J. Austral. Math. Soc. (*Series A*) **51** (1991), 33–49

A DECOMPOSITION OF INTEGER VECTORS. IV

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(Received 10 May 1989)

Communicated by J. H. Loxton

In memory of Kurt Mahler

Abstract

Given m linearly independent vectors $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^k$ and an integer $l \in [m, k]$ one proves the existence of l linearly independent vectors $\mathbf{p}_1, \dots, \mathbf{p}_l \in \mathbb{Z}^k$ or $\mathbf{q}_1, \dots, \mathbf{q}_l \in \mathbb{Z}^k$ of small size (suitably measured) such that the $\mathbf{n}_i$'s are linear combinations of $\mathbf{p}_j$'s with rational coefficients or of $\mathbf{q}_j$'s with integer coefficients.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 11 H 41.

In order to generalize the results of [10] (Part III of this series) let us introduce the following notation. Given m linearly independent vectors $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^k$ let $H(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_m)$ denote the maximum of the absolute values of all minors of order m of the matrix

$$\begin{pmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_m \end{pmatrix}$$

and $D(\mathbf{n}_1, \dots, \mathbf{n}_m)$ the greatest common divisor of these minors. Furthermore, let

$$h(\mathbf{n}) = H(\mathbf{n}) \quad \text{for } \mathbf{n} \neq \mathbf{0}, \qquad h(\mathbf{0}) = 0.$$

https://doi.org/10.1017/S1446788700033280 Published online by Cambridge University Press

DEFINITION 1. For $k \geq l \geq m$, $k > m$, let

$$c_0(k, l, m) = \sup\inf \left(\frac{D(\mathbf{n}_1, \dots, \mathbf{n}_m)}{H(\mathbf{n}_1, \dots, \mathbf{n}_m)}\right)^{\frac{k-l}{k-m}} \prod_{i=1}^{l} h(\mathbf{p}_i),$$

$$c_1(k, l, m) = \sup\inf \left(\frac{D(\mathbf{n}_1, \dots, \mathbf{n}_m)}{H(\mathbf{n}_1, \dots, \mathbf{n}_m)}\right)^{\frac{k-l}{k-m}} \prod_{i=1}^{l} h(\mathbf{q}_i),$$

where the supremum is taken over all sets of linearly independent vectors $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^k$ and the infimum is taken over all sets of linearly independent vectors $\mathbf{p}_1, \dots, \mathbf{p}_l \in \mathbb{Z}^k$ or $\mathbf{q}_1, \dots, \mathbf{q}_l \in \mathbb{Z}^k$ such that for all $i \leq m$,

$$\mathbf{n}_i = \sum_{j=1}^{l} u_{ij}\mathbf{p}_j, \qquad u_{ij} \in \mathbb{Q}, \qquad \mathbf{n}_i = \sum_{j=1}^{l} u_{ij}\mathbf{q}_j, \qquad u_{ij} \in \mathbb{Z}.$$

The Bombieri-Vaaler refinement [1] of the Siegel lemma easily leads (on the lines of the proof of (8) in [10]) to the conclusion that $c_0(k, l, m)$ is finite, first obtained by Yu. Teterin. The aim of this paper is to give bounds for $c_0(k, l, m)$ and $c_1(k, l, m)$ which are independent of k. First however we shall introduce three further series of constants, this time of geometric character.

DEFINITION 2. For a given positive integer m, let κ_m be the volume of the unit ball in $\mathbb{R}^m$,

$$g_0(m) = \sup\inf \frac{\operatorname{vol}\mathbb{P}}{\operatorname{vol}\mathbb{K}}, \qquad g_1(m) = \sup\inf \frac{\operatorname{vol}\mathbb{P}}{\operatorname{vol}\mathscr{E}(\mathbb{K})} \cdot \frac{\kappa_m}{2^m},$$

where the suprema are taken over all m-dimensional convex bodies $\mathbb{K}$ situated in $\mathbb{R}^m$, symmetric with respect to the origin, the infima are taken over all parallelopipeds containing $\mathbb{K}$ symmetric with respect to the origin and $\mathscr{E}(\mathbb{K})$ denotes the ellipsoid of the maximum volume contained in $\mathbb{K}$. (It is unique; see [7].) Clearly

$$\frac{2^m}{\kappa_m} \leq g_0(m) \leq \frac{2^m}{\kappa_m} g_1(m).$$

The best published result pertaining to $g_0(m)$, $g_1(m)$ seems to be the following inequality due to Dvoretzky and Rogers [4, Theorem 5A]:

$$g_1(m) \leq \left(\frac{m^m}{m!}\right)^{1/2}.$$

Professor A. Pełczyński who indicated to me the paper [4] has improved the above inequality by showing together with S. J. Szarek that (see [9, Proposition 2.1])

$$g_1(m)^2 \leq \binom{\frac{m(m+1)}{2}}{m} \left(\frac{2}{m+1}\right)^m$$

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and, on the other hand, they have proved that (*ibid.*, Section 6)

$$g_1(m)^2 \geq \frac{2m}{m+1}.$$

For $m \leq 2$ the two bounds coincide and give

$$g_1(1) = 1, \ g_1(2) = \sqrt{\frac{4}{3}}.$$

According to [9, Theorem 5.1], for every $\varepsilon > 0$,

$$\log g_1(m) = \frac{m}{2} + o(m^{\frac{2}{3}+\varepsilon}).$$

I am indebted to Professor Pełczyński also for the paradigm (for $l = 2$) of the proof of Lemma 1 below, which he has since proved in another way (see [9], Corollary 3.1).

We shall prove

THEOREM 1. *For all integers k, l, m satisfying $k \geq l \geq m$, $k > m > 0$,*

$$c_0(k, l, m) \leq \min\left\{(l-m+1)^{l/2} g_1(m)\gamma_l^{l/2}, \frac{l!}{m!} g_0(m), \binom{l}{m}^{l/2} l^{(l-m)/2} g_1(l)\gamma_l^{l/2}\right\}, \tag{1}$$

where γ_l is the Hermite constant. For $l = m \leq 2$ we have here equality.

