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ON THE SYMMETRIC HYPERCENTER OF A RING

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The hypercenter theorem [**6**] asserts that in a ring with no non-zero nil ideals an element commuting with a suitable power of each element of the ring must be central. In this paper we shall be concerned with a similar problem in the setting of rings with involution. Let R be a ring with involution *, let Z denote the center of R and let $S = \{x \in R | x = x^*\}$ be the set of symmetric elements in R. We define the symmetric hypercenter of R to be

$$H = \{a \in R | as^n = s^n a, n = n(a, s) \geqq 1, \text{ all } s \in S\}.$$

What can one hope to say about H? That H need not equal Z is clear. For instance, in the ring $R = F_2$ of 2×2 matrices over a field, if * is the symplectic involution, all symmetric elements are central, hence $H = R$ but $Z \neq R$. Furthermore if R is a noncommutative ring in which every symmetric element is nilpotent then even in this case $H = R$ and $Z \neq R$ follows.

Suppose that R is a prime ring with characteristic not 2 or 3. Here we will show that if R has no non-zero nil right ideals and $S \not\subset Z$, then $H = Z$ follows.

The symmetric hypercenter was first studied in [**4**]; there the authors proved that if R is a division ring then $H \cap S = Z \cap S$ provided $xx^* \notin Z$ for some $x \in R$. Another result about H is Theorem 1 in [**10**] which reads as follows: if the exponent $n(a, s) = n$ is independent of s and if R is a 2, 3-torsion free semiprime ring, then $H \cap S = Z \cap S$.

It is natural to ask if our result remains valid if one replaces the assumption "with no nil right ideals" by its two-sided version "with no nil ideals". If this were the case, then one would have a positive answer to the following question due to McCrimmon [**7**, p. 83]: let R be a ring with involution such that all symmetric elements are nilpotent; is R itself necessarily nil? (see [**1**]).

Finally we remark that if char $R = 3$, then the conclusion of our result is no more true: in fact, let $R = (GF(3))_2$ with the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a & 2c \\ 2b & d \end{pmatrix}.$$

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In this ring $S \not\subset Z$ and H coincides with the set of diagonal matrices; hence $H \neq Z$.

Throughout the paper R will denote a ring with involution * which is 2 and 3 torsion free, S will be the set of symmetric elements of R, K the set of skew elements of R, and $Z = Z(R)$ the center of R. $H = H(R)$ will denote the symmetric hypercenter of R and $H^+ = H \cap S$.

We recall that if x is a quasi-regular element of R with quasi-inverse x^* (i.e., $x + x^* + xx^* = 0$) then x is called *quasi-unitary*. If R has a unity, then clearly x is quasi-unitary if and only if $1 + x$ is unitary.

For a quasi-unitary element x the map

$$\Psi_x : y \rightarrow y + xy + yx^* + xyx^*$$

is an automorphism of R which preserves S and K and leaves the elements in Z invariant. Moreover, it is easy to establish the following remark:

Remark 1. For all quasi-unitary elements $x \in R$, $\Psi_x(H) \subset H$.

As a special case of Remark 1 that will be used later we have the

Remark 2. For all quasi-regular skew elements k, $2k(1 - k)^{-1}$ is quasi-unitary and

$$(1 - k)^{-1}(ak - ka)(1 + k)^{-1} \in H$$

for all $a \in H$.

The invariant property of H can be exploited for R a simple artinian ring viewed as $n \times n$ matrices over a division ring. We have

Remark 3. Let R be a simple artinian ring. If $S \not\subset Z$ then $H = Z$.

Proof. Let $R = D_n$, where D is a division ring. If * is symplectic then, as in [**3**, Section 6], we get the desired conclusion. Suppose that * is of transpose type. Let e_{ij} $(i, j = 1, \ldots, n)$ be the usual matrix units. Since H centralizes all symmetric idempotents, H centralizes e_{ii}, for all i; hence H consists of diagonal matrices. If D has more than 5 elements then, by [**3**, Theorem 2 and Theorem 6], $H = Z$ and we are done in this case. If $D = GF(5)$, then $R = (GF(5))_n$ is a finite ring, $H = H^+$ and by [**10**] $H = Z$.

Knowing the result for simple artinian rings, we follow the usual pattern of structure theory by proving the result for semisimple rings. We first need a lemma.

LEMMA 1. *If R is a primitive ring and $H \not\subset Z$ then R has a minimal right ideal.*

Proof. R is a dense ring of linear transformations on a vector space V over a division ring D. If $\dim_D V < \infty$ then R has a minimal right ideal. Therefore we may assume that $\dim_D V = \infty$.

Let $a \in H$, $a \notin Z$. By the proof of Lemma 2 in [**6**], there exists $v \in V$ such that v and va are linearly independent over D.

Suppose first that for all $w \notin Dv$

(1) $w(S \cap (0:v)) \not\subset Dv + Dw$,

where $(0:v) = \{x \in R | vx = 0\}$.

