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CHOQUET BOUNDARY FOR REAL FUNCTION ALGEBRAS

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Introduction. The concepts of Choquet boundary and Shilov boundary are well-established in the context of a complex function algebra (see **[2]** for example). There have been a few attempts to develop the concept of a Shilov boundary for real algebras, **[4]**, **[6]** and **[7]**. But there seems to be none to develop the concept of Choquet boundary for real algebras.

The aim of this paper is to develop the theory of Choquet boundary of a real function algebra (see Definition (1.8)) along the lines of the corresponding theory for a complex function algebra.

In the first section we define a real-part representing measure for a continuous linear functional ϕ on a real function algebra A with the property $\|\phi\| = 1 = \phi(1)$. The elements of A are functions on a compact, Hausdorff space X. The Choquet boundary is then defined as the set of those points $x \in X$ such that the real part of the evaluation functional, $\mathrm{Re}(e_x)$, has a unique real part representing measure. Several properties of the Choquet boundary are given including those that characterize the Choquet boundary (Theorem 1.17).

In the second section, we show that the closure of the Choquet boundary in X is the smallest closed boundary for A (see Theorem (2.4)). This is defined to be the Shilov boundary of A.

The third section deals with the complexification B of a real function algebra A. It is shown that the Choquet boundaries of A and B are the same. This is used to compute the Choquet boundary of the real disc algebra, (Example (3.11)). Finally we study a particular type of a real subalgebra of a complex function algebra U (Example (3.12)) and establish a certain relationship between the Choquet boundaries of the two algebras (Theorem (3.14)).

1. Choquet boundary.

As usual, **R** denotes the real line and **C** the complex plane.

Definition (1.1). Let X be a compact, Hausdorff space and τ a homeomorphism on X such that $\tau \circ \tau = \tau^2 =$ identity map on X. Then τ is called an *involution* on X or an *involutionary homeomorphism* on X.

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Definition (1.2). Let X be a compact, Hausdorff space. Then by $C(X)$ (respectively by $C_{\mathbf{R}}(X)$) we denote the complex (respectively real) Banach algebra of all continuous complex-valued (respectively real-valued) functions on X with supremum norm. Let τ be an involutionary homeomorphism on X and

$$C(X, \tau) = \{f \in C(X): \ f(\tau(x)) = \overline{f(x)} \text{ for all } x \in X\}.$$

Then $C(X, \tau)$ is a real commutative Banach algebra with identity 1. Also, $C(X, \tau)$ separates points on X, that is, for any x_1, x_2 in X with $x_1 \neq x_2$ there exists $f \in C(X, \tau)$ with $f(x_1) \neq f(x_2)$. A real function algebra on (X, τ) is a real subalgebra A of $C(X, \tau)$ such that

(i) A is uniformly closed in $C(X, \tau)$

(ii) A contains real constants

(iii) A separates points on X.

For examples of real function algebras and other details refer to [**5**].

Remark (1.3). Note that every real function algebra A is a real uniform algebra as defined in [**7**], that is, it is a real commutative Banach algebra with identity such that

$$\|f^2\| = \|f\|^2 \quad \text{for every } f \in A.$$

Conversely, a real uniform algebra can be viewed as a real function algebra as described in Section 1 of [**5**].

Definition (1.4). For a real function algebra A on (X, τ) the set of all non-zero real-linear homomorphisms of A into $\mathbf{C}$ is called the *carrier space* of A and is denoted by Φ_A. For $f \in A$, define a mapping

$$\hat{f}:\Phi_A \rightarrow \mathbf{C}$$

by $\hat{f}(\phi) = \phi(f)$ for $\phi \in \Phi_A$. $\hat{f}$ is called the *Gelfand transform* of $f \in A$. Then Φ_A is a compact, Hausdorff space with respect to the Gelfand topology, [**3**]. Observe that whenever $\phi \in \Phi_A$, the element $\overline{\phi}$ defined by $\overline{\phi}(f) = \overline{\phi(f)}$ is also in Φ_A, bar denoting complex conjugation. It is clear that each point x in X can be identified with the evaluation homeomorphism e_x defined by $e_x(f) = f(x)$ for $f \in A$. Define the map

$$\tau_0:\Phi_A \rightarrow \Phi_A$$

by $\tau_0(\phi) = \overline{\phi}$. Then X can be regarded as a subset of Φ_A and τ may be viewed as the restriction of τ_0 to X.

Definition (1.5). Let A be a real function algebra on (X, τ) and let

$$K_A = \{\phi \in A^*:\phi(1) = \|\phi\| = 1\}$$

where A^* denotes the set of all continuous linear functionals on A. It is obvious that K_A is a convex subset of the closed unit ball of A^* and K_A

contains $\mathrm{Re}(e_x)$ for each $x \in X$. It is easy to see that K_A is weak-star closed. So K_A is weak-star compact.

Remark (1.6). Note that if $\phi \in K_A$ then ϕ is a positive linear functional, that is, $\phi(f) \geqq 0$ whenever $f \geqq 0$.

Proof. Let $f \geqq 0$ and $||f|| \leqq 1$. Then $||1 - f|| \leqq 1$. Hence

$$\phi(1 - f) \leqq |\phi(1 - f)| \leqq ||\phi|| \, ||1 - f|| \leqq 1 \quad \text{or} \quad \phi(f) \geqq 0.$$

Definition (1.7). Let A be a real function algebra on (X, τ) and $\phi \in K_A$. A real part representing measure (r.p.r. measure) for ϕ is a regular Borel positive measure μ on X such that $\phi(f) = \int_X \mathrm{Re}\, f d\mu$ for all $f \in A$ and $\mu(E) = \mu(\tau(E))$ for all Borel subsets E of X.

That r.p.r. measure for $\phi \in K_A$ exists, can be seen as follows: Let

$$\mathrm{Re}\, A = \{\mathrm{Re}\, f\colon \ f \in A\}.$$

Then $\mathrm{Re}\, A$ is a subspace of $C_{\mathbf{R}}(X)$ and ϕ is a bounded linear functional on $\mathrm{Re}\, A$. Hence by applying the Riesz representation theorem to any Hahn-Banach extension of ϕ to $C_{\mathbf{R}}(X)$, we obtain a regular Borel measure μ_0 such that

$$\phi(f) = \int_X \mathrm{Re}\, f d\mu_0 \quad \text{for all } f \in A$$

and $||\phi|| = ||\mu_0||$ where $||\mu_0||$ denotes the total variation of μ_0. We may define μ by

$$\mu(E) = \frac{1}{2}[\mu_0(E) + \mu_0(\tau(E))]$$

for every Borel subset E of X. Then μ is a r.p.r. measure for ϕ. Note that since ϕ is positive and $||\phi|| = 1$, a r.p.r. measure for ϕ is a probability measure.

If ϕ is in the carrier space Φ_A, then $\mathrm{Re}\, \phi$ is in K_A and hence by the above arguments it has a r.p.r. measure. As in [**5**], we shall refer to it as r.p.r. measure for ϕ. Note that μ is a r.p.r. measure for $\phi \in \Phi_A$ if and only if it is a r.p.r. measure for $\bar{\phi}$.

Definition (1.8). Let A be a real function algebra on (X, τ). The *Choquet boundary* of A denoted by $\mathrm{Ch}(A)$ is the set of all $x \in X$ such that e_x admits a unique r.p.r. measure.

