ON THE COMMUTATIVITY OF THE RADICAL OF A GROUP ALGEBRA

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1. Introduction. Over a field of prime characteristic p the group algebra of a finite group has a non-trivial radical if and only if the order of the group is divisible by the prime p. In two earlier papers [7, 8] we have imposed certain restrictions on the radical, namely that the radical be contained in the centre of the group algebra and that the radical be of square zero, and we have considered what influence these conditions have on the structure of the group itself. These conditions are, at first sight, of different types and our present paper is an attempt to generalise them by merely assuming that the radical is commutative.

Before making the above remarks more precise we require to introduce our notations and conventions which will follow those employed in the earlier papers. Thus if G is a group and if H is a subgroup of G, we denote the order of H and index of H in G by $|H|$ and $|G:H|$ respectively. G' is the derived group of G and e is the identity of G. We consider all group algebras to be over a fixed algebraically closed field K of prime characteristic p, $A(G)$ being the group algebra of G over K and $N(G)$ being the corresponding radical. If I is a linear subspace of $A(G)$, its dimension is written as dim I. The centre of $A(G)$, that is, the subspace of $A(G)$ spanned by the class-sums of G, is denoted by $Z[A(G)]$.

We shall frequently refer to the papers mentioned above, but, for convenience, we here restate our earlier theorems.

THEOREM A. *Let G be a group. Then $N(G) \subseteq Z[A(G)]$ if and only if G is of one of the following three types:*

(i) *G has order prime to p.*
(ii) *G is abelian.*
(iii) *If P is a p-Sylow subgroup of G, then $G'P$ is a Frobenius group with G' as the regular subgroup of $G'P$ under the inner automorphisms induced by elements of P.*

THEOREM B. *Let G be a group of order $p^a m$ ($(p, m) = 1, a \geqq 1$). Then $[N(G)]^2 = \{0\}$ if and only if $p^a = 2$.*

Using both of these theorems we shall prove here the following result.

THEOREM 1. *Let G be a non-abelian group and let p be an odd prime dividing the order of G. Then the following conditions are equivalent.*

(i) *$N(G)$ is commutative.*
(ii) *$N(G) \subseteq Z[A(G)]$.*
(iii) *If P is a p-Sylow subgroup of G, then $G'P$ is a Frobenius group with G' as the regular subgroup of $G'P$ under the inner automorphisms induced by elements of P.*

If we do not restrict the prime p to be odd we are nevertheless able to prove the next theorem which is itself a necessary step in our argument leading to Theorem 1.

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THEOREM 2. *Let G be a group such that $N(G)$ is commutative. Then there exists a normal subgroup M of G of order prime to p and, if P is a p-Sylow subgroup of G, then P is abelian and $G = PM$.*

Theorem 2 implies that $G' \subseteq M$ and so, if p divides $|G|$, then p divides $|G : G'|$. We may then deduce the following from Theorem 1.

THEOREM 3. *Let G be a group such that $N(G)$ is commutative and such that $|G : G'|$ is odd. Then $N(G) \subseteq Z[A(G)]$.*

We conclude our introduction by remarking that, if a group G is such that $G'P$ is a Frobenius group, where P is a p-Sylow subgroup of G as above, then P is cyclic and G' is nilpotent [cf. Comments in **7**, p. 108].

The last theorem in our introduction is then an immediate consequence.

THEOREM 4. *Let G be a simple group such that $N(G)$ is non-trivial.* Then *$N(G)$ is not commutative.*

2. Elementary lemmas. The remaining part of the present paper is devoted to proving Theorem 2 and to deducing from this condition (iii) of Theorem 1; by Theorem A this is sufficient to establish Theorem 1 completely. We therefore assume in this and in succeeding sections that G is a group for which $N(G)$ is commutative and non-trivial.

LEMMA 1. *Let H be a normal subgroup of G, where $H = PR$, P being a p-Sylow subgroup of G and R being a normal subgroup of G of order prime to p. Then P is abelian and there exists a normal subgroup M of G of order prime to p such that $G = PM$. Further G' has order prime to p.*

Proof. Since H is normal in G, we have $N(H) \subseteq N(G)$ [**7**, Lemma 2, Corollary (i)]. From the group structure of H we have [**7**, Lemma 4] that

$$\left(\sum_{x \in R} x\right) N(P) \subseteq N(H)$$

and thus we conclude that

$$\left(\sum_{x \in R} x\right) N(P) \subseteq N(G).$$

Let $u \in G$ and let $s \in P$, $s \neq e$. Since $N(G)$ is assumed to be commutative and since $e - s \in N(P)$, it follows that

$$\left(\sum_{x \in R} x\right)(e-s)u\left(\sum_{x \in R} x\right)(e-s) = \left(\sum_{x \in R} x\right)(e-s)\left(\sum_{x \in R} x\right)(e-s)u.$$

Utilising the normality of R in G, we deduce from the above that

$$\left(\sum_{x \in R} x\right)(e-s)(e-usu^{-1}) = \left(\sum_{x \in R} x\right)(e-s)(e-s).$$

We now further suppose that u belongs to the normalizer of P in G. Thus let $usu^{-1} = s' \in P$.

