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AN ALGEBRAIC LINK CONCORDANCE GROUP FOR $(p, 2p-1)$-LINKS IN S^{2p+1}

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A concordance classification of links of $S^p \amalg S^{2p-1} \subset S^{2p+1}$, $p>1$, is given in terms of an algebraically defined group, $\Phi_\pm$, which is closely related to Levine's algebraic knot concordance group. For $p=1$, Φ_- captures certain obstructions to two component links in S^3 being concordant to boundary links, the generalized Sato–Levine invariants defined by Cochran. As a result, purely algebraic proofs of properties of these invariants are derived.

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In Gilmer and Livingston [3] a group Ψ was defined, along with a map ψ from the set of concordance classes of two component linking number 0 links in S^3 to Ψ. It was shown that Ψ contains significant new obstructions relating to boundary links. The group Ψ is an extension of an algebraically defined group Φ_- which captures abelian invariants of link concordances; the quotient Ψ/Φ_- relates to Casson–Gordon invariants. This paper begins an analysis of Ψ with a study of Φ_-, its counterpart Φ_+, and their relationship to problems of link concordances.

Levine [10] defined groups G_ε $(\varepsilon=\pm 1)$ of ε-symmetric isometric structures and proved that G_ε is isomorphic to the concordance group of knotted $(2p-1)$-spheres in S^{2p+1}, for $p>2$ $(\varepsilon=(-1)^p)$. Although G_- does not determine the classical knot concordance group $(p=1)$ (Casson and Gordon [1]), it does contain many concordance invariants. The groups Φ_ε are extensions of G_ε, and play a completely analogous role in the classification of concordances of links of $S^p \amalg S^{2p-1} \subset S^{2p+1}$. Two applications are presented:

High dimensional link concordance. There is a map from the set of concordance classes of links of $S^p \amalg S^{2p-1} \subset S^{2p+1}$ to Φ_ε which is a bijection if $p>2$, surjective if $p=1$, and a bijection onto an index 2 subgroup if $p=2$. (In the case $p=1$ we require that the link have linking number 0.) Furthermore, for $p>1$ the quotient group, $\Phi_\varepsilon/G_\varepsilon$ provides a complete set of obstructions to the existence of a concordance to a split link. The proofs are simple modifications of Levine's original argument.

A different classification of such link concordance classes has been described by Mio and Orr in unpublished work, which includes results in the full metastable range. Their result led us to the high dimensional observations here. Their classification involved the concordance group G_ε and the Kojima function of the link, η, defined in Kojima and

Yamasaki [9]. That work was inspired by work in Hacon [4] and Hacon and Mio [5]. The question of realizing obstructions was resolved by Jin [7].

Links in S^3. The quotient Φ_-/G_- provides obstructions to a classical link being concordant to a boundary link. Previously known obstructions include the Kojima function, the Sato–Levine invariant (Sato [12]), and the generalized Sato–Levine invariants, β^i, defined by Cochran [2]. The relationship between these invariants was described in Cochran [2], and it will be clear that they provide the same obstruction as does Φ_-/G_-.

In addition to providing a purely algebraic formulation of these obstructions, the present approach offers simple algebraic proofs of previously known properties. For instance, the fact that the generating function for the β^i is rational follows immediately from the existence of characteristic polynomials of linear transformations. (The proof of this result in Cochran [2] depended on the definition of η, which is geometric, and a geometric argument relating the β^i and η. The rationality of the generating function is equivalent to the sequence satisfying a linear recursion relation. A complete analysis of these recursion properties is carried out in Jin [7].)

Notation. We work throughout in the smooth category. Links will always be (ordered) links of $S^p \amalg S^{2p-1}$ in S^{2p+1}. Homology is taken with integer coefficients.

Acknowledgements. The relationship of Φ_- to Cochran's invariants is readily apparent, and hence the connection to the Kojima function. Kent Orr described to us his work with Mio showing that the Kojima function can be used to classify the higher dimension link concordance classes and it follows that the same should be true of Φ_ε; only the details of a direct argument needed to be filled in. Orr also pointed out Jin's work on recursion relations among the β^i, and it was clear that similar results would follow directly from the structure of Φ_ε.

