ON THE ZEROS OF SOLUTIONS OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Introduction. In §1 of this paper we consider the complex differential equation

$$(1) \qquad u''(z) + q(z)\,u(z) = 0, \qquad |z| < 1,$$

where $q(z)$ is a regular function in the open unit circle. We shall give a lower bound for the non-Euclidean distance of any pair of zeros of any non-trivial (i.e., not identically zero) solution $u(z)$ of (1). This theorem (Theorem 1) is a generalization of a recent theorem of Nehari (**7**, Theorem I) quoted below. The first part of our proof will use a complex technique already used elsewhere (**1**, Theorem 2.1). However, for the final step in the proof of this theorem we need a result on the (real) zeros of the (real) solutions of the real differential equation

$$(2) \qquad y''(r) + M(r)\,y(r) = 0, \qquad -1 < r < 1,$$

under certain restrictive assumptions on the function $M(r)$. This result (Lemma 1), whose proof we shall delay to the very end of our paper, is a consequence of a theorem of §2 giving a lower bound for the least positive eigenvalue of the real differential system

$$(3) \qquad y''(x) + \lambda p(x)\,y(x) = 0, \qquad y(x_0) = y(-x_0) = 0, \qquad 0 < x_0 < \infty,$$

where $p(x)$ is a real function, continuous in $-x_0 \leqslant x \leqslant x_0$, and changing sign only a finite number of times in this interval (Theorem 2). We shall, however, also obtain an upper bound for λ in the simpler, and more often considered, case where $p(x)$ is a continuous function which is non-negative in the whole interval $-x_0 \leqslant x \leqslant x_0$.

1. Non-Euclidean distance. In his first paper on this subject (**6**), Nehari made use of a fundamental relationship between the theory of the differential equation (1) and the behaviour of analytic functions $f(z)$. If we denote the Schwarzian derivative of $f(z)$ by $\{f(z), z\}$ and if we set $q(z) = \frac{1}{2}\{f(z), z\}$ then the univalence of $f(z)$ in $|z| < 1$ is equivalent to the fact that no non-trivial solution $u(z)$ of (1) has two zeros in $|z| < 1$. Bearing this in mind, we now restate the above-mentioned theorem of Nehari for differential equations (instead of stating it—as in the original— as a criterion of univalence).

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THEOREM I **(7)**. *Let $q(z)$ be regular in $|z| < 1$ and suppose there exists a function $M(r)$ satisfying*

$$|q(z)| \leqslant M(|z|), \qquad |z| < 1, \tag{4}$$

and having the following properties: (a) *$M(r)$ is positive and continuous for $-1 < r < 1$*; (b) $M(-r) = M(r)$; (c) *$(1-r^2)^2 M(r)$ is non-increasing as r varies from 0 to 1*; (d) *the differential equation*

$$y''(r) + M(r)\,y(r) = 0, \qquad -1 < r < 1, \tag{2}$$

has a (real) solution $y(r)$ which does not vanish for $-1 < r < 1$. Then no non-trivial (complex) solution $u(z)$ of the differential equation

$$u''(z) + q(z)\,u(z) = 0, \qquad |z| < 1, \tag{1}$$

has two zeros in $|z| < 1$. The right-hand side of (4) *cannot be replaced by an expression of the form $CM(|z|)$ for any constant $C > 1$.*

We shall keep conditions (a)-(c), but drop condition (d). For the statement of our theorem let us agree to denote the non-Euclidean[1] segment between two points z_1 and z_2 inside the unit circle by $[z_1 z_2]$, and to denote the non-Euclidean distance between these two points by $|[z_1 z_2]|$. We have then

$$|[z_1 z_2]| = \int \frac{|dz|}{1-|z|^2},$$

where the integration is along $[z_1 z_2]$. We now state

THEOREM 1. *Let $q(z)$ be regular in $|z| < 1$, and suppose there exists a function $M(r)$ satisfying* (4) *and having properties* (a)-(c) *of Theorem* I **(7)**. *Moreover, assume* (d′): *there exists a (necessarily even) solution $y(r)$ of* (2) *for which*

$$y(a) = y(-a) = 0, \qquad 0 < a < 1,$$
$$y(r) \neq 0 \qquad -a < r < a.$$

Let $u(z)$ be any (non-trivial) solution of the differential equation (1), *and assume that $u(z_1) = u(z_2) = 0$, $|z_1| < 1$, $|z_2| < 1$, $z_1 \neq z_2$. Then*

$$|[z_1 z_2]| \geqslant \log \frac{1+a}{1-a} = |[-a, a]|. \tag{5}$$

We remark that the existence of an (essentially unique) even solution of (2) follows from condition (b) and that, in view of the Sturm separation theorem, conditions (d) and (d′) are mutually exclusive.

