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A POLYNOMIAL MODEL FOR THE DOUBLE-LOOP SPACE OF AN EVEN SPHERE

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Abstract It is well known that $\Omega^2 S^{2n+1}$ is approximated by $\mathrm{Rat}_k(\mathbb{C}P^n)$, the space of based holomorphic maps of degree k from S^2 to $\mathbb{C}P^n$. In this paper we construct a space G_k^n which is an analogue of $\mathrm{Rat}_k(\mathbb{C}P^n)$, and prove that under the natural map $j_k : G_k^n \to \Omega^2 S^{2n}$, G_k^n approximates $\Omega^2 S^{2n}$.

Keywords: rational function; even sphere; loop space

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1. Introduction

Let $\mathrm{Rat}_k(\mathbb{C}P^n)$ denote the space of based holomorphic maps of degree k from the Riemannian sphere $S^2 = \mathbb{C} \cup \infty$ to the complex projective space $\mathbb{C}P^n$. The basepoint condition we assume is that $f(\infty) = [1, \dots, 1]$. Such holomorphic maps are given by rational functions:

$$\begin{aligned}\mathrm{Rat}_k(\mathbb{C}P^n) = \{(p_0(z), \dots, p_n(z)) :\ & \text{each } p_i(z) \text{ is a monic polynomial over } \mathbb{C} \\ & \text{of degree } k \text{ and such that there are no roots common to all } p_i(z)\}.\end{aligned}$$

There is an inclusion

$$i_k : \mathrm{Rat}_k(\mathbb{C}P^n) \hookrightarrow \Omega_k^2 \mathbb{C}P^n \simeq \Omega^2 S^{2n+1}.$$

Segal [**9**] proved that i_k is a homotopy equivalence up to dimension $k(2n-1)$. Later, the stable homotopy type of $\mathrm{Rat}_k(\mathbb{C}P^n)$ was described in [**6**] and [**7**] as follows. Let

$$\Omega^2 S^{2n+1} \underset{s}{\simeq} \bigvee_{1 \leqslant q} D_q(S^{2n-1})$$

be Snaith's stable splitting of $\Omega^2 S^{2n+1}$. Then

$$\mathrm{Rat}_k(\mathbb{C}P^n) \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q(S^{2n-1}).$$

(We can rewrite this using the fact [**5**] that $D_q(S^{2n-1}) \simeq \Sigma^{2q(n-1)} D_q(S^1)$.) In particular,

$$i_{k*} : H_*(\mathrm{Rat}_k(\mathbb{C}P^n); \boldsymbol{Z}) \to H_*(\Omega^2 S^{2n+1}; \boldsymbol{Z})$$

is injective.

Results of [**6**], [**7**] and [**9**] imply that $\mathrm{Rat}_k(\mathbb{C}P^n)$ approximates $\Omega^2 S^{2n+1}$. On the other hand, considering the double-loop space of an even sphere, we naturally encounter the following problem: how to construct spaces G_k^n which approximate $\Omega^2 S^{2n}$. Moreover, we study the stable homotopy type of G_k^n.

In special cases an answer is known. We set

$$\mathrm{RRat}_k(\mathbb{C}P^n) = \{(p_0(z), \dots, p_n(z)) \in \mathrm{Rat}_k(\mathbb{C}P^n) : \text{ each } p_i(z) \text{ has real coefficients}\}.$$

Let $\mathrm{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n)$ denote the space of continuous basepoint-preserving conjugation-equivariant maps of degree k from $\mathbb{C}P^1$ to $\mathbb{C}P^n$. It is proved in [**8**] that

$$\mathrm{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1} \quad (n \geqslant 1).$$

Hence, there is an inclusion

$$h_k : \mathrm{RRat}_k(\mathbb{C}P^n) \hookrightarrow \mathrm{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}.$$

The map h_k is a homotopy equivalence up to dimension $(k+1)(n-1)-1$. Moreover, $\mathrm{RRat}_k(\mathbb{C}P^n)$ is stably homotopy equivalent to the collection of stable summands in $\Omega S^n \times \Omega^2 S^{2n+1}$ of weight less than or equal to k. Here we define the weight of stable summands in ΩS^n as usual, but those in $\Omega^2 S^{2n+1}$ we define as being twice the usual one. Hence, in the situation where $\Omega^2 S^{2n} \simeq \Omega S^{2n-1} \times \Omega^2 S^{4n-1}$ holds, we can say that $\mathrm{RRat}_k(\mathbb{C}P^{2n-1})$ is a model which approximates $\Omega^2 S^{2n}$. Such a situation holds either (i) when it is localized at an odd prime, or (ii) when $n = 1$, 2 or 4.

