On Vortices.

By C. Chree, M.A.

Helmholtz and most, if not all, subsequent writers on vortex motions have, except in obtaining the fundamental equations, confined themselves to fluid of invariable density.

In the following paper are considered some simple systems of vortices in a compressible fluid. To show that such systems are of considerable importance it is sufficient to refer to the phenomena of cyclonic storms. It may be as well, however, to state that though the vortices are here treated as compressible, the circumstances are still so different from those found in nature that the results obtained could bear only a general resemblance at most to the phenomena of storms.

For brevity, the reader is supposed to understand and have in his hands for reference, Prof. Lamb's "Motion of Fluids," the symbols employed here having the same meaning, while the equations with suffix L refer to the equations so numbered in chapter VI. of that treatise.

When the fluid is limited the velocity must everywhere be tangential to the boundary. It is shown by Lamb how straight vortices parallel to one or more plane boundaries may be treated; and it is easily seen that a vortex of any shape in presence of an infinite plane boundary requires merely the introduction on the other side of the boundary of a vortex coinciding in position with the image of the real vortex as given by a plane mirror coinciding with the infinite plane. The direction of rotation in the imaginary vortex is such that the motion it causes tangential to the plane is in the same direction as that due to the real vortex. If the fluid be compressible it is only necessary that the imaginary fluid have at every point the same density as in the corresponding real point at the instant considered. If, for instance, xy be a boundary plane, vortices parallel to oz must be supposed to extend to infinity in both directions, and the density will be the same at any two points at equal distances from xy in a perpendicular to it.

Suppose, now, an infinite straight vortex parallel to oz, and of uniform section throughout, the density being some function of z, which we shall suppose not to change sign with z, then the motion

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is in two dimensions. For the strength of the vortex being throughout its length constan., we have only the functions N and P, of which the former is given by (33_1), or when the cross section is small by

$$N = -\frac{m}{\pi}\log r. \qquad (1).$$

Also assuming the velocity everywhere parallel to xy, and denoting by p the pressure in the fluid, and by Z the component of the external forces parallel to oz, we have

$$Z - \frac{1}{\rho}\frac{dp}{dz} = 0.$$

Provided, then, the external forces are wholly parallel to oz, and Z is a given function of z, whether $p = k\rho$ or $= k\rho^{\gamma}$ where k and γ are constants, the above equation determines the relation between the density at the plane xy and at the distance z from that plane, which will exist supposing no velocity parallel to oz. If now σ be the cross section at time t, and ρ the density at any height, supposed uniform over the cross section, the equation of continuity is simply $\frac{\delta}{\delta t}(\sigma\rho) = 0$, where, as in what follows, δ denotes differentiation following the fluid; whence we see that $-\frac{1}{\rho}\frac{\delta\rho}{\delta t}$ is $= \frac{1}{\sigma}\frac{\delta\sigma}{\delta t}$, and so is independent of z.

Thus for this elementary column we get

$$P = \frac{1}{2\pi}\log r\,\frac{\delta\sigma}{\delta t}. \qquad (2).$$

The velocity depending on P at a distance r from the axis of the column of varying density is radial, and is given by

$$\frac{dP}{dr} = \frac{1}{2\pi r}\frac{\delta\sigma}{\delta t}. \qquad (3).$$

Thus the fluid crossing a cylinder of radius r coaxial with the column of varying density, is simply $\frac{\delta\sigma}{\delta t}$ per unit of length in unit time. It follows that as much fluid leaves as enters the space between any two such cylindrical surfaces, and so the presence of a column of varying density has no direct tendency to cause variation in the density of the surrounding fluid.