Since $va \notin Dv$, from (1) we get

$$vas \notin Dv + Dva, \quad \text{for all } s \in S \cap (0:v).$$

Now, since $vas \notin Dv$, again from (1) we get $vas^2 \notin Dv + Dvas$. A repeated application of this argument leads to

$$vas^n \notin Dv + Dvas^{n-1}, \text{ for all } n \geqq 1.$$

But if m is such that $as^m = s^m a$, then

$$vas^m = vs^m a = 0,$$

a contradiction. Therefore there exists $w \notin Dv$ such that

$$w(S \cap (0:v)) \subset Dv + Dw.$$

If $Dv + Dw = V$, then V is finite dimensional and we are done. Hence there exists $x \in R$, $x \neq 0$, such that $x \in (0:v) \cap (0:w)$. Moreover, by the density theorem there exists $y \in (0:v)$ such that $wy \neq 0$. If $r \in R$, the element $c = xry^* + yr^*x^*$ lies in $(0:v) \cap S$; hence

$$wc = wyr^*x^* \in Dv + Dw, \quad \text{for all } r \in R.$$

Since $wyR = V$, then $Vx^* \subset Dv + Dw$ so that x^* induces a linear transformation of finite rank. By [**9**, Theorem p. 75] R has a minimal right ideal.

THEOREM 1. *Let R be a prime semisimple ring. If $S \not\subset Z$ then $H = Z$.*

Proof. Suppose first that R is primitive and $S \not\subset Z$. If $H \not\subset Z$, by Lemma 1 R has a minimal right ideal. This says that R is a ring of linear transformations on a vector space V over a division ring D, which space is equipped with a Hermitian or alternate form such that the elements of R are continuous with respect to this form (e.g., have adjoints); furthermore R contains all linear transformations of finite rank and the * of R is the adjoint relative to this form.

Since $H \not\subset Z$ there exists $a \in H$, $a \notin Z$. As in the proof of Lemma 2 in [**6**] there exists $v \in V$ such that v and va are linearly independent over D.

Suppose that the form (,) is Hermitian and let W be a finite dimensional non-degenerate subspace of V containing both v, va; then we may find an orthogonal basis $\{w_1, \dots, w_n\}$ for W; that is $(w_i, w_j) = \delta_{ij} d_j$ where $0 \neq d_j = d_j^* \in D$, $j = 1, \dots, n$. If $W^{\perp}$ is the orthogonal complement of W,

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then $V = W \oplus W^{\perp}$. Now, every matrix $A = (\alpha_{ij}) \in D_n$ induces a linear transformation T_A on V as follows: $T_A(w_i) = \sum \alpha_{ij}w_j \quad (i = 1, \dots, n)$ and $T_A(w) = 0$ for $w \in W^{\perp}$. Since T_A is a linear transformation of finite rank, $T_A \in R$ and so, R contains the subring

$$R^{(n)} = \{T_A | A \in D_n\} \simeq D_n.$$

Moreover the adjoint is an involution on D_n of transpose type.

Let

$$w_i a = \sum \alpha_{ij} w_j + w_i' \quad (i = 1, \dots, n)$$

where $\alpha_{ij} \in D$ and $w_i' \in W^{\perp}$, and let $\bar{a} = (\alpha_{ij})$. Then $T_{\bar{a}} \in R^{(n)}$ and, since $a \in H$, it is easy to prove that $T_{\bar{a}} \in H(R^{(n)})$ where $H(R^{(n)})$ is the symmetric hypercenter of $R^{(n)}$. By Remark 3, since * is of transpose type, $T_{\bar{a}}$ is central in $R^{(n)}$; thus

$$\bar{a} = \begin{pmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix}$$

for a suitable λ in the center of D. Now, since v, $va \in W$ we get $va = \lambda v$, and this is a contradiction. The alternate case is proved similarly.

We have proved that if R is primitive and $S \not\subset Z$ then $H = Z$.

Let now R be a prime semisimple ring and suppose that $S \not\subset Z$. It is well known that a semisimple ring is a subdirect product of primitive rings R_α; moreover, since R is 2 and 3 torsion free, we may assume that the homomorphic images R_α are still of characteristic different from 2 and 3. For every α, let P_α be a primitive ideal of R such that $R_\alpha \simeq R/P_\alpha$. Let

$$\mathscr{F} = \{P_\alpha | P_\alpha^* \subset P_\alpha \text{ and } S(R/P_\alpha) \subset Z(R/P_\alpha)\}$$

where $S(R/P_\alpha)$ are the symmetric elements of R/P_α, and set

$$A = \bigcap_{P_\alpha \in \mathscr{F}} P_\alpha \quad \text{and} \quad B = \bigcap_{P_\alpha \notin \mathscr{F}} P_\alpha.$$

Since R is prime and $AB \subset A \cap B = 0$, we must have either $A = 0$ or $B = 0$. If $A = 0$, then $S = S(R) \subset Z(R)$, a contradiction. Thus $B = 0$, and so R is a subdirect product of primitive rings R/P_α where either

$$P_\alpha^* \not\subset P_\alpha \quad \text{or} \quad S(R/P_\alpha) \not\subset Z(R/P_\alpha).$$

If $P_\alpha^* \not\subset P_\alpha$, then $I = P_\alpha + P_\alpha^*/P_\alpha$ is a non-zero ideal of R/P_α and for all $x + P_\alpha \in I$,

$$x + P_\alpha = x + x^* + P_\alpha;$$

as a consequence, if $a \in H$, then

$$(a + P_\alpha)(x + P_\alpha)^m = (x + P_\alpha)^m(a + P_\alpha),$$

for a suitable $m \geqq 1$. By [**6**, Lemma 2] or its proof, it follows that $a + P_\alpha$ centralizes I. Therefore $a + P_\alpha \in Z(R/P_\alpha)$, the center of R/P_α.