Since the elements of P can be taken to be coset representatives of R in H, the above implies that

$$(e-s)(e-s') = (e-s)(e-s)$$

and hence

$$-s'+ss' = -s+ss.$$

From considerations of linear independence over the field K it follows that, if $p \neq 2$, then $s' = s$, and, if $p = 2$, then either $s' = s$ or $s' = ss$. This last relation implies that $usu^{-1} = s^2$ and this is impossible if s has order which is a power of 2. Thus we have $usu^{-1} = s$ and consequently P is a subgroup of the centre of its normalizer. By Burnside's Theorem [**4,** Theorem 14. 3. 1], P has a normal p-complement M in G. This normal p-complement necessarily contains G'.

LEMMA 2. *Let* $x \in G'$. *Then* $(e-x)[N(G)]^2 = \{0\}$.

Proof. Any element of $[N(G)]^2$ is a linear combination of elements of the form ww', where w, $w' \in N(G)$. The lemma is proved if we show that, for $x \in G'$, $(e-x)ww' = 0$.

Let $a, b \in G$; then

$$\begin{aligned}(ab)(ww') &= a(bw)w' = a(w'(bw)) = (aw')(bw)\\ &= (bw)(aw') = b(w(aw')) = b(aw')w = (ba)(w'w)\\ &= (ba)(ww').\end{aligned}$$

Hence

$$(e-b^{-1}a^{-1}ba)ww' = 0.$$

Let $x \in G'$; then there exist commutators $c_1, c_2, \ldots, c_s \in G'$ such that $x = c_1c_2 \ldots c_s$ and then

$$(e-x)ww' = (e-c_s)ww' + (e-c_1c_2 \ldots c_{s-1})c_sww'.$$

An obvious induction argument completes the proof.

Let G' have index r in G and let

$$G = G'a_1 \cup G'a_2 \cup \ldots \cup G'a_r \quad (a_1 = e)$$

be a coset decomposition of G' in G. We now argue from Lemma 2 as we did in our earlier paper [**7,** pp. 106–107] except that, for the present, we replace $N(G)$ of the earlier discussion by $[N(G)]^2$. Thus we let I be the ideal of $A(G')$ spanned by $\sum_{y \in G'} y$ and let J be the ideal of $A(G)$ generated by I.

We deduce

LEMMA 3. $$[N(G)]^2 \subseteq J.$$

From this lemma we obtain the following lemma.

LEMMA 4. *Either* $[N(G')]^2 = \{0\}$ *or* $[N(G')]^2 = I$.

Proof. Since G' is normal in G, we have [**7,** Lemma 2, Corollary (i)] $N(G') \subseteq N(G)$ and therefore $[N(G')]^2 \subseteq [N(G)]^2 \subseteq J$.

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Hence [7, Lemma 2]

$$[N(G')]^2 \subseteq J \cap A(G') = I.$$

The lemma is then established by remarking that I is a one-dimensional subspace of $A(G)$.

Our investigation of the structure of G is facilitated by observing that Lemma 4 enables us to consider three mutually exclusive cases, the first two of which we shall show do not arise.

Case (1). $[N(G')]^2 = \{0\}$, $N(G') \neq \{0\}$.

Case (2). $[N(G')]^2 = I$.

Case (3). $N(G') = \{0\}$.

We proceed to the consideration of these cases.

3. Discussion of Case (1). We obtain the following lemma by applying Theorem B to the present situation.

LEMMA 5. *The prime $p = 2$ and the derived group G' has order $2q$, where q is odd.*

This lemma imposes conditions on G, and by purely group-theoretical arguments we shall prove Lemma 6 which, together with Lemma 1, will imply that Case (1) does not arise.

