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ZEROS OF LINEAR COMBINATIONS OF POLYNOMIALS

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The following theorem is due to J. L. Walsh (see [2, Theorem 17, 2a]):

THEOREM. *If all the zeros of* $f_1(z)=z^n+a_1z^{n-1}+\cdots+a_n$ *lie in or on the circle* C_1 *with centre* c_1 *and radius* r_1 *and if all the zeros of* $f_2(z)=z^n+b_1z^{n-1}+\cdots+b_n$ *lie in or on the circle* C_2 *with centre* c_2 *and radius* r_2, *then each zero of the polynomial*

$$h(z) = f_1(z)-\lambda f_2(z), \quad \lambda \neq 1,$$

lies in at least one of the circles Γ_k *with centre* γ_k *and radius* ρ_k, *where*

$$\gamma_k = (c_1-\omega_k c_2)/(1-\omega_k), \quad \rho_k = (r_1+|\omega_k|\, r_2)/|1-\omega_k|$$

and where the ω_k $(k=1, 2, \ldots, n)$ *are the nth roots of* λ.

As a very special case of this theorem we have:

COROLLARY. *For* $j=1, 2$, *let*

$$f_j(z) = z^n+a_{1,j}z^{n-1}+a_{2,j}z^{n-2}+\cdots+a_{n,j}$$

be a polynomial of degree n having all its zeros in $|z|\leq 1$. *If* $|\arg \lambda_j|\leq\beta<\pi/2, j=1, 2$, *then the linear combination* $\lambda_1 f_1(z)+\lambda_2 f_2(z)$ *has all its zeros in*

$$|z| \leq \frac{|\lambda_1|^{1/n}+|\lambda_2|^{1/n}}{(|\lambda_1|^{2/n}+|\lambda_2|^{2/n}-2|\lambda_1\lambda_2|^{1/n}\cos[(\pi-2\beta)/n])^{1/2}}.$$

Hence for every choice of numbers λ_1, λ_2 *such that* $|\arg\lambda_j|\leq\beta<\pi/2$, $j=1, 2$ *the polynomial* $\lambda_1 f_1(z)+\lambda_2 f_2(z)$ *has all its zeros in*

(1) $$|z| \leq \operatorname{cosec}\frac{\pi-2\beta}{2n}.$$

The number $\operatorname{cosec}(\pi-2\beta)/2n$ in (1) cannot be replaced by anything smaller. In fact, if

$$f_1(z) = \left\{z+i\exp\left(i\frac{\pi-2\beta}{2n}\right)\right\}^n, \qquad f_2(z) = \left\{z-i\exp\left(-i\frac{\pi-2\beta}{2n}\right)\right\}^n,$$

and $\lambda_2=\lambda_1 e^{-2i\beta}$ then $\lambda_1 f_1(z)+\lambda_2 f_2(z)$ vanishes for $z=\operatorname{cosec}[(\pi-2\beta)/2n]$.
We prove:

THEOREM 1. *For* $j=1, 2, \ldots, m$, *let*

$$f_j(z) = z^n+a_{1,j}z^{n-1}+a_{2,j}z^{n-2}+\cdots+a_{n,j}$$

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be a polynomial of degree n having all its zeros in $|z| \le 1$. *If* $|\arg \lambda_j| \le \beta < \pi/2$, $j=1, 2, \ldots, m$, *then the linear combination*

$$\lambda_1 f_1(z) + \lambda_2 f_2(z) + \cdots + \lambda_m f_m(z)$$

has all its zeros in $|z| \le \operatorname{cosec}[(\pi - 2\beta)/2n]$.

We also prove:

THEOREM 2. *If the polynomials*

$$f_1(z) = z^n + a_{1,1} z^{n-1} + a_{2,1} z^{n-2} + \cdots + a_{n,1},$$

$$f_2(z) = z^n + a_{1,2} z^{n-1} + a_{2,2} z^{n-2} + \cdots + a_{k,2} z^{n-k},$$

have all their zeros in $|z| \le 1$, *and* $|\arg \lambda_j| \le \beta < \pi/2$, $j=1, 2$, *then the linear combination* $\lambda_1 f_1(z) + \lambda_2 f_2(z)$ *has all its zeros in*

$$|z| \le \operatorname{cosec} \frac{\pi - 2\beta}{n+k}. \tag{2}$$

The number $\operatorname{cosec}[(\pi - 2\beta)/(n+k)]$ in (2) cannot be replaced by anything smaller. In fact, if

$$f_1(z) = \left\{z + i \exp\left(i \frac{\pi - 2\beta}{n+k}\right)\right\}^n, \qquad f_2(z) = z^{n-k}\left\{z - i \exp\left(-i \frac{\pi - 2\beta}{n+k}\right)\right\}^k,$$

and

$$\lambda_2 = \lambda_1 \left(\cos \frac{\pi - 2\beta}{n+k}\right)^{n-k} e^{-2i\beta}$$

then $\lambda_1 f_1(z) + \lambda_2 f_2(z)$ vanishes for $z = \operatorname{cosec}[(\pi - 2\beta)/(n+k)]$.