Remark (1.9). Note that this unique r.p.r. measure for e_x must be of the form $(1/2)(\delta_x + \delta_{\tau(x)}) = m_x$ (say) where δ_x denotes the pointmass at x, that is the Dirac-delta measure of x.

Observe that when $x = \tau(x)$, $m_x = \delta_x$. Also, if $x \in \mathrm{Ch}(A)$ then $\tau(x) \in \mathrm{Ch}(A)$ and vice-versa.

Now we prove a few properties of $\mathrm{Ch}(A)$. Proofs of many of these properties follow closely the analogous proofs in the complex case given in [**2**].

Throughout this section, A is a real function algebra on (X, τ) and $x \in X$.

THEOREM (1.10). *If* $x \notin \mathrm{Ch}(A)$, *there exists a* r.p.r. *measure* μ *for* e_x *with*

$$\mu(\{x, \tau(x)\}) = 0.$$

Proof. Let σ be a r.p.r. measure for e_x with $\sigma \neq m_x$ and let

$$c = \sigma(\{x, \tau(x)\}),$$

then $c < 1$.

Define

$$\mu = \frac{1}{1 - c}[\sigma - cm_x].$$

It is easily verified that

$$\int_X \mathrm{Re}\, f d\mu = \mathrm{Re}\, f(x) \quad \text{for every } f \in A,$$

that $\mu(\tau(E)) = \mu(E)$ for every Borel set E in X and that

$$\mu(\{x, \tau(x)\}) = 0.$$

Hence to finish the proof, all we need to check is that μ is a positive measure.

For this let E be any Borel subset of X. Then

Case (i). If $x \in E$ and $\tau(x) \in E$, then

$$\mu(E) = \frac{1}{1 - c}[\sigma(E) - c] \geqq 0.$$

Case (ii). If $x \in E$ and $\tau(x) \notin E$, then

$$\sigma(E) \geqq \sigma(\{x\}) \quad \text{and} \quad \sigma(\tau(E)) \geqq \sigma(\{\tau(x)\}).$$

Since $\sigma(\tau(E)) = \sigma(E)$, it follows that $2\sigma(E) \geqq c$ but

$$\mu(E) = \sigma(E) - \frac{c}{2} \geqq 0.$$

The case when $x \notin E$ but $\tau(x) \in E$ is similar to case (ii) and finally

Case (iii). If $x \notin E$ and $\tau(x) \notin E$, then

$$\mu(E) = \sigma(E) \geqq 0.$$

This shows that μ is a r.p.r. measure for e_x with

$$\mu(\{x, \tau(x)\}) = 0.$$

The next theorem provides a relation between $\mathrm{Ch}(A)$ and the extreme points of the set K_A.

THEOREM (1.11). *Let $\phi \in K_A$. Then ϕ is an extreme point of K_A if and only if $\phi = \mathrm{Re}(e_x)$ for some $x \in \mathrm{Ch}(A)$.*

Proof. Let $x \in \mathrm{Ch}(A)$ and $\mathrm{Re}(e_x) = t\psi_1 + (1 - t)\psi_2$ where $\psi_1, \psi_2 \in K_A$ and $0 < t < 1$.

Suppose that μ_1 and μ_2 are r.p.r. measures for ψ_1 and ψ_2 respectively. Then $t\mu_1 + (1 - t)\mu_2$ is a r.p.r. measure for e_x. For,

$$\begin{aligned}
&\int \mathrm{Re}\, fd(\mu_1 t + (1 - t)\mu_2) \\
&= t\int \mathrm{Re}\, fd\mu_1 + (1 - t)\int \mathrm{Re}\, fd\mu_2 \\
&= t\psi_1(f) + (1 - t)\psi_2(f) \\
&= \mathrm{Re}(e_x(f)) \\
&= \mathrm{Re}(f(x)).
\end{aligned}$$

But $x \in \mathrm{Ch}(A)$ and so

$$m_x = \frac{1}{2}(\delta_x + \delta_{\tau(x)}) = t\mu_1 + (1 - t)\mu_2$$

where $0 < t < 1$. Since μ_1 and μ_2 are positive Borel measures, it follows that $\mu_1(E) = \mu_2(E) = 0$ whenever E is a Borel subset of X and $x \notin E$, $\tau(x) \notin E$. Thus $\mu_1 = \mu_2 = m_x$ and hence $\psi_1 = \psi_2 = \mathrm{Re}(e_x)$. Thus $\mathrm{Re}(e_x)$ is an extreme point of K_A.

Conversely let ϕ be an extreme point of K_A. Let μ be a r.p.r. measure for ϕ and suppose that $x \in \mathrm{Supp}(\mu)$. Then $\tau(x) \in \mathrm{Supp}(\mu)$. (Note that $\mu(\tau(E)) = \mu(E)$ for every Borel set E contained in X.)

If for some neighbourhood U of $\{x, \tau(x)\}$ with $U = \tau(U)$ we have $\mu(U) < 1$ define θ and ψ by

$$\theta(f) = \frac{1}{\mu(U)}\int_U \mathrm{Re}\, fd\mu,$$

$$\psi(f) = \frac{1}{1 - \mu(U)}\int_{X-U} \mathrm{Re}\, fd\mu, \quad \text{for all } f \in A.$$

Then $\theta, \psi \in K_A$ and

$$\phi = \mu(U)\theta + [1 - \mu(U)]\psi.$$

Since ϕ is an extreme point of K_A, $\theta = \psi = \phi$. Thus

$$\phi(f) = \frac{1}{\mu(U)}\int_U \mathrm{Re}\, fd\mu \quad \text{for all } f \text{ in } A.$$

If $\mu(U) < 1$ for some neighbourhood U of $\{x, \tau(x)\}$ with $U = \tau(U)$ then $\mu(V) < 1$ for any smaller neighbourhood V with $(V = \tau(V))$ same properties so that

$$\phi(f) = \frac{1}{\mu(V)} \int_V \operatorname{Re} f d\mu$$

for all f in A and for arbitrarily small neighbourhoods V of $\{x, \tau(x)\}$ with $V = \tau(V)$. So,

$$\phi(f) = \frac{1}{\mu(\{x, \tau(x)\})} \int_{\{x,\tau(x)\}} \operatorname{Re} f d\mu = \operatorname{Re}(f(x))$$

by a simple calculation. Thus $\phi = \operatorname{Re}(e_x)$. This also implies that

$$\mu(\{x, \tau(x)\}) \neq 0.$$

Thus, we have shown that for every r.p.r. measure μ for e_x,

$$\mu(\{x, \tau(x)\}) \neq 0.$$

Hence by Theorem (1.10) $x \in \operatorname{Ch}(A)$.

THEOREM (1.12). *Let $x \in X$. Suppose there exist constants α, β with $0 < \alpha < \beta < 1$ such that for every neighbourhood U of $\{x, \tau(x)\}$ with $U = \tau(U)$, there exists f in A with $\|f\| \leqq 1$, $\operatorname{Re} f(x) > \beta$ and $|f(y)| < \alpha$ for all $y \notin U$. Then $x \in \operatorname{Ch}(A)$.*

Proof. Let μ be a r.p.r. measure for e_x and U a τ-invariant neighbourhood of x. Then $\tau(x) \in U$ and

$$\begin{aligned}\beta < \operatorname{Re} f(x) &= \int_X \operatorname{Re} f d\mu \\ &= \int_U \operatorname{Re} f d\mu + \int_{X-U} \operatorname{Re} f d\mu \\ &\leqq \mu(U) + \alpha\mu(X - U) \\ &= \alpha + (1 - \alpha)\mu(U)\end{aligned}$$

since $\mu(X) = 1$.

