero dimensional space with a distinguished point $e \in X$ and let $f\colon X \to X$ be a nontrivial regular homeomorphism of finite order. It is well known that if a homogeneous space contains an ω-resolvable subspace, then it itself is also ω-resolvable (see [10, Lemma 2.3]). Therefore, one may suppose that f is spectrally irreducible. Let $h\colon X \to \sum_\omega \mathbb{Z}(m)$ be a bijection guaranteed by Theorem 1.4. Denote by C the orbit in $\mathbb{Z}(m)$ (with respect to π_0) of the least possible length $s > 1$ and let

$$Y = \{x \in X : \text{ there is a coordinate of } h(x) \text{ belonging to } C\}.$$

Note that every $x \in X$ with $|O(x)| = s$ belongs to Y. For every $x \in Y$, consider the sequence of coordinates of $h(x)$ belonging to C and define $\nu(x)$ to be the number of pairs of distinct neighbouring elements in this sequence. Denote also by $\alpha(x)$ and $\gamma(x)$ the first and the last elements in the sequence. Then whenever $x, y \in Y$ and $\max \operatorname{supp}(h(x)) + 1 < \min \operatorname{supp}(h(y))$,

$$\nu(\lambda_x(y)) = \begin{cases} \nu(x) + \nu(y) & \text{if } \gamma(x) = \alpha(y); \\ \nu(x) + \nu(y) + 1 & \text{otherwise.} \end{cases}$$

We define the partition $\{Y_n : n < \omega\}$ of Y by

$$Y_n = \{x \in Y : \nu(x) \equiv 2^n \mod 2^{n+1}\}.$$

(Equivalently, Y_n consists of all $x \in Y$ such that the index of the leftmost nonzero digit in the binary expansion of $\nu(x)$ is n.)

To see that every Y_n is dense in X, let $x \in X$ and let U be an open neighbourhood of e. We have to show that $\lambda_x(U) \cap Y_n \neq \varnothing$. Put $k = 2^{n+1}$ and choose inductively $x_1, \dots, x_k \in U$ such that

(i) $|O(x_j)| = s$,

(ii) $\max \operatorname{supp}(h(x_j)) + 1 < \min \operatorname{supp}(h(x_{j+1}))$, and if $x \neq 0$, then $\max \operatorname{supp}(h(x)) + 1 < \min \operatorname{supp}(h(x_1))$,

(iii) $\lambda_{y_1} \cdots \lambda_{y_k}(e) \in U$ whenever $y_j \in O(x_j)$.

Without loss of generality one may suppose that $\gamma(x_j) = \alpha(x_{j+1})$, and that if $x \in Y$, then $\gamma(x) = \alpha(x_1)$. For every $l = 0, 1, \dots, k-1$, define $z_l \in U$ by

$$z_l = \lambda_{x_1}\lambda_{f(x_2)} \cdots \lambda_{f^l(x_{l+1})}\lambda_{f^l(x_{l+2})} \cdots \lambda_{f^l(x_k)}(e)$$

(in particular, $z_0 = \lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_k}(e)$). Then

$$h(\lambda_x(z_l)) = h(x) + h(x_1) + \pi h(x_2) + \cdots + \pi^l h(x_{l+1}) + \pi^l h(x_{l+2}) + \cdots + \pi^l h(x_k).$$

It follows that $\nu(\lambda_x(z_0)) = \nu(x)+\nu(x_1)+\cdots+\nu(x_k)$ and $\nu(\lambda_x(z_l)) = \nu(z_0)+l$. Hence, for some l, $\nu(\lambda_x(z_l)) \equiv 2^n \mod 2^{n+1}$, so $\lambda_x(z_l) \in Y_n$. ∎

The next proposition says that every nondiscrete topological group not containing an open Boolean subgroup admits a nontrivial regular homeomorphism of order 2.

