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A NOTE ON THE ZEROS OF L-FUNCTIONS ASSOCIATED TO FIXED-ORDER DIRICHLET CHARACTERS

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Abstract

We use the Weyl bound for Dirichlet L-functions to derive zero-density estimates for L-functions associated to families of fixed-order Dirichlet characters. The results improve on previous bounds given by the author when σ is sufficiently distant from the critical line.

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1. Introduction

The distribution of the nontrivial zeros of Dirichlet L-functions is of great importance in analytic number theory. The generalised Riemann hypothesis claims that these zeros all lie on the line $\frac{1}{2} + i\mathbb{R}$; however, it is possible that they could lie anywhere in the region $(0, 1) + i\mathbb{R}$. In practice, it often suffices to employ an unconditional result, known as a zero-density estimate, whose statement is not as strong as the generalised Riemann hypothesis. Let χ be a primitive Dirichlet character modulo q, and suppose that $\sigma \in (\frac{1}{2}, 1)$ and $T \in (2, \infty)$. Define $R(\sigma, T) = [\sigma, 1] + i[-T, T]$ and let $L(s, \chi)$ be the L-function associated to the character χ. Zero-density estimates are concerned with the number

$$N(\sigma, T, \chi) = \#\{\varrho \in R(\sigma, T) : L(\varrho, \chi) = 0\},$$

or rather, the average of this number over a family $\mathcal{F}$ of characters. The families $\mathcal{F} = \mathcal{O}_r$ of primitive Dirichlet characters of order r will be the primary interest. We denote by $\mathcal{O}_r(Q)$ the set of $\chi \in \mathcal{O}_r$ with conductor $q \in (Q, 2Q]$. The generalised Riemann–von Mangoldt formula [8, 18] gives the trivial bound $N(\sigma, T, \chi) \ll T \log qT$ for any primitive Dirichlet character χ, which is known to be sharp only for $\sigma = \frac{1}{2}$ (see [3, 19]).

The earliest zero-density estimates to feature an average over a family of Dirichlet L-functions are due to Bombieri [4], Vinogradov [20] and Montgomery

[14, 15]. Results containing averages over $\mathcal{O}_2(Q)$ were first given by Jutila [12] and Heath-Brown [11], both of whom followed the method laid down by Montgomery to derive his results. Montgomery's method reduces the problem to estimating mean values of the type

$$\mathfrak{S}_k(Q,T) = \sum_{\chi\in\mathcal{F}(Q)} \int_{-T}^{T} \Big| {\sum_{n\leqslant N}}' a_n\chi(n)n^{-it}\Big|^{2k}\,dt$$

and

$$\mathfrak{L}_k(Q,T) = \sum_{\chi\in\mathcal{F}(Q)} \int_{-T}^{T} |L(\tfrac{1}{2}+it,\chi)|^{2k}\,dt,$$

where $\mathcal{F}$ is the character family of interest and the prime ($'$) denotes that the sum is to be taken over square-free n. A detailed outline of the method has been given in [5]. In particular, if $\mathcal{F} = \mathcal{O}_r$ for some $r \geqslant 2$, we can show that

$$\mathfrak{L}_1(Q,T) \ll_{\varepsilon} (QT)^{1+\varepsilon} \quad \text{when } T^{2r-1} \gg Q^{2r-5}. \tag{1.1}$$

Indeed, (1.1) was proven for the case $r = 2$ in [13], for the cases $r = 3, 4, 6$ in [6], and can be proven in the remaining cases using Theorem 1.6 of [2]. The zero-density estimate of Jutila is derived using (1.1) and a suboptimal bound on $\mathfrak{S}_1(Q,T)$. In [5, 7], we used the large sieve for real characters of Heath-Brown [11] to derive an estimate for $\mathfrak{S}_1(Q,T)$ sharper than those used by Jutila and Heath-Brown to derive their results, and consequently strengthened Jutila's estimate to