LEMMA 6. *Let G be a group such that the derived group G' has order $2q$, where q is odd. Let P be a 2-Sylow subgroup of G. Then G has a normal subgroup M of odd order such that $G = PM$.*

Proof. Let $|P| = 2^a$ $(a \geqq 1)$. Since G/G' is abelian, there exists a subgroup G_0 of G containing G' such that $|G/G' : G_0/G'| = 2^{a-1}$ and thus $|G : G_0| = 2^{a-1}$. Since a 2-Sylow subgroup of G_0 has order 2, such a subgroup is clearly in the centre of its normalizer and so, by Burnside's Theorem [4, Theorem 14.3.1], G_0 has a normal subgroup M of index 2 and of odd order. Such a subgroup M of G_0 is necessarily unique and consequently, from the normality of G_0 in G, M is normal in G. Hence

$$|G : M| = |G : G_0||G_0 : M| = 2^{a-1} . 2 = 2^a$$

and from this we deduce that $G = PM$.

Lemmas 5 and 6 show that $p = 2$ and that $G = PM$, where P is a 2-Sylow subgroup of G and where M is a normal subgroup of G of odd order. This implies, by Lemma 1, that G' has order prime to 2 and this is impossible under the assumption that $N(G') \neq \{0\}$.

4. Representation theory. The discussion of Cases (2) and (3) depends on results drawn from the modular representation theory of finite groups. As a general reference we would cite the text of Curtis and Reiner [3].

In this section we write, for simplicity, $A = A(G)$ and $N = N(G)$. We can choose mutually orthogonal idempotents $e_1, e_2, \ldots, e_n$ such that

$$e = e_1 + e_2 + \ldots + e_n$$

and

$$A = Ae_1 + Ae_2 + \ldots + Ae_n,$$

where Ae_i $(i = 1, 2, \ldots, n)$ is a principal indecomposable module (indecomposable left ideal) and the sum is a direct sum of principal indecomposable modules. We adopt the convention of omitting the word " principal " when referring to principal indecomposable modules. We let $\{U_1, U_2, \ldots, U_k\}$ and $\{F_1, F_1, \ldots, F_k\}$ be complete sets of indecomposable and irreducible modules respectively, where, as usual, F_κ is isomorphic to U_κ/NU_κ and where U_κ and F_κ have, as finite dimensional vector spaces over K, the respective dimensions u_κ and f_κ $(\kappa = 1, 2, \ldots, k)$. We suppose that F_1 is the trivial irreducible module such that $gv = v$ $(g \in G, v \in F_1)$. For each i $(1 \leqq i \leqq n)$ there exists a unique κ $(1 \leqq \kappa \leqq k)$ such that Ae_i is module isomorphic to U_κ and, conversely, for each κ $(1 \leqq \kappa \leqq k)$, there exists at least one i $(1 \leqq i \leqq n)$ such that U_κ is module isomorphic to Ae_i. We make considerable use of the facts that U_κ has a unique minimal submodule and a unique maximal submodule, the latter being NU_κ, and that the unique minimal submodule is isomorphic to F_κ [**3**, p. 598].

Under the additional assumption that N is commutative, it is convenient to classify the indecomposable modules into three mutually exclusive types and to determine the irreducible modules which can appear as composition factors of the indecomposable modules.

If we consider U_κ, it is clear that one of the three following situations must arise:

(*a*) $NU_\kappa = \{0\}$.

(*b*) $NU_\kappa \neq \{0\}$, $N^2U_\kappa = \{0\}$.

(*c*) $N^2U_\kappa \neq \{0\}$.

If U_κ is of type (*a*), then U_κ is irreducible and $U_\kappa = F_\kappa$. If U_κ is of type (*b*), then $N^2U_\kappa = \{0\}$ implies that NU_κ is completely reducible [**1**, Theorem 9.4A, p. 103] and, since U_κ has a unique minimal submodule isomorphic to F_κ, we see that NU_κ is isomorphic to F_κ. Thus only two irreducible modules appear as composition factors of a composition series for U_κ and they are both isomorphic to F_κ. For this to happen the Cartan matrix $C = (c_{\kappa\lambda})$ [**3**, p. 593], which gives the multiplicities with which the irreducible modules appear as composition factors of the indecomposable modules, must be such that $c_{\kappa\kappa} = 2$, $c_{\kappa\lambda} = c_{\lambda\kappa} = 0$ $(\lambda \neq \kappa)$. This implies that $\det C$ is divisible by 2. But we know that $\det C$ is a power of p [**3**, (84. 17), p. 602] and hence, if there exists a U_κ of type (*b*), we must have $p = 2$.

We now suppose that U_κ is of type (*c*) and we shall show that the only irreducible modules appearing in U_κ are one-dimensional. The elements of N^2U_κ are linear combinations over the field K of elements of the form wv, where $w \in N^2$ and $v \in U_\kappa$.