THEOREM 2. *For all integers k, k, m satisfying $k \geq l \geq m$, $k > m > 0$ we have*

$$\frac{c_1(k, l, m)}{c_0(k, l, m)} \leq f(l) = \sup_{\mathbf{A}} \inf_{\mathbf{U}} \left(\sum_{j=1}^{l} |\delta_{ij}|\right),$$

where $[\delta_{ij}] = \mathbb{U}\mathbb{A}^{-1}$, $\mathbb{A}$ and $\mathbb{U}$ run through all lower triangular non-singular integral matrices and all lower triangular unimodular integral matrices of order l, respectively. Moreover

$$f(l) \leq \frac{(l+\lambda+1)!}{4^{l-\lambda}(2\lambda+1)!} \quad \textit{where } \lambda = \left[\frac{1+\sqrt{16l+17}}{4}\right].$$

S. Chaładus and Yu. Teterin prove in the forthcoming paper [2] that the exponent $(k-l)/(k-m)$ in the definition of $c_0(k, l, m)$ is the correct one, that is, for any smaller exponent the corresponding supremum is infinite. Moreover they give an estimate for $c_0(k, l, m)$ that depends on k and is better than (1) for $k = o(l^2)$.

Let us note that for large l the minimum on the right-hand side of (1) is equal to the first term for $m < c_1 l/\log l$, to the last term for $m > c_2 l$, where c_1, c_2 are suitable constants, $c_1 > 0$, $c_2 < 1$, provided in the latter case that γ_l, $\log(g_0(l)\kappa_l/2^l)$ are regularly growing functions and

$$\liminf_{l\to\infty} \frac{\log g_0(l) - \frac{l}{2}\log\gamma_l}{l} > \tfrac{1}{2}.$$

For $m = 1$, (1) constitutes an improvement over [8, Theorem 1] already for $l > 50$. The problem of existence of a bound for $c_0(k, l, m)$ depending only on m remains open also for $m = 1$.

Lemma 1. *If* $\mathbb{A}$ *is a parallelohedron given by the inequalities*

$$|\mathbf{a}_i\mathbf{x}| \le 1, \qquad \mathbf{a}_i \in \mathbb{R}^l \quad (1 \le i \le k)$$

then for every parallelopiped $\mathbb{P}$ *containing* $\mathbb{A}$, *symmetric with respect to* $\mathbf{0}$ *and for a suitable subset* S *of* $\{1, 2, \ldots, k\}$ *of cardinality* l *we have*

$$\operatorname{vol}\mathbb{P} \ge \operatorname{vol}\mathbb{P}_0(S),$$

where $\mathbb{P}_0(S)$ *is the parallelopiped*

$$|\mathbf{a}_i\mathbf{x}| \le 1 \qquad (i \in S).$$

Proof. We shall proceed by induction on the number n of pairs of parallel $(l-1)$ dimensional faces of $\mathbb{P}$ that do not contain $(l-1)$ dimensional faces of $\mathbb{A}$ (in the sequel, briefly, faces). If $n = 0$ the assertion is true. Suppose it is true for the case of $n-1$ pairs of parallel faces and consider a parallelopiped $\mathbb{P}$ symmetric with respect to $\mathbf{0}$ with exactly n pairs of parallel faces not containing faces of $\mathbb{A}$. Let $\mathbb{P}$ be given by the inequalities

$$|\mathbf{b}_i\mathbf{x}| \le 1, \qquad \mathbf{b}_i \in \mathbb{R}^l \quad (1 \le i \le l)$$

and let $\mathbf{b}_l\mathbf{x} = \pm 1$ be the pair of hyperplanes corresponding to one of the n pairs in question. Replacing $\mathbb{P}$ if necessary by a smaller parallelopiped we may assume that there is $\mathbf{x}_0 \in \mathbb{A}$ such that

$$\mathbf{b}_l\mathbf{x}_0 = 1. \tag{2}$$

Let $I = \{i \le k : |\mathbf{a}_i\mathbf{x}_0| = 1\}$ and let

$$\mathbf{a}_i\mathbf{x}_0 = \varepsilon_i \qquad (i \in I). \tag{3}$$

From the fact that the hyperplane $\mathbf{b}_l\mathbf{x} = 1$ is supporting $\mathbb{A}$ at $\mathbf{x}_0$ it follows that

$$\varepsilon_i\mathbf{a}_i\mathbf{t} \le 0 \qquad (i \in I) \text{ implies } \mathbf{b}_l\mathbf{t} \le 0 \quad \text{for } \mathbf{t} \in \mathbb{R}^l. \tag{4}$$

Indeed, suppose for some $\mathbf{t}_0 \in \mathbb{R}^l$ that $\varepsilon_i \mathbf{a}_i \mathbf{t}_0 \le 0$ and $\mathbf{b}_1 \mathbf{t}_0 > 0$. Then for

$$\mathbf{t}_1 = \frac{\mathbf{t}_0}{l h(\mathbf{t}_0)} \min \left\{ \min_{i \notin I} \frac{1 - |\mathbf{a}_i \mathbf{x}_0|}{h(\mathbf{a}_i)}, \min_{i \in I} \frac{2}{h(\mathbf{a}_i)} \right\}$$

we have $\pm(\mathbf{x}_0 + \mathbf{t}_1) \in \mathbf{A}$, $\mathbf{b}_1(\mathbf{x}_0 + \mathbf{t}_1) > 1$, $\mathbf{b}_1(-\mathbf{x} - \mathbf{t}_1) < -1 < 1$, and thus the hyperplane $\mathbf{b}_1 \mathbf{x} = 1$ divides $\mathbf{A}$. This contradiction proves (4). Hence by a theorem of Farkas [5, page 5] (I owe this reference to Professor S. Rolewicz. There is a related earlier statement in [8, page 45]) we have

$$\mathbf{b}_1 = \sum_{i \in I} \varepsilon_i \mathbf{a}_i \lambda_i ,$$

where

$$(5) \qquad \lambda_i \ge 0 \qquad (i \in I)$$

and by (2) and (3)

$$(6) \qquad \sum_{i \in I} \lambda_i = 1 .$$

Therefore,

$$(7) \qquad \begin{aligned} (\operatorname{vol} \mathbb{P})^{-1} &= 2^{-l} \left| \det \left(\sum_{i \in I} \varepsilon_i \mathbf{a}_i \lambda_i , \mathbf{b}_2 , \dots , \mathbf{b}_l \right) \right| \\ &= 2^{-l} \left| \sum_{i \in I} \lambda_i \det(\varepsilon_i \mathbf{a}_i , \mathbf{b}_2 , \dots , \mathbf{b}_l) \right| . \end{aligned}$$