$$\sum_{\chi\in\mathcal{O}_2(Q)} N(\sigma,T,\chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min((QT)^{(4-4\sigma)/(3-2\sigma)}, (Q^4T^3)^{1-\sigma}) \tag{1.2}$$

in [6]. Additionally, as analogues to (1.2), we showed for $T^5 \gg Q$ that

$$\sum_{\chi\in\mathcal{O}_3(Q)} N(\sigma,T,\chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min(Q^{(16-10\sigma)/9}T^{(4-4\sigma)/(3-2\sigma)}, Q^{(16-16\sigma)/(9-6\sigma)}T^{(4-4\sigma)/(3-2\sigma)}, (Q^4T^3)^{1-\sigma}) \tag{1.3}$$

and

$$\sum_{\chi\in\mathcal{O}_4(Q)} N(\sigma,T,\chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min(Q^{(5-3\sigma)/3}T^{(4-4\sigma)/(3-2\sigma)}, Q^{(5-5\sigma)/(3-2\sigma)}T^{(4-4\sigma)/(3-2\sigma)}, (Q^4T^3)^{1-\sigma}). \tag{1.4}$$

These results are derived using the large sieve for $\mathcal{O}_3$ and $\mathcal{O}_4$, respectively. Recently, Balestrieri and Rome [2] generalised the work of Baier and Young [1] and Gao and Zhao [9] to derive a large sieve estimate for general $\mathcal{O}_r$ where $r \geqslant 2$. Using (1.1) together with Theorem 1.6 of [2], we can show along similar lines to (1.3) and (1.4) that, for $T^{2r-1} \gg Q^{2r-5}$,

$$\sum_{\chi \in \mathcal{O}_r(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min(Q^{(6-4\sigma)/3} T^{(4-4\sigma)/(3-2\sigma)}, Q^{(6-6\sigma)/(3-2\sigma)} T^{(4-4\sigma)/(3-2\sigma)}, (Q^4 T^3)^{1-\sigma}). \tag{1.5}$$

If we could demonstrate that the estimate (1.1) still holds when $k = 2$, then we could unconditionally improve on these zero-density estimates. Following the method of Heath-Brown [11], we can at least show for $\mathcal{F} = \mathcal{O}_r$ that

$$\mathfrak{L}_2(Q, T) \ll_{\varepsilon} Q^{1+\varepsilon} T^{2+\varepsilon} \quad \text{when } T \gg Q, \tag{1.6}$$

which is sharp in the Q-aspect, but unfortunately not in the T-aspect. Nonetheless, we can use (1.6) to improve on (1.2), (1.3), (1.4) and (1.5) in the Q-aspect. Indeed, in [5, 7] we used (1.6) to show that

$$\sum_{\chi \in \mathcal{O}_2(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min((Q^3 T^4)^{(1-\sigma)/(2-\sigma)}, (QT)^{(3-3\sigma)/\sigma}). \tag{1.7}$$

Our results in [5, 7] for $\mathcal{O}_3$ and $\mathcal{O}_4$ can be improved by showing that Theorem 2.2 therein still holds under a weaker assumption on the relevant large sieve inequality, as in Lemma 3.1 below. Indeed, under this weaker assumption, we can strengthen the estimates in [7] to

$$\sum_{\chi \in \mathcal{O}_3(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min(Q^{(16-10\sigma)/9} T^{(4-4\sigma)/(2-\sigma)}, Q^{(13-13\sigma)/(6-3\sigma)} T^{(4-4\sigma)/(2-\sigma)}, (QT)^{(3-3\sigma)/\sigma}) \tag{1.8}$$

for $T^3 \gg Q^2$, and

$$\sum_{\chi \in \mathcal{O}_4(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min(Q^{(5-3\sigma)/3} T^{(4-4\sigma)/(2-\sigma)}, (QT)^{(4-4\sigma)/(2-\sigma)}, (QT)^{(3-3\sigma)/\sigma}), \tag{1.9}$$