In this paper we construct spaces G_k^n which approximate $\Omega^2 S^{2n}$ for all n without localization.

Definition 1.1. For $n \geqslant 1$, let G_k^n denote the space consisting of all $(n+1)$-tuples $(p_0(z), \dots, p_n(z))$ of monic polynomials over $\mathbb{C}$ of degree k and such that if $p_0(\alpha) = \cdots = p_{n-1}(\alpha) = 0$ for some $\alpha \in \mathbb{C}$, then $p_n(\alpha) \notin \mathbb{R}$.

We have $G_1^n \simeq S^{2n-2}$ (cf. Lemma 2.3 (i)). Define a map

$$j_k : G_k^n \to \Omega^2 S^{2n}$$

as follows. We embed $\mathbb{R} \hookrightarrow \mathbb{C}^{n+1}$ by

$$r \mapsto (\underbrace{0, \dots, 0}_{n \text{ times}}, r).$$

Note that $\mathbb{R}^+$, the group of positive real numbers, acts on $\mathbb{C}^{n+1} - \mathbb{R}$ so that

$$(\mathbb{C}^{n+1} - \mathbb{R})/\mathbb{R}^+ \simeq \mathbb{C}^{n+1} - \mathbb{R} \simeq S^{2n}.$$

Then j_k is defined to be the composite of maps

$$G_k^n \hookrightarrow \Omega^2((\mathbb{C}^{n+1} - \mathbb{R})/\mathbb{R}^+) \simeq \Omega^2(\mathbb{C}^{n+1} - \mathbb{R}) \simeq \Omega^2 S^{2n}.$$

For $n \geqslant 2$, let

$$\Omega^2 S^{2n} \underset{s}{\simeq} \bigvee_{1 \leqslant q} D_q(S^{2n-2})$$

be Snaith's stable splitting of $\Omega^2 S^{2n}$.

From results of [**3**] and [**5**], there is a stable homotopy equivalence

$$D_q(S^{2n-2}) \underset{s}{\simeq} S^{2q(n-1)} \vee \bigvee_{i=1}^{[q/2]} \Sigma^{2q(n-1)} D_i(S^1). \tag{1.1}$$

Our main results are the following two theorems.

Theorem A. *For $n \geqslant 2$, there is a stable homotopy equivalence*

$$G_k^n \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q(S^{2n-2}).$$

Theorem B.

(i) *For $n \geqslant 2$, the map $j_k : G_k^n \to \Omega^2 S^{2n}$ induces isomorphisms in homology groups in dimensions less than or equal to $(k+1)(2n-2)-1$. Hence, j_k induces isomorphisms in homotopy groups in dimensions less than or equal to $(k+1)(2n-2)-2$.*

(ii) *For $n \geqslant 2$, $j_{k*} : H_*(G_k^n; \boldsymbol{Z}) \to H_*(\Omega^2 S^{2n}; \boldsymbol{Z})$ is injective.*

Finally, we study the case $n = 1$. Recall that Brockett and Segal [**2**, **9**] showed that $\mathrm{RRat}_k(\mathbb{C}P^1)$ has $k+1$ connected components such that

$$\mathrm{RRat}_k(\mathbb{C}P^1) \simeq \coprod_{q=0}^{k} \mathrm{Rat}_{\min(q,k-q)}(\mathbb{C}P^1). \tag{1.2}$$

Theorem C. *There is a homotopy equivalence*

$$G_k^1 \simeq \mathrm{RRat}_k(\mathbb{C}P^1).$$

2. Proofs of Theorems A, B and C

In order to prove Theorem A, we first prove the following proposition.

