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ADDITIVE FUNCTIONALS ON LORENTZ SPACES

BY

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ABSTRACT. If (X, β, μ) is a σ-finite, non-atomic measure space, and ϕ is an increasing non-negative concave function defined on the positive real numbers, we give a set of necessary and sufficient conditions for an additive functional T on the Lorentz space N_ϕ to have an integral representation with a Caratheodory kernel. In the special case when T is statistical we classify the functional properties (enjoyed by the kernels) in terms of the Lorentz norm on the space.

1. In this note we obtain integral representation for disjointly additive functionals on certain Lorentz spaces in terms of Caratheodory kernels. When the functionals enjoy the additional property of being statistical, their kernels possess functional properties which are easily definable in terms of the Lorentz norm. Using these properties we characterize when the composition operator (generated by the kernel) mapping the Lorentz space into L_1 is completely continuous. Results of this type have been obtained by Mizel [3] for L_p spaces, by Sundaresan [6] for Orlicz spaces and Mizel and Sundaresan [4] for L_p spaces of functions which take values in a Banach space.

We start with a few definitions and establish notation useful in the following discussion.

Suppose that (X, B, μ) is a σ-finite, non-atomic positive measure space. If f is a measurable function, we denote its distribution function by λ_f. Recall that $\lambda_f(y) = \mu\{x, |f(x)| > y\}$. If ϕ is an increasing, non-negative concave function, defined on the positive real numbers and satisfies $\phi(0) = \phi(0)^+ = 0$, then we define $\|f\|_\phi = \int_0^\infty \phi(\lambda_f(y))\, dy$. The Banach space N_ϕ consists of those measurable functions for which $\|f\|_\phi$ is finite. For more details, we refer the reader to [5] and [7].

THEOREM 1. *Suppose $\mu(X) < \infty$, $\mathscr{X}$ is a normal sublattice of N_ϕ and $T: \mathscr{X} \to \mathbb{R}$ is a functional satisfying the following properties:*

(1) *If f and g are in $\mathscr{X}$ and $fg = 0$ a.e. then $T(f+g) = Tf + Tg$.*

(2) *If $f_n \xrightarrow{\mu} f$ (in measure) and $|f_n| \le |g|$ for all n and some g in $\mathscr{X}$ then $Tf_n \to Tf$.*

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(3) $\lim_{\mu(E)\to 0} T(f\psi_E)=0$ *uniformly for all* f *in a bounded subset of* $\mathscr{X}$.
Then there exists F *in* $\text{Car}(X)$, *satisfying*

$$(\dagger) \qquad \lim_{\mu(E)\to 0} \int_X F(f(x), x)\, d\mu = 0$$

uniformly on bounded subsets of N_ϕ *and* $Tf = \int_X F(f(x), x)\, d\mu$ *for all* f *in* $\mathscr{X}$.

Conversely if $F \in \text{Car}(X)$ *and satisfies* $(\dagger)$ *and* $Tf = \int_X F(f(x), x)\, d\mu$, *then* T *satisfies* (1), (2), *and* (3).

Proof. Suppose T satisfies (1), (2), and (3). By Theorem 3.2 of [7] there exists $F \in \text{Car}(X)$ for which $Tf = \int_X F(f(x), x)\, d\mu$ for all f in $\mathscr{X}$. It is obvious that F satisfies $(\dagger)$.

Conversely suppose F is in $\text{Car}(X)$ and $Tf = \int_X F(f(x), x)\, d\mu$ exists for all f in $\mathscr{X}$ then obviously T satisfies (1). By Theorem 3.2 of [7] T satisfies (2). If F satisfies $(\dagger)$ then T satisfies (3).