for $T^2 \gg Q$. Again, using Theorem 1.6 of [2], we can add to these the result

$$\sum_{\chi \in \mathcal{O}_r(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min(Q^{(6-4\sigma)/3} T^{(4-4\sigma)/(2-\sigma)}, Q^{(5-5\sigma)/(2-\sigma)} T^{(4-4\sigma)/(2-\sigma)}, (QT)^{(3-3\sigma)/\sigma}), \tag{1.10}$$

which is valid for $T \gg Q$. These estimates all improve in the Q-aspect on the corresponding estimates obtained using (1.1). In general, a sharp bound for $\mathfrak{L}_{k+1}(Q, T)$

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will always lead to a stronger zero-density result than a sharp bound for $\mathfrak{L}_k(Q,T)$. To derive our main results, we consider bounds on $\mathfrak{L}_k(Q,T)$ when k is arbitrarily large.

2. Statement of results

Unfortunately, no sharp upper bounds have been established for $\mathfrak{L}_k(Q,T)$ when $k>2$. In [6] we adapted the method of Heath-Brown [11] to show that $\mathfrak{L}_k(Q,T) \ll_\varepsilon (QT)^{k/2+\varepsilon}$, though we can obtain a better result from a much more trivial approach. Indeed, Petrow and Young [16, 17] showed that the Weyl bound

$$L(\tfrac{1}{2}+it,\chi) \underset{\varepsilon}{\ll} q^{1/6+\varepsilon}(|t|+1)^{1/6+\varepsilon} \tag{2.1}$$

holds for any Dirichlet character χ modulo q, from which we derive the trivial bound

$$\mathfrak{L}_k(Q,T) \underset{\varepsilon}{\ll} (QT)^{k/3+1+\varepsilon} \quad \text{for all } k \geqslant 1. \tag{2.2}$$

Using (2.2) to estimate $\mathfrak{L}_k(Q,T)$ for arbitrarily large k in the method of Montgomery, we derive the following result.

THEOREM 2.1. *For any $Q,T \geqslant 2$,*

$$\sum_{\chi\in O_2(Q)} N(\sigma,T,\chi) \underset{\varepsilon}{\ll} (QT)^{\varepsilon}\min((QT)^{(8-8\sigma)/3},(Q^8T^5)^{(1-\sigma)/(6\sigma-3)}),$$

where $\sigma \in (\frac{1}{2},1)$.

The above is stronger than (1.7) in the Q-aspect when $\sigma > \frac{7}{8}$, and stronger in the T-aspect for all $\sigma > \frac{1}{2}$. Additionally, it improves on (1.2) precisely when $\sigma > \frac{3}{4}$. Using the same method as is used to prove Theorem 2.1, we can show that the density conjecture

$$\sum_{\chi\in O_2(Q)} N(\sigma,T,\chi) \underset{\varepsilon}{\ll} (QT)^{2(1-\sigma)+\varepsilon} \tag{2.3}$$

is a consequence of the Lindelöf hypothesis for L-functions with real characters.

For cubic characters, we have the following analogue of Theorem 2.1.

THEOREM 2.2. *For any $Q,T \geqslant 2$,*

$$\sum_{\chi\in O_3(Q)} N(\sigma,T,\chi)$$
$$\underset{\varepsilon}{\ll} (QT)^{\varepsilon}\min(Q^{(22-16\sigma)/9}T^{(8-8\sigma)/3},Q^{4-4\sigma}T^{(8-8\sigma)/3},(Q^8T^5)^{(1-\sigma)/(6\sigma-3)}),$$

where $\sigma \in (\frac{1}{2},1)$. Furthermore, this result holds if O_3 is replaced by O_6.