Proposition 2.1. *Let p be a prime. Then, as a vector space, $H_*(G_k^n; \boldsymbol{Z}/p)$ is isomorphic to the subspace of $H_*(\Omega^2 S^{2n}; \boldsymbol{Z}/p)$ spanned by monomials of weight less than or equal to k. Here we define the weight of the (torsion-free) generators of $H_{2n-2}(\Omega^2 S^{2n}; \boldsymbol{Z})$ and $H_{4n-3}(\Omega^2 S^{2n}; \boldsymbol{Z})$ to be 1 and 2, respectively (cf. (1.1)).*

The proposition is proved as follows. First, by constructing homology classes explicitly, we find a lower bound for the mod p homology of G^n_k (cf. Proposition 2.2). Next, considering a geometrical resolution of a resultant, we construct a spectral sequence of Vassiliev type. The spectral sequence converges to the mod p homology of G^n_k and the E^1 term coincides with the lower bound. Hence, the spectral sequence collapses at the E^1 term and the lower bound is actually an upper bound (cf. Proposition 2.4).

Proposition 2.2. *Every element of $H_*(\Omega^2 S^{2n};\boldsymbol{Z}/p)$ of weight less than or equal to k is in the image of j_{k*}. Hence, these elements are a lower bound for $H_*(G^n_k;\boldsymbol{Z}/p)$.*

In order to prove Proposition 2.2, we first prove the following lemma.

Lemma 2.3.

(i) *The (torsion-free) generator of $H_{2n-2}(\Omega^2 S^{2n};\boldsymbol{Z})$ is in the image of j_{1*}.*

(ii) *The (torsion-free) generator of $H_{4n-3}(\Omega^2 S^{2n};\boldsymbol{Z})$ is in the image of j_{2*}.*

Proof.

(i) We embed $\mathbb{R} \hookrightarrow \mathbb{C}^n$ by

$$r \mapsto (\underbrace{0,\dots,0}_{n-1\text{ times}}, r).$$

Define a homeomorphism

$$f : G^n_1 \xrightarrow{\cong} \mathbb{C} \times (\mathbb{C}^n - \mathbb{R})$$

by

$$f(z+\alpha_0,\dots,z+\alpha_n) = (\alpha_0,(\alpha_1-\alpha_0,\dots,\alpha_n-\alpha_0)).$$

Then $G^n_1 \simeq S^{2n-2}$. Let u_{2n-2} be the generator of $H_{2n-2}(G^n_1;\boldsymbol{Z})$. Then it is easy to see that $j_{1*}(u_{2n-2})$ generates $H_{2n-2}(\Omega^2 S^{2n};\boldsymbol{Z})$.

(ii) Let B^n be the space consisting of all n-tuples $(p_0(z),\dots,p_{n-1}(z))$ of monic polynomials over $\mathbb{C}$ of degree 2 and such that

$$(p_0(z),\dots,p_{n-1}(z)) \neq ((z+\alpha)^2,\dots,(z+\alpha)^2) \quad \text{for any } \alpha \in \mathbb{C}.$$

There is an embedding $s : B^n \to G^n_2$ defined by

$$s(p_0(z),\dots,p_{n-1}(z)) = (p_0(z),\dots,p_n(z)),$$

where $p_n(z)$ is chosen according to $p_i(z)$ $(0 \leqslant i \leqslant n-1)$ as follows: we choose the imaginary part of the constant term of $p_n(z)$ near $+\infty$ so that $p_n(\alpha) \notin \mathbb{R}$ for any α, a root of $p_i(z)$ for some $0 \leqslant i \leqslant n-1$.