Thus

$$\mu(U) > \frac{\beta - \alpha}{1 - \alpha}$$

for any τ-invariant neighbourhood U of $\{x, \tau(x)\}$. Hence

$$\mu(\{x, \tau(x)\}) \geqq \frac{\beta - \alpha}{1 - \alpha}.$$

The theorem now follows by invoking Theorem (1.10).

THEOREM (1.13). *Let $x \in X$. Suppose there exist constants α, β with $0 < \alpha < \beta$ such that for every neighbourhood U of x where U is τ-invariant, there exists f in A with $\operatorname{Re} f \leqq 0$, $\operatorname{Re} f(x) > -\alpha$ and $\operatorname{Re} f(y) < -\beta$ for all $y \in X - U$. Then $x \in \operatorname{Ch}(A)$.*

Proof. Let U be a τ-invariant neighbourhood of $\{x, \tau(x)\}$ and μ a r.p.r. measure for e_x. Then

$$\begin{aligned} -\alpha < \operatorname{Re} f(x) &= \int_X \operatorname{Re} f d\mu \\ &= \int_U \operatorname{Re} f d\mu + \int_{X-U} \operatorname{Re} f d\mu \\ &\leqq \int_{X-U} \operatorname{Re} f d\mu \end{aligned}$$

since $\operatorname{Re} f \leqq 0$ on X

$$\begin{aligned} &< -\beta\mu(X - U) \\ &= -\beta(1 - \mu(U)) \quad \text{as } \mu(X) = 1. \end{aligned}$$

Thus

$$\mu(U) > \frac{\beta - \alpha}{\beta}$$

for any r.p.r. measure μ for e_x and any τ-invariant neighbourhood U of x. So by Theorem (1.10) $x \in \operatorname{Ch}(A)$.

Before proceeding further we introduce a notation:

Let X be a compact, Hausdorff space and τ an involutionary homeomorphism on X. Also let,

$$C_{\mathbf{R}}(X, \tau) = \{u \in C_{\mathbf{R}}(X): \quad u(\tau(x)) = u(x) \quad \text{for all } x \in X\} \quad \text{and}$$

$$C_S(X, \tau) = \{v \in C_{\mathbf{R}}(X): \quad v(\tau(x)) = -v(x) \quad \text{for all } x \in X\}.$$

Let $u \in C_{\mathbf{R}}(X)$. Then

$$u(x) = \frac{1}{2}[v(x) + w(x)]$$

where v and w are defined by

$$v(x) = \frac{1}{2}[u(x) + u(\tau(x))], \quad w(x) = \frac{1}{2}[u(x) - u(\tau(x))].$$

Thus $v \in C_{\mathbf{R}}(X, \tau)$ and $w \in C_S(X, \tau)$. Thus every element of $C_{\mathbf{R}}(X)$ can be uniquely decomposed as a sum of two elements, one from $C_{\mathbf{R}}(X, \tau)$ and the other from $C_S(X, \tau)$. If A is a real function algebra on (X, τ) and $f \in A$ then

$$\operatorname{Re} f \in C_{\mathbf{R}}(X, \tau) \quad \text{and} \quad \operatorname{Im} f \in C_S(X, \tau).$$

THEOREM (1.14). *Let* $\phi \in K_A$, $u \in C_{\mathbf{R}}(X, \tau)$ *and*

$$\alpha = \sup\{\phi(f): \quad f \in A, \operatorname{Re} f \leqq u\},$$

$$\beta = \{\inf \phi(f): \quad f \in A, \operatorname{Re} f \geqq u\}$$

and so $\alpha \leqq \beta$. *For any* γ *with* $\alpha \leqq \gamma \leqq \beta$ *there exists a* r.p.r. *measure* μ *for* ϕ *with* $\int u d\mu = \gamma$.

Proof. Replacing u by $u - \gamma$ we may assume that $\gamma = 0$. So, $\alpha \leqq 0 \leqq \beta$. Let

$$N = \{f \in C(X): \quad \operatorname{Re} f \leqq tu + \operatorname{Re} g \text{ for some } t \in \mathbf{R}, \text{ some } g \in A \text{ with } \phi(g) \leqq 0\}.$$

$$P = \{f \in C(X): \quad \operatorname{Re} f > 0\}.$$

Clearly N and P are convex cones, that is, N and P are convex sets and are also closed under addition and multiplication by real, non-negative scalars. Further we claim that N and P are disjoint sets. If $f \in N \cap P$ then $f \in C(X)$ and $\operatorname{Re} f \leqq tu + \operatorname{Re} g$ for some real t, for some $g \in A$ with $\phi(g) \leqq 0$, $\operatorname{Re} f > 0$. These imply that $tu + \operatorname{Re} g \geqq \operatorname{Re} f > 0$ or

(1) $\operatorname{Re} g > -tu$.

Case (i). Let $t > 0$. Then (1) implies

$$\operatorname{Re}\left(\frac{g}{t}\right) > -u \quad \text{or} \quad \operatorname{Re}\left(-\frac{g}{t}\right) \leqq u.$$

Hence

$$\phi\left(-\frac{g}{t}\right) \leqq \alpha \quad \text{or} \quad \phi(-g) \leqq t\alpha.$$

As $t > 0$, $\alpha \leqq 0$, $\phi(g) \geqq -t\alpha \geqq 0$.

Case (ii). Let $t < 0$. Then (1) implies that

$$\operatorname{Re}\left(-\frac{g}{t}\right) > u \quad \text{as } -t > 0.$$

Hence,

$$\phi\left(-\frac{g}{t}\right) \geqq \beta \quad \text{or} \quad \phi(g) \geqq -t\beta.$$

Thus $\phi(g) \geqq -t\beta \geqq 0$ as $t < 0$, $\beta \geqq 0$.

Case (iii). Let $t = 0$. Then, $\operatorname{Re} g \geqq \operatorname{Re} f$ implies $\phi(g) \geqq \phi(f)$ as $\phi \in K_A$ by Remark (1.6). So $\phi(f) > 0$.

Thus in all cases, $\phi(g) \geqq 0$ which is a contradiction. So N and P are disjoint sets. Hence by the Hahn-Banach separation theorem, there exists a non-zero θ in $(C(X))^*$ with $\theta(f) \leqq 0$ for f in N and $\theta(f) \geqq 0$ for f in P. Now $\theta(f) \geqq 0$ for $f \in P$ implies that θ is a positive linear functional on $C(X)$. So, we may assume that $\theta(1) = 1$. If $f \in A$ then

$$\pm(f - \phi(f)) \in N.$$

For if $g = f - \phi(f)$ then $\phi(g) = 0$ and

$$\operatorname{Re} g \leqq 0 \cdot u + \operatorname{Re} g.$$

So $g \in N$. Similarly, $-(f - \phi(f)) \in N$. Therefore, $\pm(f - \phi(f)) \in N$ and so

$$\theta(f - \phi(f)) \leqq 0 \quad \text{and} \quad \theta(\phi(f) - f)) \leqq 0$$

which yield $\theta(f) = \phi(f)$.