If $P_\alpha^* \subset P_\alpha$, then R/P_α is a primitive ring with induced involution *. Moreover $H = H(R)$ maps into the symmetric hypercenter $H(R/P_\alpha)$ of R/P_α. By the first part of the proof, since $S \not\subset Z(R/P_\alpha)$,

$$H(R/P_\alpha) = Z(R/P_\alpha).$$

Therefore we have proved that $H(R/P_\alpha) \subset Z(R/P_\alpha)$, for all α, and this forces the desired conclusion $H \subset Z$.

We continue the study of H with the following

THEOREM 2. *If R is a domain then $H^+ \subset Z$.*

Proof. Let $a \in H^+$ and $s \in S$. If R' is the subring generated by a and s then R' is still a domain with involution *.

Let $C_{R'}(s) = \{x \in R' | xs = sx\}$ be the centralizer of s in R'. $C_{R'}(s)$ is a domain stable under *; moreover, since $a \in H$, for every $t = t^* \in C_{R'}(s)$ there exists $m = m(a, s) \geqq 1$ such that

$$t^m \in C_{R'}(s) \cap C_{R'}(a) \subset Z(C_{R'}(s)).$$

By [**1**, Theorem 4] $C_{R'}(s)$ satisfies S_4, the standard identity in four variables. Now, since for a suitable integer n, $s^n \in Z(R')$, by [**11**, Theorem 2], R' satisfies a polynomial identity. Hence R' is an order in a division ring $D \simeq R' \otimes_{Z(R')} F$ where F is the field of fractions of $Z(R')$ (see Theorem 1.4.3 in [**7**]). Moreover under the induced involution the symmetric elements of D are of the form bz^{-1} where $b \in S \cap R'$ and $z \in Z(R') \cap S$. The outcome of this is that $H(R')^+ \subset H(D)^+$; hence, if $S \not\subset Z(D)$, by [**4**, Lemma 6], $H(R')^+ \subset Z(D)$. In any case $as = sa$ and by [**7**, Theorem 2.1.5.], $a \in Z(R)$ follows.

We now prove a technical result which holds in arbitrary rings, namely

THEOREM 3. *Let A be a ring with no non-zero nil right ideals. Suppose that for every positive integer n and for every choice of $a_1, a_2, \ldots, a_n \in A$ there exist positive integers $m_1 = m_1(a_1), \ldots, m_n = m_n(a_n)$, $t = t(a_1, \ldots, a_n)$ such that*

$$(a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n})^t = (a_n^{m_n} \ldots a_2^{m_2} a_1^{m_1})^t.$$

Then A is commutative.

Proof. First we remark that if $a_1, \ldots, a_n \in A$, for every non empty subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ we may take

$$m_{i_1} = \ldots = m_{i_k} = m \quad \text{where } m = m(a_{i_1}, \ldots, a_{i_n}).$$

If A is a division ring, let $a, b \in A$ and $m = m(a, b)$, $t = t(a, b)$ such that

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$$(a^m b^m a^{-m})^t = (a^{-m} b^m a^m)^t.$$

It follows that

$$a^m b^{mt} a^{-m} = a^{-m} b^{mt} a^m$$

and so,

$$a^{2m} b^{mt} = b^{mt} a^{2m}.$$

By [**8**, Theorem], A is commutative.

The commutativity condition imposed on A goes through when passing to subrings or to homomorphic images; therefore, in order to prove the theorem for a semisimple ring, using standard structure theory, it is enough to do so for $n \times n$ matrices over a division ring. Suppose $n > 1$. For e_{ij} the usual matrix units, let $a = e_{11}$, $b = e_{11} + e_{12}$. Then, for all $m \geqq 1$, $a^m b^m = b$ and $b^m a^m = a$; hence, if t is any positive integer,

$$(a^m b^m)^t = b \neq a = (b^m a^m)^t.$$

Thus $n = 1$ and by the division ring case the theorem is proved in case A is semisimple.

In the general case, let $a \in A$ be such that $a^2 = 0$. If $x \in A$, let $n = n(a, x)$, $t = t(a, x)$ be such that

$$(((1 + a)ax(1 - a))^n (ax)^n)^t = ((ax)^n ((1 + a)ax(1 - a))^n)^t.$$

Recalling that $1 - a = (1 + a)^{-1}$, we get

$$((1 + a)(ax)^n (1 - a)(ax)^n)^t = ((ax)^n (1 + a)(ax)^n (1 - a))^t$$

and, since $a^2 = 0$,

$$(ax)^{2nt} = ((ax)^{2n} - (ax)^{2n} a)^t.$$

From this last equality it follows that $(ax)^{2nt} a = 0$. Therefore, aA is a nil right ideal of A. By the hypothesis placed on A, it follows that $a = 0$. We have shown that A has no non-zero nilpotent elements. Since any such ring is a subdirect product of domains (see [**7**, Theorem 1.1.1]), we may assume A to be a domain.

