Bull. Austral. Math. Soc. **78** (2008), 97–106
doi:10.1017/S0004972708000506

SCHREIER CONDITIONS ON CHIEF FACTORS AND RESIDUALS OF SOLVABLE-LIKE GROUP FORMATIONS

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(Received 11 September 2007)

Abstract

Let α be a formation of finite groups which is closed under subgroups and group extensions and which contains the formation of solvable groups. Let G be any finite group. We state and prove equivalences between conditions on chief factors of G and structural characterizations of the α-residual and the α-radical of G. We also discuss the connection of our results to the generalized Fitting subgroup of G.

2000 *Mathematics subject classification*: 20D25, 20E34.

Keywords and phrases: Schreier's conjecture, residual, radical, chief factors, generalized Fitting subgroup.

1. Introduction

Let G be a finite group such that each chief factor of G is simple nonabelian. It can be proved from Schreier's conjecture (which states that $\mathrm{out}(T)$ is solvable if T is simple nonabelian) that $G = \mathrm{Soc}(G)$, where $\mathrm{Soc}(G)$ is the subgroup generated by the minimal normal subgroups of G. Theorem 6 extends this result under the assumption that certain nonabelian chief factors of G satisfy a suitable 'Schreier property'. Similar ideas underlie Theorem 8. Here the 'Schreier property' is assumed to hold for a family of chief factors which includes the abelian ones. A special case of Theorem 8 is obtained when G is a group for which every chief factor M/N, where M is contained in the solvable radical R of G, is cyclic. In this case, Theorem 8 gives $C_G(L) = R$ where L is the solvable residual of G (compare to [4]).

The proof of Theorem 6 utilizes a generalization of a characteristic subgroup which plays a role in various algorithms of computational group theory [1, Ch. 10, 6, Ch. 6]. This subgroup (denoted N_2 in [6]) is the preimage in G of $\mathrm{Soc}(G/R)$ where R is the solvable radical of G. Theorem 4 provides another way of looking at N_2 which naturally leads us to consider the generalized Fitting subgroup $F^*(G)$. We do this in Section 3 where we prove (see Theorem 24) that $F^*(G) = N_2 \cap C_G(R)F(G)$ where $F(G)$ is the Fitting subgroup of G. We also give in this section a short proof, based on basic consequences of our concepts, of the well-known fact that $C_G(F^*(G)) \leq F^*(G)$.

Throughout the paper we consider finite groups. The Greek letter α will denote a formation of finite groups, that is, a property of finite groups which has a residual and is closed under homomorphic images. We recall that α has a residual if for every group G there exists a normal subgroup $L_\alpha(G)$ such that $G/L_\alpha(G)$ is α, and, for any $N \trianglelefteq G$ such that G/N is α, $L_\alpha(G) \leq N$. Henceforth we denote the α-residual of G by $L_\alpha(G)$. In addition, we assume that α is closed under group extensions and under subgroups and that α contains the formation of solvable groups. Three particular examples of formations which satisfy all of our assumptions are the following:

(1) the formation of solvable groups;
(2) the formation of π-solvable groups, where π is any fixed set of primes;
(3) the formation of π-separable groups, where π is any fixed set of primes.

Our assumptions imply that α has a radical (see Lemma 10). This means that for every group G there exists a normal subgroup which is α and contains each normal α-subgroup of G. Henceforth we denote the α-radical of G by $R_\alpha(G)$.

The main objects of interest in the present paper are given by the following definitions.

DEFINITION 1. Let G be a group and $K \leq G$. We say that K is *minimal* α' *normal in* G if:

(1) K is a normal subgroup of G;
(2) K is not α;
(3) if $N \trianglelefteq G$ and $N < K$ then N is α.

Note that since α is closed under group extensions, a minimal α' normal subgroup of G is necessarily perfect.

DEFINITION 2. Let G be a group. $L_0(\alpha, G)$ is the subgroup of G which is generated by all of the minimal α' normal subgroups of G. If there are none (that is, G is α) then $L_0(\alpha, G) = 1$.