REMARKS. (1) T is norm-continuous. To see this, suppose f, f_n belong to $\mathscr{X}$ and $f_n \to f$ in the N_ϕ-norm. If $g_n(x) = F(f_n(x), x)$ and $g(x) = F(f(x), x)$ then each subsequence of $\{g_n\}$ possesses a subsequence $\{g_{n'}\}$ satisfying $g_{n'}(x) \to g(x)$ a.e. (This follows by a standard argument from the fact that F is continuous in the first variable.) And hence $g_n \xrightarrow{\mu} g$ (in measure). Now by (3), if $\varepsilon > 0$ $\exists$ $\delta > 0$ such that whenever $\mu(E) < \delta$, $|\int_E g_n(x)\, d\mu| < \varepsilon$ and $|\int_E g(x)\, d\mu| < \varepsilon$. By Egoroff's theorem, $\exists$ E_0 such that $\mu(E_0) < \delta$ and $\|(g_n - g)\,|_{X\setminus E_0}\|_\infty \to 0$ as $n \to \infty$

$$\begin{aligned}
\left|\int_X g_n(x)\, d\mu - \int_X g(x)\, d\mu\right| &\le \left|\int_{X\setminus E_0} (g_n - g)(x)\, d\mu\right| \\
&\quad + \left|\int_{E_0} g_n(x)\, d\mu\right| + \left|\int_{E_0} g(x)\, d\mu\right| \\
&\le \mu(X)\, \|(g_n - g)|_{X\setminus E_0}\|_\infty \\
&\quad + \left|\int_{E_0} g_n(x)\, d\mu\right| + \left|\int_{E_0} g(x)\, d\mu\right| \le 3\varepsilon
\end{aligned}$$

for n sufficiently large. Thus $Tf_n \to Tf$.

(2) We show by an example that condition (3) of Theorem 1 does not follow from (1) and (2). For that purpose, consider $X = [0, 1]$ with μ = Lebesgue measure, and $\phi(t) = t^{1/2}$. We consider a subspace of N_ϕ defined as $\mathscr{X} = \{f, f \in N_\phi$ such that $f^3 \in L_1\}$. Then $L_\infty[0, 1] \subseteq \mathscr{X}$ and if $F(\alpha, x) = \alpha^3$, $F \in \text{Car}(X)$. If $Tf = \int_0^1 f^3\, dx$, T satisfies (1) and (2). We show that T does not satisfy (3), nor is it norm-continuous. If $f_n = n\psi_{[0,1/n^2]}$, $E_n = [0, 1/n^3]$ and $h_n = n\psi_{[0,1/n^3]}$, we have $\|f_n\|_\phi = 1$, $f_n\psi_{E_n} = h_n$, $\|h_n\|_\phi = 1/n^{1/2}$ and $T(f_n\psi_{E_n}) = T(h_n) = 1$. Since $\mu(E_n) \to 0$ and $\|h_n\|_\phi \to 0$ this proves both of our claims.

(3) If X, μ and ϕ are as in the previous remark let $\mathscr{Y} = N_\phi$ and define

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$Tf=\int_0^1 f^2\,dx$ for $f\in\mathscr{Y}$. By [5, Theorem 5.6(b)], T is well-defined and norm-continuous. If $f_n=n\psi_{[0,1/n^2]}$ and $E_n=[0,1/n^2]$ then $\|f_n\|_\Phi=1$, $f_n\psi_{E_n}=f_n$, $\mu(E_n)\to 0$ but $T(f_n\psi_{E_n})=1$. Thus condition (3) of Theorem 1 cannot be replaced by the norm-continuity of T.

(4) We point out by an example that when $\mu(X)=\infty$, norm-continuity of T is not a consequence of the other conditions. If $X=(0,\infty)$ with Lebesque measure μ and $\phi(t)=\sqrt{t}$ let $\mathscr{X}=N_\Phi\cap L_1$. Then $\mathscr{X}$ is a normal sublattice of N_Φ and $Tf=\int f\,d\mu$ is a linear functional on $\mathscr{X}$ which satisfies (2) by dominated convergence theorem. Also $|T(f\psi_E)|=|\int_E f\,d\mu|\le\mu(E)\,\|f\|_2$ by Hölder's inequality. By 5.6(b) of [5], $\|f\|_2\le\|f\|_{N_\Phi}$ and hence condition (3) is satisfied. However if $f_n=(1/n)\psi_{[0,n]}$ then $\|f_n\|_{N_\Phi}=(\sqrt{n}/n)\to 0$ but $Tf_n=1$.

The original version of theorem 1 had a set of conditions given in [3, Theorem 1] some of which were proved superfluous in [7]. We thank the referee for bringing [7] to our notice.