Let now $a, b \in A$ non-zero and $n = n(a)$, $m = m(b)$, $t = t(a, b)$ such that

$$(a^n b^m)^t = (b^m a^n)^t.$$

We call A_0 the subring generated by a^n and b^m and we remark that in order to complete the proof of the theorem, it is enough to prove that A_0 is commutative. In fact, if this is the case, by [**8**, Theorem] A will be commutative.

Now, $Z(A_0) \neq 0$, in fact, from

$$a^n(a^nb^m)^t = a^n(b^ma^n)^t = (a^nb^m)^ta^n \quad \text{and}$$
$$b^m(a^nb^m)^t = (b^ma^n)^tb^m = (a^nb^m)^tb^m$$

it follows that $(a^nb^m)^t$ commutes with a^n and b^m; hence

$$0 \neq (a^nb^m)^t \in Z(A_0).$$

Let A_1 be the localization of A_0 at $Z(A_0) - \{0\}$. A_1 is still a domain whose center is a field; moreover A_1 satisfies all the hypotheses placed on A. Let J be the Jacobson radical of A_1 and suppose $J \neq 0$. Let $0 \neq c \in J$ and $d \in A_1$. If $r = r(c)$, $s = s(d)$, $u = u(c, d)$ are such that

$$((1 + c)^rd^s(1 + c)^{-r})^u = ((1 + c)^{-r}d^s(1 + c))^u,$$

we get

$$(1 + c)^rd^{su}(1 + c)^{-r} = (1 + c)^{-r}d^{su}(1 + c)^r$$

and so,

$$(1 + c)^{2r}d^{su} = d^{su}(1 + c)^{2r}.$$

By the hypercenter theorem, $(1 + c)^{2r} \in Z(A_1)$. Since $Z(A_1)$ is a field, it follows that c is invertible in A_1, and this contradicts $c \in J$. Thus A_1 is semisimple and by the first part of the proof A_1 and so A_0 is commutative.

In the rest of the paper R will be a prime ring with no non-zero nil right ideals. In this general setting, we start to study H^+ by investigating its zero divisors. The first result in this direction is given by the following:

LEMMA 2. *H^+ has no non-zero nilpotent elements.*

Proof. Let $a \in H^+$ be such that $a^2 = 0$. If $x \in R$, $ax^* + xa$ is a symmetric element; let $m = m(a, x)$ be such that

$$a(ax^* + xa)^m = (ax^* + xa)^ma.$$

Since $a^2 = 0$, we get $a(xa)^m = (ax^*)^ma$; thus $a(xa)^m \in S$.

For every positive integer n, let $x_1, \dots, x_n$ be elements of R and $m_1, \dots, m_n$ the corresponding integers such that

$$a(x_1a)^{m_1}, \dots, a(x_na)^{m_n} \in S.$$

For a suitable integer $m = m(x_1, \dots, x_n)$,

$$a((x_1a)^{m_1} \dots (x_na)^{m_n})^m \in S.$$

We have

$$a((x_1a)^{m_1} \dots (x_na)^{m_n})^m = ((x_1a)^{m_1} \dots (x_na)^{m_n})^{*m}a$$
$$= ((ax_n^*)^{m_n} \dots (ax_1^*)^{m_1})^ma = ((ax_n)^{m_n} \dots (ax_1)^{m_1})^ma$$
$$= a((x_na)^{m_n} \dots (x_1a)^{m_1})^m.$$

Let now $R_1 = Ra/r_R(a) \cap Ra$ where $r_R(a) = \{x \in R | ax = 0\}$. Since R has no non-zero nil right ideals, then R_1 has no non-zero nil right ideals; moreover the above equality says that R_1 satisfies the hypotheses of Theorem 3. Hence R_1 is commutative. This says that $axaya - ayaxa$ is a generalized polynomial identity for R. By [**2**, Proposition 6] R contains a *-closed prime subring R_0 containing a, which is an order in 2×2 matrices over a field F. Since

$$H(R_0)^+ \supset H(R)^+ \cap R_0$$

then $a \in H(R_0)^+$; moreover if F_2 is endowed with the involution induced by the one in R_0, then $a \in H(F_2)^+$. By Remark 3, $a \in F$ and since $a^2 = 0$ we deduce $a = 0$.

The invariance of H and the conclusion of Lemma 2 together imply that H^+ centralizes all square-zero skew elements. In fact we have the

LEMMA 3. *Let $a \in H^+$. If $k \in K$ is such that $k^2 = 0$ then $ak = ka$.*

Proof. Since k is a quasi-unitary element with quasi-inverse $-k$, then

$$(1 + k)a(1 - k) \in H^+ \quad \text{and} \quad (1 - k)a(1 + k) \in H^+.$$

Since R is 2-torsion free we deduce that

$$kak \in H^+ \quad \text{and} \quad ka - ak \in H^+.$$

Since $(kak)^2 = 0$, by Lemma 2 we must have $kak = 0$ giving

$$(ka - ak)^2 = 0.$$

Again, by Lemma 2, $ka - ak = 0$.