When α is the formation of solvable groups, $L_0(\alpha, G)R_\alpha(G) = N_2$ (see above and Proposition 18).

DEFINITION 3. Let G be a group.

(1) For any $M \trianglelefteq G$

$$\mathrm{Inn}_G(M) \stackrel{\mathrm{def}}{=} C_G(M)M.$$

For any $M, N \trianglelefteq G$, $N < M$, we define $\mathrm{Inn}_G(M/N)$ to be the preimage in G of $\mathrm{Inn}_{G/N}(M/N)$.

(2) For any $M \trianglelefteq G$,

$$\mathrm{Out}_G(M) \stackrel{\mathrm{def}}{=} G/\mathrm{Inn}_G(M) = G/(C_G(M)M).$$

For any $M, N \trianglelefteq G$, $N < M$, we define

$$\mathrm{Out}_G(M/N) \stackrel{\mathrm{def}}{=} G/\mathrm{Inn}_G(M/N) \cong \mathrm{Out}_{G/N}(M/N).$$

Note that for $M \trianglelefteq G$, $\mathrm{Inn}_G(M)$ is the set of all elements of G which act on M (by conjugation) like inner automorphisms of M. It can also be verified that $\mathrm{Out}_G(M)$ is embedded in $\mathrm{Out}(M)$.

We prove and then use the following characterization of $L_0(\alpha, G)R_\alpha(G)$.

THEOREM 4. *Let G be a group. Then*

$$L_0(\alpha, G)R_\alpha(G) = \bigcap_{\substack{M/N \text{ is a non-}\alpha \text{ chief} \\ \text{factor of } G}} \mathrm{Inn}_G(M/N). \tag{1}$$

Theorem 4 brings to mind the characterization of the generalized Fitting subgroup $F^*(G)$ (see [2]) as the set of all elements $g \in G$ such that g acts as an inner automorphism on all chief factors of G, that is,

$$F^*(G) = \bigcap_{\substack{M/N \text{ is a chief} \\ \text{factor of } G}} \mathrm{Inn}_G(M/N). \tag{2}$$

In Section 3 we make some further observations on the formal similarity between $L_0(\alpha, G)R_\alpha(G)$ and $F^*(G)$.

DEFINITION 5. Let G be a group and let $M, N \trianglelefteq G$. Then M/N has the *α-Schreier property in G* if $\mathrm{Out}_G(M/N)$ is α.

Note that if α is the formation of solvable groups and M/N is a simple nonabelian chief factor of G, then $\mathrm{Out}(M/N)$ is solvable by Schreier's conjecture and hence M/N has the α-Schreier property in G.

The next two theorems characterize certain α-Schreier properties of chief factors. For the first theorem note that $L_0(\alpha, G) \leq L_\alpha(G)$ holds for any group G (see Corollary 15).

THEOREM 6. *Let G be a group. Then the following conditions are equivalent.*

(a) $L_0(\alpha, G) = L_\alpha(G)$.
(b) *Every chief factor of G of the form $M/R_\alpha(G)$ has the α-Schreier property in G.*
(c) *Every non-α chief factor of G has the α-Schreier property in G.*

REMARK 7. Note that $R_\alpha(G) = 1$ implies that $L_0(\alpha, G) = \mathrm{Soc}(G)$ (this follows easily from the definition of $L_0(\alpha, G)$). Using this, one can verify that the result mentioned at the beginning of the introduction is a special case of Theorem 6.

For the next result note that $C_G(L_\alpha(G)) \leq R_\alpha(G)$ for any group G (see Lemma 22).