Also it can be proved that $\pm u \in N$ by taking $t = 1$, $g = 0$ in the definition of the set N. So $\theta(u) = 0$. Let μ_0 be the representing measure for θ. Define

$$\mu(E) = \frac{1}{2}[\mu_0(E) + \mu_0(\tau(E))]$$

for every Borel subset E of X. Then for any $w \in C_{\mathbf{R}}(X, \tau)$,

$$\int_X w d\mu = \int_X w d\mu_0.$$

In particular,

$$\int_X u d\mu = \int_X u d\mu_0 = \theta(u) = 0$$

and for $f \in A$,

$$\int_X \operatorname{Re} f d\mu = \int_X \operatorname{Re} f d\mu_0 = \operatorname{Re}(\theta(f)) = \operatorname{Re} \phi(f) = \phi(f).$$

Thus μ is a r.p.r. measure for ϕ.

THEOREM (1.15). *Let $\phi \in K_A$. Then ϕ admits a unique* r.p.r. *measure μ if and only if for every $u \in C_{\mathbf{R}}(X, \tau)$,*

$$\sup\{\phi(f): f \in A, \operatorname{Re} f \leqq u\} = \int_X u d\mu$$
$$= \inf\{\phi(f): f \in A, \operatorname{Re} f \geqq u\}.$$

Proof. Let

$$\alpha_u = \sup\{\phi(f): f \in A, \operatorname{Re} f \leqq u\},$$
$$\beta_u = \inf\{\phi(f): f \in A, \operatorname{Re} f \geqq u\}$$

where $u \in C_{\mathbf{R}}(X, \tau)$. Then, for every r.p.r. measure σ for ϕ,

$$\phi(f) = \int_X \operatorname{Re} f d\sigma \leqq \int_X u d\sigma \quad \text{for all } f \text{ in } A.$$

Hence

$$\alpha_u \leqq \int_X u d\sigma.$$

Similarly

$$\int_X u d\sigma \leqq \beta_u.$$

Thus

$$\alpha_u \leqq \int_X u d\mu \leqq \beta_u.$$

Hence if $\alpha_u = \beta_u$ for every $u \in C_{\mathbf{R}}(X, \tau)$ there can exist only one r.p.r. measure μ for ϕ such that

$$\alpha_u = \beta_u = \int_X u d\mu.$$

Conversely if for some $u \in C_{\mathbf{R}}(X, \tau)$, $\alpha_u < \beta_u$, we can find γ_1, γ_2 such that $\alpha_u \leqq \gamma_1 \leqq \gamma_2 \leqq \beta_u$. Then by Theorem (1.14) there exist r.p.r. measures μ_1 and μ_2 for ϕ such that

$$\gamma_1 = \int_X u d\mu_1 \neq \int_X u d\mu_2 = \gamma_2.$$

COROLLARY (1.16). *Let $x \in X$. Then $x \in \mathrm{Ch}(A)$ if and only if for every $u \in C_{\mathbf{R}}(X, \tau)$,*

$$\sup\{\operatorname{Re} f(x): \ f \in A, \operatorname{Re} f \leqq u\} = u(x)$$
$$= \inf\{\operatorname{Re} f(x): \ f \in A, \operatorname{Re} f \geqq u\}.$$

Proof. This follows by applying Theorem (1.15) to the special case $\phi = \operatorname{Re}(e_x) \in K_A$.

The following theorem should be compared to Theorem (2.2.6) of [**2**].

THEOREM (1.17). *Let $x \in X$. Then the following statements are equivalent*:

(i) $x \in \mathrm{Ch}(A)$.

(ii) *If μ is a* r.p.r. *measure for e_x then $\mu(\{x\}) > 0$.*

(iii) *For every α, β with $0 < \alpha < \beta$ and for every τ-invariant neighbourhood U of x there exists f in A with $\operatorname{Re} f \leqq 0$, $\operatorname{Re} f(x) > -\alpha$ and $\operatorname{Re} f(y) < -\beta$ for all $y \in X - U$.*

(iv) *There exist α, β with $0 < \alpha < \beta$ such that for every τ-invariant neighbourhood U of x there exists $f \in A$ with $\operatorname{Re} f \leqq 0$, $\operatorname{Re} f(x) > -\alpha$ and $\operatorname{Re} f(y) < -\beta$ for all $y \in X - U$.*

(v) *For all $u \in C_{\mathbf{R}}(X, \tau)$,*

$$\sup\{\operatorname{Re} f(x): \ f \in A, \operatorname{Re} f \leqq u\} = u(x)$$
$$= \inf\{\operatorname{Re} f(x): \ f \in A, \operatorname{Re} f \geqq u\}.$$

Proof. Let (i) hold. If $x \in \mathrm{Ch}(A)$, m_x is the only r.p.r. measure for e_x. Hence (ii) follows.

(ii) implies (i) is Theorem (1.10).

We will now prove that (i) implies (iii). Let $x \in \mathrm{Ch}(A)$, U a τ-invariant neighbourhood of x and α, β be such that $0 < \alpha < \beta$. By Urysohn's

Lemma, there exists $w \in C_{\mathbf{R}}(X)$ such that $w \leqq 0$, $w(x) = 0$ and $w < -\sqrt{\beta}$ on $X - U$. Define a function u by

$$u(s) = -w(s)w(\tau(s)) \quad \text{for all } s \in X.$$

Then $u \in C_{\mathbf{R}}(X, \tau)$. Since $w(x) = 0$, we have $u(x) = 0$. Since $w(s) < -\sqrt{\beta}$ for all $s \in X - U$ and since $U = \tau(U)$, $w(\tau(s)) < -\sqrt{\beta}$. Hence $u(s) < -\beta$ for all $s \in X - U$. By Corollary (1.16),

$$\sup\{\operatorname{Re} f(x): \ f \in A, \operatorname{Re} f \leqq u\} = u(x) = 0 > -\alpha.$$

Hence there exists $f \in A$ such that $\operatorname{Re} f \leqq u$ and $\operatorname{Re} f(x) > -\alpha$. Since $u < -\beta$ on $X - U$, $\operatorname{Re} f \leqq u < -\beta$ on $X - U$.

That (iii) implies (iv) is obvious.

(iv) implies (i) is Theorem (1.13).

(i) and (v) are equivalent in view of Corollary (1.16).

Remark (1.18). At this point the reader may ask when the Choquet boundary of a real function algebra on (X, τ) is the whole of X. Theorem (1.20) gives a sufficient condition for this.

Definition (1.19). A real function algebra A on (X, τ) is called a *real Dirichlet algebra* if $\operatorname{Re} A$ is dense in $C_{\mathbf{R}}(X, \tau)$.

THEOREM (1.20). *If A is a real Dirichlet algebra on (X, τ) then* $\operatorname{Ch}(A) = X$.

Proof. Let $x \in X$ be any point and μ_1, μ_2 be r.p.r. measures for e_x. Then,

$$\int_X \operatorname{Re} f d\mu_1 = \int_X \operatorname{Re} f d\mu_2 = \operatorname{Re} f(x) \quad \text{for all } f \in A.$$

Hence for all $u \in C_{\mathbf{R}}(X, \tau)$,

$$\int_X u d\mu_1 = \int_X u d\mu_2,$$

since $\operatorname{Re} A$ is dense in $C_{\mathbf{R}}(X, \tau)$. It is also easy to see that

$$\int_X v d\mu_1 = \int_X v d\mu_2 = 0 \quad \text{for all } v \in C_S(X, \tau).$$

Let $w \in C_{\mathbf{R}}(X)$. Then, w can be written uniquely as $w = u + v$ where $u \in C_{\mathbf{R}}(X, \tau)$, $v \in C_S(X, \tau)$. Hence

$$\int_X w d\mu_1 = \int_X w d\mu_2 \quad \text{for all } w \in C_{\mathbf{R}}(X),$$

that is, $\mu_1 = \mu_2$. Thus e_x admits a unique r.p.r. measure. Hence, $x \in \operatorname{Ch}(A)$.

