Construction of strongly regular graphs using affine designs

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Suppose there exist a balanced incomplete block design with
$\lambda = 1$ and an affine resolvable balanced incomplete block
design, the two designs having the same replication number.
Combining these designs we construct two strongly regular
graphs. This is applied to give a new family of design graphs
$\big((v, k, \lambda)\text{-graphs}\big)$. Finally, we show that for any prime power
n there are two non-isomorphic design graphs with
$v = n^2(n+2)$, $k = n(n+1)$ and $\lambda = n$.

We have discussed design graphs, under the name "(v, k, λ)-graphs",
in (for example) [6]. In this paper we give a construction method for the
more general strongly regular graphs; we then discuss the particular
cases when the graphs constructed are design graphs. We get two known
classes of graphs and infinitely many new graphs. In particular we obtain
the graphs with parameters $v = n^2(n+2)$, $k = n(n+1)$, $\lambda = n$, n a
prime power, which were discussed by Ahrens and Szekeres in [1]. We prove
the existence of a pair of non-isomorphic graphs with these parameters for
every suitable n.

The first two sections contain definitions which are needed later.
General references for these are [3] and [5]. Material on strongly
regular graphs appears in [2].

1. Basic definitions

A balanced incomplete block design (BIB design) with parameters

(v, b, r, k, λ) is a scheme for selecting b k-sets (called blocks) from a v-set of objects (called treatments) such that any treatment belongs to r blocks and any pair of treatments belongs to λ blocks. A symmetric BIB design, or (v, k, λ)-configuration, is one for which $v = b$. BIB designs necessarily satisfy

$$(1) \qquad\qquad \lambda(v-1) = r(k-1) , \quad bk = vr .$$

An Hadamard matrix of order $4n$ is a square matrix of order $4n$ with elements ± 1 and with any two rows orthogonal. It exists if and only if a $(4n-1, 2n-1, n-1)$-configuration exists.

Graphs in this paper are finite and undirected, without loops or multiple edges. Thus a graph $\underline{G} = (V, X)$ is a set V of vertices together with a set X of edges, or unordered pairs of vertices. If $(x, y) \in X$ then we say x is adjacent to y, written $x \sim y$. If U is a subset of V, then the spanning subgraph of U in $\underline{G}(V, X)$ is $\underline{G}(U, Y)$, where Y consists of all members of X with both vertices in U. The complement of $\underline{G}(V, X)$ is $\underline{G}(V, \overline{X})$, where $\overline{X}$ consists of all possible edges which are not in X. The union of graphs is defined by

$$\underline{G}(V_1, X_1) \cup \underline{G}(V_2, X_2) = \underline{G}(V_1 \cup V_2, X_1 \cup X_2) .$$

Write $S_x = \{y : y \in V, y \sim x\}$. Then $|S_x|$ is the valency of x. $\underline{G}$ is called regular if $|S_x| = k$ for every x in V; if also constants λ_1 and λ_2 exist such that

$$|S_x \cap S_y| = \lambda_1 \quad \text{when} \quad x \sim y ,$$
$$|S_x \cap S_y| = \lambda_2 \quad \text{otherwise,}$$

then it is called a strongly regular graph with parameters $(v; k; \lambda_1, \lambda_2)$. In particular when $\lambda_1 = \lambda_2 = \lambda$, $\underline{G}$ is called a design graph or (v, k, λ)-graph; in this case the vertices of $\underline{G}$ as treatments and the sets S_x as blocks form a (v, k, λ)-configuration. The complement of a strongly regular graph with parameters $(v; k; \lambda_1, \lambda_2)$ is strongly regular with parameters

$$(2) \qquad\qquad (v; v-k-1; v-2k+\lambda_2-2, v-2k+\lambda_1) .$$

The complete graph and null graph on a certain set of vertices are

the graphs with, respectively, all vertices adjacent and no vertices adjacent. The k-partite graph $\underline{\underline{K}}_{n_1,n_2,\ldots,n_k}$ is a graph on $n_1 + n_2 + \ldots + n_k$ vertices constructed as follows: the vertices are partitioned into subsets called components of sizes n_1, n_2, ..., and n_k ; the induced subgraph of any component is null; vertices in different components are always adjacent. If $n_1 = n_2 = \ldots = n_k = n$, we denote the graph $\underline{\underline{K}}_n^k$. For example, $\underline{\underline{K}}_3^2$ is shown.