THEOREM 8. *Let G be a group. Then the following conditions are equivalent.*

(a) $C_G(L_\alpha(G)) = R_\alpha(G)$.
(b) *Every chief factor M/N of G such that $M \leq R_\alpha(G)$ has the α-Schreier property in G.*

2. Proofs of Theorems 4, 6 and 8

LEMMA 9. *Let G be a group. Then $L_\alpha(L_\alpha(G)) = L_\alpha(G)$. In particular, $L_\alpha(G)$ is perfect.*

PROOF. This follows from the fact that α is closed under extensions, and from the fact that α contains the formation of solvable groups. □

LEMMA 10. *The property α has a radical.*

PROOF. It is sufficient to prove that if $N_1, N_2 \trianglelefteq G$ and N_1, N_2 are α then N_1N_2 is α. Consider $N_1N_2/N_1 \cong N_2/(N_1 \cap N_2)$. Since α is inherited by quotients, N_1N_2/N_1 is α. Now, since N_1 is α and α is inherited by extensions, N_1N_2 is α. □

LEMMA 11. *Let G be a group and $N \trianglelefteq G$. Then $R_\alpha(N) = R_\alpha(G) \cap N$.*

PROOF. This follows easily from the assumption that α is closed under subgroups. □

LEMMA 12. *Let G be a group and let $N \trianglelefteq G$ be such that $N \le R_\alpha(G)$. Then $R_\alpha(G/N) = R_\alpha(G)/N$. In particular, $R_\alpha(G/R_\alpha(G)) = 1$.*

PROOF. $R_\alpha(G)/N$ is α because α is closed under homomorphic images, and so $R_\alpha(G)/N \le R_\alpha(G/N)$. On the other hand, let $M/N = R_\alpha(G/N)$. Then, since both M/N and N are α and α is closed under extensions, M is α and hence $M \le R_\alpha(G)$. This shows that $R_\alpha(G/N) \le R_\alpha(G)/N$. □

LEMMA 13. *Let G be a group and $N \trianglelefteq G$. Then $L_\alpha(G/N) = L_\alpha(G)N/N$.*

PROOF. Let $K/N = L_\alpha(G/N)$. Then $(G/N)/(K/N) \cong G/K$ is α and so $L_\alpha(G) \le K$ and $L_\alpha(G)N \le K$. This proves that $L_\alpha(G)N/N \le L_\alpha(G/N)$. On the other hand, $G/(L_\alpha(G)N)$ is α since α is closed under homomorphic images. Thus $L_\alpha(G/N) \le L_\alpha(G)N/N$. □

LEMMA 14. *Let G be a group and $N \trianglelefteq G$. Let K be minimal α' normal in G. Then either $K \le N$ or KN/N is minimal α' normal in G/N. It follows from Definition 2 that $L_0(\alpha, G)N/N \le L_0(\alpha, G/N)$.*

PROOF. Suppose that $K \nleq N$. Then $1 < KN/N \trianglelefteq G/N$. We also have $K \cap N < K$, and hence (Definition 1(3)) $K \cap N$ is α. Hence, using the fact that α is closed under extensions, $KN/N \cong K/K \cap N$ is α if and only if K is α. But K is not α (Definition 1(2)), hence KN/N is not α. Let $M/N < KN/N$ be a normal subgroup of G/N. We shall prove that M/N is α. Now $M = M \cap KN = (M \cap K)N$. Supposing that $M \cap K = K$ yields that $M = KN$ – a contradiction. Hence $M \cap K < K$. Since $M \cap K \trianglelefteq G$ and K is minimal α' normal in G, we get (Definition 1(3)) that $M \cap K$ is α. Thus, $M/N = (M \cap K)N/N \cong (M \cap K)/(K \cap N)$ is α. Combining all of the above, we have proved that KN/N is minimal α' normal in G/N. □

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COROLLARY 15. *Let G be a group. Then $L_0(\alpha, G) \leq L_\alpha(G)$.*

PROOF. Note that if a group H is α then $L_0(\alpha, H) = 1$. Hence, if we choose $N = L_\alpha(G)$ in Lemma 14, we get $L_0(\alpha, G)L_\alpha(G)/L_\alpha(G) \leq L_0(\alpha, G/L_\alpha(G)) = 1$. The claim follows. □