Example (1.21). Let $A = C(X, \tau)$. Obviously, A is a real Dirichlet algebra. Hence in view of the above Theorem (1.20), $\operatorname{Ch}(A) = X$.

Remark (1.22). Since for a real function algebra A on (X, τ) X can be identified with a subset of Φ_A by the map $x \rightarrow e_x$ we can regard $\mathrm{Ch}(A)$ as a subset of Φ_A. Let M_A be the maximal ideal space of A. Then for each f in A, $\mathrm{Re}\,\hat{f}$ and $|\hat{f}|$ are well-defined real-valued functions on M_A. It was shown in [5] that the smallest topology on M_A making $\mathrm{Re}\,\hat{f}$ continuous for all $f \in A$ is the same as the smallest topology on M_A making $|\hat{f}|$ continuous for all $f \in A$ and that M_A is a compact, Hausdorff space with respect to this topology. Let $T:\Phi_A \rightarrow M_A$ be defined by

$$T(\phi) = \phi^{-1}(\{0\}), \quad \phi \in \Phi_A.$$

Note that $T(\phi) = T(\bar{\phi})$ for all $\phi \in \Phi_A$.

COROLLARY (1.23). *Let $y \in M_A$. Then the following statements are equivalent*:

(i) $y \in T(\mathrm{Ch}(A))$.

(ii) *For every α, β with $0 < \alpha < \beta$ and for every neighbourhood V of y in M_A there exists f in A such that* $\mathrm{Re}\,\hat{f} \leqq 0$, $\mathrm{Re}\,\hat{f}(y) > -\alpha$ *and* $\mathrm{Re}\,\hat{f}(z) < -\beta$ *for all $z \in M_A - V$.*

(iii) *There exists α, β with $0 < \alpha < \beta$ and for every neighbourhood V of y in M_A there exists $f \in A$ such that* $\mathrm{Re}\,\hat{f} \leqq 0$, $\mathrm{Re}\,\hat{f}(y) > -\alpha$ *and* $\mathrm{Re}\,\hat{f}(z) < -\beta$ *for all $z \in M_A - V$.*

(iv) *For all $w \in C_{\mathbf{R}}(M_A)$*

$$\begin{aligned}\sup\{\mathrm{Re}\,\hat{f}(y) : f \in A, \mathrm{Re}\,\hat{f} \leqq w\} &= w(y)\\ &= \inf\{\mathrm{Re}\,\hat{f}(y) : f \in A, \mathrm{Re}\,\hat{f} \geqq w\}.\end{aligned}$$

2. Shilov boundary.

Definition 2.1. Let A be a real function algebra on (X, τ) and $S \subset X$. S is called a *Choquet set* (respectively a *boundary*) if $S = \tau(S)$ and if $\mathrm{Re}\, f$ (respectively $|f|$) assumes its maximum on S for all $f \in A$.

Remark (2.2). Choquet set and boundary of a real commutative Banach algebra with unit were defined in [6] as subsets of M_A. Our proof of the following theorem (Theorem (2.3)) is similar to that of an analogous Theorem in [6].

THEOREM (2.3). (i) *Every boundary for A is a Choquet set for A.*

(ii) *Every closed Choquet set for A is a boundary for A.*

Proof. (i) follows from the fact that

$$\mathrm{Re}\, f = \log|\exp(f)| \quad \text{for all } f \in A.$$

(ii) Let S be a closed Choquet set for A. If possible, let S be not a boundary for A. Then there exists $f \in A$, $\epsilon < 1$ and $y \in X$ such that $|f| \leqq \epsilon$ on S and $|f(y)| = 1$. Since for each positive integer n,

$$|\mathrm{Re}(f^n)| \leqq |f^n| \leqq \epsilon^n$$

on S and S is a Choquet set for A, $|\mathrm{Re}(f^n)| \leqq \epsilon^n$ on X and in particular at y. Since $|f(y)| = 1$ let $f(y) = \exp(ia)$ for some real number a. Thus

$$|\mathrm{Re}(f^n)(y)| = |\cos na| \leqq \epsilon^n$$

for each positive integer n. But as $n \to \infty$, $\epsilon^n \to 0$ while $\cos na$ does not and hence we have a contradiction. Thus S is a boundary for A.

THEOREM (2.4). $\mathrm{cl}(\mathrm{Ch}(A))$ = *closure of* $\mathrm{Ch}(A)$ *is the smallest closed boundary as well as the smallest closed Choquet set for* A.

Proof. First we shall prove that $\mathrm{cl}(\mathrm{Ch}(A))$ is a boundary. Recall that

$$K_A = \{\phi \in A^*: \quad \phi(1) = \|\phi\| = 1\}.$$

Since K_A = convex hull of $\mathrm{ext}(K_A)$ where $\mathrm{ext}(K_A)$ is the set of all extreme points of K_A we see that for all f in A

$$\sup\{|\mathrm{Re}\, f(x)|: \quad x \in X\} \leqq \sup\{|\phi(f)|: \quad \phi \in K_A\}$$

as $\mathrm{Re}\, e_x \in K_A$

$$\begin{aligned} &= \sup\{|\phi(f)|: \quad \phi \in \mathrm{conv}(\mathrm{ext}\, K_A)\}, \\ &= \sup\{|\phi(f)|: \quad \phi \in \mathrm{cl}(\mathrm{ext}\, K_A)\} \\ &= \sup\{|\mathrm{Re}\, f(x)|: \quad x \in \mathrm{cl}(\mathrm{Ch}(A))\}, \end{aligned}$$

in view of Theorem (1.11). Thus $\mathrm{cl}(\mathrm{Ch}(A))$ is a Choquet set and since it is closed it is a boundary for A.

Now we prove that $\mathrm{cl}(\mathrm{Ch}(A))$ is contained in every closed boundary. In view of Theorem (2.3), it suffices to prove that $\mathrm{Ch}(A)$ is contained in every closed Choquet set for A. Let $x \in \mathrm{Ch}(A)$ and U a τ-invariant neighbourhood of x. Then by Theorem (1.17) for all α, β with $0 < \alpha < \beta$ and for all τ-invariant neighbourhoods of x there exists $f \in A$ such that $\mathrm{Re}\, f \leqq 0$, $\mathrm{Re}\, f(x) > -\alpha$ and $\mathrm{Re}\, f(y) < -\beta$ for all $y \in X - U$. Thus Choquet boundary of A is contained in every closed Choquet set for A and hence in every closed boundary for A. This proves the theorem.

Definition (2.5). $\mathrm{cl}(\mathrm{Ch}(A))$ which is the smallest closed boundary and Choquet set for A, which exists by Theorem (2.4) is called the *Shilov boundary* for A and is denoted by $S(A)$.

Remark (2.6). The concept of Shilov boundary for a real commutative Banach algebra A has been defined in **[4]**, **[6]** and **[7]**. In **[6]** and **[7]** the Shilov boundary of A is a subset of the maximal ideal space M_A of A whereas in **[4]** it is a subset of the carrier space Φ_A of A. For a real function algebra, Shilov boundary as defined in **[4]** coincides with our definition whereas the Shilov boundary as defined in **[6]** and **[7]** is the image of the Shilov boundary as defined above under the map T. However, our approach is entirely different.