Proposition 3.2 *Let G be a nondiscrete topological group not containing an open Boolean subgroup. Suppose that for every element $x \in G$ of order 2, the conjugation $G \ni y \mapsto xyx^{-1} \in G$ is a trivial local automorphism. Then the inversion $G \ni y \mapsto y^{-1} \in G$ is a nontrivial regular homeomorphism.*

In order to prove Proposition 3.2, we need the following.

Lemma 3.3 *Let X be a homogeneous space with a distinguished point $e \in X$ and let $f\colon X \to X$ be a homeomorphism of finite order n with $f(e) = e$. Suppose that for every $x \in X \setminus \{e\}$ with $|O(x)| = s < n$, there is a homeomorphism g_x of a neighborhood U of e onto a neighborhood of x with $g_x(e) = x$ such that $f^s g_x(y) = g_x f^s(y)$ for all $y \in U$. Then f is regular. In particular, if for every $x \in X \setminus \{e\}$, $|O(x)| = n$, then f is regular.*

Proof Consider an arbitrary orbit in X distinct from $\{e\}$ and enumerate it as $\{x_i : i < s\}$, where $x_{i+1} = f(x_i)$ for $i = 0, \dots, s-2$ and $f(x_{s-1}) = x_0$. If $s = n$, choose as g_{x_0} any homeomorphism of a neighborhood U of e onto a neighborhood of x_0 with $g_{x_0}(e) = x_0$. If $s < n$, choose g_{x_0} in addition such that $f^s g_{x_0}(y) = g_{x_0} f^s(y)$ for all $y \in U$. For every $i = 1, \dots, s-1$, put $g_{x_i} = f^i g_{x_0} f^{-i}|_U$. Then for every $i = 0, \dots, s-1$ and $y \in U$,

$$f g_{x_i}(y) = f f^i g_x f^{-i}(y) = f^{i+1} g_x f^{-(i+1)} f(y).$$

If $i < s-1$, then $f^{i+1} g_x f^{-(i+1)} f(y) = g_{x_{i+1}} f(y)$, so $f g_{x_i}(y) = g_{x_{i+1}} f(y)$. Hence, it remains only to check that $f g_{x_{s-1}}(y) = g_{x_0} f(y)$. If $s = n$, then

$$f g_{x_{s-1}}(y) = f^s g_{x_0} f^{-s} f(y) = \mathrm{id}_X g_{x_0} \mathrm{id}_X f(y) = g_{x_0} f(y).$$

If $s < n$, then

$$f g_{x_{s-1}}(y) = f^s g_{x_0} f^{-s} f(y) = g_{x_0} f^s f^{-s} f(y) = g_{x_0} f(y).$$ ∎

Proof of Proposition 3.2 Let f denote the inversion. It is clear that f is a homeomorphism of order 2 and that $B = \{x \in G : x^2 = e\}$ is the set of fixed points of f, in particular, $f(e) = e$. We first show that B is not a neighborhood of e, and so f is nontrivial. Indeed, assume the contrary. Then there exists a neighborhood U of e such that $U^2 \subseteq B$. For every $x, y \in U$, $(xy)^2 = e$ and also $xyyx = e$, consequently, $xy = yx$. But then for every $x_1, \dots, x_n \in U$, $(x_1 \cdots x_n)^2 = x_1^2 \cdots x_n^2 = e$, and so $x_1 \cdots x_n \in B$. Hence, the subgroup generated by U is open and contained in B, which is a contradiction.

To see that f is regular, let $x \in G \setminus \{e\}$ and $|O(x)| < 2$. Then x is a fixed point, and so $x \in B$. But then there is a neighborhood U of e such that $xyx^{-1} = y$ for all $y \in U$, that is $xy = yx$. Define $g_x \colon U \to xU$ by $g_x(y) = xy$. We have that

$$fg_x(y) = (xy)^{-1} = (yx)^{-1} = x^{-1}y^{-1} = xy^{-1} = g_x f(y).$$

Hence, by Lemma 3.3, f is regular. ■

Combining Theorem 3.1 and Proposition 3.2, we obtain the following.