LEMMA 16. *Let K be minimal α' normal in G. Then $KR_\alpha(G)/R_\alpha(G)$ is a non-α chief factor of G.*

PROOF. $KR_\alpha(G)/R_\alpha(G) \cong K/R_\alpha(K)$ (Lemma 11). Since K is not α, $K/R_\alpha(K)$ is not α. Thus $KR_\alpha(G)/R_\alpha(G)$ is non-α. Suppose to the contrary that $KR_\alpha(G)/R_\alpha(G)$ is not a chief factor of G. Then there exists $N \trianglelefteq G$ such that $R_\alpha(G) < N < KR_\alpha(G)$. We have $N = N \cap KR_\alpha(G) = (N \cap K)R_\alpha(G)$. Note that $N \cap K \trianglelefteq G$. Assuming that $N \cap K$ is α leads to $N \cap K \leq R_\alpha(G)$ which gives $N = (N \cap K)R_\alpha(G) = R_\alpha(G)$, contradicting $R_\alpha(G) < N$. Hence, $N \cap K$ is a normal non-α subgroup of G contained in K. Since K is minimal α' normal in G we get $N \cap K = K$, giving $N = KR_\alpha(G)$ – a contradiction. □

LEMMA 17. *Let G be a group. Suppose $M/R_\alpha(G)$ is a chief factor of G. Then $L_\alpha(M)$ is a minimal α' normal subgroup of G. Furthermore, $M = L_\alpha(M)R_\alpha(G)$.*

PROOF. Note that $L_\alpha(M)$ is a normal non-α subgroup of G. For if $L_\alpha(M)$ is α, then, since $M/L_\alpha(M)$ is α, we get that M is α and $M \leq R_\alpha(G)$, contradicting the assumption that $M/R_\alpha(G)$ is a chief factor of G. Let $N \trianglelefteq G$ be such that $N \leq L_\alpha(M)$. Then $R_\alpha(G) \leq NR_\alpha(G) \leq M$. Since $M/R_\alpha(G)$ is a chief factor of G, either $N \leq R_\alpha(G)$ or $NR_\alpha(G) = M$. The first possibility implies that N is α. The second possibility implies that $M/N \cong R_\alpha(G)/R_\alpha(N)$ is α and hence $N \geq L_\alpha(M)$, implying $N = L_\alpha(M)$. This concludes the proof that $L_\alpha(M)$ *is* minimal α' normal in G. Moreover, repeating the last argument with $N = L_\alpha(M)$ gives $M = L_\alpha(M)R_\alpha(G)$. □

PROPOSITION 18. *Let G be a group. Then*

$$L_0(\alpha, G)R_\alpha(G)/R_\alpha(G) = \mathrm{Soc}(G/R_\alpha(G)).$$

PROOF. *Set* $R_\alpha(G) = R$. Let K be minimal α' normal in G. Then, by Lemma 16, KR/R is minimal normal in G/R. Hence, $KR/R \leq \mathrm{Soc}(G/R)$. Since $L_0(\alpha, G)$ is generated by all minimal α' normal subgroups of G we get $L_0(\alpha, G)R/R \leq \mathrm{Soc}(G/R)$.

For the reverse inclusion, let $M \trianglelefteq G$ be such that M/R is minimal normal in G/R. By Lemma 17, $M = L_\alpha(M)R$, and $L_\alpha(M)$ is minimal α' normal in G. Hence, $M/R \leq L_0(\alpha, G)R/R$. Thus $\mathrm{Soc}(G/R) \leq L_0(\alpha, G)R/R$. □

LEMMA 19. *Let G be a group and let N be a minimal normal subgroup of G. Then* $\mathrm{Soc}(G) \leq C_G(N)N$.