$$\underline{\underline{K}}_3^2$$
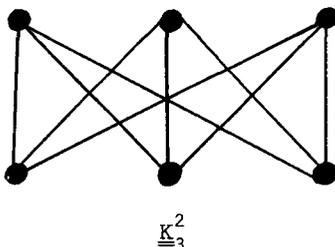
$\underline{\underline{K}}_n^k$ is strongly regular with parameters

(3) $\qquad\qquad \bigl(kn; \ (k-1)n; \ (k-2)n, \ (k-1)n\bigr) \ .$

2. Affine resolvable designs

A BIB design with parameters $(v, \ b, \ r, \ k, \ \lambda)$ is called affine resolvable if its set of blocks can be partitioned into r subsets or "parallel classes" of size n such that each treatment occurs in exactly one block in each parallel class and such that two blocks from different parallel classes have μ common points for some fixed μ . It is known [3, p. 73] that the parameters of an affine resolvable BIB design are

(4) $v = n^2\mu$, $b = n^2\mu + n(t+\mu)$, $r = n\mu + t + \mu$, $k = n\mu$, $\lambda = t + \mu$,

where the non-negative integer t satisfies $\mu = (n-1)t + 1$. Consequently we shall denote such a design as an $AR(n, \mu)$. We shall refer to the blocks and treatments as lines and planes; lines in the same parallel class are called parallel, and clearly have no common points.

The designs $AR(2, \mu)$ are precisely the family of designs discussed

in [7]; consequently an AR(2, μ) exists if and only if there is an
Hadamard matrix of order 4μ (that is, probably for all μ).

An AR(n, 1) is precisely an affine plane of parameter n , so it is
known to exist when n is a prime power and known not to exist for certain
values such as $n = 6$. More generally, if we take the points and
hyper-planes of affine d-space over GF(q) as points and lines we obtain
an AR$\left(q, q^{d-2}\right)$ [3, p. 72].

3.   Construction of strongly regular graphs

THEOREM 1. *Suppose there exists an* AR(n, μ) *and there exists a*
BIB *design* B *with parameters* (v, b, r, k, 1) , *where*

(5) $r = n\mu + t + \mu$, $\mu = (n-1)t + 1$.

Then there is a strongly regular graph G *with parameters*

(6) $\left(vn^2\mu;\ (v-1)n\mu;\ (v-k)\mu+(k-2)n,\ (v-k)\mu\right)$;

moreover the vertices of G *can be partitioned into* v *subsets of size*
$n^2\mu$ *whose induced subgraphs are null.*

Proof. First select v disjoint sets A^1, A^2, ..., A^v of size
$n^2\mu$; the $vn^2\mu$ objects will be the vertices of $\underline{G}$. Then construct
designs $\underline{\underline{A}}^1$, $\underline{\underline{A}}^2$, ..., $\underline{\underline{A}}^v$, each of which is an AR(n, μ) (these are not
necessarily isomorphic), with A^α as the set of points of $\underline{\underline{A}}^\alpha$. Whenever
$1 \le \alpha \le v$ and $1 \le \pi \le r$, the set of lines in the π-th parallel class of
$\underline{\underline{A}}^\alpha$ is denoted L_π^α .

With the i-th block $B_i = \{\beta, \gamma, ..., \delta\}$ of $\underline{B}$ we associate a
symbol $\left[\beta_\rho, \gamma_\sigma, ..., \delta_\tau\right]$, where ρ, σ, ..., τ are integers chosen in
some way that each α_π , where $1 \le \alpha \le v$ and $1 \le \pi \le r$, will occur in
exactly one of the complete set of b symbols. (This can be done since
each α occurs in exactly r blocks of $\underline{B}$.) Corresponding to each block
we construct a graph $\underline{\underline{G}}_i$ which is the union of n k-partite graphs of
type $\underline{\underline{K}}_{n\mu}^k$ on disjoint sets of vertices: if B_i has symbol

$\left[\beta_\rho, \gamma_\sigma, \ldots, \delta_\tau\right]$ then the first components of the n $\underset{=n\mu}{K^k}$ are the n

lines in L_ρ^β , the second components are the lines of L_σ^γ, $\ldots$, and the

k-th components are the lines of L_τ^δ .