The first part of the proof of this theorem follows closely the proof of Theorem I **(7)**. That is, we assume that a non-trivial solution $u(z)$ of (1) vanishes at the points z_1, z_2 inside the unit circle, and consider the circle which passes through these two points and is orthogonal to $|z| = 1$; let us denote its whole

[1]The non-Euclidean distance refers to the Klein-Poincaré hyperbolic geometry.

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arc inside the unit circle by C. C is, therefore, the non-Euclidean straight line containing $[z_1 z_2]$. Without loss of generality we may assume that C lies in the upper half-plane and is symmetric with respect to the imaginary axis. Indeed, this position can always be achieved by a rotation $\zeta = \alpha z$, $|\alpha| = 1$, which transforms (1) into

$$u_1''(\zeta) + \alpha^{-2} q(\zeta/\alpha)\, u_1(\zeta) = 0,$$

where $u_1(\zeta) = u(z)$. But clearly $\alpha^{-2} q(z/\alpha)$ is, together with $q(z)$, majorized by $M(|z|)$. We assume therefore that C is already in this symmetrical position, and denote the imaginary point of C (which may or may not lie in $[z_1 z_2]$ by $i\,\beta$, $0 \leqslant \beta < 1$. The linear transformation

$$w = \frac{z - i\,\beta}{1 + i\,\beta\, z}$$

of $|z| < 1$ onto $|w| < 1$ carries C onto the line segment $-1 < w < 1$, and $[z_1 z_2]$ onto a segment $r_1 \leqslant w \leqslant r_2$, $-1 < r_1 < r_2 < 1$. Now, setting

$$U(w) = (1 - i\,\beta\, w)\, u\left(\frac{w + i\,\beta}{1 - i\,\beta\, w}\right)$$

we see that $U(w)$ satisfies the differential equation

$$U''(w) + \frac{(1-\beta^2)^2}{(1 - i\,\beta\, w)^4}\, q\left(\frac{w + i\,\beta}{1 - i\,\beta\, w}\right) U(w) = 0, \tag{6}$$

and, moreover, $U(r_1) = U(r_2) = 0$. We have, for real w,

$$\left|\frac{1-\beta^2}{(1 - i\,\beta\, w)^2}\right| = \frac{1-\beta^2}{1+\beta^2 w^2}\,; \quad \left|\frac{w + i\,\beta}{1 - i\,\beta\, w}\right| = \left(\frac{w^2+\beta^2}{1+\beta^2 w^2}\right)^{\frac{1}{2}}.$$

Using (4), it follows that for real w the factor of $U(w)$ in (6) is majorized by

$$\left(\frac{1-\beta^2}{1+\beta^2 w^2}\right)^2 M\left(\left(\frac{\beta^2+w^2}{1+\beta^2 w^2}\right)^{\frac{1}{2}}\right).$$

Now, writing r instead of the real w, we compare (6) for $-1 < r < 1$ with its real majorant

$$y''(r) + \left(\frac{1-\beta^2}{1+\beta^2 r^2}\right)^2 M\left(\left(\frac{\beta^2+r^2}{1+\beta^2 r^2}\right)^{\frac{1}{2}}\right) y(r) = 0, \qquad -1 < r < 1. \tag{7}$$

We now use the fact (**7**; **11**, Theorem 4.1; **1**, Corollary 1.2) that if a real solution $y(r)$ of the majorant equation (7) is non-vanishing on an interval $r_3 < r < r_4$, $-1 < r_3 < r_4 < 1$, then no (complex) solution $U(w)$ of the majorized equation (6) can have two zeros inside this interval.