3. Complexification. Let A be a real function algebra on (X, τ). Define

$$B = \{f + ig: \ f, g \in A\}.$$

It is seen that for f, g in A $\|f + ig\| = \|f - ig\|$ so that

$$\|f\|, \|g\| \leqq \|f + ig\| \leqq \|f\| + \|g\|$$

which shows that B is uniformly closed in $C(X)$. So B is a complex function algebra on X and may be regarded as the complexification of A, [**5**]. For definition and properties of the Choquet boundary of a complex function algebra refer to [**2**]. In this section we prove that the Choquet boundaries of A and B coincide.

Definition (3.1). Let M_B, the maximal ideal space of B, be identified with the space Φ_B of all non-zero complex homomorphisms of B as usual. Define $\alpha:\Phi_A \to \Phi_B$ by

$$\alpha(\phi)(f + ig) = \phi(f) + i\phi(g) \quad \text{for } \phi \in \Phi_A, f, g \in A.$$

Then, α is a bijection and $\alpha(\phi)|_A = \phi$.

Definition (3.2). Let μ be a Borel measure on X. Define a measure μ_τ on X by $\mu_\tau(E) = \mu(\tau(E))$ for all Borel subsets E of X. If h is μ-measurable then it can be proved that h is also μ_τ-measurable and in a straight forward manner, one has

$$\int_X h d\mu_\tau = \int_X (h \circ \tau) d\mu.$$

Also note that $(\mu_\tau)_\tau = \mu$.

THEOREM (3.3). *Let A be a real function algebra on (X, τ), B its complexification and $\phi \in \Phi_A$. Suppose μ is a representing measure for $\alpha(\phi)$. Then μ_τ is a representing measure for $\alpha(\bar{\phi})$.*

Proof. Let $f + ig \in B$ where $f, g \in A$. Then

$$\begin{aligned}\int_X (f + ig) d\mu_\tau &= \int_X f d\mu_\tau + i \int_X g d\mu_\tau \\ &= \int_X (f \circ \tau) d\mu + i \int_X (g \circ \tau) d\mu \\ &= \int_X \bar{f} d\mu + i \int_X \bar{g} d\mu \\ &= \overline{\left(\int_X f d\mu\right)} + i\overline{\left(\int_X g d\mu\right)}\end{aligned}$$

as μ is a real measure

$$= \overline{\phi(f)} + i\overline{\phi(g)}$$
$$= (\alpha(\overline{\phi}))(f + ig).$$

COROLLARY (3.4). *Under the hypotheses of Theorem* (3.3) $(1/2)[\mu + \mu_\tau]$ *is a* r.p.r. *measure for* ϕ.

COROLLARY (3.5). $\alpha(\phi)$ *has a unique representing measure if and only if* $\alpha(\overline{\phi})$ *has a unique representing measure.*

COROLLARY (3.6). $x \in \mathrm{Ch}(B)$ *if and only if* $\tau(x) \in \mathrm{Ch}(B)$.

Next we will prove that $\mathrm{Ch}(A) = \mathrm{Ch}(B)$.

THEOREM (3.7). $\mathrm{Ch}(A) = \mathrm{Ch}(B)$.

Proof. First we will show that $\mathrm{Ch}(A) \subset \mathrm{Ch}(B)$. If possible assume that $x \notin \mathrm{Ch}(B)$. Then by Theorem (2.3.4), Chapter II of [**2**], there exists a representing measure μ for e_x with $\mu(\{x\}) = 0$. Let

$$\sigma = \frac{\mu + \mu_\tau}{2}.$$

Then σ is a r.p.r. measure for e_x by Corollary (3.4) and

$$\sigma(\{x\}) = \frac{1}{2}[\mu(\{x\}) + \mu(\{\tau(x)\})] = \frac{1}{2}\mu(\{\tau(x)\}).$$

Case (a). Let $x = \tau(x)$. Then $\sigma(\{x\}) = 0$ and hence $x \notin \mathrm{Ch}(A)$ by Theorem (1.10).

Case (b). Let $x \neq \tau(x)$. As A separates points on X, there exists a function $h \in A$ such that $h(x) \neq h(\tau(x))$. We may assume that

$$h(x) = i, \quad h(\tau(x)) = -i.$$

Therefore, $\int_X h d\mu = i$ but

$$\int_X h d\delta_{\tau(x)} = -i.$$

Thus $\mu \neq \delta_{\tau(x)}$. Hence

$$\mu(\{\tau(x)\}) = c < 1$$

and so

$$\sigma(\{x\}) = \frac{1}{2}c < \frac{1}{2}.$$

But $m_x(\{x\}) = 1/2$.

Thus we have shown that there exists a r.p.r. measure $\sigma \neq m_x$ for e_x. Hence $x \notin \mathrm{Ch}(A)$. Thus $\mathrm{Ch}(A)$ is contained in $\mathrm{Ch}(B)$.

Conversely let $x \in \mathrm{Ch}(B)$. Suppose U is a τ-invariant neighbourhood of x. Choose $\epsilon > 0$ such that $0 < \epsilon < 1/3$. Thus

$$0 < \epsilon < \frac{1}{2}(1 - \epsilon) < 1.$$

We will show that there exists $f \in A$ such that $\|f\| \leqq 1$, $\operatorname{Re} f(x) > (1/2)(1 - \epsilon)$ and $|f(y)| < \epsilon$ for all $y \in X - U$. This will imply that $x \in \mathrm{Ch}(A)$ by Theorem (1.12).

Case (a). Let $x = \tau(x)$. As $x \in \mathrm{Ch}(B)$ by Theorem (2.3.4), Chapter II of [**2**], there exists $f + ig \in B$ such that

$$\|f + ig\| \leqq 1, \quad f(x) + ig(x) = 1 \quad \text{and}$$

$$|f(y) + ig(y)| < \epsilon \quad \text{for all } y \in X - U.$$

Now

$$f(x) - ig(x) = f(\tau(x)) - ig(\tau(x)) = \overline{f(x) + ig(x)} = 1.$$

Hence we have $f(x) = 1$. Let $y \in X - U$. Then $\tau(y) \in X - U$. Hence

$$|f(y) + ig(y)| < \epsilon \quad \text{and} \quad |f(\tau(y)) - ig(\tau(y))| < \epsilon,$$

that is,

$$|f(y) - ig(y))| < \epsilon.$$

Thus,

$$|f(y)| \leqq \frac{1}{2}[\,|f(y) + ig(y)| + |f(y) - ig(y)|\,] < \epsilon$$

clearly $f \in A$ and $\|f\| \leqq \|f + ig\| \leqq 1$.

Case (b). $x \neq \tau(x)$. As X is Hausdorff, there exists a neighbourhood V of x such that $\tau(x) \notin V$. Let $W = U \cap V$. Then W is a neighbourhood of x. As before, there exists a function $f + ig \in B$ such that

$$\|f + ig\| \leqq 1, \quad f(x) + ig(x) = 1 \quad \text{and}$$

$$|f(y) + ig(y)| < \epsilon \quad \text{for all } y \in X - W.$$

Clearly $f \in A$, $\|f\| \leqq 1$. If $y \in X - U$ both y and $\tau(y)$ are in $X - W$ and hence as above $|f(y)| < \epsilon$.