The immediate advantage of the above result, as compared to (1.3) and (1.8), is that there is no restriction on the relation between Q and T. Additionally, Theorem 2.2 is stronger than (1.3) in the Q-aspect when $\sigma > \frac{5}{6}$ and in the T-aspect when $\sigma > \frac{3}{4}$, and stronger than (1.8) in the Q-aspect when $\sigma > \frac{9}{10}$ and in the T-aspect for all $\sigma > \frac{1}{2}$. The

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fact that Theorem 2.2 holds for $\mathcal{O}_6$ as well as $\mathcal{O}_3$ is a direct result of Theorem 1.5 of [1]. For $\mathcal{O}_4$, we derive the following slightly stronger result.

THEOREM 2.3. *For any $Q, T \geqslant 2$,*

$$\sum_{\chi \in \mathcal{O}_4(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min(Q^{(7-5\sigma)/3} T^{(8-8\sigma)/3}, Q^{(11-11\sigma)/3} T^{(8-8\sigma)/3}, (Q^8 T^5)^{(1-\sigma)/(6\sigma-3)}),$$

where $\sigma \in (\frac{1}{2}, 1)$.

This improves the Q-aspect of (1.4) whenever $\sigma > \frac{9}{11}$ and (1.9) whenever $\sigma > \frac{9}{10}$. In the T-aspect, improvements are made in the same regions as with Theorems 2.1 and 2.2. Similarly, we have the following result.

THEOREM 2.4. *For any $Q, T \geqslant 2$ and any integer $r \geqslant 2$,*

$$\sum_{\chi \in \mathcal{O}_r(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{\varepsilon} \min(Q^{(8-6\sigma)/3} T^{(8-8\sigma)/3}, Q^{(14-14\sigma)/3} T^{(8-8\sigma)/3}, (Q^8 T^5)^{(1-\sigma)/(6\sigma-3)}),$$

where $\sigma \in (\frac{1}{2}, 1)$.

In the Q-aspect, this improves on (1.5) when $\sigma > \frac{5}{6}$ and on (1.10) when $\sigma > \frac{9}{10}$.

REMARK 2.5. The reason for Theorem 2.1 containing the minimum of two quantities, as opposed to the minima of three quantities in Theorems 2.2, 2.3 and 2.4, essentially comes down to the fact that the available large sieve inequality for $\mathcal{O}_2$ is optimal, whereas this is not the case for $\mathcal{O}_r$ when $r > 2$. The available large sieve inequalities for $\mathcal{O}_r$ with $r > 2$ are given as minima of four quantities, two of which are used in the proofs of the last three theorems above. Note, however, that the last term in the minima of the above results is derived using the large-moduli approach of Montgomery (see [5, Theorem 3.1.3]), and as a result is independent of the large sieve inequalities (see (4.5) below).

3. Lemmas

In this section we present the prerequisites in terms of an arbitrary family $\mathcal{F}$ of primitive Dirichlet characters. To estimate $\mathfrak{S}_1(Q, T)$, we consider the polynomials $\Delta(Q, T, N)$ such that

$$\mathfrak{S}_1(Q, T) \ll_{\varepsilon} (QN)^{\varepsilon} \Delta(Q, T, N) \sideset{}{'}\sum_{n \leqslant N} |a_n|^2$$

for all $Q, T, N \geqslant 2$ and any sequence $(a_n)_{n \leqslant N}$ of complex numbers. In practice, a bound for $\Delta(Q, T, N)$ can easily be obtained from the corresponding large sieve estimate, as in [5]. The method of Montgomery can then be summarised in the following two results.

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LEMMA 3.1. *Suppose that $X, Y \geqslant 2$ are such that $X \ll Y \ll (QT)^A$ for some absolute constant A. Then*

$$\sum_{\chi\in\mathcal{F}(Q)} N(\sigma,T,\chi) \ll_{\varepsilon} (QT)^{\varepsilon}((\mathfrak{L}_k(Q,T)\Delta(Q,T,X)^k Y^{k(1-2\sigma)})^{1/(k+1)} + \Delta(Q,T,X)X^{1-2\sigma} + \Delta(Q,T,Y)Y^{1-2\sigma})$$

for any $k \geqslant 1$, where the implied constant does not depend on k.