Corollary 3.4 ([7]) *Every countable nondiscrete topological group not containing an open Boolean subgroup is ω-resolvable.*

The second application deals with the Stone–Čech compactification βS of a discrete semigroup S. We take the points of βS to be the ultrafilters on S and identify the principal ultrafilters with the points of S. The topology of βS has a base of subsets of the form $\overline{A} = \{p \in \beta S : A \in p\}$, where $A \subseteq S$. The operation of S can be naturally extended to βS by $pq = \lim_{x\to p} \lim_{y\to q} xy$, where $x, y \in S$. This makes βS a compact right topological semigroup (*i.e.,* for each $q \in \beta S$, the right shift $\beta S \ni p \mapsto p\cdot q \in \beta S$ is continuous) with S contained in the topological center (*i.e.,* for each $a \in S$, the left shift $\beta S \ni a \mapsto a \cdot q \in \beta S$ is continuous). For $p, q \in \beta S$, the ultrafilter pq has a base of subsets $\bigcup\{xB_x : x \in A\}$ where $A \in p$ and $B_x \in q$. As any compact right topological semigroup, βS has idempotents. An elementary introduction to the semigroup βS can be found in [3].

Now let X be a nondiscrete local left topological group and let βX_d be the Stone–Čech compactification of X as a discrete space. Denote by $\mathrm{Ult}(X)$ the closed subspace in βX_d of all nonprincipal ultrafilters on X converging to $e \in X$. As in the case of βS, the partial operation of X can be naturally extended to βX_d by $pq = \lim_{x\to p} \lim_{y\to q} xy$, where $x, y \in X$, making βX_d a compact right topological partial semigroup. The product pq is defined if and only if

$$\{x \in X : \{y \in X : xy \text{ is defined}\} \in q\} \in p,$$

in particular, if $q \in \mathrm{Ult}(X)$. Clearly if $p, q \in \mathrm{Ult}(X)$, then also $pq \in \mathrm{Ult}(X)$. Hence, $\mathrm{Ult}(X)$ is a closed subsemigroup in βX_d. It is called the *ultrafilter semigroup* of X.

Suppose that $f \colon X \to X$ is a local automorphism and let $\overline{f} \colon \beta X_d \to \beta X_d$ be the continuous extension of f, that is, for every $p \in \beta X_d$, $\overline{f}(p)$ is the ultrafilter on X with a base of subset $f(A)$, where $A \in p$. Then $\overline{f}|_{\mathrm{Ult}(X)}$ is an automorphism on $\mathrm{Ult}(X)$ (see [3, Theorem 4.21]), in particular, if $p \in \mathrm{Ult}(X)$ is an idempotent, so is $\overline{f}(p) \in \mathrm{Ult}(X)$. We will write $f(p)$ instead of $\overline{f}(p)$.

Our second application is the following result.

Theorem 3.5 *Let X be a countable nondiscrete zero dimensional local left topological group, let $f \colon X \to X$ be a local automorphism of finite order, and let $p \in \mathrm{Ult}(X)$ be an idempotent. If $f(p) \neq p$, then the subset $\{p, f(p)\} \subseteq \mathrm{Ult}(X)$ algebraically generates the free product of one-element semigroups $\{p\}$ and $\{f(p)\}$.*

https://doi.org/10.4153/CJM-2009-038-x Published online by Cambridge University Press

Proof Consider an arbitrary relation

$$p_1 \cdots p_k = q_1 \cdots q_s$$

in $\mathrm{Ult}(X)$, where $p_i, q_j \in \{p, f(p)\}$, $p_i \neq p_{i+1}$ and $q_j \neq q_{j+1}$. We have to prove that $p_1 = q_1$ and $k = s$.

