Canad. Math. Bull. Vol. **29** (4), 1986

A CATEGORICAL CHARACTERIZATION OF THE FOUR COLOUR THEOREM

BY

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ABSTRACT. The surjectivity of epimorphisms in the category of planar graphs and edge-preserving maps follows from and is implied by the Four Colour Theorem. The argument that establishes the equivalence is not combinatorially complex.

1. **Introduction**. Category theory tells us: to learn about A, study pairs of maps out of A. The Four Colour Theorem of graph theory is studied here by considering pairs of planar homomorphisms whose common domain is a minimal 5-chromatic planar graph. It is proved that epimorphisms are surjective in the category of planar graphs and homomorphisms. This proposition is equivalent to the Four Colour Theorem in the sense that the two results follow from one another using arguments that are not combinatorially complex. Assaults on the Four Colour Theorem, including the computer-assisted proof in [1], make use of a minimal 5-chromatic planar graph, therefore new properties of this presumably non-existent graph may be of interest. It is proved below that such a graph is rigid, i.e. admits no nontrivial homomorphisms, or has a non-trivial automorphism with a fixed point.

The graphs are undirected graphs without loops or multiple edges. A *graph* X is a pair $(V(X), E(X))$ of sets; $V(X)$ the vertex set and $E(X)$ the edge set. The elements of $E(X)$ are certain unordered pairs $[x, x']$ of elements of $V(X)$. A *homomorphism* ϕ: $X \to Y$ is a function from $V(X)$ to $V(Y)$ which maps edges to edges. Thus ϕ induces a map $\phi^{\#}: E(X) \to E(Y)$ by the rule $\phi^{\#}[x, x'] = [\phi x, \phi x']$; ϕ is a *full* homomorphism if the graph $\phi X = (\phi V(X), \phi^{\#} E(X))$ is an induced subgraph of Y. The category of graphs and homomorphisms is denoted by $\mathcal{G}$; $\mathcal{P}$ denotes the full subcategory of planar graphs. We require the notion of a strict monomorphism.

DEFINITION. A monomorphism $\phi: X \to Y$ in a category $\mathcal{H}$ is *strict* if and only if for every morphism $\psi: Z \to Y$ which has the property that $f\psi = g\psi$ for all morphisms f, g: $Y \to Y'$ such that $f\phi = g\phi$ (i.e. ψ equalizes any pair of morphisms that ϕ equalizes), there exists a morphism $h: Z \to X$ such that $\phi h = \psi$. (These are the *regular* monomorphisms of [3].)

Received by the editors October 9, 1984, and, in revised form, May 23, 1985.

AMS Subject Classification (1980): 05C15

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https://doi.org/10.4153/CMB-1986-067-8 Published online by Cambridge University Press

All concepts relating to the theory of categories, unless stated otherwise, will be used as in [6].

2. **The Category** $\mathcal{G}$. In [2] it is shown that the monomorphisms of $\mathcal{G}$ are the injective homomorphisms and that the epimorphisms are the surjective homomorphisms. The first assertion depends on the existence of a one-point graph. Thus, in the full subcategory of graphs without isolated points, monomorphisms need not be injective (see [2]). The second assertion is proved in the following manner: assuming that $\phi: X \to Y$ is not surjective and that $y \notin \phi X$, one constructs Y' identical to Y except that y is replaced by two vertices y_1, y_2 which have the same neighbours as y. Using these one readily constructs two homomorphisms $f_1, f_2: Y \to Y'$ satisfying $f_1\phi = f_2\phi$ and $f_1 \neq f_2$, which demonstrates that ϕ is not epi. What is noteworthy for our purposes is that this pushout construction does not carry over into $\mathcal{P}$; the graph Y' constructed from a planar graph Y is not necessarily planar. This difficulty may be circumvented by employing K_5^- in place of Y'. (K_n^- is a complete graph K_n with one edge deleted). However, the proof depends on the Four Colour Theorem.

Another way of demonstrating the surjectivity of epimorphisms in $\mathcal{G}$ is to observe that the functor from the category of sets to $\mathcal{G}$ which associates with any set X the complete graph structure on X is right adjoint to the forgetful functor. Epis are surjective in the category of sets and adjointness preserves epimorphisms. (The same argument works in many other categories which admit "indiscrete" constructions of this sort: e.g. the indiscrete topology, the All relation in posets, etc.).

LEMMA 1. In $\mathcal{G}$, the following are equivalent:
(1) ϕ is a strict monomorphism,
(2) ϕ is an extremal monomorphism,
(3) ϕ is a full monomorphism.

PROOF. (1) $\Rightarrow$ (2) holds in any category, and is part of the folklore.

(2) $\Rightarrow$ (3) Let $\phi: X \to Y$ be extremal; take Z to be the subgraph of Y induced by ϕX and consider the factorization given below.

$$\begin{tikzcd} X \arrow[rr, "\phi \text{ extremal}"] \arrow[dr, "\bar{\phi}\text{ epi}"'] & & Y \\ & Z \arrow[ur, "\subseteq"'] & \end{tikzcd}$$

($\bar{\phi}$ differs from ϕ only by virtue of its codomain.) Since $\bar{\phi}$ is surjective on the vertex set, it is epi and therefore iso. Since $Z \simeq \phi X$, ϕ is full.