1. Definition and properties of Φ

We follow the notation of Stoltzfus [13] in his description of the integral knot concordance group. The group itself was first defined by Levine in [10]. The notion of isometric structure first appeared in Kervaire's work [8].

Definition. An *enhanced ε-symmetric isometric structure* is a quadruple $(M, \langle\,,\rangle, t, \lambda)$ where M is a finitely generated free Z module, $\langle\,,\rangle$ is an ε-symmetric bilinear form on M, t is a linear endomorphism of M, and λ is an element in M, satisfying:

(i) $\langle\,,\rangle$ is nonsingular, and

(ii) $\langle tx, y\rangle + \langle x, ty\rangle = \langle x, y\rangle$, for all x, y in M.

Definition. An enhanced isometric structure is *metabolic* if there is a submodule N of M such that:

(i) $N = N^{\perp}$,

(ii) N is invariant under t, and

(iii) $\lambda \in N$.

Definition. Φ_ε is the Witt group of equivalence classes of enhanced ε-isometric structures modulo metabolic structures.

The proof that this is well defined, and the basic properties of the Witt group, follow as in Stoltzfus [13]. (Note that $-(M, \langle\,,\rangle, t, \lambda) = (M, -\langle\,,\rangle, t, \lambda)$. A metabolizer for the sum $(M, \langle\,,\rangle, t, \lambda) \oplus (M, -\langle\,,\rangle, t, \lambda)$ is given by $N = \{(x, x) \mid x \in M\}$. Also note that $(M, \langle\,,\rangle, t, \lambda) = (M, \langle\,,\rangle, t, -\lambda)$; a metabolizer for the difference is given by $N = \{(x, -x) \mid x \in M\}$.)

The definition of G_ε is the same as for Φ_ε, only without reference to λ. There is an inclusion of G_ε into Φ_ε which sends $(M, \langle\,,\rangle, t)$ to $(M, \langle\,,\rangle, t, 0)$, and this inclusion is split by the forgetful map. We will want to understand the quotient, $\Phi_\varepsilon/G_\varepsilon$. To do so, first note that the functions $a_{ij}\,(M, \langle\,,\rangle, t, \lambda) = \langle t^i\lambda, t^j\lambda\rangle$, $i, j \geqq 0$, are well defined on Φ_ε and vanish on G_ε. These functions are a complete (but, as we will see, redundant) set of invariants for the quotient:

Theorem 1. *The class $(M, \langle\,,\rangle, t, \lambda)$ is in the image of G_ε if and only if $a_{i,j} = 0$ for all i, j.*

Proof. Suppose that $a_{i,j} = 0$ for all i, j. Let L be the smallest (t invariant) summand of M containing $\{t^i\lambda\}$, $i \geqq 0$. Then $(M, \langle\,,\rangle, t, \lambda)$ is Witt equivalent to $(L^{\perp}/L, \langle\,,\rangle, t, 0)$, where $\langle\,,\rangle$ and t are induced by those on M. The proof is the same as the usual proof that Witt classes have anisotropic representatives (Stoltzfus [13]). (A metabolizer for $(M, \langle\,,\rangle, t, \lambda) \oplus -(L^{\perp}/L, \langle\,,\rangle, t, 0)$ is given by $\{(x, x + L) \mid x \in L^{\perp}\}$.)

2. Link concordance invariants

Let (K, J) be a link of $S^p \amalg S^{2p-1}$ in S^{2p+1}, with linking number 0 if $p = 1$. An obstruction theory argument shows that J bounds a $2p$-manifold V in the complement of K.

Definition. $\phi(K, J) = (H_p(V)/\text{torsion}, \langle\,,\rangle, t, \lambda) \in \Phi_\varepsilon$, where $\langle\,,\rangle$ is the standard intersection form on $H_p(V)$; t is defined by $\langle tx, y\rangle = \operatorname{lk}(x^+, y)$, ($x^+$ is the positive push off of x) for all $x, y \in H_p$; and λ is defined by $\langle \lambda, y\rangle = \operatorname{lk}(K, y)$ for all y in $H_p(V)$.

In earlier work $\operatorname{lk}(x, y^+)$ was used instead of $\operatorname{lk}(x^+, y)$. Our choice is consistent with standard conventions on orientations and intersection forms, and also with recent work on boundary link invariants, such as Cochran [2]. Note also that t could be defined without reference to intersection forms or linking forms via duality; t is the composition of i^+ with the Alexander and Poincaré duality maps.)