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PROOF. If M is a minimal normal subgroup of G then either $M = N$ or $M \leq C_G(N)$. The claim follows. $\square$

LEMMA 20. *Let G be a group such that $R_\alpha(G) = 1$. Then*

$$\mathrm{Soc}(G) = \bigcap_{\substack{N \text{ is minimal} \\ \text{normal in } G}} C_G(N)N.$$

PROOF. Set

$$M = \bigcap_{\substack{N \text{ is minimal} \\ \text{normal in } G}} C_G(N)N.$$

We show that $\mathrm{Soc}(G) = M$. By Lemma 19, $\mathrm{Soc}(G) \leq M$. Let N be a minimal normal subgroup of G. Then, using $N \leq \mathrm{Soc}(G) \leq M \leq C_G(N)N$, we obtain $M = M \cap C_G(N)N = (M \cap C_G(N))N$, implying

$$M/(M \cap C_G(N)) = (M \cap C_G(N))N/(M \cap C_G(N)) \cong N/C_G(N) \cap N.$$

Since α contains the formation of solvable groups, $R_\alpha(G) = 1$ implies that the solvable radical of G is 1 and hence N is nonabelian. Therefore $C_G(N) \cap N = 1$, and we have proved that $M/C_M(N) \cong N$. Let $N_1, \ldots, N_t$ be minimal normal subgroups of G such that $\mathrm{Soc}(G) = N_1 \times \cdots \times N_t$. Now $M/(C_M(N_1) \cap \cdots \cap C_M(N_t))$ can be embedded in $(M/C_M(N_1)) \times \cdots \times (M/C_M(N_t))$, which is isomorphic (see above) to $\mathrm{Soc}(G)$. However,

$$C_M(N_1) \cap \cdots \cap C_M(N_t) = M \cap C_G(\mathrm{Soc}(G)).$$

Now $Z(\mathrm{Soc}(G)) = 1$, forcing $C_G(\mathrm{Soc}(G)) = 1$. Thus M itself can be embedded in $\mathrm{Soc}(G)$. Since $\mathrm{Soc}(G) \leq M$, we get $\mathrm{Soc}(G) = M$. $\square$

COROLLARY 21. *Let G be a group. Then*

$$L_0(\alpha, G)R_\alpha(G) = \bigcap_{\substack{\text{all } M \text{ such that } M/R_\alpha(G) \\ \text{is a chief factor of } G}} \mathrm{Inn}_G(M/R_\alpha(G)).$$

PROOF. Set $R_\alpha(G) = R$. Let M/R be a chief factor of G. Then $\mathrm{Inn}_G(M/R)/R = C_{G/R}(M/R)(M/R)$. Thus

$$\left(\bigcap_{\substack{\text{all } M \text{ such that } M/R \\ \text{is a chief factor of } G}} \mathrm{Inn}_G(M/R) \right) \Big/ R = \bigcap_{\substack{K \text{ is minimal} \\ \text{normal in } G/R}} C_{G/R}(K)K$$
$$= \mathrm{Soc}(G/R)$$
$$= L_0(\alpha, G)R/R,$$

where the second equality is justified by Lemma 20 and the third by Proposition 18. $\square$

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PROOF OF THEOREM 4. Let M/N be a non-α chief factor of G. We show that $L_0(\alpha, G)R_\alpha(G) \le \mathrm{Inn}_G(M/N)$. First note that $R_\alpha(G)N/N \cap M/N = 1$ and hence $R_\alpha(G) \le C_G(M/N) \le \mathrm{Inn}_G(M/N)$ ($C_G(M/N)$ is the preimage in G of $C_{G/N}(M/N)$). Next, let K be minimal α' normal in G. Then, by Lemma 14, either $K \le N$ (implying that $K \le \mathrm{Inn}_G(M/N)$) or KN/N is minimal α' normal in G/N. In the second case, since M/N is minimal normal in G/N and non-α, either $KN/N = M/N$ or $KN/N \cap M/N = 1$. In both cases $KN/N \le \mathrm{Inn}_G(M/N)$, and $L_0(\alpha, G) \le \mathrm{Inn}_G(M/N)$ follows. Thus we have proved the inclusion of $L_0(\alpha, G)R_\alpha(G)$ in the right-hand side of (1). Equality now follows from Corollary 21. □

PROOF OF THEOREM 6.