(3) $\Rightarrow$ (1) Take $\phi: X \to Y$ a full monomorphism and $\psi: Z \to Y$ as in the definition of strict. Provided that $\psi Z \subseteq \phi X$ one may define h by $h = \phi^{-1}\psi$. (Note that ϕ full mono $\Rightarrow \phi^{-1}$ preserves edges.) If there exists $y \in \psi Z - \phi X$, Y' may be constructed as before; that is two copies y_1, y_2 of y with the same adjacencies as y replace y in Y'. The morphisms $f_1, f_2: Y \to Y'$ have the properties $f_1\phi = f_2\phi$ and $f_1\psi \neq f_2\psi$. This violates the hypothesis on ψ.

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3. **The Category $\mathcal{P}$**

THEOREM 1. Epimorphisms are surjective in $\mathcal{P}$. (The proof uses the Four Colour Theorem.)

PROOF. Suppose that $\alpha: P \to Q$ is not surjective and that $q \notin \alpha Q$. (It may be assumed that $|Q| \geq 2$.) In K_5^- let us take the vertex set to be $\{1, 2, 3, 4, 5\}$ with $[4, 5]$ the deleted edge. Since Q is colourable in 4 or fewer colours there is a (planar) homomorphism f: $Q \to K_5^-$ such that $V(fQ) \subseteq \{1, 2, 3, 4\}$ and $fq = 4$. Construct g: $Q \to K_5^-$ by putting $g = f$ on $Q - \{q\}$ and $g(q) = 5$. Then $g\alpha = f\alpha$ and $f \neq g$. Therefore α is not epi.

THEOREM 2. In $\mathcal{P}$ the following are equivalent. (The proof uses the Four Colour Theorem.)

(1) ϕ is a strict monomorphism

(2) ϕ is an extremal monomorphism

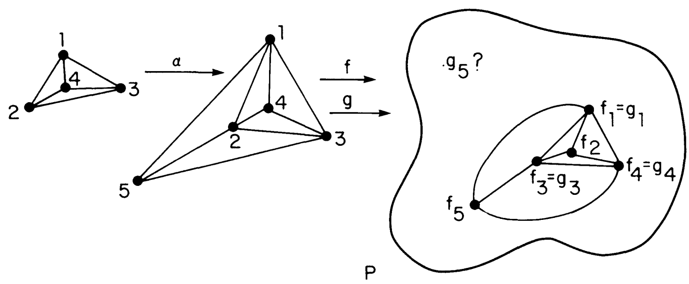
(3) ϕ is a full monomorphism.

PROOF. (3) $\Rightarrow$ (1). (The remaining implications are proved as in Lemma 1). It is necessary to verify that $\psi Z \subseteq \phi X$. If $y \in \psi Z$ and $y \notin \phi X$ construct two morphisms f, g: $Y \to K_5^-$ as in the proof of Theorem 2 with y in the role of q. Then $f\phi = g\phi$ and $f\psi \neq g\psi$. This violates the hypothesis on ψ.

REMARK. Theorem 1 follows from Theorem 2 in a fairly general sense: In every category with pushouts and an epi-mono factorization system $(E, \mathcal{M})$, the statement "$\mathcal{M}$ = strict monos" implies the statement "E = epimorphisms". (See [3] and [5]). So, in the category $\mathcal{P}$, take E = surjective homomorphisms and $\mathcal{M}$ = full embeddings.

EXAMPLE. A category in which planar epimorphisms are not surjective.

Let $\mathcal{M}$ be the category of Planar Graphs and injective homomorphisms. Consider the inclusion α of K_4 into K_5^-. If f, g: $K_5^- \to P$ are any morphisms in $\mathcal{M}$ satisfying $f\alpha = g\alpha$ then $f = g$. This follows from the following observations: fK_5^- and gK_5^- are copies in P of the maximal planar graph K_5^-. If $f_5 \neq g_5$ the two sets of vertices $\{f_4, f_5, g_5\}$ and $\{f_1, f_2, f_3\}$ determine a copy of $K_{3,3}$ inside P. Since P is planar $f_5 = g_5$ and α is epi.



THEOREM 3. The Four Colour Theorem follows from the assumption that epimorphisms in $\mathcal{P}$ are surjective.

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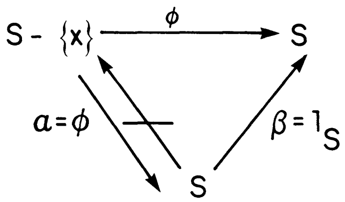
PROOF. Let S be a critically 5-chromatic graph of minimum cardinality; all of the properties assumed of S used in this section can be found in [4].