Theorem 2. *$\phi(K, J)$ is a well defined concordance invariant.*

Proof. Given a concordance from (K, J) to (K', J') with J bounding V and J' bounding V', one uses an obstruction theory argument to construct an embedded manifold W^{2p+1} in $S^{2p+1} \times I$ with boundary V, V', and the concordance of the J's. W can be arranged to be in the complement of the concordance of the K's. A metabolizer for the direct sum is given by the smallest summand which includes the kernel of the inclusion of $H_p(V \cup V')$ into $H_p(W)$.

Given two links, (K, J) and (K', J'), separated by a $2p$-sphere, S, it is clear that for any choice of Seifert manifolds for J and J' in the complements of K and K', one can find bands to form the connected sum of links so that the bands miss the Seifert manifolds and the core of each intersects S in exactly one point. (A more general construction of band connect sums is given in Cochran [2].) The following result is immediate.

Theorem 3. *ϕ is additive for band sums as described above.*

Seifert matrices. If a knot J bounds a Seifert manifold V, the Seifert form Θ on $H_p(V)$/torsion can be defined by $\Theta(x, y) = \mathrm{lk}(x^+, y)$. If V is a Seifert matrix representing Θ with respect to some basis, then $V - V^t$ represents the intersection form of V. A quick calculation shows that t is represented by $(V^t - V)^{-1} V^t$.

3. High dimensional classification

The proof that for $p > 2$ Φ_ε classifies links (of the appropriate dimension) up to concordance is the same as Levine's original argument (Levine [10]), with a few simple additions. The case of $p = 2$ is also completely analogous, and is omitted. Let (K, J) be a link of $S^p \amalg S^{2p-1} \subset S^{2p+1}$. Let J bound a Seifert manifold V in the complement of K.

Theorem 4. *For $p > 1$, if $\phi(K, J) = 0$ then (K, J) is strongly slice.*

Proof. *Step 1* (make V simple). A cobordism W of V to a simple manifold V_0 can be embedded in B^{2p+2}. Properly embed a copy of $K \times I$ into $B^{2p+2} - W$ with $K \times \{0\}$ mapping to K. Now use Hirsch's engulfing theorem (Hirsch [6]) to embed a $(2p+2)$-ball B_0 into B^{2p+2} so that $B_0 \cap (W \cup K \times I) = V_0 \cup K \times \{1\}$. The difference $B^{2p+2} - B_0$ provides the concordance to the link (K', J') in ∂B_0, where $K' = K \times \{1\}$ and $J' = \partial V_0$.

Step 2 (surger V_0). The pair $(K', \partial V_0)$ determines an element $\phi(K', \partial V_0)$ in Φ'. The form is Witt equivalent to $\phi(K, J) = 0$. Hence, one can surger V_0 to reduce it to a disk. That surgery can be performed ambiently, via the Whitney trick, in B_0. Unfortunately, the added handles may intersect a slice disk, D, for K in B_0.

As $\phi(K', \partial V_0) = 0$, each added handle intersects D algebraically 0 times. The Whitney trick can hence be performed to eliminate the intersections of the handles with D.

Theorem 5. *For $p>1$, links (K,J) and (K',J') are concordant if and only if $\phi(K,J)=\phi(K',J')$.*

Proof. Form the band connected sum of the links (K,J) and $-(K',J')$ as described in the previous section. The resulting link has ϕ trivial and is hence strongly slice. Now add a $2p+1$ handle to B^{2p+2} along S to construct $S^{2p+1}\times I$. The slice disks are easily modified to yield the desired concordances.

Theorem 6. *For $p\neq 2$, every enhanced isometric structure is realized by a link.*

(The theorem remains true for $p=2$, with the usual restriction to an index 2 subgroup of Φ_ε.)

Proof. Note that by Kervaire [8] any isometric structure is realized by a knotted sphere bounding a simple Seifert manifold, V. By the Hurewicz theorem and general position any homology class in $H_p(S^{2p+1}-V)$ is represented by a sphere; one constructs K to correspond to a given λ.