COROLLARY 1. *If in addition to conditions* (1) *through* (3) *T is statistical, then there exists a continuous $G:\mathbb{R}\to\mathbb{R}$, $G(0)=0$ for which $Tf=\int_X G\circ f\,d\mu$ for all f in N_Φ.*

Proof. Suppose T is statistical i.e. f, g are equimeasurable $\Rightarrow Tf=Tg$. (See [6], p. 269). Now if $\alpha\in\mathbb{R}$, $f=\alpha\psi_{E_1}$, $f=\alpha\psi_{E_2}$ where $0<\mu(E_1)=\mu(E_2)<\infty$. Then f and g are equimeasurable. Hence

$$\int_{E_1} F(\alpha,x)\,d\mu=\int_{E_2} F(\alpha,x)\,d\mu.$$

Thus if $G_\alpha(\cdot)$ denotes the function $F(\alpha,\cdot)$ we have $\int_{E_1} G_\alpha(x)\,d\mu=\int_{E_2} G_\alpha(x)\,d\mu$ wherever $0<\mu(E_1)=\mu(E_2)<\infty$. We claim that G_α is constant a.e. If not, we can find intervals I_1 and I_2, $I_1=(\alpha_1,\beta_1)$, $I_2=(\alpha_2,\beta_2)$, $\beta_1<\alpha_2$ such that $G_\alpha^{-1}(I_1)$ and $G_\alpha^{-1}(I_2)$ have positive measure. μ is assumed to be non-atomic, hence we can find sets E_1 and E_2, $0<\mu(E_1)=\mu(E_2)$, $E_1\subseteq G_\alpha^{-1}(I_1)$, $E_2\subseteq G_\alpha^{-1}(I_2)$, $\int_{E_1} G_\alpha(x)\,d\mu\le\beta_1\mu(E_1)$ and $\int_{E_2} G_\alpha(x)\,d\mu\ge\alpha_2\mu(E_2)$. In particular, $\int_{E_1} G_\alpha(x)\,d\mu\ne\int_{E_2} G_\alpha(x)\,d\mu$. This establishes the claim. Now if $G(\alpha)=G_\alpha(x)$ then we have: $G:\mathbb{R}\to\mathbb{R}$, $Tf=\int_X G\circ f\,d\mu$, $G(0)=0$ ($F(0,\cdot)=0$ a.e.). Since F is continuous in the first variable, G is continuous.

2. In this section we restrict ourselves to the case where $\phi(x)=x^{1/p}$, $p>1$.

LEMMA 1. *Suppose $G:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying $G(0)=0$. Then* (†) $\lim_{\mu(E)\to 0}\int_E G\circ f\,d\mu=0$ *uniformly on bounded subsets of N_Φ if and only if $|G(\alpha)|/\alpha^p\to 0$ as $\alpha\to\infty$.*

Proof. Suppose (†) is true. If $\alpha_n\to\infty$ then since μ is non-atomic, we may choose measurable sets E_n such that $\mu(E_n)=1/\alpha_n^p$. Let $f_n=\alpha_n\psi_{E_n}$, $\|f_n\|_\Phi=\alpha_n(\mu(E_n))^{1/p}=1$. $\int_{E_n} G\circ f_n\,d\mu=G(\alpha_n)\mu(E_n)=G(\alpha_n)/\alpha_n^p\to 0$ as $n\to\infty$. Since

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$\{\alpha_n\}$ was chosen arbitrarily we have: $\lim_{\alpha\to\infty}|G(\alpha)|/\alpha^p=0$. Conversely suppose $|G(\alpha)|/\alpha^p\to 0$ as $\alpha\to\infty$. Let $\varepsilon>0$. Choose α_0 such that for every $\alpha\ge\alpha_0$, $|G(\alpha)|\le\varepsilon\alpha^p$. Let $\sup\{|G(y)|, |y|\le\alpha_0\}=M_0$. Now if E is a measurable subset of X of positive, finite measure and if f is an arbitrary element of N_Φ, we have:

$$\int_E |G\circ f|\,d\mu=\int_{E_1}|G\circ f|\,d\mu+\int_{E_2}|G\circ f|\,d\mu$$

where

$$E_1=\{x, x\in E, |f(x)|\le\alpha_0\},\qquad E_2=\{x, x\in E, |f(x)|>\alpha_0\}.$$