Suppose we have a single thin straight filament parallel to oz, its vorticity being of strength m and its cross section at time t being σ; then the velocity at distance r from the axis is $\frac{dP}{dr}$ along and $\frac{dN}{dr}$ perpendicular to r. Thus, taking the axis of z along the axis of the

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filament, the component velocities parallel to rectangular axes of x and y are

$$u = -\frac{my}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{x}{r^2} \qquad (4),$$

$$v = \frac{mx}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{y}{r^2} \qquad (5).$$

If the vortex, though of larger cross section, be circular and ρ and ζ be both functions only of the distance from the centre the same formulae will apply.

Unless $\frac{\delta\sigma}{\delta t}$ be constant the stream lines due to such a vortex vary in position with the time; if, however, $\frac{\delta\sigma}{\delta t}$ be constant they are equiangular spirals of the type

$$r = ae^{\frac{1}{2m}\frac{\delta\sigma}{\delta t}\phi} \qquad (6),$$

denoting by ϕ an angle measured from some fixed plane through oz.

Let us next consider two thin vortices similar to the last—viz., the vortex m_1, σ_1 with its centre at the point (x_1, y_1), and the vortex m_2, σ_2 with its centre at (x_2, y_2). Then for the motion of the vortices we have

$$\frac{\delta x_2}{\delta t} = -\frac{m_1}{\pi r^2}(y_2 - y_1) + \frac{1}{2\pi}\frac{\delta\sigma_1}{\delta t}\frac{x_2 - x_1}{r^2} \qquad (7),$$

$$\frac{\delta y_2}{\delta t} = \frac{m_1}{\pi r^2}(x_2 - x_1) + \frac{1}{2\pi}\frac{\delta\sigma_1}{\delta t}\frac{y_2 - y_1}{r^2} \qquad (8),$$

$$\frac{\delta x_1}{\delta t} = -\frac{m_2}{\pi r^2}(y_1 - y_2) + \frac{1}{2\pi}\frac{\delta\sigma_2}{\delta t}\frac{x_1 - x_2}{r^2} \qquad (9),$$

$$\frac{\delta y_1}{\delta t} = \frac{m_2}{\pi r^2}(x_1 - x_2) + \frac{1}{2\pi}\frac{\delta\sigma_2}{\delta t}\frac{y_1 - y_2}{r^2} \qquad (10),$$

where $r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

These equations would also apply approximately to the case of two circular vortices of larger cross section, provided r were great compared to the radius of either.

From (7) – (10) we get

$$(x_2 - x_1)\frac{\delta}{\delta t}(x_2 - x_1) + (y_2 - y_1)\frac{\delta}{\delta t}(y_2 - y_1) = \frac{1}{2\pi}\frac{\delta}{\delta t}(\sigma_1 + \sigma_2)$$

$$\therefore \qquad r^2 = a^2 + \frac{1}{\pi}\{\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2\} \qquad (11),$$

if at the time $t = o$, $r = a$, $\sigma_1 = {}_0\sigma_1$, and $\sigma_2 = {}_0\sigma_2$.

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Again, from (7) – (10) it is easy to prove

$$\frac{\delta}{\delta t}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \frac{m_1 + m_2}{\pi(x_2 - x_1)^2} \qquad (12).$$

Suppose, now, the line joining the vortices to be at time t inclined at an angle ϕ to its original direction, which is taken as axis of x; then the above equation becomes

$$\frac{1}{\sec^2\phi}\frac{\delta}{\delta t}(\tan\phi) = \frac{m_1 + m_2}{\pi r^2},$$

whence
$$\phi = \frac{m_1 + m_2}{\pi}\int_0^t \frac{dt}{a^2 + (\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2)|\pi} \qquad (13.)$$