Let us denote by C the extended centroid of R and let $Q = RC$ stand for the central closure of R.

The next lemma gives us some information about the right annihilator of elements of H^+.

LEMMA 4. *Let $a = a^* \in Q$ be such that, for all $s \in S \cap R$, $as^m = s^m a$ where $m = m(a, s) \geqq 1$ is an integer. If t is a symmetric or skew element of Q such that $t^2 = 0$ and $at = 0$, then either $a^3 = 0$ or $t = 0$.*

Proof. Suppose $t \in S$ and let $U = U^*$ be an ideal of R such that $aUt \subset R$ and $a^2Ut \subset R$. If $x \in U$, the element $k = axt - tx^*a$ (if $t \in K$, $k = axt + tx^*a$) is a skew element of R; moreover $k^3 = 0$ and $(ak - ka)^3 = 0$. Since k is a quasi-unitary element of R, the element

$$b = (1 + k)^{-1}(ak - ka)(1 - k)^{-1}$$

still commutes with suitable powers of elements of $S \cap R$. Moreover, since $b \in R$, $b \in H^+$. But

$$b^2 = (1 + k)^{-1}(ak - ka)(1 + k^2)(ak - ka)(1 - k)^{-1}$$
$$= (1 + k)^{-1}(ak - ka)^2(1 - k)^{-1}$$

and

$$b^3 = (1 + k)^{-1}(ak - ka)^3(1 - k)^{-1} = 0.$$

By Lemma 2 we must have $b = 0$. Now

$$0 = ab = a^3xt,$$

i.e., $a^3Ut = 0$ and the primeness of R proves the lemma.

At this stage we would like to prove that H^+ centralizes all square-zero symmetric elements. Unfortunately this seems still out of hand. One step in this direction is the following:

LEMMA 5. *If* $s \in S$ *is such that* $s^2 = 0$ *then* $sH^+s = 0$.

Proof. Let $a \in H^+$. If k is a skew element of R, then

$$sks \in K \quad \text{and} \quad (sks)^2 = 0.$$

By Lemma 3 $asks = sksa$ giving $sasks = 0$. Let $sas = t$. For $x \in R$, $x - x^* \in K$ and so,

$$t(x - x^*)s = 0;$$

this implies $txs = tx^*s$. Now, if $x, y \in R$

$$txtys = t(xty)^*s = ty^*tx^*s = tytxs.$$

We have shown that for all $x, y \in R$

(2) $\quad txtys = tytxs.$

Moreover, taking * we also get

(3) $\quad sxtyt = sytxt.$

By [**11**, Lemma 3], if $txt \neq 0$, there exists $\lambda = \lambda(x)$ in the extended centroid C of R such that $txs = \lambda s$. Substituting in (3) (recall that $t = sas$) we obtain

$$(sxt - \lambda s)yt = 0, \quad \text{for all } y \in R.$$

Since R is prime and $t \neq 0$ this forces $sxt = \lambda s = txs$. Therefore, for all $x \in R$, either $sxt = txs$ or $txt = 0$. Since (2) holds and R is prime, $txt = 0$ forces $txs = 0$ and so, $sx^*t = sxt = 0$. We have proved that $sxt = txs$, for all $x \in R$.

Now, if $t \neq 0$, by [**11**, Lemma 3], there exists $\mu \in C$ such that $t = \mu s$ and, recalling that $tKs = 0$, we get $sKs = 0$. By [2, Proposition 6] there exists a *-closed prime subring R_0 of R containing s, and R_0 is an order in $fQf \simeq C_2$, for some symmetric idempotent f in Q.

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First we claim that $af = fa$. In fact, since R_0 satisfies a polynomial identity, by a theorem of Posner $fQf \simeq R_0 \otimes_{Z(R_0)} F$ where F is the field of fractions of $Z(R_0)$. Moreover, under the induced involution, the symmetric elements of fQf are of the form bz^{-1} with $b \in R_0 \cap S$ and $z \in Z(R_0) \cap S$. Thus, since $a \in H(R)^+$ and $f = f^2 = f^* \in fQf$, we have that $af = fa$.

Notice that $af = fa \in H(fQf)^+$. In fact, let $b = b^* \in R_0$, and m such that $ab^m = b^m a$. Since $b \in fQf$, $b = fb = bf$; hence

$$afb^m = ab^m = b^m a = b^m fa = b^m af.$$

Since R_0 is an order in fQf, by the remark made above, we get

$$af \in H(fQf)^+.$$

Being $fQf \simeq C_2$ by Remark 3 af and so, a centralizes all elements in fQf; hence $as = sa$ and so, $sas = 0$.

Let $p = \text{char } R$. We now define a subset H_p^+ of H^+ which will play an important role in what follows. H_p^+ is defined to be equal to H^+ in case char $R = p = 0$ and $H_p^+ = \{a^p | a \in H^+\}$ otherwise.

The next lemma tells us that H_p^+ centralizes all square-zero symmetric elements.