Since $B^n \cong \mathbb{C}^{2n} - \mathbb{C} \simeq S^{4n-3}$, there is an element $v_{4n-3} \in H_{4n-3}(G^n_2;\boldsymbol{Z})$. It is easy to see that $j_{2*}(v_{4n-3})$ generates $H_{4n-3}(\Omega^2 S^{2n};\boldsymbol{Z})$. This completes the proof of Lemma 2.3. □

Proof of Proposition 2.2. By an argument quite similar to that found in [**1**], we have a loop sum

$$* : H_i(G_k^n; \boldsymbol{Z}/p) \otimes H_j(G_{k'}^n; \boldsymbol{Z}/p) \to H_{i+j}(G_{k+k'}^n; \boldsymbol{Z}/p)$$

and the first Dyer–Lashof operation

$$Q_1 : H_i(G_k^n; \boldsymbol{Z}/p) \to H_{ip+p-1}(G_{kp}^n; \boldsymbol{Z}/p)$$

that are compatible with those in $H_*(\Omega^2 S^{2n}; \boldsymbol{Z}/p)$.

The structure of $H_*(\Omega^2 S^{2n}; \boldsymbol{Z}/p)$ is given as follows (cf. [**4**]).

(i) For $p = 2$,

$$H_*(\Omega^2 S^{2n}; \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2[u_{2n-2}, Q_1(u_{2n-2}), \dots, Q_1 \cdots Q_1(u_{2n-2}), \dots].$$

(ii) For an odd prime p,

$$H_*(\Omega^2 S^{2n}; \boldsymbol{Z}/p) \cong H_*(\Omega S^{2n-1}; \boldsymbol{Z}/p) \otimes H_*(\Omega^2 S^{4n-1}; \boldsymbol{Z}/p).$$

Moreover, $H_*(\Omega S^{2n-1}; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p[u_{2n-2}]$ and

$$\begin{aligned} H_*(\Omega^2 S^{4n-1}; \boldsymbol{Z}/p) \cong {} & \bigwedge(v_{4n-3}, Q_1(v_{4n-3}), \dots, Q_1 \cdots Q_1(v_{4n-3}), \dots) \\ & \otimes \boldsymbol{Z}/p[\beta Q_1(v_{4n-3}), \dots, \beta Q_1 \cdots Q_1(v_{4n-3}), \dots], \end{aligned}$$

where β is the mod p Bockstein operation.

By Lemma 2.3, u_{2n-2} is in the image of j_{1*} and v_{4n-3} is in the image of j_{2*}. Hence, from the structure of $H_*(\Omega^2 S^{2n}; \boldsymbol{Z}/p)$, every element of $H_*(\Omega^2 S^{2n}; \boldsymbol{Z}/p)$ of weight less than or equal to k is constructed in $H_*(G_k^n; \boldsymbol{Z}/p)$. This completes the proof of Proposition 2.2. □

Proposition 2.4. *The lower bound of Proposition 2.2 is actually an upper bound.*

Proof. We prove the proposition along the lines of [**10**, p. 151]. For a locally compact space X, let $\bar{X}$ denote the one-point compactification of X, $\bar{X} = X \cup \{\infty\}$, and let $\bar{H}_*(X; \boldsymbol{Z})$ be the Borel–Moore homology group $\bar{H}_*(X; \boldsymbol{Z}) = \tilde{H}_*(\bar{X}; \boldsymbol{Z})$.

We regard $\mathbb{C}^{k(n+1)}$ as the space consisting of all $(n+1)$-tuples $(p_0(z), \dots, p_n(z))$ of monic polynomials over $\mathbb{C}$ of degree k. Let Σ_k^n be the complement of G_k^n in $\mathbb{C}^{k(n+1)}$. Thus

$$\begin{aligned} \Sigma_k^n = \{(p_0(z), \dots, p_n(z)) \in \mathbb{C}^{k(n+1)} : {} & p_0(\alpha) = \cdots = p_{n-1}(\alpha) = 0 \\ & \text{and } p_n(\alpha) \in \mathbb{R} \text{ for some } \alpha \in \mathbb{C}\}. \end{aligned}$$

From the Alexander duality, there is a natural isomorphism

$$\tilde{H}^*(G_k^n; \boldsymbol{Z}) \cong \bar{H}_{2k(n+1)-1-*}(\Sigma_k^n; \boldsymbol{Z})$$

and so we study $\bar{H}_*(\Sigma_k^n; \boldsymbol{Z})$.