Also since $f(x) + ig(x) = 1$ we have

$$\operatorname{Re} f(x) - \operatorname{Im} g(x) = \operatorname{Re}((f + ig)(x)) = 1.$$

Further,

$$|\operatorname{Re} f(x) + \operatorname{Im} g(x)| = |\operatorname{Re}((f - ig)(x))|$$
$$\leqq |(f - ig)(x)| = |(f + ig)(\tau(x))| < \epsilon$$

as $\tau(x) \notin W$. Hence

$$\begin{aligned}&|\mathrm{Re}\, f(x)|\\ &\geqq \frac{1}{2}[\,|\mathrm{Re}\, f(x) - \mathrm{Im}\, g(x)| - |\mathrm{Re}\, f(x) + \mathrm{Im}\, g(x)|\,]\\ &> \frac{1}{2}(1 - \epsilon).\end{aligned}$$

Replacing f by $-f$ if necessary, we have $\mathrm{Re}\, f(x) > (1/2)(1 - \epsilon)$.

COROLLARY (3.8). $S(A) = S(B)$.

Remark (3.9). We have pointed out earlier (Remark (2.6)) that in **[6]** and **[7]** the Shilov boundary is a subset of M_A. Let

$$cx^*: M_B \rightarrow M_A$$

be the restriction map. Then Proposition (2.1) of **[6]** is equivalent to the assertions

$$cx^*(S(B)) = \tau(S(A)) \quad \text{and}$$
$$(cx^*)^{-1}(\tau(S(A))) = S(B).$$

This fact follows immediately from Corollary (3.8) by noting that we have identified M_B with Φ_B and hence $cx^* = \tau \circ \alpha^{-1}$.

Remark (3.10). Let A be a real function algebra on (X, τ), and X a metrisable space. By Corollary (2.2.7), Chapter II of **[2]**, $\mathrm{Ch}(B)$ is a G_δ set. Hence $\mathrm{Ch}(A)$ is a G_δ set by Theorem (3.7).

Examples. We can use Theorem (3.7) to compute the Choquet boundaries of those real function algebras whose complexifications are well-known complex function algebras. This technique is illustrated in the following example.

Example (3.11). (Real disc algebra). Let D be the closed unit disc in the complex plane. Define $\tau: D \rightarrow D$ by $\tau(z) = \bar{z}$ for all $z \in D$. Let

$$A = \{f \in C(D, \tau): \text{ The restriction of } f \text{ to the interior of } D \text{ is analytic}\}.$$

Then A is a real function algebra on (D, τ) and its complexification B is the well-known disc algebra. Since $\mathrm{Ch}(B)$ = unit circle in the complex plane (Chapter II, **[2]**) we obtain $\mathrm{Ch}(A)$ = unit circle in the complex plane by Theorem (3.7).

In the above example, we computed the Choquet boundary of a real algebra through that of its complexification. In the example to follow we construct a real function algebra from any given complex function algebra and a relationship between the Choquet boundaries of these two function algebras is established.

Example (3.12). Let U be a complex function algebra defined on a compact, Hausdorff space X. Let $\{z_1, z_2, \dots, z_q\}$ be a specified finite subset of q points in X and D_k a continuous point derivation of U at z_k for each k. Construct a subset A_q of U as follows:

$$A_q = \{f \in U: \ f(z_k) \text{ and } D_k(f) \text{ are real for } 1 \leqq k \leqq q\}.$$

Then A_q is a real uniform algebra. Further A_q can be viewed as a real function algebra on (Y, τ) where Y and τ are defined as follows: Let Y consist of two copies of the set X identified at the prescribed points $z_1, z_2, \dots, z_q$. Thus $Y = X \times \{0, 1\}$ where $\{0, 1\}$ has the discrete topology and Y has the usual product topology. Define $\tau: Y \to Y$ by $\tau(x, 0) = (x, 1)$ and $\tau(x, 1) = (x, 0)$ for all $x \in X$. Then τ is an involutionary homeomorphism on Y. Note that $(z_i, 0) = (z_i, 1)$ for all $i = 1, 2, \dots, q$. Hence $z_1, z_2, \dots, z_q$ are all fixed points of τ. Define

$$f(x, 0) = f(x) \quad \text{and} \quad f(x, 1) = \overline{f(x)}$$

for all $x \in X$ and $f \in A_q$. Then

$$f(\tau(x, 0)) = f(x, 1) = \overline{f(x)}$$

for all $x \in X$, $f \in A_q$ and

$$f(\tau(x, 1)) = f(x, 0) = f(x)$$

for all $x \in X$, $f \in A_q$. Thus A_q can be regarded as a real function algebra on (Y, τ). Further we may identify $X \times \{0\}$ with X and regard X as a subset of Y. With this convention $Y = X \cup \tau(X)$. We now investigate the relationship between the Choquet boundaries of U and A_q. It turns out that $\mathrm{Ch}(U) = \mathrm{Ch}(A_q) \cap X$. The proof of this assertion depends on the following lemma, which seems to be essentially known.

LEMMA (3.13). *Let B be a complex function algebra and $\phi \in \Phi_B$. Suppose $\phi \in \mathrm{Ch}(B)$. Then there exists no non-zero point derivation at ϕ.*

Proof. Let $\phi \in \mathrm{Ch}(B)$. Then ϕ is a peak point in the weak sense by Theorem (2.3.4), Chapter II of [**2**]. In view of a remark made after the proof of Theorem (2.3.5), Chapter II of [**2**], the kernel of ϕ has an approximate identity. Therefore there exists no non-zero point derivation at ϕ by Corollary (1.6.6), Chapter I of [**2**].

THEOREM (3.14).

(a) $\mathrm{Ch}(U) = \mathrm{Ch}(A_q) \cap X$

(b) $\mathrm{Ch}(A_q) = \mathrm{Ch}(U) \cup \tau(\mathrm{Ch}(U))$,

that is $\mathrm{Ch}(A_q)$ *is the union of two copies of* $\mathrm{Ch}(U)$ *identified at* $\{z_1, z_2, \dots, z_q\} \cap \mathrm{Ch}(U)$.

Proof. (a) Suppose $x \in \mathrm{Ch}(A_q) \cap X$. Let V be a neighbourhood of x in X and $0 < \alpha < \beta$. Let $W = V \cup \tau(V)$. Then W is a τ-invariant neighbourhood of x. Hence by Theorem (1.17), there exists $f \in A_q$ with $\mathrm{Re}\, f \leqq 0$, $\mathrm{Re}\, f(x) > -\alpha$ and $\mathrm{Re}\, f(y) < -\beta$ for all $y \in X - V$ as $X - V$ is contained in $Y - W$. Hence $x \in \mathrm{Ch}(U)$ by Theorem (2.2.6), Chapter II of [**2**].

Next, let $x \in \mathrm{Ch}(U)$. Obviously $x \in X$.

Claim. For every neighbourhood V of x in X and every $\alpha > 0$ there exists $h \in A_q$ such that $h(x) = 1$ and $|h(y)| < \alpha$ for all $y \in X - V$.

Let V be a neighbourhood of x and $\alpha > 0$.

Case (i). Suppose $x \notin \{z_1, z_2, \dots, z_q\}$. As $x \in \mathrm{Ch}(U)$, by Theorem (2.3.4), Chapter II of [**2**], there exists $f \in U$ with $\|f\| \leqq 1$, $f(x) = 1$ and $|f(y)| < 1$ for all $y \in X - V$. As X is Hausdorff, we can so choose the neighbourhood V that $z_1, z_2, \dots, z_q \notin V$. So $|f(z_k)| < 1$ for $1 \leqq k \leqq q$.