We define $\underline{\underline{G}} = \cup \underline{\underline{G}}_i$, and shall show that it has the desired

properties.

Two vertices in A^α cannot lie in different components of the one

k-partite graph, so they are not adjacent. Hence the various A^α are v

disjoint subsets of size $n^2\mu$ whose induced subgraphs are null. If

$x \in A^\alpha$ then the r graphs in $\underline{\underline{G}}_i$ containing x have no common points

outside of A^α (as $\underline{B}$ has last parameter 1); from (3) x is adjacent

to $(k-1)n\mu$ points of each $\underline{\underline{G}}_i$ containing it; so x has valency

$r(k-1)n\mu = (v-1)n\mu$, for any x .

If x and y belong to A^α then they occur together in $\mu + t$

lines of $\underline{\underline{A}}^\alpha$, so they occur in the same component in $\mu + t$ k-partite

graphs, and have $(k-1)n\mu$ common vertices adjacent to them both in each

one; these vertices are all distinct. When α belongs to any other

block B_j of $\underline{\underline{B}}$, x and y are in different k-partite graphs in $\underline{\underline{G}}_i$;

if β is any other element of B_j then the lines of $\underline{\underline{A}}^\beta$ which are

adjacent to x and y are parallel, so they have no common points. So,

in total, x and y are mutually adjacent to $(k-1)n\mu(\mu+t)$ vertices.

Say x belongs to A^β and y to A^γ , $\beta \neq \gamma$. There will be one

block B_i of $\underline{\underline{B}}$ which contains both β and γ : either x and y are

in the same k-partite graph in $\underline{\underline{G}}_i$, whence they are adjacent and have

$(k-2)n\mu$ mutually adjacent vertices in $\underline{\underline{G}}_i$; or not, whence they are not

adjacent and have no mutually adjacent vertices in $\underline{\underline{G}}_i$. This eliminates

k of the A^α . Each of x and y are adjacent to one line $(n\mu$

points) in any $\underline{A}^\alpha$ where $\alpha \notin B_i$; these two lines of $\underline{A}^\alpha$ are not parallel, since they correspond to different blocks, so they have μ common points by the properties of an $AR(n, \mu)$. So x and y have $\mu(v-k)$ further mutually adjacent vertices.

Summarizing, two adjacent vertices have $\mu(v-k) + n(k-2)$ vertices mutually adjacent; two non-adjacent vertices have either $\mu(v-k)$ or $\mu(\mu+t)n(k-1)$. But applying (1) and (5) to $\underline{B}$,

$$v - k = r(k-1) - (k-1)$$
$$= n(t+\mu)(k-1) ,$$

so the graph is strongly regular.

THEOREM 2. *Suppose there exist designs as in Theorem 1, which further satisfy*

(7) $$k - 2 = \mu(n-2) .$$

Then there is a strongly regular graph $\underline{H}$ *with parameters*

(8) $$\left(vn^2\mu; \ (v+n-1)n\mu-1; \ x+n^2\mu-2, \ x+2n\mu\right)$$

where $x = (v-2-\mu n+2\mu)\mu$. *Furthermore the vertices of* $\underline{H}$ *can be partitioned into* v *subsets of size* $n^2\mu$ *whose induced subgraphs are complete.*