Let $x, y \in G$; then, from the proof of Lemma 2,

$$(xy)(wv) = ((xy)w)v = ((yx)w)v = (yx)(wv),$$

and consequently the linear transformations $t \to (xy)t$ and $t \to (yx)t$ $(t \in N^2U_\kappa)$ induced by xy and yx in the submodule N^2U_κ of U_κ are identical. That is, the linear transformations induced by x and y in N^2U_κ commute. Hence we infer that the only irreducible modules appearing in N^2U_κ are one-dimensional. We can argue further however. We remark first that, as U_κ has a unique minimal submodule isomorphic to F_κ, this submodule must appear in N^2U_κ and hence F_κ is one-dimensional. Suppose, for the sake of argument, that F_λ appears as an irreducible constituent of U_κ, where $\lambda \neq \kappa$. Then $c_{\kappa\lambda} \geqq 1$ and, since C is a symmetric matrix, F_κ appears

$c_{\kappa\lambda}$ times in U_λ. Let us consider to which type this U_λ belongs. Since U_λ has F_κ as an irreducible constituent where $\kappa \neq \lambda$, it follows that U_λ must itself be of type (c). By our remarks above this implies that F_λ is one-dimensional.

We summarise our results in the following lemma.

LEMMA 7. *Let G be a group such that $N = N(G)$ is commutative. Let $\{U_1, U_2, \ldots, U_k\}$ and $\{F_1, F_2, \ldots, F_k\}$ be complete sets of indecomposable and irreducible modules of $A = A(G)$ as above. Then U_κ $(1 \leqq \kappa \leqq k)$ belongs to one of the following three types:*

(a) U_κ is irreducible.

(b) U_κ has two irreducible constituents both isomorphic to F_κ.

(c) U_κ is not of type (a) or (b) and the only irreducible constituents appearing in U_κ are one-dimensional.

If there exists a κ $(1 \leqq \kappa \leqq k)$ such that U_κ is of type (b), then $p = 2$.

It is of some importance to determine the dimensions of the indecomposable modules and we prove therefore the following lemma.

LEMMA 8. *Let G be a group of order $p^a m$ $((p, m) = 1, a \geqq 1)$ such that $N = N(G)$ is commutative. Let $A = Ae_1 + Ae_2 + \ldots + Ae_n$ be as above and suppose that U_1 has dimension p^a. Then, for each i, Ae_i is an indecomposable module, and, if Ae_i is of type (a) above, then* $\dim Ne_i = 0$, *and, if Ae_i is of type* (c), *then* $\dim Ne_i = p^a - 1$.

Proof. If Ae_i is of type (a), then Ae_i is irreducible and so $Ne_i = \{0\}$.

If Ae_i is of type (c), then we suppose that Ae_i is isomorphic to U_κ of type (c) and this implies that Ae_i/Ne_i is isomorphic to the one-dimensional irreducible module F_κ. Hence

$$\dim Ne_i = \dim Ae_i - \dim Ae_i/Ne_i = u_k - 1.$$

We now show that $u_\kappa = p^a$. We know that the tensor product module $U_1 \otimes F_\kappa$ has U_κ as a direct summand [**2**, p. 579; **5**, p. 413]. But we have $\dim (U_1 \otimes F_\kappa) = \dim U_1 = p^a$ and we also have that p^a divides u_κ [**3**, (65.17), p. 439]. Hence

$$U_1 \otimes F_\kappa = U_\kappa \quad \text{and} \quad u_\kappa = p^a.$$

5. Discussion of Case (2). We now suppose that $[N(G')]^2 = I$. Since G' is normal in G, it follows that $N(G') \subseteq N(G)$ [**7**, Lemma 2, Corollary (i)] and so $N(G')$ is commutative. We therefore apply to G' the arguments applied to G in the previous section. In order to emphasise that we are considering G' rather than G we write $\tilde{A} = A(G')$ and $\tilde{N} = N(G')$. We let

$$\{\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_l\} \quad \text{and} \quad \{\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_l\}$$

be complete sets of indecomposable and irreducible modules, respectively, of G'. In a similar manner we have

$$e = \tilde{e}_1 + \tilde{e}_2 + \ldots + \tilde{e}_t$$

and

$$\tilde{A} = \tilde{A}\tilde{e}_1 + \tilde{A}\tilde{e}_2 + \ldots + \tilde{A}\tilde{e}_t$$

with the obvious interpretations. From these relations we have

$$\tilde{N}^2 = \tilde{N}^2\tilde{e}_1 + \tilde{N}^2\tilde{e}_2 + \ldots + \tilde{N}^2\tilde{e}_t.$$

In the present case we have dim $\tilde{N}^2 = \dim [N(G')]^2 = 1$ and hence there exists j $(1 \leqq j \leqq t)$ such that dim $\tilde{N}^2\tilde{e}_j = 1$ and, for $i \neq j$ $(i = 1, 2, \ldots, t)$, $\tilde{N}^2\tilde{e}_i = \{0\}$. Thus the indecomposable modules of G' are of types (*a*) or (*b*) with the exception of one which is of type (*c*).