LEMMA 6. *Let $a \in H_p^+$. If $s \in S$ is such that $s^2 = 0$ then $as = sa$.*

Proof. Let $b \in H^+$. Since, by Lemma 5, $sH^+s = 0$, $bs - sb$ is a square-zero skew element of R. Hence, by Lemma 3, b commutes with $bs - sb$. Now, if char $R = p \neq 0$, then $b^p s = sb^p$ and we are done. In case char $R = 0$ let m be such that

$$b(b + s)^m = (b + s)^m b.$$

Since $sH^+s = 0$ we get

$$b(b^m + b^{m-1}s + \ldots + sb^{m-1}) = (b^m + b^{m-1}s + \ldots + sb^{m-1})b.$$

Hence $b^m s = sb^m$. Recalling that b commutes with $bs - sb$, we obtain

$$0 = b^m s - sb^m = mb^{m-1}(bs - sb).$$

Since char $R = 0$ and b is not nilpotent, it follows by Lemma 4 that $bs = sb$.

A slight generalization of Lemma 6 is the following

LEMMA 7. *Let $a \in H_p^+$. If $x \in R$ is such that $x^2 = xx^* = x^*x = 0$ then $ax = xa$.*

Proof. The conditions imposed on x imply

$$(x + x^*)^2 = (x - x^*)^2 = 0.$$

By Lemma 3 and Lemma 6 we get

$$a(x + x^*) = (x + x^*)a \quad \text{and} \quad a(x - x^*) = (x - x^*)a$$

resulting in

$$x^*a - ax^* = ax - xa = - x^*a + ax^*.$$

Thus $ax - xa = 0$.

We are now in a position to prove that the elements of H^+ are algebraic over the extended centroid provided the ring R has non-zero symmetric nilpotent elements.

LEMMA 8. *If R has non-zero symmetric nilpotent elements, then for all $a \in H_p^+$, there exists $\lambda = \lambda^* \in C$ such that $(a - \lambda)^3 = 0$.*

Proof. Let $s \neq 0$ be a symmetric element of R such that $s^2 = 0$. If $x \in R$ then $y = sxs$ satisfies $y^2 = yy^* = y^*y = 0$ and so, by Lemma 7 $asxs = sxsa$. By [**11**, Lemma 3] there exists $\lambda \in C$ such that $as = \lambda s$ and, since $as = sa$, $\lambda = \lambda^*$ is symmetric. Therefore $(a - \lambda)s = 0$ and by Lemma 4 $(a - \lambda)^3 = 0$, as wished.

Before proving our main result we need a lemma on invariant subrings whose proof is due to Herstein. If B is a ring, let $J(B)$ denote the Jacobson radical of B.

LEMMA 9. *Let B be a prime ring which is not a domain in which $J(B) \neq 0$. Suppose that A is a subring of B such that $(1 + x)A(1 + x)^{-1} \subset A$ for all $x \in J(B)$. If $A \not\subset Z(B)$ and A does not contain a non-zero ideal of B, then $A \cap J(B)$ has non-zero nilpotent elements.*

Proof. We note first that $A_1 = A \cap J(B) \neq 0$. In fact, if not, for $a \in A$ and $x \in J = J(B)$,

$$(ax - xa)(1 + x)^{-1} = (1 + x)a(1 + x)^{-1} - a \in A_1 \quad \text{implies}$$
$$(ax - xa)(1 + x)^{-1} = 0$$

and so, $ax = xa$. Thus A centralizes the non-zero ideal J and by the primeness of B, $A \subset Z$.

Suppose first that no element of A_1 is a zero-divisor in J. Let $a \in A_1$ and let $x \in J$, $x \neq 0$, be a left zero-divisor in J. Then, from

$$(ax - xa)(1 + x)^{-1} \in A_1 \quad \text{and} \quad (aax - axa)(1 + ax)^{-1} \in A_1$$

we get

$$a(ax - xa)((1 + x)^{-1} - (1 + ax)^{-1}) \in A_1;$$

hence

$$a(ax - xa)(1 + x)^{-1}(1 - a)x(1 + ax)^{-1} \in A_1.$$

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Conjugating this last element by $1 + ax$ we get

$$c = (1 + ax)^{-1}a(ax - xa)(1 + x)^{-1}(1 - a)x \in A_1.$$

Now, c is a left zero-divisor in J since x is; thus

$$a(ax - xa)(1 + x)^{-1}(1 - a)x = 0.$$

From $(1 - a)x \neq 0$ and $a(ax - xa)(1 + x)^{-1} \in A_1$ we then get

$$a(ax - xa)(1 + x)^{-1} = 0.$$

This implies $a(ax - xa) = 0$ and so, $ax - xa = 0$. We have shown that a centralizes all left zero divisors in J. Notice that if $x \in J$ is a left zero-divisor, so is every element in the left ideal Jx. Therefore a centralizes Jx forcing $a \in Z$. We have proved that $A_1 \subset Z$. This easily leads to the contradiction $A \subset Z$.

Therefore there exists $0 \neq a \in A_1$ which is a zero-divisor in J. Let $0 \neq x \in J$, with $ax = 0$. For all $r \in B$,

$$xra = (axr - xra)(1 + xr)^{-1} \in A_1 \quad \text{and} \quad (xra)^2 = 0.$$

Since B is prime, $xra \neq 0$ for some $r \in B$. This establishes the lemma.