For fixed β, $0 \leqslant \beta < 1$, let us now denote any pair of consecutive zeros of any solution of (7) by $a_1(\beta)$ and $a_2(\beta)$, $-1 < a_1(\beta) < a_2(\beta) < 1$, and set

$$d(\beta) = \text{g.l.b.} \int_{a_1(\beta)}^{a_2(\beta)} \frac{dr}{1-r^2}\,,$$

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where the g.l.b. is taken over all such pairs $a_1(\beta)$, $a_2(\beta)$. At this stage, without having yet used assumption (c), we have proved that

$$\|[z_1z_2]\| = \|[r_1r_2]\| \geqslant \underset{0\leqslant\beta<1}{\text{g.l.b.}}\ d(\beta).$$

However, as noted in (**1**, (2.3) and (2.4) for $\beta_0 = 1$), assumption (c) is equivalent to the inequality

$$\left(\frac{1-\beta^2}{1+\beta^2r^2}\right)^2 M\left(\left(\frac{\beta^2+r^2}{1+\beta^2r^2}\right)^{\frac{1}{2}}\right) \leqslant M(r), \qquad 0 \leqslant \beta < 1,\ -1 < r < 1.$$

As equation (7) is therefore majorized by equation (2), it follows that a lower bound for $\|[z_1z_2]\|$ is given by

$$\text{g.l.b.} \int_{a_1}^{a_2} \frac{dr}{1-r^2}$$

where the g.l.b. is taken over all pairs a_1, a_2, $-1 < a_1 < a_2 < 1$, of consecutive zeros of all non-trivial solutions $y(r)$ of equation (2). We now use

LEMMA 1. *Let $M(r)$ fulfil conditions* (a) *to* (d′) *of Theorem* 1. *Let a_1, a_2, $-1 < a_1 < a_2 < 1$, be any pair of consecutive zeros of any non-trivial solution $y(r)$ of* (2). *Then*

$$\int_{a_1}^{a_2} \frac{dr}{1-r^2} \geqslant \log\frac{1+a}{1-a}, \tag{8}$$

where a is defined by condition (d′). *If, in addition, $(1 - r^2)^2 M(r)$ is strictly decreasing for $0 < r < a$, then we have strict inequality in* (8) *except for the case $a_1 = -a$, $a_2 = a$.*

The proof of Lemma 1 which, as stated, will be given in §2 completes the proof of Theorem 1.

We remark that for a function $M(r)$ fulfilling conditions (a)-(c), condition (d′) will be implied by

$$\lim_{r\to 1} (1 - r^2)^2 M(r) > 1.$$

This will follow from the proof of Lemma 1, and may also be seen by using the Sturm comparison theorem for (2) and the differential equation

$$y''(r) + \frac{1+4\gamma^2}{(1-r^2)^2} y(r) = 0, \qquad \gamma > 0,$$

having the solutions

$$(1-r^2)^{\frac{1}{2}} \sin\left\{\gamma \log\left(\frac{1+r}{1-r}\right) - C\right\}, \quad -\infty < C < \infty, \tag{9}$$

which are oscillatory for $-1 < r < 1$. On the other hand, if

$$\lim_{r\to 1} (1 - r^2)^2 M(r) \leqslant 1$$

then conditions (a)-(c) are compatible either with (d) or (d′). Indeed, for

$M(r) = 1/(1-r^2)^2$ and $M(r) = \pi^2/4$, equation (2) has solutions which do not vanish on $-1 < r < 1$ (cf. Nehari (**6**, Theorems I and II)), while examples (ii) and (iii) below, with

$$\lim_{r\to 1} (1-r^2)^2 M(r) = 1 \text{ and } 0 \text{ respectively},$$

illustrate Theorem 1, i.e., correspond to (d′).

With respect to the sharpness of Theorem 1, we remark that if $q(z)$ is an even function, real on the real segment $-1 < z < 1$, which attains its maximum on the real axis for each $|z| = C$, $0 < C < 1$, and if in addition $(1-r^2)^2 q(r)$ is non-increasing for $0 < r < 1$, then—taking $q(r)$ as the $M(r)$ of Theorem 1—there exists a solution $u(z)$ of (1) and zeros z_1, z_2 of $u(z)$ such that we have equality in (5). Indeed, $-a$ and a (defined by (d′)) are zeros of any even solution of (1). It follows that the inequality (5) is the best possible of its kind.

The following examples deal with functions $M(r)$ for which such corresponding functions $q(z)$ are readily found.