If, then, the law of variation of σ_1 and σ_2 be given, the distance of the vortices and the direction of the line joining them follow at once from (11) and (13). Suppose, for instance,

$$\left.\begin{array}{l}\sigma_1 = {}_0\sigma_1(1 + \gamma_1 t)\\ \sigma_2 = {}_0\sigma_2(1 + \gamma_2 t)\end{array}\right\} \qquad (14),$$

where γ_1 and γ_2 are constants; then, if $\dfrac{2(m_1 + m_2)}{{}_0\sigma_1\gamma_1 + {}_0\sigma_2\gamma_2}$ be denoted by B, we get

$$r^2 = a^2\left\{1 + \frac{2(m_1 + m_2)}{\pi a^2 \mathrm{B}}t\right\} \qquad (15),$$

and
$$\phi = \frac{\mathrm{B}}{2}\log\left\{1 + 2\frac{(m_1 + m_2)}{\pi a^2 \mathrm{B}}t\right\} \qquad (16),$$

whence
$$r^2 = a^2 e^{2\phi/\mathrm{B}} \qquad (17).$$

In this case the path of one vortex relative to the other is an equiangular spiral, and the time of the n^{th} complete revolution of the line joining them is

$$t_n = \frac{\pi a^2 \mathrm{B}}{2(m_1 + m_2)}(e^{4\pi/\mathrm{B}} - 1)e^{4(n-1)\pi/\mathrm{B}} \qquad (18).$$

To find the absolute motion, let X, Y be the co-ordinates of the centre of inertia of the masses m_1 at (x_1, y_1) (x_2, y_2); so that

$$(m_1 + m_2)\mathrm{X} = m_1 x_1 + m_2 x_2,$$
$$(m_1 + m_2)\mathrm{Y} = m_1 y_1 + m_2 y_2.$$

Then from (7)—(10) we get

$$\left.\begin{array}{l}(m_1 + m_2)\dfrac{\delta \mathrm{X}}{\delta t} = \dfrac{1}{2\pi}\dfrac{x_2 - x_1}{r^2}\left(m_2\dfrac{\delta\sigma_1}{\delta t} - m_1\dfrac{\delta\sigma_2}{\delta t}\right)\\ (m_1 + m_2)\dfrac{\delta \mathrm{Y}}{\delta t} + \dfrac{1}{2\pi}\dfrac{y_2 - y_1}{r^2}\left(m_2\dfrac{\delta\sigma_1}{\delta t} - m_1\dfrac{\delta\sigma_2}{\delta t}\right)\end{array}\right\} \qquad (19.)$$

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If $m_2 \frac{\delta\sigma_1}{\delta t} - m_1 \frac{\delta\sigma_2}{\delta t}$ vanish, which includes the more special case when σ_1 and σ_2 remain constant, we have

$$X = \text{constant}, \quad Y = \text{constant}.$$

In the general case we could not integrate the expressions for X and Y; but if the relations (14) hold, we have

$$(m_1 + m_2) \frac{\delta X}{\delta t} = \frac{A}{r} \cos\phi,$$

$$(m_1 + m_2) \frac{\delta Y}{\delta t} = \frac{A}{r} \sin\phi,$$

where

$$A = \frac{m_2\, {}_0\sigma_1 \gamma_1 - m_1\, {}_0\sigma_2 \gamma_2}{2\pi}.$$

By means of (15) we may regard r as the variable instead of t, and so employing (17) get

$$(m_1 + m_2) \frac{dX}{dr} \frac{r\,\delta r}{\delta t} = A \cos\left(B \log \frac{r}{a}\right);$$

but

$$\frac{\delta r}{r \delta t} = \frac{m_1 + m_2}{\pi B},$$

$$\therefore \quad X = \int C \cos\left(B \log \frac{r}{a}\right) dr,$$

where

$$C = \frac{\pi A B}{(m_1 + m_2)^2}.$$

Integrating this by parts, and supposing when $t = o$ and $r = a$ that $X = o = Y$, we get

$$X = \frac{C}{1 + B^2}\left[r\cos\left(B\log\frac{r}{a}\right) - a + Br\sin\left(B\log\frac{r}{a}\right)\right\}\Bigg] \qquad (20).$$

Similarly we find

$$Y = \frac{C}{1 + B^2}\left[r\sin\left(B\log\frac{r}{a}\right) + B\left\{a - r\cos\left(B\log\frac{r}{a}\right)\right\}\right] \qquad (21).$$

Since r and ϕ have been already determined the motion is in this case completely investigated.