Thus

$$\int_E|G\circ f|\,d\mu\le M_0\mu(E)+\varepsilon\int_E|f(x)|^p\,d\mu\le M_0\mu(E)+\varepsilon\,\|f\mid E\|_p^p$$

where $\|f\|_p$ denotes the L_p-norm of f. (By 5.9 of [5], $\|f\|_p\le\|f\|_\Phi\ \forall f\in N_\Phi$). In particular, $\int_E|G\circ f|\,d\mu\le M_0\mu(E)+\varepsilon\,\|f\|_\Phi^p$. Now if $\mu(E)<\delta=\varepsilon/M_0$ we have, $\int_E|G\circ f|\,d\mu\le\varepsilon[1+\|f\|_\Phi^p]$. Hence $\lim_{\mu(E)\to 0}\int_E|G\circ f|\,d\mu=0$ uniformly on bounded subsets of N_Φ.

Definition. If $G:\mathbb{R}\to\mathbb{R}$ $(G(0)=0)$ is a continuous function with the property that whenever $f\in N_\Phi$, $G\circ f\in L_1$, we say that the composition operator $A:f\to G\circ f$ is bounded if whenever $\sup\{\|f\|_p, f\in S\}<\infty$, we have $\sup\{\|G\circ f\|_1, f\in S\}<\infty$ for an arbitrary subset of S of N_Φ.

Lemma 2. *If $\mu(X)<\infty$ then A is bounded if and only if $|G(\alpha)|/\alpha^p=0(1)$, $|\alpha|\to\infty$.*

Proof. Suppose A is bounded. We claim that there exist real numbers k and α_0 such that $|F(\alpha)|/\alpha^p\le k$ for $\alpha\ge\alpha_0$. Suppose not. Choose $\{\alpha_n\}$ such that $\alpha_n\to\infty$ and $|G(\alpha_n)|/\alpha_n^p\to\infty$. Since μ is non-atomic, we may choose a sequence E_n of measurable sets satisfying $\mu(E_n)=1/\alpha_n^p$. If $f_n=\alpha_n\psi_{E_n}$, then $\|f_n\|_\Phi=1$ but $\|G\circ f_n\|_1=\|G(\alpha_n\psi_{E_n})\|_1=|G(\alpha_n)|\,\mu(E_n)=|G(\alpha_n)|/\alpha_n^p\to\infty$. This contradicts the fact that A is bounded and establishes our claim. We remark that this argument is valid even if $\mu(X)=\infty$. Conversely suppose that there exist real numbers k and α_0 such that $|G(\alpha)|/\alpha^p\le k$ for $\alpha\ge\alpha_0$. Let $M_0=\sup\{|G(\alpha)|, |\alpha|\le\alpha_0\}$. We know that $\|G\circ f\|_1=\int_0^\infty\lambda_{G\circ f}(y)\,dy$. (See p. 137 of [5]). If $y>0$, $\lambda_{G\circ f}(y)=\mu\{x, |G(f(x))|>y\}$, if $E_y=\{x, |f(x)|\le\alpha_0, y<|G(f(x))|\le M_0\}$, and $F_y=\{x, |f(x)|>\alpha_0, y<|G(f(x))|<k\,|f(x)|^p\}$ then

$$\lambda_{G\circ f}(y)\le\mu(E_y)+\mu(F_y).$$

But $\mu(F_y)\le\lambda_f(y/k)^{1/p}$. Hence

$$\int_0^\infty\mu(F_y)\,dy\le kp\int_0^\infty\lambda_f(z)z^{p-1}\,dz=kp\,\|f\|_p^p\le kp\,\|f\|_\Phi^p$$

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On the other hand $0 \le \mu(E_y) \le 0$ if $y > M_0$. Hence

$$\int_0^\infty \mu(E_y)\,dy \le \int_0^{M_0} \mu(E_y)\,dy \le M_0\mu(X).$$

Thus

$$\int_0^\infty \lambda_{G\circ f}(y)\,dy \le M_0\mu(X) + kp\,\|f\|_\Phi^p$$

i.e.

$$\|G\circ f\|_1 \le M_0\mu(X) + kp\,\|f\|_\Phi^p\ \forall\, f \in N_\Phi.$$

In particular, A is bounded.