PROOF. We demonstrated the case $k = 1$ in [6]. The remaining cases follow similarly, by using Hölder's inequality to derive the estimate

$$\#\mathcal{R}_2 \ll_{\varepsilon} (QT)^{\varepsilon} Y^{k(1-2\sigma)/(k+1)} \times \Big(\sum_{(\varrho,\chi)\in\mathcal{R}_2} |M_X(\tfrac{1}{2}+it_\varrho,\chi)|^2\Big)^{k/(k+1)} \Big(\sum_{(\varrho,\chi)\in\mathcal{R}_2} |L(\tfrac{1}{2}+it_\varrho,\chi)|^{2k}\Big)^{1/(k+1)},$$

where $\mathcal{R}_2$ is as defined in [6]. □

LEMMA 3.2. *For any $Q, T \geqslant 2$,*

$$\sum_{\chi\in\mathcal{F}(Q)} N(\sigma,T,\chi) \ll_{\varepsilon} (QT)^{\varepsilon}((\mathfrak{L}_k(Q,T)^2 Q^{2k} T^k)^{(1-\sigma)/(2-k+\sigma(2k-2))} + (Q^2T)^{(1-\sigma)/(2\sigma-1)})$$

for any $k \geqslant 1$, where the implied constant does not depend on k.

PROOF. The case $k = 1$ is shown in [6]. The remaining cases follow similarly using the estimate

$$\#\{(\varrho,\chi) \in \mathcal{R}_2 : |L(\tfrac{1}{2}+it_\varrho,\chi)| \geqslant V\} \ll_{\varepsilon} (QT)^{\varepsilon} V^{-2k} \mathfrak{L}_k(Q,T)$$

to derive a bound for $\#\mathcal{R}_2$, where $\mathcal{R}_2$ is defined as in [6]. □

In this paper, we consider the case where k is taken arbitrarily large. The above lemmas are used to derive the following two results, from which our main results follow.

LEMMA 3.3. *Suppose that $\eta, \vartheta \geqslant 0$ are constants such that the bound*

$$L(\tfrac{1}{2}+it,\chi) \ll_{\varepsilon} q^{\eta+\varepsilon}(|t|+1)^{\vartheta+\varepsilon}$$

holds for all $\chi \in \mathcal{F}$, where q is the conductor of χ. Then for any $Q, T \geqslant 2$,

$$\sum_{\chi\in\mathcal{F}(Q)} N(\sigma,T,\chi) \ll_{\varepsilon} (QT)^{\varepsilon}(Q^{2\eta}T^{2\vartheta}\Delta(Q,T,X)Y^{1-2\sigma} + \Delta(Q,T,X)X^{1-2\sigma} + \Delta(Q,T,Y)Y^{1-2\sigma}),$$

where $X, Y \geqslant 2$ are as in Lemma 3.1.

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PROOF. Using the trivial bound $\#\mathcal{F}(Q) \ll Q^2$ and integrating trivially over $t \in [-T, T]$, the hypothesis gives

$$\mathfrak{L}_k(Q,T)^{1/(k+1)} \ll_\varepsilon Q^{(2+2k\eta)/(k+1)+\varepsilon} T^{(1+2k\vartheta)/(k+1)+\varepsilon}$$

for any integer $k \geqslant 1$. Consequently,

$$\begin{aligned}\mathfrak{L}_k(Q,T)^{1/(k+1)} &\Delta(Q,T,X)^{k/(k+1)} Y^{k(1-2\sigma)/(k+1)} \\ &\ll_\varepsilon (QT)^{(A+2)/(k+1)+\varepsilon} Q^{2\eta} T^{2\vartheta} \Delta(Q,T,X) Y^{1-2\sigma},\end{aligned}$$

where A is as in Lemma 3.1. As the implied constant does not depend on k, we may take k to be sufficiently large that $(A+2)/(k+1) \leqslant \varepsilon$. The result then follows by Lemma 3.1. □

LEMMA 3.4. *Let $\eta, \vartheta \geqslant 0$ be as in Lemma 3.3 and suppose that $Q, T \geqslant 2$. Then*

$$\sum_{\chi \in \mathcal{F}(Q)} N(\sigma, T, \chi) \ll_\varepsilon (Q^{2+4\eta} T^{1+4\vartheta})^{(1-\sigma)/(2\sigma-1)+\varepsilon}$$

whenever $\sigma \geqslant \frac{1}{2} + \varepsilon$.