We see from (11) that in every case the vortices approach to or recede from one another according as the sum of their cross sections is diminishing or increasing, the rate of approach being independent either of the directions or magnitudes of their vorticities. Since σ_1 and σ_2 must be positive, the distance of the vortices can never be less than $\sqrt{a^2 - \frac{1}{\pi}({}_0\sigma_1 + {}_0\sigma_2)}$; and as we have practically assumed the distance a great compared to the radius of either vortex,

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it follows that the distance can experience only a comparatively small diminution. In particular, in applying the results deduced on the hypothesis (14) we must limit t so that $1+\gamma_1 t$ and $1+\gamma_2 t$ are both positive.

A special interest as will be seen attaches to the case $m_2=-m_1$. The equation (11) still holds, but in place of (13) we have $\phi=0$ and so $y_2=y_1$, the vortices being supposed initially to have the centres of their bases in the axis of x. It follows at once from the equations that $(x_2-x_1)^2=r^2=a^2+\frac{1}{\pi}(\sigma_1+\sigma_2-{}_0\sigma_1-{}_0\sigma_2)$ (22).

while $$y_2=y_1=\frac{m_1}{\pi}\int\frac{\delta t}{\sqrt{a^2+\frac{1}{\pi}(\sigma_1+\sigma_2-{}_0\sigma_1-{}_0\sigma_2)}} \qquad (23).$$

If we suppose $\frac{\delta\sigma_1}{\delta t}=\frac{\delta\sigma_2}{\delta t}$ then the equations give $\frac{\delta x_2}{\delta t}=-\frac{\delta x_1}{\delta t}$, or the vortices have equal but opposite velocities in the line joining them; and if the point half-way between them be taken as origin, we have

$$x_2=-x_1=\tfrac{1}{2}a\sqrt{1+2(\sigma_1-{}_0\sigma_1)/\pi a^2} \qquad (24).$$

$$y_2=y_1=\frac{m_1}{\pi a}\int\frac{\delta t}{\sqrt{1+2(\sigma_1-{}_0\sigma_1)/\pi a^2}} \qquad (25).$$

This answers to the case when a single vortex exists parallel to a boundary plane, taken as that of yz, the original distance being $\frac{a}{2}$. If, again, while $m_2=-m_1$ we have $\sigma_1+\sigma_2$ constant then $r^2=a^2$; and, taking the original position of the middle point of the line joining the vortices as origin, we get

$$x_2=\frac{a}{2}+\frac{1}{2\pi a}(\sigma_1-{}_0\sigma_1),$$

$$x_1=-\frac{a}{2}+\frac{1}{2\pi a}(\sigma_1-{}_0\sigma_1),$$

and $$y_2=y_1=\frac{m_1}{\pi a}t.$$

If, while still supposing $m_2=-m_1$ we suppose the relations (14) to hold, it is easy to prove that the vortices move along straight lines; viz. m_2 along the line $y=\frac{2m_1}{{}_0\sigma_1\gamma_1}\left(x-\frac{a}{2}\right)$ and m_1 along $y+\frac{2m_1}{{}_0\sigma_2\gamma_2}\times\left(x+\frac{a}{2}\right)=0$, the origin being at the initial position of the middle point of the line joining the vortices.