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Let $I : \mathbb{C} \to \mathbb{C}^k$ be the Veronese embedding $I(z) = (z, z^2, \dots, z^k)$. Let $f = (p_0(z), \dots, p_n(z)) \in \Sigma_k^n$ and suppose that $p_0(z), \dots, p_{n-1}(z)$ have at least d distinct common roots $\{\alpha_1, \dots, \alpha_d\} \subset \mathbb{C}$ which satisfy $p_n(\alpha_i) \in \mathbb{R}$ $(1 \leqslant i \leqslant d)$. We denote by $\Delta(f, \{\alpha_1, \dots, \alpha_d\}) \subset \mathbb{C}^k$ the open simplex in $\mathbb{C}^k$ with vertices $\{I(\alpha_1), \dots, I(\alpha_d)\}$. (Note that since $d \leqslant k$, the points $I(\alpha_1), \dots, I(\alpha_d)$ are in general position.) Define a geometrical resolution $\tilde{\Sigma}_k^n$ of Σ_k^n by

$$\tilde{\Sigma}_k^n = \bigcup_{f \in \Sigma_k^n;\ \{\alpha_1, \dots, \alpha_d\}} \{f\} \times \Delta(f, \{\alpha_1, \dots, \alpha_d\}) \subset \Sigma_k^n \times \mathbb{C}^k.$$

The first projection defines an open proper map $\pi : \tilde{\Sigma}_k^n \to \Sigma_k^n$, and this induces a map between the one-point compactification spaces

$$\bar{\pi} : \bar{\tilde{\Sigma}}_k^n \to \bar{\Sigma}_k^n.$$

It is known [**10**] that the map $\bar{\pi}$ is a homotopy equivalence. Define subspaces $F_s \subset \bar{\tilde{\Sigma}}_k^n$ by

$$F_s = \begin{cases} \{\infty\} \cup \bigcup_{f \in \Sigma_k^n;\ \{\alpha_1, \dots, \alpha_d\},\ d \leqslant s} \{f\} \times \Delta(f, \{\alpha_1, \dots, \alpha_d\}) & \text{if } s \geqslant 1, \\ \{\infty\} & \text{if } s = 0. \end{cases}$$

There is an increasing filtration

$$F_0 = \{\infty\} \subset F_1 \subset F_2 \subset \cdots \subset F_k = \bar{\tilde{\Sigma}}_k^n \simeq \bar{\Sigma}_k^n$$

and this induces a spectral sequence

$$E_{s,t}^1 = \bar{H}_{s+t}(F_s - F_{s-1}; \boldsymbol{Z}) \ \Rightarrow\ \bar{H}_{s+t}(\tilde{\Sigma}_k^n; \boldsymbol{Z}) \cong \bar{H}_{s+t}(\Sigma_k^n; \boldsymbol{Z}).$$

$F_s - F_{s-1}$ is the space of a fibre bundle which is a fibred product of the following two bundles. The two bundles have common base $C_s(\mathbb{C})$, where $C_s(\mathbb{C})$ denotes the configuration space of unordered s-tuples of distinct points in $\mathbb{C}$.

(i) The first bundle has an open $(s-1)$-dimensional simplex as a fibre.

(ii) The second bundle is an affine $((\mathbb{C}^{k-s})^{n+1} \times \mathbb{R}^s)$ bundle. The fibre over a collection $\{\alpha_1, \dots, \alpha_s\} \in C_s(\mathbb{C})$ consists of $((p_0(z), \dots, p_n(z)), (r_1, \dots, r_s))$, where $\deg p_i(z) = k$ $(0 \leqslant i \leqslant n)$, $p_i(z)$ $(0 \leqslant i \leqslant n-1)$ has roots $\alpha_1, \dots, \alpha_s$ and $p_n(\alpha_j) = r_j$ $(1 \leqslant j \leqslant s)$.