(i) $$M(r) = \frac{1+4\gamma^2}{(1-r^2)^2}, \qquad \gamma > 0.$$

In this case we have, for any pair a_1, a_2 of consecutive zeros of any real solution $y(r)$—given by (9)—of equation (2), $|[a_1a_2]| = \pi/2\gamma$. Sharpness is shown by $q(z) = (1+4\gamma^2)/(1-z^2)^2$. This case was considered earlier (**10**, Theorem 3); however, the bound given there was not sharp. It is of interest to note that in this case, equations (2) and (7) are identical so that the result would follow without an application of Lemma 1.

(ii) $$M(r) = \frac{1}{(1-r^2)^2} + \frac{\nu(\nu+1)}{1-r^2},$$

where ν is an even positive integer. The even solution of (2) is, in this case, $(1-r^2)^{\frac{1}{2}} P_\nu(r)$, where $P_\nu(r)$ is the Legendre polynomial of degree ν. Denoting its least positive zero by α_ν, we obtain the bound $\log [(1+\alpha_\nu)/(1-\alpha_\nu)]$. Sharpness is shown by $q(z) = [1/(1-z^2)^2] + \nu(\nu+1)/(1-z^2)$.

(iii) $$M(r) = \frac{\nu(\nu+1)}{1-r^2},$$

where ν is an odd positive integer larger than 1. In this case the even solution of (2) is $(1-r^2)\, P_\nu'(r)$, where $P_\nu(r)$ is again the νth Legendre polynomial. Denoting the least positive zero of $P_\nu'(r)$ by β_ν, we obtain the bound $\log [(1+\beta_\nu)/(1-\beta_\nu)]$; $q(z)$ is given by $\nu(\nu+1)/(1-z^2)$.

For examples (ii) and (iii) it was shown earlier (**1**, Corollary 2.1 and Corollary 4.1) that no solution of (1) has two zeros in the circle $|z| < \alpha_\nu$ or $|z| < \beta_\nu$ respectively, a fact that now follows from Theorem 1.

2. Bounds for the least positive eigenvalue. The statement of Theorem 2 uses notions which are defined in the books of Hardy, Littlewood and Pólya

(**3**, chap. X), and of Pólya and Szegö (**9**, chap. VII). Moreover, its proof relies on a theorem of these books (**3**, Theorem 378, and **9**, p. 153). For completeness we shall restate this material (cf. **9**, pp. 151–153), but only for the case needed here, i.e., for real functions defined and continuous on the closed segment $\langle -x_0, x_0\rangle$, $0 < x_0 < \infty$.

Let $f(x)$ and $g(x)$ be two such functions. They are called *similarly ordered* if for each pair of points x_1, x_2 of the above integral we have

$$[f(x_1) - f(x_2)]\cdot[g(x_1) - g(x_2)] \geqslant 0;$$

f and g are called *oppositely ordered* if f and $-g$ are similarly ordered. Consider now, for each real u, the set of points x in $\langle -x_0, x_0\rangle$ for which $f(x) \geqslant u$ and denote its measure by $M(u)$. Let $N(u)$ be related to g as $M(u)$ is to f. If, for each real u, we have $M(u) = N(u)$ then we say that f and g are *equimeasurable*. We now quote the special case of the above-mentioned theorem as

LEMMA 2. *If f, f_1, f_2, g, g_1, and g_2 are real continuous functions defined on $\langle -x_0, x_0\rangle$, $0 < x_0 < \infty$, f_1 and g_1 are similarly ordered, f_2 and g_2 oppositely ordered, f, f_1 and f_2 are equimeasurable, and also g, g_1 and g_2 are equimeasurable, then*

$$\int_{-x_0}^{x_0} f_2 g_2 dx \leqslant \int_{-x_0}^{x_0} f\, g\, dx \leqslant \int_{-x_0}^{x_0} f_1 g_1 dx.$$

Let $f(x)$ be defined as above. Let $f(x)$, $f^+(x)$ and $f^-(x)$ be equimeasurable, and in addition let $f^+(x)$ and x^2 be similarly ordered, and $f^-(x)$ and x^2 be oppositely ordered. The uniquely defined and continuous functions $f^+(x)$ and $f^-(x)$ are called *the rearrangement of $f(x)$ in symmetrically increasing respectively decreasing order*. $f^-(x)$ is an even function decreasing (i.e., non-increasing) for $0 < x < x_0$. The connection between $f^+(x)$ and $f^-(x)$ is given by $f^+(x) = f^-(x_0 - x)$ for $0 \leqslant x \leqslant x_0$, and $f^+(x) = f^+(-x)$ for $-x_0 \leqslant x \leqslant 0$.