Regarding λ_i as variables restricted by the conditions (5) and (6), we easily see that the right-hand side of (7) takes the maximum for $\lambda_i = 1$ if $i = i_0$, $\lambda_i = 0$ otherwise. Hence

$$(8) \qquad \operatorname{vol} \mathbb{P} \ge \operatorname{vol} \mathbb{P}_1 ,$$

where $\mathbb{P}_1$ is the parallelopiped

$$|\mathbf{a}_{i_0} \mathbf{x}| \le 1 , \qquad |\mathbf{b}_i \mathbf{x}| \le 1 \qquad (2 \le i \le l) .$$

However $\mathbb{P}_1$ contains $\mathbf{A}$ and it has only $n - 1$ pairs of parallel faces that do not contain faces of $\mathbf{A}$. Thus by the inductive assumption there exists a set $S \subset \{1, 2, \dots, k\}$ of cardinality l and with the property

$$\operatorname{vol} \mathbb{P}_1 \ge \operatorname{vol} \mathbb{P}_0(S) .$$

In view of (8) this gives

$$\operatorname{vol} \mathbb{P} \ge \operatorname{vol} \mathbb{P}_0(S)$$

and concludes the inductive argument.

https://doi.org/10.1017/S1446788700033280 Published online by Cambridge University Press

LEMMA 2. *For all linearly independent vectors* $\mathbf{c}_1, \dots, \mathbf{c}_l \in \mathbb{R}^k$ *the domain*

$$\mathbb{C} : h(\mathbf{c}_1 x_1 + \cdots + \mathbf{c}_l x_l) \le 1$$

satisfies

$$\operatorname{vol}\mathbb{C} \ge \frac{2^l}{g_0(l)H(\mathbf{c}_1, \dots, \mathbf{c}_l)}, \qquad \operatorname{vol}\mathscr{E}(\mathbb{C}) \ge \frac{\kappa_l}{g_1(l)H(\mathbf{c}_1, \dots, \mathbf{c}_l)}.$$

PROOF. Put

(9) $$\mathbf{a}_i = [c_{1i}, c_{2i}, \dots, c_{li}] \qquad (1 \le i \le k).$$

Then

$$\mathbb{C} = \{x \in \mathbb{R}^l : |\mathbf{a}_i \mathbf{x}| \le 1 \text{ for all } i \le k\}$$

and clearly $\mathbb{C}$ is a convex body symmetric with respect to $\mathbf{0}$. By Definition 2

$$\operatorname{vol}\mathbb{C} \ge g_0(l)^{-1} \inf \operatorname{vol}\mathbb{P}, \qquad \operatorname{vol}\mathscr{E}(\mathbb{C}) \ge g_1(l)^{-1} 2^{-l} \kappa_l \inf \operatorname{vol}\mathbb{P},$$

where the infimum is taken over all parallelopipeds $\mathbb{P}$ symmetric with respect to $\mathbf{0}$ and containing $\mathbb{C}$. However by Lemma 1 the infimum can be replaced by the minimum taken over the finite set of all parallelopipeds

$$\mathbb{P}_0(S), \qquad |\mathbf{a}_i \mathbf{x}| \le 1 \qquad (i \in S),$$

where S runs through all subsets of $\{1, \dots, k\}$ of cardinality l. Since

$$\operatorname{vol}\mathbb{P}_0(S) = 2^l |\det\{\mathbf{a}_i : i \in S\}|^{-1}$$

we have by (9) that

$$\min \operatorname{vol}\mathbb{P}_0(S) = 2^l H(\mathbf{c}_1, \dots, \mathbf{c}_l)^{-1}$$

and the lemma follows.

LEMMA 3. *If for all linearly independent vectors* $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^k$ *such that* $D(\mathbf{n}_1, \dots, \mathbf{n}_m) = 1$ *there exist linearly independent vectors* $\mathbf{p}_1, \dots, \mathbf{p}_l \in \mathbb{Z}^k$ *such that*

$$\mathbf{n}_i = \sum_{j=1}^{l} u_{ij}\mathbf{p}_j, \qquad u_{ij} \in \mathbb{Q}$$

and

$$\prod_{j=1}^{l} h(\mathbf{p}_j) \le cH(\mathbf{n}_1, \dots, \mathbf{n}_m)^{(k-l)/(k-m)}$$

then $c_0(k, l, m) \le c$.

PROOF. Consider m linearly independent vectors $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^k$ and let $\mathscr{N}$ be the linear space spanned by them over $\mathbb{R}$. Further, let $\mathbf{b}_1, \dots, \mathbf{b}_m$

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be a basis of the lattice $\mathscr{N} \cap \mathbb{Z}^k$ and $\mathbf{c}_1, \ldots, \mathbf{c}_{k-m} \in \mathbb{Z}^k$ linearly independent vectors perpendicular to $\mathscr{N}$. Since $\mathscr{N} \cap \mathbb{Z}^k$ is the lattice of all solutions $\mathbf{x} \in \mathbb{Z}^k$ of the system $\mathbf{c}_i\mathbf{x} = 0$ $(1 \le i \le k-m)$, we have by the known theorem [3, page 53] that

$$D(\mathbf{b}_1, \ldots, \mathbf{b}_m) = 1. \tag{10}$$

On the other hand clearly

$$\begin{pmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_m \end{pmatrix} = \mathbb{A} \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_m \end{pmatrix}, \tag{11}$$

where $\mathbb{A}$ is an integral square matrix of order m. It follows from (11) that

$$D(\mathbf{n}_1, \ldots, \mathbf{n}_m) = |\det \mathbb{A}| D(\mathbf{b}_1, \ldots, \mathbf{b}_m),$$
$$H(\mathbf{n}_1, \ldots, \mathbf{n}_m) = |\det \mathbb{A}| H(\mathbf{b}_1, \ldots, \mathbf{b}_m)$$

and by (10)

$$H(\mathbf{b}_1, \ldots, \mathbf{b}_m) = \frac{H(\mathbf{n}_1, \ldots, \mathbf{n}_m)}{D(\mathbf{n}_1, \ldots, \mathbf{n}_m)}. \tag{12}$$