Putting all the pieces together we can now prove that $H^+ \subset Z$.

LEMMA 10. $H^+ \subset Z$.

Proof. By Theorem 1 we may assume that the Jacobson radical $J(R)$ of R is non-zero. Suppose first that the involution is positive definite in R, i.e., $xx^* = 0$ implies $x = 0$.

If R is a domain, by Theorem 2 we are done; hence, we may assume that R has non-zero nilpotent elements. Let $x \in R$ be such that $x^2 = 0$. If $a \in H^+$ let m be such that

$$a(xx^*)^m = (xx^*)^m a;$$

then $x^2 = 0$ implies $xa(xx^*)^m = 0$. Since $*$ is positive definite, we get either

$$xa(xx^*)^{m/2} = 0 \quad \text{or} \quad xa(xx^*)^{m-1/2}x = 0$$

according as m is even or odd. A repeated application of this argument leads to $xax = 0$.

Now let $x, y \in R$ be such that $xy = 0$. For all $r \in R$, $(yrx)^2 = 0$ so

$$yrxayrx = 0;$$

this says that $xayR$ is a nil right ideal of R of bounded exponent. Since R is prime we get, by a result of Levitzki, that $xay = 0$. We have proved that

$$H^+ \subset T = \{a \in R | xy = 0 \text{ implies } xay = 0\}.$$

We remark that T is a subring of R such that

$$(1 + x)T(1 + x)^{-1} \subset T \quad \text{for all } x \in J(R).$$

Now, if $T \subset Z$, $H^+ \subset Z$ and we are done. On the other hand, since R is prime T cannot contain a non-zero ideal of R. Therefore by Lemma 9 we may assume that $T \cap J(R)$ has non-zero nilpotent elements. Moreover, by the first part of the proof of Lemma 9 in [**5**] we know that all right annihilators in $J = J(R)$ of elements of T are linearly ordered, that is, if a, $b \in T$ then either $r_J(a) \subset r_J(b)$ or $r_J(b) \subset r_J(a)$. Let $x \neq 0$ in $T \cap J$ be such that $x^2 = 0$. Since $T^* = T$, then xx^*, $x^*x \in T \cap J$. Thus either

$$r_J(xx^*) \subset r_J(x^*x) \quad \text{or} \quad r_J(x^*x) \subset r_J(xx^*).$$

In either case $xx^*xx^* = 0$. Since * is positive definite we get $x = 0$, a contradiction.

Suppose now that * is not positive definite. By [**7**, Theorem 2.2.1] either $S \subset Z$ or S has non-zero nilpotent elements. If the first possibility occurs, we are done; therefore we may assume that there exists $s \in S$ such that $s \neq 0$ and $s^2 = 0$.

Let $a \in H_p^+$. By Lemma 8 there exists $\lambda = \lambda^* \in C$ such that $(a - \lambda)^3 = 0$ and we may clearly assume that $\lambda \neq 0$. Let $U = U^*$ be an ideal of R such that $0 \neq \lambda^i U \subset R$, for $i = 1, \ldots, 4$ (see [**7**, Lemma 2.4.1]). Now, since R is prime

$$V = U \cap J(R) \neq 0.$$

If $V \cap K = 0$ then for all $x \in V$, $x = x^*$ forcing $V \subset S$. Take now $x, y \in V$; we have:

$$xy = (xy)^* = y^*x^* = yx.$$

Thus V and so R is commutative. In this case there is nothing to prove.

Therefore we may assume that $V \cap K \neq 0$. Let $k \in V \cap K$ and set

$$b = (a - \lambda)^2.$$

The element $c = (1 + k)^{-1}(ak - ka)(1 - k)^{-1}$ lies in H^+. Since also $bcb \in H^+$ and $(bcb)^2 = 0$, by Lemma 2, $bcb = 0$. Similarly $bc^2b = 0$ and so, the element $bc + cb \in H^+$ is square zero. Lemma 2 then says that $bc + cb = 0$, i.e., $bc = -cb \in K$. Since $(bc)^2 = 0$, by Lemma 3, $bc^2 = cbc$; on the other hand $bc = -cb$ implies $cbc = -bc^2$. Therefore $cbc = 0$. Now, since H^+ has no nilpotent elements, a repeated application of Lemma 4 forces either $b = 0$ or $c = 0$.

If $c = 0$, then $ak = ka$, for all $k \in V \cap K$. Let $x \in V$; $x - x^* \in V \cap K$, hence

$$a(x - x^*) = (x - x^*)a;$$

this says that $ax - xa$ is a skew element of V, so,

$$a(ax - xa) = (ax - xa)a.$$

At this point it is not difficult to prove that a centralizes V and so, a is central in R.

If $b = 0$, then $(a - \lambda)^2 = 0$ and by applying the argument above to the element $a - \lambda$, we obtain $a - \lambda = 0$ and so, a is central in R.

Now, if char $R = 0$, $H^+ = H_0^+ = Z$ and we are done. We may therefore assume that char $R = p \neq 0$. In this case we have just seen that $Z = Z(R) \neq 0$.