(a) implies (c): $L_0(\alpha, G) = L_\alpha(G)$ implies that $L_0(\alpha, G)R_\alpha(G) = L_\alpha(G)R_\alpha(G)$. By Theorem 4(1),

$$L_\alpha(G)R_\alpha(G) = \bigcap_{\substack{M/N \text{ is a non-}\alpha \text{ chief} \\ \text{factor of } G}} \mathrm{Inn}_G(M/N).$$

Thus, if M/N is a non-α chief factor of G, then $L_\alpha(G) \le \mathrm{Inn}_G(M/N)$ and hence $\mathrm{Out}_G(M/N) = G/\mathrm{Inn}_G(M/N)$ is α.

(c) implies (b): Trivial.

(b) implies (a): We assume that every chief factor of G of the form $M/R_\alpha(G)$ has the α-Schreier property in G. Hence, for any such chief factor, $L_\alpha(G) \le \mathrm{Inn}_G(M/R_\alpha(G))$. Thus, by Corollary 21, $L_\alpha(G) \le L_0(\alpha, G)R_\alpha(G)$. Since $L_0(\alpha, G) \le L_\alpha(G)$ (Corollary 15), we obtain $L_\alpha(G) = L_0(\alpha, G)(L_\alpha(G) \cap R_\alpha(G))$. From this we get that $L_\alpha(G)/L_0(\alpha, G)$ is α. Since, by Lemma 9, $L_\alpha(L_\alpha(G)) = L_\alpha(G)$, we obtain $L_\alpha(G) = L_0(\alpha, G)$ as required. □

LEMMA 22. *Let G be a group. Then $C_G(L_\alpha(G)) \le R_\alpha(G)$.*

PROOF. Set $L = L_\alpha(G)$. Clearly $C_G(L) \trianglelefteq G$, so it is sufficient to prove that $C_G(L)$ is α. Indeed, $C_G(L)L/L \cong C_G(L)/(L \cap C_G(L))$ is α, and $L \cap C_G(L)$ is abelian and hence α. Therefore, $C_G(L)$ is α. □

PROOF OF THEOREM 8.

(a) implies (b): Suppose that $C_G(L_\alpha(G)) = R_\alpha(G)$. Let M/N be a chief factor of G such that $M \le R_\alpha(G)$. Then every element of M commutes with every element of $L_\alpha(G)$. Hence, $C_{G/N}(M/N) \ge L_\alpha(G)N/N$. Thus, by Lemma 13, $(G/N)/C_{G/N}(M/N)$ is α and M/N has the α-Schreier property in G.

(b) implies (a): Suppose that every chief factor M/N of G such that $M \le R_\alpha(G)$ has the α-Schreier property in G. By Lemma 22, we may assume that $R_\alpha(G) > 1$. Let N be a minimal normal subgroup of G such that $N \le R_\alpha(G)$. By Lemma 12, $R_\alpha(G/N) = R_\alpha(G)/N$. Hence, if $(M/N)/(K/N) \cong M/K$ is a chief factor of G/N and $M/N \le R_\alpha(G/N)$, then M/K is a chief factor of G such that $M \le R_\alpha(G)$.