By the assumption of the lemma there exist linearly independent vectors $\mathbf{p}_1, \ldots, \mathbf{p}_l \in \mathbb{Z}^k$ and a matrix $\mathbb{U} \in \mathscr{M}_{m,l}(\mathbb{Q})$ such that

$$\begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_m \end{pmatrix} = \mathbb{U} \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_l \end{pmatrix}, \tag{13}$$

and

$$\prod_{j=1}^{l} h(\mathbf{p}_j) \le cH(\mathbf{b}_1, \ldots, \mathbf{b}_m)^{(k-l)/(k-m)}. \tag{14}$$

It follows from (11) and (13) that

$$\begin{pmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_m \end{pmatrix} = \mathbb{A}\mathbb{U} \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_l \end{pmatrix},$$

while from (12) and (14) that

$$\prod_{j=1}^{l} h(\mathbf{p}_j) \le \left(\frac{H(\mathbf{n}_1, \ldots, \mathbf{n}_m)}{D(\mathbf{n}_1, \ldots, \mathbf{n}_m)} \right)^{(k-l)/(k-m)}.$$

Thus by Definition 1, $c_0(k, l, m) \le c$.

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Lemma 4. *Let* $\mathbb{K}$ *be a convex domain symmetric with respect to* $\mathbf{0}$ *in the linear subspace* $\mathscr{L} : x_1 = \cdots = x_m = 0$ *of* $\mathbb{R}^k$, *not containing in its interior any point of the lattice* $\mathscr{L} \cap \mathbb{Z}^k$ *except* $\mathbf{0}$ *and let* $\| \ \|_{\mathbb{K}}$ *be the corresponding distance function. Let* $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^k$ *and* $\mathscr{N}$ *be the linear space spanned by* $\mathbf{n}_1, \dots, \mathbf{n}_m$ *over* $\mathbb{R}$. *If* $\Delta = \det(n_{ij})_{i,j\le m} \neq 0$ *and* $D(\mathbf{n}_1, \dots, \mathbf{n}_m) = 1$ *there exist vectors* $\mathbf{n}_{m+1}, \dots, \mathbf{n}_k \in \mathbb{Z}^k$ *such that* $\mathbf{n}_1, \dots, \mathbf{n}_k$ *are linearly independent and*

$$\prod_{i=m+1}^{k} \|(\mathbf{n}_i + \mathscr{N}) \cap \mathscr{L}\|_{\mathbb{K}} \le 2^{k-m}(\operatorname{vol}\mathbb{K})^{-1}|\Delta|^{-1}.$$

Remark. Since $\Delta \neq 0$ we have $\mathscr{N} \cap \mathscr{L} = \{\mathbf{0}\}$, and hence $(\mathbf{n}_i + \mathscr{N}) \cap \mathscr{L}$ consists of one point and $\|(\mathbf{n}_i + \mathscr{N}) \cap \mathscr{L}\|_{\mathbb{K}}$ means the distance from this point to $\mathbf{0}$ measured through $\mathbb{K}$.

Proof. If $|\Delta| = 1$ the desired conclusion follows directly from Minkowski's second theorem. Indeed by that theorem applied to the domain $\mathbb{K}$ there exist linearly independent vectors $\mathbf{n}_{m+1}, \dots, \mathbf{n}_k \in \mathscr{L} \cap \mathbb{K}$ such that

$$\prod_{i=m+1}^{k} \|\mathbf{n}_i\|_{\mathbb{K}} \le 2^{k-m}(\operatorname{vol}\mathbb{K})^{-1}.$$

Since $\mathscr{N} \cap \mathscr{L} = \{\mathbf{0}\}$ we have $(\mathbf{n}_i + \mathscr{N}) \cap \mathscr{L} = \{\mathbf{n}_i\}$ $(m < i \le k)$ and $\mathbf{n}_1, \dots, \mathbf{n}_k$ are linearly independent. Therefore assume that $|\Delta| > 1$. Let $\Delta_i(\mathbf{x})$ be the determinant of the matrix obtained from $(n_{ij})_{i,j\le m}$ by replacing the ith row by the first m coordinates of the vector $\mathbf{x}$.

Let us take a real number $r > 1$ and consider in $\mathbb{R}^k$ the domain

$$\mathbb{D}_r(\mathbb{K}) : \max_{1\le\mu\le m} |\Delta_\mu(\mathbf{x})| + |\Delta|^r \|\mathbf{x}\Delta - \sum_{\mu=1}^{m} \mathbf{n}_\mu \Delta_\mu(\mathbf{x})\|_{\mathbb{K}}^{(k-m)r} \le |\Delta|^{(k-m)r}.$$

Then $\mathbb{D}_r(\mathbb{K})$ is convex and symmetric with respect to $\mathbf{0}$. In order to compute its volume we make the affine transformation

$$\frac{\Delta_\mu(\mathbf{x})}{\Delta^{(k-m)r}} = y_\mu \qquad (\mu = 1, \dots, m), \qquad x_\mu = y_\mu \qquad (\mu = m+1, \dots, k).$$

This transformation has Jacobian equal to $\Delta^{(k-m)rm-m+1}$ and it transforms $\mathbb{D}_r(\mathbb{K})$ into

$$\mathbb{D}'_r(\mathbb{K}) : \max_{1\le\mu\le m} |y_\mu| + |\Delta|^r \|[\mathbf{0}, y_{m+1}, \dots, y_k] - \sum_{\mu=1}^{m} \mathbf{n}'_\mu y_\mu \Delta^{(k-m)r-1}\|_{\mathbb{K}}^{(k-m)r} \le 1,$$

where $\mathbf{n}'_\mu$ is the projection of $\mathbf{n}_\mu$ on $\mathscr{L}$. Clearly

$$\begin{aligned}\operatorname{vol}\mathbb{D}_r(\mathbb{K}) &= |\Delta|^{(k-m)rm-m+1}\operatorname{vol}\mathbb{D}'_r(\mathbb{K}) \\ &= |\Delta|^{(k-m)rm-m+1}\operatorname{vol}\mathbb{K}\int_{\max_{1\le\mu\le m}|y_\mu|\le 1} dy_1 dy_2\cdots dy_m\left(\frac{1-\max|y_\mu|}{|\Delta|^r}\right)^{1/r} \\ &= 2^m|\Delta|^{((k-m)r-1)m}\operatorname{vol}\mathbb{K}\int_0^1 mt^{m-1}(1-t)^{1/r}\,dt.\end{aligned}$$

Put $\int_0^1 mt^{m-1}(1-t)^{1/r}\,dt = I_{r,m}$.