Corollary 7. *For $p>1$, a link (K,J) of $S^p\amalg S^{2p-1}\subset S^{2p+1}$ is concordant to a split link if and only if $a_{i,j}\phi(K,J)=0$ for all $i,j\geqq 0$.*

Proof. The vanishing of the $a_{i,j}$ implies that $\phi(K,J)$ is Witt equivalent to an isometric structure with $\lambda=0$. Hence, the link is concordant to a link with corresponding enhanced isometric structure having $\lambda=0$. In the above construction K can be taken to be trivial.

4. Connections with Cochran's invariants

We first give a very brief sketch the definition of the Sato–Levine invariant and Cochran's generalization of it in the classical dimension. See Cochran [2] for details. Let K bound a manifold W in the complement of J. Let l_1 be the transverse intersection of W with the Seifert manifold, V, for J. The Sato–Levine invariant β^1 is defined to be $\mathrm{lk}(l_1,l_1^+)$. Now, replace K in the original link with l_1^+. Repeating the previous construction gives an l_2, and corresponding β^2. Iterate the procedure to define an infinite sequence of invariants.

In Cochran [2] a method of computing the β^i in terms of V is given without proof. As that method provides the connection between the algebra here and previous geometric methods we will quickly fill in the argument. Recall first that the linking number of two disjoint closed curves, x and y, in S^3 is computed by letting x bound an immersed surface F transverse to y, and computing the intersection number of y with F. In the situation of links (K,J) in S^3, let K bound a surface W transverse to the Seifert surface V for J, as in the previous paragraph, and let l_1 be the intersection of W and V. The intersection number of curves on V with l_1 then gives the linking number with K. Hence, l_1 is a representative of λ and $\beta^1=\langle t(\lambda),\lambda\rangle$.

Similarly, a representative of $t(x)$, where $x \in H_1(V)$, satisfies $\langle t(x), y\rangle = \mathrm{lk}(x^+, y)$ for all y. In particular, $t(\lambda)$ satisfies $\langle t(\lambda), y\rangle = \mathrm{lk}(l_1^+, y)$ for all y. Hence, $t(\lambda)$ is represented by the intersection of a surface bounded by l_1^+ with V, l_2 in the previous section. Continuing, we see that $t^i(\lambda)$ is represented by l_{i+1}. Finally, $\beta^i = \mathrm{lk}(l_i^+, l_i) = \mathrm{lk}(t^{i-1}(\lambda)^+, t^{i-1}(\lambda)) = \langle t^i(\lambda), t^{i-1}(\lambda)\rangle = a_{i-1,i}$.

5. $\Phi_\varepsilon/G_\varepsilon$, recursion relations among the $a_{i,j}$, and generating functions

As noted, the $a_{i,j}$ form a complete set of invariants of $\Phi_\varepsilon/G_\varepsilon$. Here we develop their properties. From the definition we have the relations:

$$a_{i+1,j} + a_{i,j+1} = a_{i,j} \tag{*}$$

and

$$a_{i,j} = \varepsilon a_{j,i}. \tag{**}$$

View the functions $a_{i,j}$ as defined on the upper right quadrant of the integer lattice. Relation (*) states that two consecutive entries on a horizontal row determine the entry immediately above the first. Hence:

Fact 1. *The entries $a_{i,0}$ determine all the $a_{i,j}$.*

In the case that $\varepsilon = -1$, (**) implies that the diagonal entries $a_{i,i}$ are 0. If $\varepsilon = +1$ (*) and (**) imply that $a_{i,i} = 2a_{i+1,i}$. Next note that $a_{i,i}$ and $a_{i,i-1}$ together determine $a_{i+1,i-1}$. Continuing this "downward propagation" quickly yields:

Fact 2. *The entries $a_{i+1,i}$ determine all the $a_{i,j}$.*

Recursion. Suppose that the characteristic polynomial of t is given by $p = \sum_{k=0}^n \alpha_k x^k$. (Note that $\alpha_n = 1$. Also, using the Seifert matrix description, one has that up to a power of x, p is given by the $x^n \Delta_J(1 - 1/x)$ for some n, where Δ_J is the Alexander polynomial.) It follows that $\langle p(t)t^i\lambda, t^j\lambda\rangle = 0$. Hence, $\sum_{k=0}^n \alpha_k a_{i+k,j} = 0$ for all i,j. Since s consecutive $a_{i+1,i}$ determine $2s$ consecutive $a_{i,j}$ for fixed j, the given length $n+1$ recurrence relation among the $a_{i,j}$ determines a length $[(n/2)+1)]$ linear recurrence relation among the $a_{i+1,i}$.