We may now state

THEOREM 2. *Let $p(x)$ be continuous and not identically zero*[2] *for $-x_0 \leqslant x \leqslant x_0$, $0 < x_0 < \infty$, and let $p^+(x)$ and $p^-(x)$ be the rearrangement of $p(x)$ in symmetrically increasing resp. decreasing order. Consider the three differential systems*

(10)	$y''(x) + \lambda p(x)\, y(x) = 0,$	$y(\pm x_0) = 0;$
(10^+)	$u''(x) + \lambda^+ p^+(x)\, u(x) = 0,$	$u(\pm x_0) = 0;$
(10^-)	$v''(x) + \lambda^- p^-(x)\, v(x) = 0,$	$v(\pm x_0) = 0;$

denote their least positive eigenvalues also by λ, λ^+ and λ^- respectively. Then $\lambda^- \leqslant \lambda$ even if $p(x)$ changes sign finitely often, while $\lambda \leqslant \lambda^+$ holds if $p(x) \geqslant 0$.

Proof. We shall use, in addition to Lemma 2, the minimum property of the least positive eigenvalue. Since the first half of this theorem deals with

[2]If $p(x) \leqslant 0$ throughout the interval, then the differential systems (10) have no positive eigenvalues; $p(x)$ is therefore assumed to be positive somewhere in the interval.

the polar case—where $p(x)$ may change sign—we shall give an explicit statement of this property (**5**, pp. 214–215). Consider all functions[3] $y(x)$ of class D' on $-x_0 \leqslant x \leqslant x_0$ such that $y(\pm x_0) = 0$, and such that

$$\int_{-x_0}^{x_0} py^2 dx > 0.$$

Then the least positive eigenvalue of the system (10) is given by

$$\lambda = \min \frac{\int_{-x_0}^{x_0} y'^2 dx}{\int_{-x_0}^{x_0} py^2 dx},$$

where the minimum is taken over the above class. This minimum is obtained for a solution of the system (10), and we shall denote this eigenfunction corresponding to the least positive eigenvalue by $y(x)$. We also use the fact (see Ince (**4**, p. 237) or Bôcher (**2**, p. 176)) that this first eigenfunction does not vanish for $-x_0 < x < x_0$.

It follows that $y(x)$ may be assumed to be positive for $-x_0 < x < x_0$; the same therefore holds true for $y^-(x)$, the rearrangement of $y(x)$ in symmetrically decreasing order. $y^-(x)$ vanishes, together with $y(x)$, at $\pm x_0$. Moreover, it is easily seen that $y^-(x)$ is continuous and that its derivative may have discontinuities only for those values of the ordinate y^- for which y had extrema. Since by hypothesis $p(x)$ changes sign only finitely often, $y(x)$ has only a finite number of inflection points and of extrema, and it follows that $y^-(x)$ is in D'. We then have

$$\lambda = \frac{\int_{-x_0}^{x_0} y'^2 dx}{\int_{-x_0}^{x_0} py^2 dx} \geqslant \frac{\int_{-x_0}^{x_0} (y^-)'^2 dx}{\int_{-x_0}^{x_0} p^-(y^-)^2 dx} \geqslant \min \frac{\int_{-x_0}^{x_0} v'^2 dx}{\int_{-x_0}^{x_0} p^- v^2 dx} = \lambda^-. \tag{11$^-$}$$

To justify the first inequality sign we remark that $\{y^-(x)\}^2$ is, together with $y^-(x)$, symmetrically decreasing; $p^-(x)$ and $\{y^-(x)\}^2$ are therefore similarly ordered, and it follows from Lemma 2, that

$$\int_{-x_0}^{x_0} py^2 dx \leqslant \int_{-x_0}^{x_0} p^-(y^-)^2 dx.$$

[3]Kamke (**5**) states the minimum property only for comparison functions of class C'; however an elementary argument extends the validity of the result to this wider class of functions.