Let $\lambda_i = \inf\{\lambda : \dim\lambda\mathbb{D}_r(\mathbb{K})\cap\mathbb{Z}^k \ge i\}$ $(1\le i\le k)$. By Minkowski's second theorem there exist linearly independent points $\mathbf{m}_1, \dots, \mathbf{m}_k$ such that

(15) $$\mathbf{m}_i \in \lambda_i\mathbb{D}_r(\mathbb{K})\cap\mathbb{Z}^k$$

and

(16) $$\prod_{i=1}^k \lambda_i \le 2^k\operatorname{vol}\mathbb{D}_r(\mathbb{K})^{-1} = 2^{k-m}I_{r,m}^{-1}(\operatorname{vol}\mathbb{K})^{-1}|\Delta|^{(1-(k-m)r)m}.$$

We shall show that

(17) $$\lambda_i = |\Delta|^{1-(k-m)r} \qquad (1\le i\le m)$$

and

(18) $$\mathbf{m}_i \in \mathscr{N} \qquad (1\le i\le m).$$

Indeed, for $i\le m$, $\mu\le m$ we have

$$\Delta_\mu(\mathbf{n}_i) = \Delta \quad \text{if } \mu = i, \quad 0 \text{ otherwise};$$

$$\Delta\mathbf{n}_i = \sum_{\mu=1}^m \mathbf{n}_\mu\Delta_\mu(\mathbf{n}_i),$$

and hence

(19) $$\mathbf{n}_i \in |\Delta|^{1-(k-m)r}\mathbb{D}_r(\mathbb{K}) \qquad (1\le i\le m).$$

On the other hand, if $\mathbf{x}\in\lambda\mathbb{D}_r(\mathbb{K})\cap\mathbb{Z}^k$ and $\mathbf{x}\notin\mathscr{N}$ we have $\Delta\mathbf{x} \ne \sum_{\mu=1}^m \mathbf{n}_\mu\Delta_\mu(\mathbf{x})$, and thus by the assumption about $\mathbb{K}$, $\|\Delta\mathbf{x} - \sum_{\mu=1}^{n_\mu}\mathbf{n}_\mu\Delta_\mu(\mathbf{x})\|_{\mathbb{K}} \ge 1$ and by the definition of $\mathbb{D}_r(\mathbb{K})$,

(20) $$\lambda^{(k-m)r}|\Delta|^{(k-m)r} \ge |\Delta|^r; \qquad \lambda \ge |\Delta|^{-1+\frac{1}{k-m}} > |\Delta|^{1-(k-m)r}.$$

If $\mathbf{x}\in\lambda\mathbb{D}_r(\mathbb{K})\cap\mathbb{Z}^k$ and $\mathbf{x}\in\mathscr{N}$ we have $\Delta\mathbf{x} = \sum_{\mu=1}^m \mathbf{n}_\mu\Delta_\mu(\mathbf{x})$ and thus by the assumption that $D(\mathbf{n}_1, \dots, \mathbf{n}_m) = 1$ we have $\Delta_\mu(\mathbf{x}) \equiv 0 \pmod{\Delta}$, and hence either $\mathbf{x} = \mathbf{0}$ or $\max_{1\le\mu\le m} |\Delta_\mu(\mathbf{x})| \ge |\Delta|$, which by the definition of $\mathbb{D}_r(\mathbb{K})$ implies

$$\lambda \ge |\Delta|^{1-(k-m)r}. \tag{21}$$

The claims (17) and (18) follow from (19), (20) and (21).

From (16) and (17) we infer that

$$\prod_{i=m+1}^{k} \lambda_i \le 2^{k-m} (\operatorname{vol} \mathbb{K})^{-1} I_{r,m}^{-1}$$

and since by (15)

$$|\Delta|^r \left\| \Delta \mathbf{m}_i - \sum_{\mu=1}^{x} \mathbf{n}_\mu \Delta_\mu(\mathbf{m}_i) \right\|_{\mathbf{K}}^{(k-m)r} \le |\Delta|^{(k-m)r} \lambda_i^{(k-m)r}$$

we obtain

$$\prod_{i=m+1}^{k} \left\| \mathbf{m}_i - \Delta^{-1} \sum_{\mu=1}^{m} \mathbf{n}_\mu \Delta_\mu(\mathbf{m}_i) \right\|_{\mathbf{K}} \le 2^{k-m} (\operatorname{vol} \mathbb{K})^{-1} |\Delta|^{-1} I_{r,m}^{-1}. \tag{22}$$

Moreover, by (18), $\mathbf{n}_1, \dots, \mathbf{n}_m, \mathbf{m}_{m+1}, \dots, \mathbf{m}_k$ are linearly independent. For every $r > 1$ there corresponds a certain choice of vectors $\mathbf{m}_i \in \mathbb{Z}^k$, however the set of values which we can obtain on the left-hand side of (22) is discrete. Therefore there exist vectors $\mathbf{n}_i$ $(m < i \le k)$ such that $\mathbf{n}_i$ $(1 \le i \le k)$ are linearly independent and

$$\prod_{i=m+1}^{k} \left\| \mathbf{n}_i - \Delta^{-1} \sum_{\mu=1}^{m} \mathbf{n}_\mu \Delta_\mu(\mathbf{n}_i) \right\|_{\mathbf{K}} \le 2^{k-m} (\operatorname{vol} \mathbb{K})^{-1} |\Delta|^{-1} \lim_{r\to\infty} I_{r,m}^{-1}.$$

However

$$\left\{ \mathbf{n}_i - \Delta^{-1} \sum_{\mu=1}^{m} \mathbf{n}_\mu \Delta_\mu(\mathbf{n}_i) \right\} = (\mathbf{n}_i + \mathscr{N}) \cap \mathscr{L}$$

and

$$\lim_{r\to\infty} I_{r,m} = \int_0^1 m t^{m-1}\, dt = 1,$$

which proves the lemma.