PROOF. The result follows from Lemma 3.2 in much the same manner as Lemma 3.3 from Lemma 3.1. □

4. Proof of the main results

To derive our main results from the above lemmas, we will employ the Weyl bound (2.1). Note that the last term in the minima of the theorems follows by taking $(\eta, \vartheta) = (\frac{1}{6}, \frac{1}{6})$ in Lemma 3.4, and thus it suffices to use Lemma 3.3 to prove the remaining terms.

PROOF OF THEOREM 2.1. As in [6, 7], we can deduce by Corollary 1 of [11] that

$$\Delta(Q,T,N) \ll QT + N.$$

For appropriate $\eta, \vartheta \geqslant 0$, Lemma 3.3 then gives

$$\begin{aligned}\sum_{\chi \in O_2(Q)} N(\sigma, T, \chi) &\ll_\varepsilon (QT)^\varepsilon (Q^{2\eta} T^{2\vartheta} (QT + X) Y^{1-2\sigma} + QTX^{1-2\sigma} + Y^{2-2\sigma}) \\ &\ll_\varepsilon Q^{(2+4\eta)(1-\sigma)+\varepsilon} T^{(2+4\vartheta)(1-\sigma)+\varepsilon} \qquad (4.1)\end{aligned}$$

on taking $X = QT$ and $Y = Q^{1+2\eta} T^{1+2\vartheta}$, from which the assertion follows on taking $(\eta, \vartheta) = (\frac{1}{6}, \frac{1}{6})$. □

PROOF OF THEOREM 2.2. As in [6, 7], we can show using Theorem 1.4 of [1] that

$$\Delta(Q,T,N) \ll \min(Q^{5/3}T + N, Q^{11/9}T + Q^{2/3}N).$$

By Lemma 3.3, we see for appropriate $\eta, \vartheta \geqslant 0$ that

$$\begin{aligned}\sum_{\chi\in O_3(Q)} N(\sigma,T,\chi) &\ll_\varepsilon (QT)^\varepsilon(Q^{2\eta}T^{2\vartheta}\min(Q^{5/3}T+X, Q^{11/9}T+Q^{2/3}X)Y^{1-2\sigma}\\ &\qquad + \min(Q^{5/3}TX^{1-2\sigma}+Y^{2-2\sigma}, Q^{11/9}TX^{1-2\sigma}+Q^{2/3}Y^{2-2\sigma}))\\ &\ll_\varepsilon \min(Q^{(10/3+4\eta)(1-\sigma)+\varepsilon}, Q^{2/3+(10/9+4\eta)(1-\sigma)+\varepsilon})T^{(2+4\vartheta)(1-\sigma)+\varepsilon}, \end{aligned} \tag{4.2}$$

where in the first term of the minimum we have taken

$$X = Q^{5/3}T \quad\text{and}\quad Y = Q^{5/3+2\eta}T^{1+2\vartheta},$$

and in the second we have taken

$$X = Q^{5/9}T \quad\text{and}\quad Y = Q^{5/9+2\eta}T^{1+2\vartheta}.$$

The desired result follows from taking $(\eta,\vartheta) = (\frac{1}{6},\frac{1}{6})$ in (4.2). □