Proof of Theorem 1. For $j=1, 2, \ldots, m$, all the zeros of the polynomial

$$g_j(z) = z^n f_j(1/z) = 1 + a_{1,j} z + a_{2,j} z^2 + \cdots + a_{n,j} z^n$$

lie in the circular region $|z| \ge 1$. According to a result of Dieudonné [1, p. 7] there exists a function $\phi_j(z)$ holomorphic and of modulus at most 1 in $|z| < 1$ such that

$$g_j(z) = \{1 - z\phi_j(z)\}^n. \tag{3}$$

For any given z in the disk $|z| < \sin[(\pi - 2\beta)/2n]$, the point $1 - z\phi_j(z)$ lies in the disk $|z-1| < \sin[(\pi - 2\beta)/2n]$ and hence in the sector

$$-\frac{\pi - 2\beta}{2n} < \theta < \frac{\pi - 2\beta}{2n}.$$

It follows that each of the functions $g_j(z)$ maps the disk $|z| < \sin[(\pi - 2\beta)/2n]$ into the sector $-(\pi - 2\beta)/2 < \theta < (\pi - 2\beta)/2$. Thus, if $|\arg \lambda_j| \le \beta$, then

$$\operatorname{Re} \lambda_j g_j(z) > 0, \quad j = 1, 2, \ldots, m$$

for $|z| < \sin [(\pi-2\beta)/2n]$. Hence

$$\mathrm{Re} \sum_{j=1}^{m} \lambda_j g_j(z) > 0$$

if $|z| < \sin [(\pi-2\beta)/2n]$ and $|\arg \lambda_j| \le \beta, j=1, 2, \ldots, m$. This proves that

$$G(z) = \lambda_1 g_1(z)+\lambda_2 g_2(z)+\cdots+\lambda_m g_m(z)$$

does not vanish in $|z| < \sin [(\pi-2\beta)/2n]$ and so

$$f(z) = z^n G(1/z) = \lambda_1 f_1(z)+\lambda_2 f_2(z)+\cdots+\lambda_m f_m(z)$$

has all its zeros in $|z| \le \operatorname{cosec} [(\pi-2\beta)/2n]$.

Proof of Theorem 2. Consider $g_1(z)=z^n f_1(1/z)$ which is a polynomial of degree n and $g_2(z)=z^n f_2(1/z)$ which is a polynomial of degree k. The zeros of $g_1(z)$, $g_2(z)$ lie in $|z| \ge 1$. Hence according to Dieudonné's result mentioned above

$$g_1(z) = \{1-z\phi_1(z)\}^n, \qquad g_2(z) = \{1-z\phi_2(z)\}^k,$$

where the functions $\phi_1(z)$, $\phi_2(z)$ are holomorphic and of modulus at most 1 in $|z|<1$. For any given z in $|z| < \sin [(\pi-2\beta)/(n+k)]$ the point $1-z\phi_1(z)$ lies in the disk $|z-1| < \sin [(\pi-2\beta)/(n+k)]$ and hence in the sector

$$-\frac{\pi-2\beta}{n+k} < \theta < \frac{\pi-2\beta}{n+k}.$$

Consequently, $\{1-z\phi_1(z)\}^n$ is a point of the sector

$$-\frac{n(\pi-2\beta)}{n+k} < \theta < \frac{n(\pi-2\beta)}{n+k}.$$

For the same reason $\{1-z\phi_2(z)\}^k$ is a point of the sector

$$-\frac{k(\pi-2\beta)}{n+k} < \theta < \frac{k(\pi-2\beta)}{n+k}.$$

It follows that if $|\arg \lambda_j| \le \beta, j=1, 2$, then for a given z in $|z| < \sin [(\pi-2\beta)/(n+k)]$, the two points $\lambda_1 g_1(z)$, $\lambda_2 g_2(z)$ simultaneously belong to at least one of the half planes

$$-\frac{n(\pi-2\beta)}{n+k}-\beta < \theta < \frac{k(\pi-2\beta)}{n+k}+\beta, \qquad -\frac{k(\pi-2\beta)}{n+k}-\beta < \theta < \frac{n(\pi-2\beta)}{n+k}+\beta.$$

Hence if $|\arg \lambda_j| \le \beta$, $j=1, 2$, then $\lambda_1 g_1(z)+\lambda_2 g_2(z)$ does not vanish in $|z| < \sin [(\pi-2\beta)/(n+k)]$, i.e. $\lambda_1 f_1(z)+\lambda_2 f_2(z)$ has all its zeros in

$$|z| \le \operatorname{cosec} [(\pi-2\beta)/(n+k)].$$

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References

1. J. Dieudonné, *Sur quelques propriétés des polynômes*, Actualités Sci. Indust. No. 114, Hermann, Paris, 1934.

2. M. Marden, *Geometry of polynomials*, Math. Surveys No. 3, Amer. Math. Soc., Providence, R.I., 1966.

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