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The methods here employed will also give the motion when any number of vortices exist, but the difficulty of solving the equations increases rapidly with the number of vortices. As an example suppose the planes of zx and zy to form two infinite boundaries at right angles. Let a straight vortex filament of strength m parallel to oz have at time t the cross section σ, and let (x, y) be the centre of the cross section by the plane xy. Then the planes may be supposed non-existent and the fluid infinite if we introduce three new vortices all of cross section σ. One of these has its centre at $(-x, -y)$ and is of strength m, the other two are each of strength $-m$ and have their centres at $(x, -y)$ and $(-x, y)$ respectively. The velocities of the original vortex are

$$\left.\begin{aligned} u &= \frac{m}{2\pi}\frac{x^2}{y(x^2+y^2)} + \frac{1}{4\pi}\frac{\delta\sigma}{\delta t}\frac{2x^2+y^2}{x(x^2+y^2)} \\ v &= -\frac{m}{2\pi}\frac{y^2}{x(x^2+y^2)} + \frac{1}{4\pi}\frac{\delta\sigma}{\delta t}\frac{x^2+2y^2}{y(x^2+y^2)} \end{aligned}\right\} \quad (26).$$

To determine completely the path described would in the general case be difficult. If, however, $\frac{\delta\sigma}{\delta t} = \text{constant} = km$, and (r, ϕ) denote the polar co-ordinates answering to (x, y), it is not difficult to prove that the path of the vortex is the curve

$$\left(\frac{r}{r_0}\right)^{1+k^2}\left\{\frac{\sin 2\phi - k\cos 2\phi}{\sin 2\phi_0 - k\cos 2\phi_0}\right\}^{1+3k^2/4} = e^{k(\phi-\phi_0)/2} \quad (27),*$$

where r_0, ϕ_0 are the original values of r, ϕ.

In any application of the preceding results to the case of vortices in the earth's atmosphere it must be observed that the vortices are here supposed to exist in a fluid limited only by infinite planes and not revolving as a whole about an axis. In the case of the earth, the vortices are at distances apart comparable with the earth's radius, and the vortey motion is directly influenced by the earth's rotation; the effect also of the rotation in modifying the atmospheric density in different latitudes is of great importance. Further, the motion considered here is only in two dimensions; while in cyclonic storms the velocity and even the direction of the wind seem often to vary at

* For special case of incompressible fluid see a paper by Professor Greenhill in the "Quarterly Journal of Mathematics," Vol. XV.

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different altitudes above the ground, and some observers assert that in the centre of the disturbance ascending vertical currents often exist. In connection with this point it may be of interest to refer to Buys Ballot's law for cyclonic storms. In these there is a central area where the barometer is low and the wind blows round this area. According to the law in question the wind does not blow perpendicularly to the line joining the observer to the point where the barometer is lowest, but is more or less directed towards the centre of the depression. Now, in accordance with the results we have obtained, if the motion were in two dimensions this law would be true only if the section of the vortex were contracting, in which case the density would be increasing and the barometer rising at the centre of the depression. Further, the magnitude of the radial velocity would be proportional to the rate of variation with the time in the height of the barometer. If the barometer were falling throughout the area of the disturbance the direction of the wind would be on the whole outwards from the centre. Thus, supposing Buys Ballot's law well founded, we must conclude either that vertical currents do exist in the centre of cyclonic storms, or else that cyclonic depressions fill up in much less time than they take to form. It should also be noticed that the rate of fluctuation of the barometer at any one station affords no clue to the law of fluctuation of the density at the centre of the disturbance. A rapid fall, for instance, might mean merely that the storm had a rapid motion of translation, or that the density diminished rapidly in approaching the centre of the depression.

A Theorem in Algebra.

By J. L. Mackenzie.

If we have given two equations $\phi(x) = 0$ and $\psi(y) = 0$, it is possible to express in the form of a determinant the equation whose roots are $f(x, y)$, where f is any given rational integral function.

Let a_r, β_r, be the sums of the r^{th} powers of the roots of the given equations, and s_r of the required equation. Then

$$s_r = \{f(a_1, b_1)\}^r + \{f(a_2, b_1)\}^r + \ldots\ldots$$
$$+ \{f(a_1, b_2)\}^r + \{f(a_2, b_2)\}^r + \ldots\ldots \text{ \&c.}$$

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