That

$$\int_{-x_0}^{x_0} y'^2 dx \geqslant \int_{-x_0}^{x_0} (y^-)'^2 dx, \qquad y(\pm x_0) = 0,$$

i.e., that under symmetrization the one-dimensional Dirichlet integral decreases, follows (analogous to the two-dimensional case; see (**9**, Note A.3)) from the well-known fact that the arc length decreases under symmetrization. The minimum in the fourth term of (11^-) is taken over all functions $v(x)$ of class D' such that $v(\pm x_0) = 0$ and such that

$$\int_{-x_0}^{x_0} p^- v^2 dx > 0.$$

To prove that $\lambda \leqslant \lambda^+$ under the hypothesis $p(x) \geqslant 0$ (but $p(x) \not\equiv 0$), let $u(x)$ be a fixed first eigenfunction of (10^+). Since (10^+) is a symmetric differential system, $u(x)$ is an even function of class C'', which we may assume to be positive for $-x_0 < x < x_0$. It follows, moreover, from (10^+) that $u(x)$ is concave from below and therefore symmetrically decreasing. Here we use the fact that $p^+(x)$ is, together with $p(x)$, non-negative for $-x_0 < x < x_0$. We have now

$$(11^+) \qquad \lambda^+ = \frac{\int_{-x_0}^{x_0} u'^2 dx}{\int_{-x_0}^{x_0} p^+ u^2 dx} \geqslant \frac{\int_{-x_0}^{x_0} u'^2 dx}{\int_{-x_0}^{x_0} p u^2 dx} \geqslant \min \frac{\int_{-x_0}^{x_0} y'^2 dx}{\int_{-x_0}^{x_0} p y^2 dx} = \lambda,$$

the first inequality sign in (11^+) now following from the fact that $\{u(x)\}^2$ is, together with $u(x)$, symmetrically decreasing so that $p^+(x)$ and $\{u(x)\}^2$ are oppositely ordered.

Theorem 2 includes a result announced by Pokornyi (**8**), and proved elsewhere (**1**, Lemma 5.2):

Let $p(x)$ be continuous and non-negative on the interval $-x_0 \leqslant x \leqslant x_0$, $p(x) = p(-x)$, and let $p(x)$ be non-increasing for $0 \leqslant x \leqslant x_0$. Suppose $y''(x) + p(x)y(x) = 0$ has a solution which does not vanish on $-x_0 < x < x_0$. Set

$$p_1(x) = \begin{cases} p(x_0 - x), & 0 \leqslant x \leqslant x_0, \\ p_1(-x), & -x_0 \leqslant x \leqslant 0; \end{cases}$$

then the equation $y_1''(x) + p_1(x)y_1(x) = 0$ has a solution with the same property.

In our notation this result is equivalent to the inequality $\lambda^+ \geqslant \lambda^-$ and follows therefore—for non-negative $p(x)$—from Theorem 2.

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We finally remark that Theorem 2 can be generalized to two dimensions, i.e., to the equation

$$\Delta u(x, y) + \lambda p(x, y)\, u(x, y) = 0.$$

We intend to deal with this and related material in another paper.

For the proof of Lemma 1 we need an intermediate step which is a consequence of the first half of Theorem 2. For completeness, however, we shall also state and prove the analogous consequence of the second half of Theorem 2.

LEMMA 3. *Let $p(x)$ have the following properties:*

(a) *$p(x)$ is continuous for $-\infty < x < \infty$;*

(b) $p(-x) = p(x)$;

(c) *$p(x)$ is non-increasing for $0 < x < \infty$;*

(d′) *the even solution $y(-x) = y(x)$ of the differential equation*

$$y''(x) + p(x)\, y(x) = 0, \qquad -\infty < x < \infty, \tag{12}$$

vanishes for finite x, with its least positive zero at $x = \alpha$.