Lemma 5. *If* $m < k$, $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^k$, $D(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_m) = 1$ *there exist vectors* $\mathbf{n}_{m+1}, \dots, \mathbf{n}_k \in \mathbb{Z}^k$ *such that* $\mathbf{n}_1, \dots, \mathbf{n}_k$ *are linearly independent and for each* $l \in [m, k]$ *the domain* $\mathbb{D} \subset \mathbb{R}^l : h(\sum_{i=1}^{l} x_i \mathbf{n}_i) \le 1$ *satisfies*

$$\operatorname{vol} \mathbb{D} \ge \frac{2^l m!}{g_0(m) l!} H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{-(k-l)/(k-m)} \tag{23}$$

and

$$\operatorname{vol}\mathscr{E}(\mathbb{D}) \geq \max\left\{\frac{\kappa_l}{g_1(m)(l-m+1)^{l/2}}, \frac{\kappa_l}{g_1(l)}\binom{l}{m}^{-1/2} l^{(m-l)/2}\right\} \times H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{(k-l)/(k-m)}. \tag{24}$$

PROOF. Without loss of generality we may assume that $H(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_m) = |\Delta|$, where $\Delta = \det(n_{ij})_{i,j\leq m}$. By Lemma 4 applied with $\mathbb{K} = \{\mathbf{x} \in \mathscr{L} : h(\mathbf{x}) \leq 1\}$ there exists vectors $\mathbf{n}_{m+1}, \dots, \mathbf{n}_k \in \mathbb{Z}^k$ such that $\mathbf{n}_1, \dots, \mathbf{n}_l$ are linearly independent and

$$\prod_{i=m+1}^{k} h(\mathbf{n}_i') \leq |\Delta|^{-1}, \quad \text{where } \{\mathbf{n}_i'\} = (\mathbf{n}_i + \mathscr{N}) \cap \mathscr{L} \quad (m < i \leq k). \tag{25}$$

Permuting the vectors $\mathbf{n}_i$ if necessary we may assume that the sequence $h(\mathbf{n}_i')$ is nondecreasing. Then (25) implies

$$\prod_{i=m+1}^{l} h(\mathbf{n}_i') \leq H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{-(l-m)/(k-m)}. \tag{26}$$

In order to prove (23) let us write explicitly

$$\mathbf{n}_i' = \mathbf{n}_i - \sum_{\mu=1}^{m} a_{i\mu}\mathbf{n}_\mu \qquad (m < i \leq l).$$

Then

$$\sum_{i=1}^{l} x_i\mathbf{n}_i = \sum_{\mu=1}^{m} \mathbf{n}_\mu \left(x_\mu + \sum_{i=m+1}^{l} a_{i\mu}x_i\right) + \sum_{i=m+1}^{l} x_i\mathbf{n}_i'$$

and

$$h\left(\sum_{i=1}^{l} x_i\mathbf{n}_i\right) \leq h\left(\sum_{\mu=1}^{m} \mathbf{n}_\mu \left(x_\mu + \sum_{i=m+1}^{l} a_{i\mu}x_i\right)\right) + \sum_{i=m+1}^{l} |x_i| h(\mathbf{n}_i').$$

It follows by a change of variables that

$$\operatorname{vol}\mathbb{D} \geq \int_{\mathbb{D}_0} dx_{m+1}\cdots dx_l \operatorname{vol}\left\{\mathbf{x} \in \mathbb{R}^m : h\left(\sum_{\mu=1}^{m} x_\mu\mathbf{n}_\mu\right) \leq 1 - \sum_{i=m+1}^{l} |x_i| h(\mathbf{n}_i')\right\},$$

where $\mathbb{D}_0$ is the domain $\sum_{i=m+1}^{l} |x_i| h(\mathbf{n}_i') \leq 1$. However by Lemma 2,

$$\operatorname{vol}\left\{\mathbf{x} \in \mathbb{R}^m : h\left(\sum_{\mu=1}^{m} x_\mu\mathbf{n}_\mu\right) \leq c\right\} \geq \frac{2^m c^m}{g_0(m)H(\mathbf{n}_1, \dots, \mathbf{n}_m)},$$

and hence

$$\operatorname{vol}\mathbb{D} = \frac{2^m}{g_0(m)H(\mathbf{n}_1, \dots, \mathbf{n}_m)} \int_{\mathbb{D}_0} \left(1 - \sum_{i=m+1}^{l} |x_i| h(\mathbf{n}_i')\right)^m dx_{m+1} \cdots dx_l$$

$$= \frac{2^l m!}{g_0(m) l! H(\mathbf{n}_1, \dots, \mathbf{n}_m)} \prod_{i=m+1}^{l} h(\mathbf{n}_i')^{-1}$$

and (23) follows from (26).

In order to prove the part of (24) corresponding to the first term of the maximum on the right-hand side, let $\mathbb{D}_1$ be the domain $h(\sum_{i=1}^m x_i \mathbf{n}_i) \le 1$. The ellipsoid $\mathscr{E}(\mathbb{D}_1)$ is given by the inequality $F_1(x_1, \dots, x_m) \le 1$, where F_1 is a positive definite quadratic form.

Since $\mathscr{E}(\mathbb{D}_1) \subset \mathbb{D}_1$ we have for all $\mathbf{x} \in \mathbb{R}^m$,

$$\sqrt{F_1(x_1, \dots, x_m)} = \|\mathbf{x}\|_{\mathscr{E}(\mathbb{D}_1)} \ge \|\mathbf{x}\|_{\mathbb{D}_1} = h\left(\sum_{i=1}^{m} x_i \mathbf{n}_i\right). \tag{28}$$

By virtue of Lemma 2, we have

$$\operatorname{vol}\mathscr{E}(\mathbb{D}_1) \ge \kappa_m g_1(m)^{-1} H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{-1}.$$