PROOF OF THEOREM 2.3. As in [6, 7], Lemma 2.10 of [9] can be used to show that

$$\Delta(Q,T,N) \ll \min(Q^{3/2}T + N, Q^{7/6}T + Q^{2/3}N).$$

Then, by Lemma 3.3, for appropriate $\eta, \vartheta \geqslant 0$,

$$\begin{aligned}\sum_{\chi\in O_4(Q)} N(\sigma,T,\chi) &\ll_\varepsilon (QT)^\varepsilon(Q^{2\eta}T^{2\vartheta}\min(Q^{3/2}T+X, Q^{7/6}T+Q^{2/3}X)Y^{1-2\sigma}\\ &\qquad + \min(Q^{3/2}TX^{1-2\sigma}+Y^{2-2\sigma}, Q^{7/6}TX^{1-2\sigma}+Q^{2/3}Y^{2-2\sigma}))\\ &\ll_\varepsilon \min(Q^{(3+4\eta)(1-\sigma)+\varepsilon}, Q^{2/3+(1+4\eta)(1-\sigma)+\varepsilon})T^{(2+4\vartheta)(1-\sigma)+\varepsilon}, \end{aligned} \tag{4.3}$$

where in the first term of the minimum we have taken

$$X = Q^{3/2}T \quad\text{and}\quad Y = Q^{3/2+2\eta}T^{1+2\vartheta},$$

and in the second we have taken

$$X = Q^{1/2}T \quad\text{and}\quad Y = Q^{1/2+2\eta}T^{1+2\vartheta}.$$

The assertion then follows from taking $(\eta,\vartheta) = (\frac{1}{6},\frac{1}{6})$ in (4.3). □

PROOF OF THEOREM 2.4. It follows from Theorem 1.6 of [2] that

$$\Delta(Q,T,N) \ll \min(Q^{2}T + N, Q^{4/3}T + Q^{2/3}N).$$

Then, by Lemma 3.3, for appropriate $\eta, \vartheta \geqslant 0$,

$$\begin{aligned}\sum_{\chi\in O_4(Q)} N(\sigma,T,\chi) &\ll_\varepsilon (QT)^\varepsilon(Q^{2\eta}T^{2\vartheta}\min(Q^{2}T+X, Q^{4/3}T+Q^{2/3}X)Y^{1-2\sigma}\\ &\qquad + \min(Q^{2}TX^{1-2\sigma}+Y^{2-2\sigma}, Q^{4/3}TX^{1-2\sigma}+Q^{2/3}Y^{2-2\sigma}))\\ &\ll_\varepsilon \min(Q^{(4+4\eta)(1-\sigma)+\varepsilon}, Q^{2/3+(4/3+4\eta)(1-\sigma)+\varepsilon})T^{(2+4\vartheta)(1-\sigma)+\varepsilon}, \end{aligned} \tag{4.4}$$

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where in the first term of the minimum we have taken

$$X = Q^2 T \quad \text{and} \quad Y = Q^{2+2\eta} T^{1+2\vartheta},$$

and in the second we have taken

$$X = Q^{2/3} T \quad \text{and} \quad Y = Q^{2/3+2\eta} T^{1+2\vartheta}.$$

The proof is complete on taking $(\eta, \vartheta) = (\frac{1}{6}, \frac{1}{6})$ in (4.4). □

It is clear from (4.1) how the density conjecture for real characters (2.3) follows from the Lindelöf hypothesis. However, an analogous result of the same strength cannot be established for $\mathcal{O}_3$, $\mathcal{O}_4$ or $\mathcal{O}_r$ using (4.2), (4.3) or (4.4), respectively. Additionally, it is clear that the bound derived from Lemma 3.4 has no dependence on the character family $\mathcal{F}$. Indeed, using the above method, we can show that

$$\sum_{q \leqslant Q} \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (QT)^{\varepsilon} \min(Q^{(14-14\sigma)/3} T^{(8-8\sigma)/3}, Q^{(8-8\sigma)/(6\sigma-3)} T^{(5-5\sigma)/(6\sigma-3)}), \tag{4.5}$$

which improves on Theorem 12.2 of [14], and confirms the density conjecture for $\sigma > \frac{11}{12}$. Heath-Brown [10], however, was able to show that the density conjecture holds in the larger range $\sigma > \frac{11}{14}$.

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