Let $\alpha_1, \alpha_2, -\infty < \alpha_1 < \alpha_2 < \infty$ be any pair of consecutive zeros of any non-trivial solution of (12). *Then*

$$\alpha_2 - \alpha_1 \geqslant 2\alpha. \tag{13}$$

If, in addition, $p(x)$ is strictly decreasing for $0 < x < \alpha$ then we have strict inequality in (13) *except for the case $\alpha_1 = -\alpha$, $\alpha_2 = \alpha$.*

Moreover, if we keep conditions (b) *and* (d′) *but replace* (a) *and* (c) *by* (a′): *$p(x)$ is non-negative and continuous for $-\infty < x < \infty$, and* (c′): *$p(x)$ is non-decreasing for $0 < x < \infty$, then we have*

$$\alpha_2 - \alpha_1 \leqslant 2\alpha. \tag{13′}$$

In this case strict inequality holds in (13′) *(except for $\alpha_1 = -\alpha$, $\alpha_2 = \alpha$) if $p(x)$ is strictly increasing for $0 < x < \alpha$.*

We begin the proof of the first half of the lemma with the remark that the properties of $p(x)$ imply $p(0) > 0$. Now take any fixed real $c \neq 0$, and consider the function $q(x) = p(x + c)$ for the interval $-\alpha \leqslant x \leqslant \alpha$ only, where α is defined by (d′). We now compare the three differential systems

$$y''(x) + \lambda q(x)\, y(x) = 0, \qquad y(\pm\alpha) = 0, \tag{14}$$

$$v''(x) + \lambda^- q^-(x)\, v(x) = 0, \qquad v(\pm\alpha) = 0, \tag{14$^-$}$$

and

$$Y''(x) + \lambda^0 p(x)\, Y(x) = 0, \qquad Y(\pm\alpha) = 0, \tag{14°}$$

and denote their least positive eigenvalues also by λ, λ^- and λ^0 respectively. These eigenvalues will all exist since—at least for all constants c which we need to consider—each of the three functions $q(x)$, $q^-(x)$ and $p(x)$ is somewhere positive in $-\alpha \leqslant x \leqslant \alpha$. This is true of $p(x)$ by our first remark; as far as $q(x)$ is concerned (and hence also $q^-(x)$) we need only consider constants c which are such that $q(x) = p(x + c)$ is somewhere positive in $-\alpha \leqslant x \leqslant \alpha$.

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For, if $p(x) \leqslant 0$ for $-\alpha + c \leqslant x \leqslant \alpha + c$ no solution of (12) can have two zeros in this interval, i.e., α_1 and α_2 are not defined in this case.

In (14^-) $q^-(x)$ is the rearrangement of $q(x)$ (considered only for $-\alpha \leqslant x \leqslant \alpha$) in symmetrically decreasing order. It follows from (b) and (c) that

$$q^-(x) \leqslant p(x), \qquad -\alpha \leqslant x \leqslant \alpha. \tag{15}$$

If, in addition, $p(x)$ is strictly decreasing for $0 < x < \alpha$, then we have strict inequality in (15) for some subintervals of $-\alpha \leqslant x \leqslant \alpha$. By using the minimum property for (14^-) and (14^0) it follows from (15) that $\lambda^- \geqslant \lambda^0$ with strict inequality if $p(x)$ is strictly decreasing. By Theorem 2 it follows that $\lambda \geqslant \lambda^-$, hence we have $\lambda \geqslant \lambda^0$ with strict inequality if $p(x)$ is strictly decreasing.

We now show that condition (d′) implies $\lambda^0 = 1$. Let $Y(x)$ be an even solution of (12). We have $Y(\pm\alpha) = 0$ and $Y(x) \neq 0$ for $-\alpha < x < \alpha$. Since

$$\int_{-\alpha}^{\alpha} pY^2dx = \int_{-\alpha}^{\alpha} Y'^2dx > 0$$

it follows from the minimum property of the least positive eigenvalue that $\lambda^0 \leqslant 1$. On the other hand, if $w(x)$ is any function of class D' on $-\alpha \leqslant x \leqslant \alpha$ such that $w(\pm\alpha) = 0$, then $Y(x) \neq 0$ for $-\alpha < x < \alpha$ implies (cf. **1**, Lemma 1.1)

$$\int_{-\alpha}^{\alpha} w'^2dx \geqslant \int_{-\alpha}^{\alpha} pw^2dx, \tag{16}$$

so that $\lambda^0 \geqslant 1$.