However

$$\operatorname{vol}\mathscr{E}(\mathbb{D}_1) = \frac{\kappa_m}{\sqrt{d(F_1)}},$$

and thus

$$\sqrt{d(F_1)} \le g_1(m) H(\mathbf{n}_1, \dots, \mathbf{n}_m). \tag{29}$$

Consider now the quadratic form

$$F(x_1, \dots, x_l) = (l - m + 1) F_1\left(\dots, x_\mu + \sum_{i=m+1}^{l} a_{i\mu} x_i, \dots\right) + (l - m + 1) \sum_{i=m+1}^{l} x_i^2 h^2(\mathbf{n}_i').$$

For all $\mathbf{x} \in \mathbb{R}^l$ we have by the Cauchy inequality, by (28) and (27), that

$$\sqrt{F(x_1, \dots, x_l)} \ge \sqrt{F_1\left(\dots, x_\mu + \sum_{i=m+1}^{l} a_{i\mu} x_i, \dots\right)} + \sum_{i=m+1}^{l} |x_i| h(\mathbf{n}_i') \ge h\left(\sum_{\mu=1}^{m} \mathbf{n}_\mu \left(x_\mu + \sum_{i=m+1}^{l} a_{i\mu} x_i\right)\right) + \sum_{i=m+1}^{l} |x_i| h(\mathbf{n}_i') \ge h\left(\sum_{i=1}^{l} x_i \mathbf{n}_i\right),$$

and thus the ellipsoid

$$E : F(x_1, \dots, x_l) \le 1$$

is contained in $\mathbb{D}$ and by the definition of $\mathscr{E}(\mathbb{D})$,

$$\operatorname{vol}\mathscr{E}(\mathbb{D}) \ge \operatorname{vol}\mathbb{E} = \frac{\kappa_l}{\sqrt{d(F)}}. \tag{30}$$

Since F is obtained from the quadratic form

$$(l-m+1)\left(F_1 + \sum_{i=m+1}^{l} x_i^2 h^2(\mathbf{n}_i')\right)$$

by a unimodular substitution, we have

$$\sqrt{d(F)} = (l-m+1)^{l/2}\sqrt{d(F_1)}\prod_{i=m+1}^{l} h(\mathbf{n}_i')$$

and by (26), (29) and (30),

$$\operatorname{vol}\mathscr{E}(\mathbb{D}) \ge \kappa_l(l-m+1)^{-l/2}H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{-(k-l)/(k-m)}.$$

In order to prove the remaining part of (24) note that

$$H(\mathbf{n}_1, \dots, \mathbf{n}_l) = H(\mathbf{n}_1, \dots, \mathbf{n}_m, \mathbf{n}_{m+1}', \dots, \mathbf{n}_l').$$

Let M be a minor of order of l of the matrix

$$\begin{pmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_m \\ \mathbf{n}_{m+1}' \\ \vdots \\ \mathbf{n}_l' \end{pmatrix}$$

and S the set of indices of the columns of M. Developing M according to the first m rows we obtain from the Laplace theorem

$$|M| \le H(\mathbf{n}_1, \dots, \mathbf{n}_m)\sum |M_{j_1, \dots, j_{l-m}}|, \tag{31}$$

where $M_{j_1, \dots, j_{l-m}}$ is the minor of

$$\begin{pmatrix} \mathbf{n}_{m+1}' \\ \vdots \\ \mathbf{n}_l' \end{pmatrix}$$

consisting of the columns $j_1, \dots, j_{l-m}$, while $\{j_1, \dots, j_{l-m}\}$ runs through all subsets of S of cardinality $l-m$.

By the generalized Hadamard inequality [1, formula (2.6)]

$$\sum M^2_{j_1,\dots,j_{l-m}} \le \prod_{i=m+1}^{l} \sum_{j\in S} n'^2_{ij} \le l^{l-m} \prod_{i=m+1}^{l} h(\mathbf{n}'_i)^2,$$

and hence, by the Cauchy inequality,

$$\sum |M_{j_1,\dots,j_{l-m}}| \le \binom{l}{m}^{1/2} l^{(l-m)/2} \prod_{i=m+1}^{l} h(\mathbf{n}'_i). \tag{32}$$

The inequalities (26), (31) and (32) give

$$|M| \le \binom{l}{m}^{1/2} l^{(l-m)/2} H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{(k-l)/(k-m)},$$

and hence by the arbitrary choice of M

$$H(\mathbf{n}_1, \dots, \mathbf{n}_l) \le \binom{l}{m}^{1/2} l^{(l-m)/2} H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{(k-l)/(k-m)}.$$

Now Lemma 2 applied with $\mathbb{C} = \mathbb{D}$ implies

$$\operatorname{vol} \mathscr{E}(\mathbb{D}) \ge \frac{\kappa_l}{g_1(l)} \binom{l}{m}^{-1/2} l^{(m-l)/2} H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{-(k-l)/(k-m)}.$$

Proof of Theorem 1. Let $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^k$ be linearly independent and $D(\mathbf{n}_1, \dots, \mathbf{n}_m) = 1$. Let $\mathbf{n}_{m+1}, \dots, \mathbf{n}_l$ be vectors the existence of which is asserted in Lemma 5 and consider the domain $\mathbb{D} : h(\sum_{j=1}^{l} x_j \mathbf{n}_j) \le 1$. Let

$$\mu_i = \min\{\mu : \dim \mu\mathbb{D} \cap \mathbb{Z}^l \ge i\} \qquad (1 \le i \le l).$$

By Minkowski's second theorem there exist linearly independent vectors $\mathbf{y}_i = [y_{i1}, \dots, y_{il}]$ $(1 \le i \le l)$ such that

$$\mathbf{y}_i \in \mu_i \mathbb{D} \cap \mathbb{Z}^l \tag{33}$$

and

$$\prod_{i=1}^{l} \mu_i \le 2^l (\operatorname{vol} \mathbb{D})^{-1}. \tag{34}$$

By another theorem of Minkowski (see [8, §51] or [6, §18, Theorem 3]),

$$\prod_{i=1}^{l} \mu_i \le \Delta(\mathscr{E}(\mathbb{D}))^{-1}, \tag{35}$$

where $\Delta(\mathscr{E}(\mathbb{D}))$ is the critical determinant of $\mathscr{E}(\mathbb{D})$ and by the definition of the Hermite constant

$$\Delta(\mathscr{E}(\mathbb{D}))^{-1} = \gamma_l^{l/2} \frac{\kappa_l}{\operatorname{vol} \mathscr{E}(\mathbb{D})} \tag{35}$$