We now consider $\tilde{U}_1$ and we shall show that $\tilde{U}_1$ is the indecomposable module of type (*c*). Since $\tilde{U}_1$ is not irreducible, $\tilde{U}_1$ can only be of types (*b*) or (*c*). If $\tilde{U}_1$ is of type (*b*), then $\tilde{U}_1$ contains two irreducible constituents both isomorphic to $\tilde{F}_1$. Thus $p = 2$ and $\tilde{u}_1 = 2$. Further, since 2 divides $\tilde{u}_1$ to the first power only, we must have $|G'| = 2q$, where q is odd [**3**, (65.17), p. 439]. By Theorem B, this latter condition implies that $[N(G')]^2 = \{0\}$ and this contradicts our supposition that dim $[N(G')]^2 = 1$. Hence we have shown that $\tilde{U}_1$ is of type (*c*) and that $\tilde{U}_1$ is the only indecomposable module of type (*c*). This implies that the indecomposable modules all belong to different blocks [**2**, p. 562; **3**, p. 604] and so, in particular, $\tilde{U}_1$ is in a block by itself. Consequently [**2**, p. 587; **3**, p. 610] G' has a normal subgroup M and a p-Sylow subgroup Q such that $G' = QM$, where $Q \cap M = \{e\}$. M is necessarily unique in G' and so it follows that M is normal in G. Let P be a p-Sylow subgroup of G containing Q and let $H = PM$. Then $H = PQM = PG'$ and therefore H is normal in G. This implies, by Lemma 1, that G' has order prime to p and this is impossible if $N(G') \neq \{0\}$. We have therefore established that Case (2) does not arise.

6. Discussion of Case (3). We have now shown that this is the only case which arises; this proves that p does not divide $|G'|$. By elementary group theory we are able to deduce Theorem 2.

We now seek to establish that Theorem 1 is valid and, as we remarked before, we do this by proving that condition (i) of Theorem 1 implies condition (iii) of the same theorem.

Suppose henceforth that p is an odd prime and that $|G| = p^a m$ $((p, m) = 1,\ a \geqq 1)$. Lemma 7 implies that the indecomposable modules of G are of types (*a*) and (*c*). Since, by Theorem 2, there exists a normal subgroup M of G of order prime to p and with index p^a, it follows that $u_1 = p^a$ [**2**, p. 587]. We now apply Lemma 8 to the decompositions

$$A(G) = A = Ae_1 + Ae_2 + \ldots + Ae_n$$

and

$$N(G) = N = Ne_1 + Ne_2 + \ldots + Ne_n$$

and thereby obtain the result that, if Ae_i is of type (*a*), then dim $Ne_i = 0$; and, if Ae_i is of type (*c*), then dim $Ne_i = p^a - 1$. We wish ultimately to obtain dim $N(G)$ and hence we require to count those Ne_i for which $Ne_i \neq \{0\}$. If $Ne_i \neq \{0\}$ then Ae_i is isomorphic to an indecomposable module U_κ of type (*c*). Since the corresponding F_κ is of dimension 1, an isomorphic copy of U_κ appears exactly once in the expression $A = Ae_1 + Ae_2 + \ldots + Ae_n$ for A as a direct sum of indecomposable modules. Conversely, for each λ for which F_λ has dimension 1 there exists exactly one isomorphic copy of U_λ in the above decomposition. Hence

$$\dim N(G) = \alpha(p^a - 1),$$

where α is the number of F_λ of dimension 1. But the number α is equal to $|G : G'P|$, where P is a p-Sylow subgroup of G [**2**, p. 588]. Hence we obtain finally

$$\dim N(G) = |G : G'P|\,(p^a - 1).$$

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As $G/G'P$ is an abelian group of order prime to p, we have [7, Lemma 7] that

$$\dim N(G'P) = \frac{1}{|G:G'P|} \dim N(G) = p^a - 1.$$

We now impose the further condition that G is non-abelian and so $G' \neq \{e\}$. The above equality now implies [6, Theorem 2] that $G'P$ is a Frobenius group such that P acts as a group of regular automorphisms on G', as in condition (iii) of Theorem 1. This completes our proof.

Part of this work was carried out while I was a visiting lecturer in the Mathematics Department of the University of Western Australia, to the members of which I would like to express my thanks.

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