LEMMA 3. *Suppose $\mu(X)=\infty$. Then A is bounded if and only if*

$$\sup\left\{\frac{|G(\alpha)|}{\alpha^p}, \alpha \in \mathbb{R}\right\} < \infty.$$

Proof. Suppose A is bounded. We already know that $|G(\alpha)|/|\alpha|^p = 0(1)$; $|\alpha| \to \infty$. (as in the first part of the proof of Lemma 2). Now suppose that $\exists\,\{\alpha_n\}$ such that $|\alpha_n| \to 0$ and $|G(\alpha_n)|/|\alpha_n^p| \to \infty$. Since $\mu(X)=\infty$ we may choose measurable sets E_n satisfying $\mu(E_n) = 1/\alpha_n^p$ $(\to\infty)$. Once again if $f_n = \alpha_n \psi_{E_n}$ then $\|f_n\|_\Phi = 1$ but $\|G\circ f_n\|_1 = |G(\alpha_n)|\,\mu(E_n) \to \infty$. Since $|G(\alpha)|/|\alpha^p|$ is continuous on $(0,\infty)\cup(-\infty,0)$ we have:

$$\sup\left\{\left|\frac{G(\alpha)}{\alpha^p}\right|, \alpha \in \mathbb{R}\right\} < \infty.$$

Conversely suppose $|G(\alpha)| \le M\,|\alpha|^p\ \forall\,\alpha$. Then

$$\lambda_{G\circ f} = \mu\{x, y < |(G\circ f)(x)| \le M\,|f(x)|^p\} \le \lambda_f((y/M)^{1/p}).$$

Hence as in the proof of Lemma 2 we have

$$\|G\circ f\|_1 \le \int_0^\infty \lambda_f\left(\left(\frac{y}{M}\right)^{1/p}\right)dy \le Mp\int_0^\infty z^{p-1}\lambda_f(z)\,dz = Mp\,\|f\|_p^p \le Mp\,\|f\|_\Phi^p.$$

In particular, A is bounded. As a simple consequence of the characterization of compact sets in L_1, the preceding three lemmas and Theorem 9 of [2], we have the following two theorems.

THEOREM 2. *If $\mu(X) < \infty$ then A is completely continuous if and only if*

$$\frac{|G(\alpha)|}{|\alpha^p|} \to 0 \quad \text{as} \quad |\alpha| \to \infty.$$

THEOREM 3. *If $\mu(X) = \infty$ then A is completely continuous if and only if*
(1)

$$\left|\frac{G(\alpha)}{\alpha^p}\right| \to 0 \quad \text{as} \quad |\alpha| \to \infty$$

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and

(2)

$$\sup\left\{\left|\frac{G(\alpha)}{\alpha^p}\right|, \alpha \in \mathbb{R}\right\} < \infty.$$

3. Note that if

$$\phi(x) = \ln(1+x) \quad \text{then} \quad \sup_{u>0} \frac{\phi(u)}{u} = \lim_{u \to 0^+} \frac{\phi(u)}{u} = 1$$

(1.15 of [5] establishes the first equality.) By 5.9 of [5], $\|f\|_1 \le \|f\|_\phi$ whenever $f \in N_\phi$ and arguments identical to the ones in all the three lemmas of the previous section establish the following.

THEOREM 4. (1) *If $\mu(X) < \infty$ then A is completely continuous if and only if*

$$\left|\frac{G(\alpha)}{\alpha}\right| \to 0 \quad as \quad |\alpha| \to \infty$$

(2) *If $\mu(X) = \infty$ then A is completely continuous if and only if*

(a)

$$\sup\left\{\left|\frac{G(\alpha)}{\alpha}\right|, \alpha \in \mathbb{R}\right\} < \infty$$

and

(b)

$$\left|\frac{G(\alpha)}{\alpha}\right| \to 0 \quad as \quad |\alpha| \to \infty.$$

In conclusion it might be mentioned that the more general problem of finding conditions on the Caratheodory function F (whose existence was establised in Theorem 1) which will make the composition operator A completely continuous was not discussed here. A similar problem for Orlicz spaces was discussed in [1].

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