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(see [6, formula (37.6)]). Let us put

$$(37)\qquad \mathbf{p}_i = \sum_{j=1}^{l} y_{ij}\mathbf{n}_j \qquad (1 \le i \le l).$$

It follows from the definition of $\mathbb{D}$ and from (34)–(37) that $h(\mathbf{p}_i) = \mu_i$, hence by (34)–(37)

$$\prod_{i=1}^{l} h(\mathbf{p}_i) \le \min\{2^l(\operatorname{vol}\mathbb{D})^{-1}, \gamma_l^{l/2}\kappa_l(\operatorname{vol}\mathscr{E}(\mathbb{D}))^{-1}\}$$

and by Lemma 5

$$\prod_{i=1}^{l} h(\mathbf{p}_i) \le \min\left\{\frac{l!}{m!}g_0(m), (l-m+1)^{l/2}g_1(m)\gamma_l^{l/2}, \binom{l}{m}^{1/2} l^{(l-m)/2} g_1(l)\gamma_l^{l/2}\right\} H(\mathbf{n}_1, \dots, \mathbf{n}_m)^{(k-l)/(k-m)}.$$

Moreover since $\mathbf{y}_1, \dots, \mathbf{y}_l$ are linearly independent the system (37) can be solved with respect to $\mathbf{n}_1, \dots, \mathbf{n}_l$ and we obtain

$$\mathbf{n}_i = \sum_{j=1}^{l} u_{ij}\mathbf{p}_j, \qquad u_{ij} \in \mathbb{Q} \quad (1 \le i \le l).$$

Since $\mathbf{n}_i$ $(1 \le i \le l)$ are linearly independent so are $\mathbf{p}_j$ $(1 \le j \le l)$ and we obtain from (37) and Lemma 3 that

$$(38)\quad c_0(k, l, m) \le \min\left\{(l-m+1)^{l/2}g_1(m)\gamma_l^{l/2}, \frac{l!}{m!}g_0(m), \binom{l}{m}^{1/2} l^{\frac{l-m}{2}} g_1(l)\gamma_l^{l/2}\right\},$$

which proves the first part of the theorem.

In order to prove the second part let us observe that if $l = m = 1$ the right-hand side of (38) equals 1, while it immediately follows from the definition of $c_0(k, l, m)$ that $c_0(k, 1, 1) \ge 1$. If $l = m = 2$ the right hand side of (38) equals $\frac{4}{3}$, since

$$g_1(2) = \sqrt{\frac{4}{3}}, \qquad \gamma_2 = \sqrt{\frac{4}{3}}, \qquad g_0(2) \ge \frac{4}{3}.$$

On the other hand, consider the following vectors in $\mathbb{Z}^k$ $(k \ge 3)$

$$\mathbf{n}_1 = [2t, 4t+1, 2t, 0, \dots, 0], \quad \mathbf{n}_2 = [4t-1, 2t, -2t, 0, \dots, 0] \quad (t \in \mathbb{N}).$$

We have here

$$H(\mathbf{n}_1, \mathbf{n}_2) = 12t^2 + 2t, \qquad D(\mathbf{n}_1, \mathbf{n}_2) = 1.$$

Hence, if

$$\mathbf{n}_i = \sum_{j=1}^{2} u_{ij}\mathbf{p}_j, \quad u_{ij} \in \mathbb{Q}, \mathbf{p}_j \in \mathbb{Z}^k \quad (1 \le i, j \le 2)$$

we have

$$\mathbf{p}_j = \mathbf{n}_1 x_j + \mathbf{n}_2 y_j, \qquad [x_j, y_j] \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \quad (1 \le j \le 2).$$

If $x_j = y_j$ we have $|p_{j2}| > 6t$, otherwise $|p_{j3}| \ge 2t$, and thus $h(\mathbf{p}_j) \ge 2t$ $(1 \le j \le 2)$. If for an $\varepsilon > 0$ we have

$$h(\mathbf{p}_1)h(\mathbf{p}_2) \le (\tfrac{4}{3} - \varepsilon)H(\mathbf{n}_1, \mathbf{n}_2) = (\tfrac{4}{3} - \varepsilon)(12t^2 + 2t)$$

then for $t > t_0(\varepsilon)$

$$h(\mathbf{p}_1)h(\mathbf{p}_2) < (16 - 10\varepsilon)t^2 \tag{39}$$

and since $h(\mathbf{p}_j) \ge 2t$ we obtain $h(\mathbf{p}_j) < (8 - 5\varepsilon)t^2$ $(1 \le j \le 2)$. Hence for $t > t_1(\varepsilon)$, by consideration of the first three coordinates of $\mathbf{p}_j$

$$|2x_j + 4y_j| \le 7, \qquad |4x_j + 2y_i| \le 7, \qquad |2x_j - 2y_j| \le 7;$$

$|x_j| \le 1$, $|y_j| \le 1$ and since $[x_j, y_j] \ne [0, 0]$, $h(\mathbf{p}_j) \ge 4t - 1$ $(1 \le j \le 2)$. It follows that

$$h(\mathbf{p}_1)h(\mathbf{p}_2) \ge 16t^2 - 8t + 1,$$

which for $t > \max\{t_0(\varepsilon), t_1(\varepsilon), \varepsilon^{-1}\}$ contradicts (39). This shows that $c_0(k, 2, 2) = \frac{4}{3}$ and completes the proof of the theorem.

Proof of Theorem 2. The proof does not differ essentially from the proof of [10, Theorem 2]. In formula (14) and in the fourth displayed formula on page 701 there, one has to replace $c_0(k, l)$ by $c_0(k, l, m)$ and $h(\mathbf{n})^{(k-l)/(k-m)}$ by $\left(\frac{H(\mathbf{n}_1, \dots, \mathbf{n}_m)}{D(\mathbf{n}_1, \dots, \mathbf{n}_m)}\right)^{(k-l)/(k-m)}$.

Note added in proof

Yu. Teterin has remarked that Lemma 4 holds under a weaker assumption, namely that $\operatorname{vol}\mathbb{K} < \infty$. To see this, it suffices to apply the original formulation to the body of $\lambda\mathbb{K}$ for suitable λ.

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https://doi.org/10.1017/S1446788700033280 Published online by Cambridge University Press