Proof. Construct $\underline{G}$ as in Theorem 1, and in addition make all points in the same A^α adjacent. Call the new graph $\underline{H}$. If S_x is the set of vertices adjacent to x in $\underline{G}$, then the set adjacent to x in $\underline{H}$ is

$$S_x \cup \left(A^\alpha \backslash \{x\}\right) ,$$

where $x \in A^\alpha$. As S_x and A^α are disjoint, this set has order

$$|S_x| + |A^\alpha| - 1 = (v+n-1)n\mu - 1 ,$$

so the graph is regular of the required valency. If $x \in A^\alpha$ and $y \in A^\beta$ the set of vertices adjacent to both x and y is

$$\Big[S_x \cup (A^\alpha \backslash \{x\})\Big] \cap \Big[S_y \cup (A^\beta \backslash \{y\})\Big]$$

$$= \Big[S_x \cap S_y\Big] \cup \Big[(A^\alpha \backslash \{x\}) \cap S_y\Big] \cup \Big[S_x \cap (A^\beta \backslash \{y\})\Big] \cup \Big[(A^\alpha \backslash \{x\}) \cap (A^\beta \backslash \{y\})\Big] \; .$$

The four sets enclosed by square brackets are disjoint. If $\alpha = \beta$, then the second and third sets have no elements and the fourth has $n^2\mu - 2$. If $\alpha \neq \beta$ then the fourth is empty; the second and third have $n\mu$ elements each if x and y are non-adjacent and $n\mu - 1$ each if they are adjacent, since S_x includes one line of A^β which contains y if and only if $x \sim y$. The size of $S_x \cap S_y$ is given in Theorem 1. With a little arithmetic we obtain the parameters shown in (8), provided (7) holds.

Naturally the theorems imply the existence of the complements of $\underline{G}$ and $\underline{H}$ which are strongly regular; their parameters may be deduced from (6), (8) and (2). In the complement of $\underline{G}$ the A^i induce complete graphs, while in the complement of $\underline{H}$ they induce null graphs.

4. Design graphs

The graph $\underline{G}$ of the preceding section is a design graph if and only if $k = 2$. In that case $\underline{B}$ always exists.

COROLLARY   3. *If there is an* $AR(n, \mu)$ *, then there is a design graph with parameters*

$$\big(n^2\mu(n\mu+t+\mu+1), \; n\mu(n\mu+t+\mu), \; n\mu(\mu+t)\big) \; ,$$

where $\mu = (n-1)t + 1$. *In particular*

   (i)   if there is an affine plane of parameter n *, then there is a graph*

(9) $\big(n^2(n+2), \; n(n+1), \; n\big)$;

   (ii)   if there is an Hadamard matrix of order 4μ *, then there is a graph*

(10) $(16\mu^2, \; 8\mu^2-2\mu, \; 4\mu^2-2\mu)$;

   (iii) if n *is a prime power greater than* 2 *and* d *is an integer greater than* 2 *, then there is a graph*

(11)
$$\left(\frac{n^d (n^d + n - 2)}{n-1}, \frac{n^{d-1}(n^d - 1)}{n-1}, \frac{n^{d-1}(n^{d-1} - 1)}{n+1} \right) .$$

The parameters (11) all seem to be new.

Part *(ii)* of the corollary appears in [4]. Note that a graph with parameters (10) implies the existence of a design graph

$$(16\mu^2 - 1, \ 8\mu^2, \ 4\mu^2)$$

by Theorem 4 of [6].

Graphs with parameters (9) with n a prime power were constructed by Ahrens and Szekeres in [1]. Their graphs all have the following property: if x and y are any two adjacent vertices and z and t are mutually adjacent to x and y, then z is adjacent to t.

In our construction for (9) the parameters of $\underline{B}$ include $v = n + 2$ and $k = 2$. There are n lines in each parallel class of each $\underline{A}^\alpha$. Suppose x and y are two given adjacent vertices in our graph and $z \in A^\beta$ and $t \in A^\gamma$ are adjacent to both x and y. Suppose the symbol corresponding to the block $\{\beta, \gamma\}$ of $\underline{B}$ is $\left[\beta_\sigma, \gamma_\tau \right]$. Since $n \geq 2$, we have enough freedom to ensure that the lines containing z and t in L_σ^β and L_τ^γ do not lie in the same 2-partite graph, ensuring that our graph is not isomorphic to the Ahrens-Szekeres solution.

COROLLARY. *If n is a prime power, there are at least two non-isomorphic graphs of parameters* (9).

The other graphs constructed in Section 3 do not yield design graphs, except that the complement of the graph $\underline{H}$ of Theorem 2 is a graph of parameters (10) when $n = k = 2$.

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