It remains to show that $\lambda \geqslant 1$ implies (13), and that $\lambda > 1$ implies strict inequality in (13). First, if $\lambda = 1$, then the corresponding eigenfunction of the system (14) has consecutive zeros at $x = \pm\alpha$, and the corresponding solution of (12) has $\alpha_2 - \alpha_1 = (\alpha + c) - (-\alpha + c) = 2\alpha$ so that (13) is satisfied. Now, suppose that $\lambda > 1$ and that $\alpha_2 - \alpha_1 > 2\alpha$ is not satisfied. Then for an appropriate $c \neq 0$ the solution of the equation

$$y''(x) + q(x)\,y(x) = 0, \qquad q(x) = p(x + c),$$

which vanishes at $x = -\alpha$ would vanish again at x', where $-\alpha < x' \leqslant \alpha$. Now define $y_1(x)$ by

$$y_1(x) = \begin{cases} y(x), & -\alpha \leqslant x \leqslant x', \\ 0, & x' \leqslant x \leqslant \alpha. \end{cases}$$

We now have

$$0 < \int_{-\alpha}^{\alpha} y_1'^2dx = \int_{-\alpha}^{\alpha} qy_1^2\,dx.$$

By the preceding inequality (and since $\lambda > 1$) it follows that

$$0 < \frac{\int_{-\alpha}^{\alpha} y_1'^2dx}{\int_{-\alpha}^{\alpha} qy_1^2dx} < \lambda.$$

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But this contradicts the minimum property of the least positive eigenvalue λ of the system (14), and thus proves the first half of our lemma. The proof of the second half is analogous but somewhat simpler since $p(x)$ is non-negative. The properties of $p(x)$ now imply that $p(x)$ is not identically zero. We now compare (14) with

$$(14^+) \qquad u''(x) + \lambda^+ q^+(x)\, u(x) = 0, \qquad u(\pm\alpha) = 0,$$

and (14^0), using now $q^+(x) \geqslant p(x)$, $-\alpha \leqslant x \leqslant \alpha$, and the other half of Theorem 2. In this case, $p(x) \geqslant 0$ and (d′) imply $\lambda^0 = 1$ by a direct application of the Sturm comparison theorem; moreover $\lambda^+ \geqslant 1$ ($\lambda^+ > 1$) implies (13′) (with strict inequality if $p(x)$ is strictly increasing) also follows from the Sturm comparison theorem. This completes the proof of Lemma 3.

Suppose now that $M(r)$ has the properties (a)-(d′) of Lemma 1. Set $g(r) = (1 - r^2)^2 M(r)$; by (c) $g(r)$ is non-increasing for $0 < r < a$. Let $y(r)$ be any solution of the differential equation

$$(2) \qquad y''(r) + M(r)\, y(r) = 0, \qquad -1 < r < 1.$$

Set

$$x = \tfrac{1}{2} \log \frac{1 + r}{1 - r}, \qquad -1 < r < 1,$$

and define

$$(17) \qquad Y(x) = (e^x + e^{-x})\, y\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right).$$

$Y(x)$ is then a solution of the differential equation

$$(12) \qquad Y''(x) + p(x)\, Y(x) = 0, \qquad -\infty < x < \infty,$$

where

$$p(x) = g\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) - 1.$$

$p(x)$ has the properties (a)-(d′) of Lemma 3. The even solutions $y(r)$ of (2) transform into the even solutions $Y(x)$ of (12). The numbers a and α defined by conditions (d′) are connected by

$$\alpha = \tfrac{1}{2} \log \frac{1 + a}{1 - a},$$

and $g(r)$ strictly decreasing implies $p(x)$ is also strictly decreasing. If a_1 and a_2 are any pair of consecutive zeros of a solution $y(r)$ of (2), then, setting

$$\alpha_i = \tfrac{1}{2} \log \frac{1 + a_i}{1 - a_i}, \qquad i = 1, 2,$$

α_1 and α_2 will be the corresponding consecutive zeros of the corresponding solution $Y(x)$ of (12). We have now

$$\alpha_2 - \alpha_1 = \int_{\alpha_1}^{\alpha_2} dx = \int_{a_1}^{a_2} \frac{dx}{dr}\, dr = \int_{a_1}^{a_2} \frac{dr}{1 - r^2} = |[a_1 a_2]|.$$

By this equality it follows that (13) implies (8). We have thus proved Lemma 1 and with it the proof of Theorem 1 is complete.

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