By localizing at $Z - \{0\}$, we obtain a prime ring R' with induced involution for which $H(R') = H(R)_{Z-\{0\}}$. For every $a \in H(R)^+$, $a^p \in Z$; hence $H(R')$ consists of invertible elements. Moreover if $J(R') = 0$, by Theorem 1, $H^+ \subset Z$.

By working with R' instead of R thus we may assume that H^+ consists of invertible elements.

Let $a \in H^+$ and $k \in K \cap J(R)$. The element $(1 + k)^{-1}(ak - ka)(1 - k)^{-1}$ lies in H^+, hence, if non-zero, it is invertible. But $(1 + k)^{-1}(ak - ka)(1 - k)^{-1}$ lies in $J(R)$ and no element of $J(R)$ can be invertible. We must conclude $ak - ka = 0$, that is, a centralizes all skew elements in $J(R)$. As before we deduce a central in R.

LEMMA 11. *If $I = I^*$ is an ideal of R such that $H^+ \cap I = 0$ then $H \cap I = 0$.*

Proof. Let $H^- = H \cap K$. Since $I = I^*$ it is enough to show that $H^- \cap I = 0$. Let $a \in H^- \cap I$. For all $k \in K$ with $k^2 = 0$,

$$(1 + k)a(1 - k) \in H^- \cap I \quad \text{and}$$

$$(1 - k)a(1 + k) \in H^- \cap I.$$

Thus $2(ak - ka) \in H^- \cap I$. Since $4(ak - ka)^2$ and a^2 lie in $H^+ \cap I$, we obtain

$$4akaka = 4a(ak - ka)^2 = 0.$$

Therefore, since R is 2-torsion free, ak is a nilpotent element of R.

Let now $r \in R$ and let $m \geqq 1$ be an even integer such that

$$a(ra - ar^*)^m = (ra - ar^*)^m a.$$

Since $a^2 = 0$, $a(ra)^m = (ar^*)^m a \in K$ and, as in proof of Lemma 2, we deduce that $axaya - ayaxa$ is a generalized polynomial identity for R. As in [**2**, Proposition 6] there exists a *-closed prime subring R_o containing a, which is an order in 2×2 matrices over a field F. Clearly

$$a \in H(R_o) \subset H(F_2).$$

Moreover, since ak is nilpotent for every square-zero skew in R, this property still holds in R_o, and so in F_2.

Now, if * is of transpose type, by Remark 3, $H(F_2) = F$ forcing $a = 0$. On the other hand, if * is symplectic, by the proof of Proposition 9 in [**2**], $a = 0$ follows. With this the lemma is proved.

We are now in a position to prove our main theorem.

THEOREM 4. *Let R be a prime ring with involution with characteristic not* 2 *or* 3. *If R has no non-zero nil right ideals and $S \not\subset Z$ then $H = Z$.*

Proof. Suppose $S \not\subset Z$. If $Z \cap S = 0$, then Lemma 10 implies $H^+ = 0$ and, by Lemma 11, $H = 0$ follows. Suppose now that $Z \cap S \neq 0$. By localizing at $Z - \{0\}$, we may clearly assume that R is a prime ring whose center is a field. Moreover, if J is the Jacobson radical of R, by Theorem 1, we may also assume that $J \neq 0$.

Since $H^+ \cap J$ consists of invertible elements, we must have $H^+ \cap J = 0$ and, by Lemma 11, $H \cap J = 0$. Now, $K \cap J = 0$ implies that J, and so R, is commutative. Thus we may assume that $K \cap J \neq 0$. Let $k \in K \cap J$ and $x \in H$; then

$$(1 + k)^{-1}(xk - kx)(1 - k)^{-1} \in H \cap J = 0$$

forcing $xk = kx$; thus H centralizes all skew elements in J. As in the proof of Lemma 10, this implies that H centralizes J forcing $H \subset Z$.

REFERENCES

1. M. Chacron, *A generalization of a theorem of Kaplansky and rings with involution*, Mich. Math. J. *20* (1973), 45-53.

2. ——— *A commutativity theorem for rings with involution*, Can. J. Math. *30* (1978), 1121-1143.

3. ——— *Unitaries in simple artinian rings*, Can. J. Math. *31* (1979), 542-557.

4. M. Chacron and I. N. Herstein, *Powers of skew and symmetric elements in division rings*, Houston J. Math. *1* (1975), 15-27.

5. B. Felzenszwalb and A. Giambruno, *Centralizers and multilinear polynomials in noncommutative rings*, J. London Math. Soc. *19* (1979), 417-428.

6. I. N. Herstein, *On the hypercenter of a ring*, J. Algebra *36* (1975), 151-157.

7. ——— *Rings with involution* (U. of Chicago Press, Chicago, 1976).

8. ——— *A commutativity theorem*, J. Algebra *38* (1976), 112-118.

9. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Coll. Publ. *37* (1976).

10. P. Misso, *Commutativity conditions on rings with involution*, Can. J. Math. *34* (1982), 17-22.

11. M. Smith, *Rings with an integral element whose centralizer satisfies a polynomial identity*, Duke Math. J. *42* (1975), 137-149.

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