Without loss of generality one may suppose that f is spectrally irreducible. Let $h\colon X \to \sum_\omega \mathbb{Z}(m)$ be a bijection guaranteed by Theorem 1.4. Denote by C the set of all nonfixed points in $\mathbb{Z}(m)$ (with respect to π_0) and let

$$Y = \{x \in X : \text{ there is a coordinate of } h(x) \text{ belonging to } C\}.$$

(Equivalently, Y consists of all nonfixed points in X.) Note that $\overline{Y} \cap \mathrm{Ult}(X)$ is a subsemigroup containing p and $f(p)$. For every $x \in Y$, consider the sequence of coordinates of $h(x)$ belonging to C and denote $\alpha(x)$ and $\gamma(x)$ the first and the last elements in this sequence. Then for every $u, v \in \overline{Y} \cap \mathrm{Ult}(X)$, $\alpha(uv) = \alpha(u)$.

Indeed, let $\alpha(u) = c \in C$ and let $A = \{x \in Y : \alpha(x) = c\}$. Then $A \in u$. For every $x \in A$, put $n(x) = \max\operatorname{supp}(h(x)) + 1$ and $U_x = h^{-1}(Z_{n(x)})$. We have that $\bigcup_{x\in A} xU_x \in uv$ and for every $y \in U_x$, $\alpha(xy) = \alpha(x) = c$, so $\alpha(uv) = c$.

Similarly, $\gamma(uv) = \gamma(v)$, and if $f(u) \neq u$, then $\alpha(u) \neq \alpha(f(u))$ and $\gamma(u) \neq \gamma(f(u))$. Applying α and γ to the relation gives $\alpha(p_1) = \alpha(q_1)$ and $\gamma(p_k) = \gamma(q_s)$, so $p_1 = q_1$ and $p_k = q_s$.

We now show that $k = s$. Define the subset $F \subseteq C^2$ by

$$F = \{(\gamma(q), \alpha(q)) : q \in \{p, f(p)\}\}$$

and let n be any integer $\geq \max\{k, s\}$. For every $x \in X$, consider the sequence of coordinates of $h(x)$ belonging to C and define $\nu(x) \in \mathbb{Z}(n)$ to be the number modulo n of pairs of neighbouring elements in this sequence other than pairs from F. Then for every $u, v \in \mathrm{Ult}(X)$,

$$\nu(uv) = \begin{cases} \nu(u) + \nu(v) & \text{if } (\gamma(u), \alpha(v)) \in F \\ \nu(u) + \nu(v) + 1 & \text{otherwise.} \end{cases}$$

It follows from this that

$$\nu(p_1 \cdots p_k) = \nu(p_1) + \cdots + \nu(p_k) + k - 1$$

and

$$\nu(q_1 \cdots q_s) = \nu(q_1) + \cdots + \nu(q_s) + s - 1.$$

Also we have that for every $q \in \{p, f(p)\}$, $\nu(qq) = 2\nu(q)$. Consequently, since q is an idempotent, $\nu(q) = \nu(qq) = 2\nu(q)$. Hence, $\nu(q) = 0$. Finally, we obtain that $\nu(p_1 \cdots p_k) = k - 1$ and $\nu(q_1 \cdots q_s) = s - 1$, so $k = s$. ∎

Corollary 3.6 *Let G be a countably infinite Abelian group and let $B = \{x \in G : 2x = 0\}$. Then for every idempotent $p \in \overline{G \setminus B}$ in βG, the subset $\{p, -p\}$ algebraically generates the free product of one-element semigroups $\{p\}$ and $\{-p\}$.*

Proof Take any totally bounded group topology $\mathcal{T}$ on G and let $X = (G, \mathcal{T})$. Then $\mathrm{Ult}(X)$ holds all the idempotents of βG except for 0 [9, Lemma 3]. Define $f\colon G \to G$ by $f(x) = -x$ and apply Theorem 3.5. ∎

Note that in the case $G = \mathbb{Z}$, a much stronger result than Corollary 3.6 is known to be true. For every idempotent $p \in \beta\mathbb{Z} \setminus \{0\}$, all expressions of the form

$$a_1 \cdot p + a_2 \cdot p + \cdots + a_n \cdot p$$

are distinct, where $\langle a_1, a_2, \ldots, a_n \rangle$ denotes a sequence in $\mathbb{Z} \setminus \{0\}$ in which any two successive terms are different [4, Corollary 4.2].

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