Let $f(z_k) = c_k$. Define the functions f_k by

$$f_k = \frac{f - c_k}{1 - \bar{c}_k f} \frac{f - \bar{c}_k}{1 - c_k f} \quad \text{for } 1 \leqq k \leqq q.$$

Then $f_k \in U$ for $1 \leqq k \leqq q$ as $|c_k| < 1$ for $1 \leqq k \leqq q$. Also $f_k(z_k) = 0$ for all $k = 1, 2, \dots, q$. Construct a function g as follows:

$$g = \prod_{k=1}^{q} f_k.$$

Then $g \in U$, $\|g\| \leqq 1$, $g(z_k) = 0$ for each $1 \leqq k \leqq q$ and

$$g(x) = \prod_{k=1}^{q} f_k(x) = 1.$$

Define h by setting $h = g^2$. Then, $h \in U$, $\|h\| \leqq 1$, $h(z_k) = 0$ for $1 \leqq k \leqq q$. Further

$$D_k(h) = D_k(g^2) = 2D_k(g)g(z_k) = 0 \quad \text{for } 1 \leqq k \leqq q.$$

Thus $h \in A_q$ with $\|h\| \leqq 1$, $h(x) = g^2(x) = 1$.

Next let $y \in X - V$. Then $|h(y)| = |g^2(y)| < 1$ since

$$\begin{aligned} |g(y)| &= |f_1(y)|\,|f_2(y)| \dots |f_q(y)| \\ &= \left|\frac{f(y) - c_1}{1 - \bar{c}_1 f(y)}\right| \dots \left|\frac{f(y) - c_q}{1 - \bar{c}_q f(y)}\right| < 1 \end{aligned}$$

as $|f(y)| < 1$ and $|c_k| < 1$ for $1 \leqq k \leqq q$. By taking sufficiently higher powers of h, if necessary, we see that $|h(y)|$ can be made less than any $\alpha > 0$.

Case (ii). $x \in \{z_1, z_2, \ldots, z_q\}$. We may assume $x = z_1$. As $x \in \mathrm{Ch}(U)$ as before by Theorem (2.3.4), Chapter II of **[2]** there exists a function $f \in U$ such that $\|f\| \leqq 1$, $f(x) = 1$ and $|f(y)| < 1$ for all $y \in X - V$. Since $x \in \mathrm{Ch}(U)$, by Lemma (3.13) there exists no non-zero point derivation at x.

If $q = 1$, define $h = f$. So, $D_1(h) = D_1(f) = 0$. Thus $D_1(h)$ and $h(z_1) = h(x) = f(x)$ are real and hence $h \in A_q$. Thus $h \in A_q$ with $h(x) = 1$, $\|h\| \leqq 1$ and by taking sufficiently higher powers of h, $|h(y)|$ can be made $<\alpha$ for any $\alpha > 0$.

If $2 \leqq k \leqq q$, we proceed as follows: In view of the Hausdorff nature of X we can choose V so that $z_2, z_3, \ldots, z_q \notin V$. Hence $|f(z_i)| < 1$ for $i = 2, 3, \ldots, q$. Let $f(z_k) = c_k$ for $k = 2, 3, \ldots, q$. Define f_k's as before by

$$f_k = \frac{f - c_k}{1 - \bar{c}_k f} \, \frac{f - \bar{c}_k}{1 - c_k f}.$$

As $|c_k| < 1$ for all $k = 2, 3, \ldots, q$ and $f(x) = f(z_1) = 1$, f_k is well-defined. Also $f_k \in U$, $\|f_k\| \leqq 1$ for $2 \leqq k \leqq q$. Construct another function g by

$$g = \prod_{k=2}^{q} f_k.$$

Then $g \in U$, $\|g\| \leqq 1$, $g(z_k) = 0$ for $2 \leqq k \leqq q$, $g(x) = 1$ and $|g(y)| < 1$ for all $y \in X - V$ and for all $k = 2, 3, \ldots, q$.

Set $h = g^2$. Then $h \in U$, $\|h\| \leqq 1$, $h(z_k) = 0$ for $2 \leqq k \leqq q$,

$$h(x) = h(z_1) = g^2(x) = 1 = \text{real}.$$

Also for $k = 2, 3, \ldots, q$,

$$D_k(h) = D_k(g^2) = 2D_k(g)g(z_k) = 0.$$

Further $|h(y)| < 1$ for all $y \in X - V$. By taking sufficiently higher powers of h, we see that $|h(y)| < \alpha$ for any $\alpha > 0$.

Thus the claim is proved in all the cases. Now to prove that $x \in \mathrm{Ch}(A_q)$ we proceed as follows:

Let W be a neighbourhood of x in Y such that $\tau(W) = W$ and $\alpha > 0$. Then $V = X \cap W$ is a neighborhood of x in X. Hence by the above claim there is $h \in A_q$ such that $\|h\| \leqq 1$, $h(x) = 1$ and $|h(y)| < \alpha$ for all $y \in X - V$. Then it is easy to see that $|h(y)| < \alpha$ for all $y \in Y - W$. Hence $x \in \mathrm{Ch}(A_q)$ by Theorem (1.12).

Proof of (b). Let $x \in \mathrm{Ch}(U) \cup \tau(\mathrm{Ch}(U))$. So $x \in \mathrm{Ch}(U)$ or $x \in \tau(\mathrm{Ch}(U))$. $x \in \mathrm{Ch}(U)$ implies $x \in \mathrm{Ch}(A_q)$ in view of Theorem 3.14 (a). If $x \in \tau(\mathrm{Ch}(U))$, then this implies $\tau(x) \in \mathrm{Ch}(U)$. Hence $\tau(x) \in \mathrm{Ch}(A_q)$ by Theorem 3.14 (a).

So $x \in \mathrm{Ch}(A_q)$. Thus $\mathrm{Ch}(U) \cup \tau(\mathrm{Ch}(U))$ is contained in $\mathrm{Ch}(A_q)$.

On the other hand let $x \in \mathrm{Ch}(A_q)$. Then $x \in Y$. So $x \in X$ or $x \in \tau(X)$, that is $x \in X$ or $\tau(x) \in X$. If $x \in X$, by 3.14 (a), $x \in \mathrm{Ch}(U)$. Otherwise $\tau(x) \in \mathrm{Ch}(U)$ which implies

$$x \in \tau(\mathrm{Ch}(U)).$$

So, $\mathrm{Ch}(A_q)$ is contained in $\mathrm{Ch}(U) \cup \tau(\mathrm{Ch}(U))$. Hence (b) is proved.

COROLLARY (3.15).

(a) $S(U) = X \cap S(A_q)$

(b) $S(A_q) = S(U) \cup \tau(S(U))$.

Remark (3.16). From Corollary 3.15 (b), we have

$$\begin{aligned} T(S(A_q)) &= T(S(U)) \cup T(\tau(S(U)) \\ &= T(S(U)) \cup T(S(U)) \end{aligned}$$

since $T(\tau(S(U))) = T(S(U))$

$$\begin{aligned} &= T(S(U)) \\ &= S(U) \end{aligned}$$

by definition of T.

But $T(S(A_q))$ is nothing but the Shilov boundary of A_q as defined in [7]. Hence the Shilov boundary of A_q (as defined in [7]) and the Shilov boundary of U are the same. This result has been proved in [7] (Proposition 2.2) under the additional hypothesis that the Dirichlet deficiency of U be finite whereas we have made no such assumption.

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