Consider a real s-dimensional vector bundle over $C_s(\mathbb{C})$ with fibre over a collection $\{\alpha_1, \dots, \alpha_s\} \in C_s(\mathbb{C})$ being the space of functions on its points. The local system of the vector bundle is locally isomorphic to $\boldsymbol{Z}$ but changes the orientation over the loops defining odd permutations. Note that the bundles (i) and (ii) have this local system. Hence, by the Thom and Poincaré isomorphisms,

$$E_{s,t}^1 = \begin{cases} H^{2(k-s)(n+1)+3s-t-1}(C_s(\mathbb{C}); \boldsymbol{Z}) & 1 \leqslant s \leqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $1 \leqslant *$. From the Alexander duality, we have

$$\begin{aligned}\dim H_*(G_k^n; \boldsymbol{Z}/p) &\leqslant \sum_{s=1}^{k} \dim H_{*-2s(n-1)}(C_s(\mathbb{C}); \boldsymbol{Z}/p) \\ &= \sum_{s=1}^{k} \dim H_*(\Sigma^{2s(n-1)}(C_s(\mathbb{C}) \vee S^0); \boldsymbol{Z}/p).\end{aligned}$$

Since $D_s(S^{2n-2}) \simeq \Sigma^{2s(n-1)}(C_s(\mathbb{C}) \vee S^0)$ (cf. [**5**]), we have

$$\dim H_*(G_k^n; \boldsymbol{Z}/p) \leqslant \sum_{s=1}^{k} \dim H_*(D_s(S^{2n-2}); \boldsymbol{Z}/p).$$

This completes the proof of Proposition 2.4, and, consequently, of Proposition 2.1. □

Proof of Theorem A. Let f_k be the stable map given by the composite of maps

$$f_k : G_k^n \xrightarrow{j_k} \Omega^2 S^{2n} \underset{s}{\simeq} \bigvee_{1 \leqslant q} D_q(S^{2n-2}) \to \bigvee_{q=1}^{k} D_q(S^{2n-2}).$$

Note that f_k is compatible with the homology splitting by weights. Then, using Proposition 2.1, we see that f_k induces an isomorphism in homology, hence is a stable homotopy equivalence. This completes the proof of Theorem A. □

Proof of Theorem B.

(i) Among elements of $H_*(\Omega^2 S^{2n}; \boldsymbol{Z}/p)$ which are not contained in $\operatorname{Im} j_{k*}$, the element of least degree is u_{2n-2}^{k+1} (cf. Theorem A). Hence, the homological assertion holds. Since G_k^n and $\Omega^2 S^{2n}$ are simply connected for $n \geqslant 2$, the homotopical assertion follows from the Whitehead Theorem.

(ii) Part (ii) is clear from Theorem A. This completes the proof of Theorem B.

□

Proof of Theorem C. Let $(p_0(z), p_1(z)) \in G_k^1$. If $p_0(\alpha) = 0$, then we have $p_1(\alpha) \in H_+$ or H_-, where H_+ (respectively, H_-) is the open upper (respectively, lower) half-plane. If $p_1(\alpha) \in H_+$ (respectively, H_-), then we give the sign '+' (respectively, '−') to α. Let X_k be the space of unordered collections $\{\alpha_1, \dots, \alpha_k\}$ of k points in $\mathbb{C}$ such that each α_i has sign '+' or '−' with the following condition: if α_i and α_j have the same sign, then we allow $\alpha_i = \alpha_j$, but if they have opposite sign, then we do not allow $\alpha_i = \alpha_j$. It is clear that $G_k^1 \simeq X_k$. Let $\{\beta_1, \dots, \beta_q, \gamma_1, \dots, \gamma_{k-q}\} \in X_k$, where β_i has sign '+' and γ_i has sign '−'. We consider a pair of polynomials $(q_0(z), q_1(z))$ defined by

$$q_0(z) = \prod_{i=1}^{q} (z - \beta_i) \quad \text{and} \quad q_1(z) = \prod_{i=1}^{k-q} (z - \gamma_i).$$

Using the division algorithm we change $(q_0(z), q_1(z))$ to an element of

$$\mathrm{Rat}_{\min(q,k-q)}(\mathbb{C}P^1).$$

Then we see that X_k has $k+1$ connected components so that

$$X_k \simeq \coprod_{q=0}^{k} \mathrm{Rat}_{\min(q,k-q)}(\mathbb{C}P^1).$$

Now Theorem C follows from (1.2). This completes the proof of Theorem C. □

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