Let $x \in S$. The inclusion map $\phi: S - \{x\} \to S$ is a non surjective epimorphism. For suppose that $f, g: S \to P$ are such that $f\phi = g\phi$ and $f = g$ on $S - \{x\}$, but $fx \neq gx$. Now any morphism out of S is injective (if f is not injective fS is of smaller cardinality and hence 4-colourable. A 4-colouring $\gamma: fS \to K_4$ induces a 4-colouring of S by composition.). The vertex gx falls inside a triangular face of the maximal planar (triangulated) graph fS, in any of the equivalent embeddings. Now gx can be adjacent to at most 3 vertices in fS, and therefore also adjacent to at most 3 vertices in gS. This is a contradiction, as all vertices in S have degree ≥ 5. Thus $fx = gx$ and ϕ is epi.

THEOREM 4. The Four Colour Theorem follows from the assumption that the extremal monomorphisms in $\mathcal{P}$ are the full injective morphisms.

PROOF. The mapping ϕ of Theorem 3 is a full injective morphism, but not extremal, as the diagram illustrates.



4. **Automorphisms of S**. Let f, g be a pair of planar morphisms out of S. If the vertex sets $V(fS)$ and $V(gS)$ do not coincide, the argument of Theorem 3 shows that all vertices of fS lie in the same face of gS, since S has no separating triangle. Thus fS and gS have at most a triangle in common. The other possibility is that $V(fS) = V(gS)$. In this case, if $f \neq g$ then f and g determine a non-trivial automorphism of S, namely fg^{-1}. Since S has a non-trivial full subgraph inclusion in S of which is epi, it has one which is of minimum cardinality. Call this graph S'. In the following lemma uniqueness of S' is not claimed.

LEMMA 2. The graph S' has these properties.
(1) S' is not a triangle, edge, or point
(2) S' has at least 2 non adjacent vertices
(3) S' does not contain a copy of K_4^-.

PROOF. (1) Inclusion of a triangle $K_3 \subseteq S$ is not epi. To see this construct a planar graph P from 2 copies of S, one in the outer face of the triangle, one in the inner, having the triangle in common, and so obtain a pair of unequal morphisms equalized by the inclusion. (2) S contains no copy of K_4. S' is not K_n for $n = 1, 2, 3$ by (1). (3) Suppose $K_4^- \subseteq S'$. By dropping from S' a vertex x which has degree 3 in K_4^- a graph $S - \{x\}$ of smaller cardinality is obtained, inclusion of which cannot be epi. Therefore there exists a pair of planar morphisms f, g out of S with $f = g$ on $S' - \{x\}$ but $fx \neq gx$.

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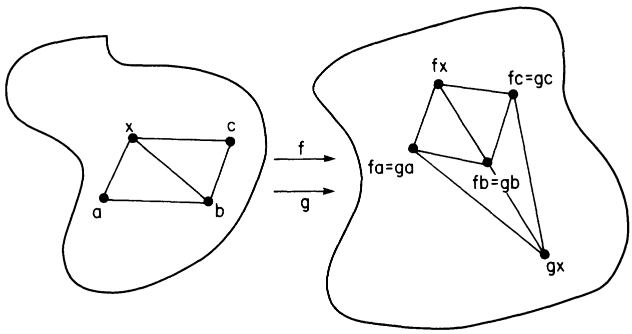

If the morphism pair f, g is of the type which has $V(fS) = V(gS)$ then S has a separating 4-cycle, which is impossible. (See [3] p. 192 *ff.*) In this case it is also impossible that gx falls into a triangular face of fS. That triangle would have vertex set $\{fa, fb, fc\}$. The existence of the edge $[fa, fc]$ implies that S contains a copy of K_4, contradicting the lemma.

THEOREM 5. *S is rigid or S has a nontrivial automorphism with a fixed point.*

PROOF. Let x, y be non adjacent points of S' as guaranteed by the lemma. Inclusion ϕ of $S' - \{x\}$ in S is not epi, therefore there exists a pair of planar homomorphisms f, g out of s such that $f\phi = g\phi$ and $f \neq g$. It must be that $fx \neq gx$ otherwise f and g agree on S' and inclusion of S' is epi. It may be that S is rigid and the only pairs of maps f, g out of S with $f \neq g$ are those which agree on a K_3 subgraph. In this case fS and gS are two copies of S each lying in a triangular face of the other and having at most that triangle in common. Assume now that S is not rigid; without loss of generality we may suppose that the pair of homomorphisms f, g is not of the type where f and g agree on a K_3. Thus gx does not fall into a face of fS, else all of gS falls into the same face. This would imply that $gy = fy$ is separated from fS by a triangle, which is impossible. (Recall that gS is a copy of S and therefore has no separating triangle.) It follows that the vertex sets $V(fS)$ and $V(gS)$ coincide, and that $h = fg^{-1}$ is a nontrivial automorphism of S with fixed point y.

REMARK. The author conjectures that the alternative of non-rigidity can be eliminated. An automorphism fixed on y permutes the first neighbourhood circuit of y cyclically or anti-cyclically. It appears that, at least in the cyclic case, one can use such an automorphism to 4-colour S.

ACKNOWLEDGEMENTS. The author is indebted to the referee for many useful suggestions, in particular for the remark following Theorem 2 and the associated references.

https://doi.org/10.4153/CMB-1986-067-8 Published online by Cambridge University Press

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