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It follows from this and the definition of α-Schreier property (Definition 5) that condition (b) of the theorem holds for the group G/N. Hence, by induction,

$$C_{G/N}(L_\alpha(G/N)) = R_\alpha(G/N),$$

and so $C_{G/N}(L_\alpha(G)N/N) = R_\alpha(G)/N$. We get that $[L_\alpha(G), R_\alpha(G)] \leq N$. Moreover, since N has the α-Schreier property in G, we get that $G/C_G(N)N$ is α. Since

$$(G/C_G(N))/(C_G(N)N/C_G(N)) \cong G/(C_G(N)N),$$

and α is closed under extensions, we have that $G/C_G(N)$ is α which implies that $L_\alpha(G) \leq C_G(N)$. Thus,

$$[R_\alpha(G), L_\alpha(G), L_\alpha(G)] = 1.$$

Hence, by the three-subgroups lemma,

$$[L_\alpha(G), L_\alpha(G), R_\alpha(G)] = 1.$$

But $L_\alpha(G)$ is perfect (Lemma 9), hence $[L_\alpha(G), R_\alpha(G)] = 1$ and $C_G(L_\alpha(G)) = R_\alpha(G)$. □

3. Some comments on $F^*(G)$

We begin by noting a formal similarity between $L_0(\alpha, G)R_\alpha(G)$ and $F^*(G)$. Recall that $F^*(G) = E(G)F(G)$, where $E(G)$ is the layer of G. The following property of $E(G)$ whose proof is omitted (see [3, Section 6.5]) is useful for our purposes (compare with Definition 1).

LEMMA 23. *Let G be a group. Then $E(G)$ is generated by all subgroups $1 < K \trianglelefteq G$ such that K is perfect, and for all $N \trianglelefteq G$ such that $N < K$ we have $N \leq Z(K)$ (henceforth such K will be called a minimal z' normal subgroup of G).*

Thus, $L_0(\alpha, G)$, which is generated by the minimal α' normal subgroups of G, resembles $E(G)$, and $F(G)$, which is the nilpotent radical of G, resembles $R_\alpha(G)$ (note that $F(G) \leq R_\alpha(G)$, and if α is the formation of solvable groups then $E(G) \leq L_0(\alpha, G)$).

A more direct connection between these subgroups is given by the following.

THEOREM 24. *Let G be a group. Denote $F_\alpha(G) = R_\alpha(G) \cap F^*(G)$. Then*

$$F^*(G) = L_0(\alpha, G)R_\alpha(G) \cap C_G(R_\alpha(G))F_\alpha(G).$$

In particular, $F^*(G) = N_2 \cap C_G(R)F(G)$, where R is the solvable radical of G.

The proof of this theorem requires the following lemma.

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LEMMA 25. *Set*

$$E_\alpha(G) = \prod_{\substack{K \text{ is minimal } \alpha' \text{normal in } G \\ R_\alpha(K)=Z(K)}} K,$$

$$L_1(\alpha, G) = \prod_{\substack{K \text{ is minimal } \alpha' \text{normal in } G \\ R_\alpha(K)\neq Z(K)}} K.$$

Then $L_0(\alpha, G) = E_\alpha(G)L_1(\alpha, G)$, $F^*(G) = E_\alpha(G)F_\alpha(G)$ *(see Theorem 24) and* $E_\alpha(G) \leq C_{E(G)}(R_\alpha(G))$.

PROOF. $L_0(\alpha, G) = E_\alpha(G)L_1(\alpha, G)$ is obvious from the definitions. Let K be minimal α' normal in G such that $R_\alpha(K) = Z(K)$. By Lemma 23, $K \leq E(G)$, from which $E_\alpha(G) \leq E(G)$ follows. Thus $E_\alpha(G)F_\alpha(G) \leq F^*(G)$. In order to prove the reverse inclusion, note that $F(G) \leq R_\alpha(G)$ implies $F(G) \leq F_\alpha(G)$. Next, let K be a minimal z' normal subgroup of G (see Lemma 23). If K is α then $K \leq R_\alpha(G) \cap F^*(G) = F_\alpha(G)$. If K is not α then $R_\alpha(K) < K$ and we get $R_\alpha(K) = Z(K)$. This proves that $E(G) \leq E_\alpha(G)F_\alpha(G)$ and concludes the proof that $E_\alpha(G)F_\alpha(G) = F^*(G)$. Finally, let K be minimal α' normal in G such that $R_\alpha(K) = Z(K)$. Since K is not α, then $K \cap R_\alpha(G) < K$ implying $[K, R_\alpha(G)] \leq K \cap R_\alpha(G) \leq Z(K)$, hence $[K, R_\alpha(G), K] = 1$. Thus, by the three-subgroups lemma [3, 1.5.6], $[K, K, R_\alpha(G)] = 1$. Since K is perfect this implies that $[K, R_\alpha(G)] = 1$, leading to $K \leq C_{E(G)}(R_\alpha(G))$. Thus $E_\alpha(G) \leq C_{E(G)}(R_\alpha(G))$. □