Generating functions. Let $f_j(t)$ be the generating function for the sequence $a_{i,j}$; that is, $f_j(t) = \sum_{i=0}^\infty a_{i,j} t^i$. Note that * implies that $f_{j+1}(t) = \mathrm{Pos}((1 - t^{-1}) f_j(t))$, where Pos( ) represents the terms in the power series with non-negative exponents. If p is the degree n characteristic polynomial of t, the recursion relation it determines among the $a_{i,j}$ is equivalent to the condition that $t^n(p(1/t)) f_j(t)$ is a finite polynomial. As an application we have the special case of Theorem 7.1 of Cochran [2].

Theorem 8. *If the Alexander polynomial of J is trivial, then $a_{i+1,i} = 0$ for i large.*

Proof. The Alexander polynomial is of the form t^n. We then have that $p(t)=t^n(1-1/t)^k$, for some n and k. By the above discussion, $(t-1)^k f_0(t)$ is a (finite) polynomial. This implies that $f_j(t)$ is a polynomial for all $j \geqq k$. Finally, noting that the degree of $f_{k+1}(t)$ is less than or equal to the degree of $f_k(t)$ for all k completes the argument.

6. Genus and the length of recursion relations

The genus of a Seifert manifold for a knot can be defined to be half the dimension of the middle dimensional homology. This is an integer for p odd because a skew symmetric nonsingular form has even rank. For p even, the form is even, symmetric and nonsingular, so the signature is divisible by 8, and the rank is again even.

Suppose that J bounds a genus g Seifert manifold in the complement of K. The characteristic polynomial of t is then of degree at most $2g+1$, and hence the linear recursion relation among the $a_{i,j}$ is of length at most $2g+1$, for any fixed j. This implies that the $a_{i+1,i}=\beta^i$ satisfy a linear recursion relation of length $g+1$. The following theorem is then immediate.

Theorem 9. *If J bounds a genus g Seifert manifold in the complement of K, and $\beta^i=0$ for $1 \leqq i \leqq g$, then $\beta^i=0$ for all i.*

Corollary 10. *For $p>1$, if J bounds a genus g Seifert manifold in the complement of K, and $\beta^i=0$ for $1 \leqq i \leqq g$, then (K, J) is concordant to a split link.*

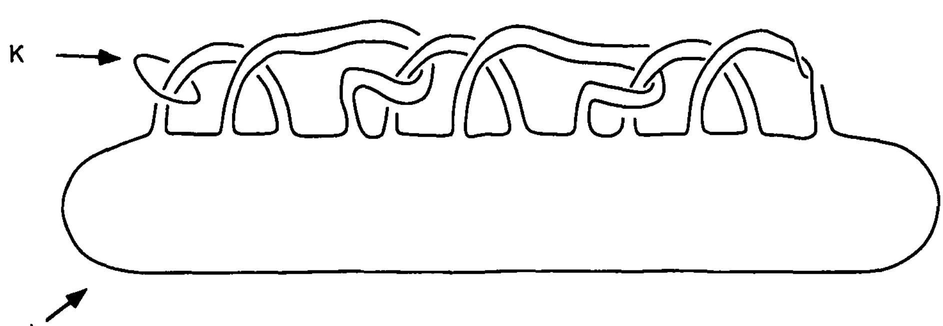

FIGURE 1

Figure 1 gives an interesting example of linking. Note that both K and J are unknotted in S^3, that K bounds a genus 1 surface in the complement of J, and that J bounds a genus g surface in the complement of K. (The figure illustrates the case $g=3$.) Our observation is that (K, J) is not even concordant to a link (K', J') with J' bounding a surface of genus less than g in the complement of K'.

For the link in Figure 1, it is easily computed that $\beta^i=0$ for $i<g$, and $\beta^g=1$. (In fact $\beta^i=0$ for $i>g$ also.) Hence the recursion relation for the β^i must be of length at least

$g+1$, providing the obstruction to finding a lower genus surface bounded by J' in the complement of K'.

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