PROOF OF THEOREM 24. We use the notation of Lemma 25. Since $F_\alpha(G) \leq R_\alpha(G)$ and $E_\alpha(G) \leq C_G(R_\alpha(G))$ (Lemma 25),

$$\begin{aligned} &L_0(\alpha, G)R_\alpha(G) \cap C_G(R_\alpha(G))F_\alpha(G) \\ &\quad = E_\alpha(G)(L_1(\alpha, G)R_\alpha(G) \cap C_G(R_\alpha(G))F_\alpha(G)). \end{aligned}$$

Since $F^*(G) = E_\alpha(G)F_\alpha(G)$ (Lemma 25) it is sufficient to prove that $L_1(\alpha, G)R_\alpha(G) \cap C_G(R_\alpha(G)) \leq F_\alpha(G)$. In fact, it is sufficient to prove that $L_1(\alpha, G)R_\alpha(G) \cap C_G(R_\alpha(G))$ is α since then it is contained in $Z(R_\alpha(G))$ and hence in $F_\alpha(G)$. Assume to the contrary that $L_1(\alpha, G)R_\alpha(G) \cap C_G(R_\alpha(G))$ is not α. Then it must contain a minimal α' normal subgroup of G, say K. Since $K \leq C_G(R_\alpha(G))$, then $R_\alpha(K) = Z(K)$.

Let N be any minimal α' normal subgroup of G such that $R_\alpha(N) \neq Z(N)$ (such N's generate $L_1(\alpha, G)$). Then $K \not\leq N$ and hence $K \cap N \leq Z(K)$. It follows that $[K, N, K] = 1$ and by the three-subgroups lemma $[K, K, N] = 1$ and (K is perfect) $[K, N] = 1$. Thus $[K, L_1(\alpha, G)] = 1$. But $K \leq L_1(\alpha, G)R_\alpha(G) \cap C_G(R_\alpha(G))$ now implies that $K \leq Z(L_1(\alpha, G)R_\alpha(G))$, contradicting the fact that K is not α. □

We close this section with a proof that utilizes our Definition 1 of the following well-known fact. We only need Lemma 23.

https://doi.org/10.1017/S0004972708000506 Published online by Cambridge University Press

FACT. *For any group G, $C_G(F^*(G)) \leq F^*(G)$.*

PROOF. In this proof the property α is solvability and $R_\alpha(K) = R(K)$. Recall [5, 7.67] that if $H \trianglelefteq G$ is solvable and H centralizes $F(G)$ then $H \leq F(G)$. From this and from the fact that $C_G(F^*(G)) \leq C_G(F(G))$ it easily follows that the claim holds if $C_G(F^*(G))$ is solvable. Otherwise, $C_G(F^*(G))$ is a normal nonsolvable subgroup of G, and hence contains a minimal solvable$'$ (that is, nonsolvable) normal subgroup K. Now $R(K) < K \leq C_G(F(G))$, and hence, by the same result mentioned above, $R(K) \leq F(G)$. Since $K \leq C_G(F(G))$ we get $R(K) = Z(K)$. Hence (Lemma 23) $K \leq E(G)$. However, $K \leq C_G(E(G))$, hence $K \leq Z(E(G))$, contradicting the fact that K is nonsolvable. □

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