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ATOMIC SPACES AND SPECTRA

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1. The subject-matter of this paper is in some sense known; but we will try to organise, explain and reprove it, and to give examples.

In essence, a space or spectrum X is "atomic" if a map $f: X \to X$ may be proved to be an equivalence by a simple, computable test applied in one dimension; this goes back to [4] (published as [5]) and first appeared in print in [12]. That it is useful to prove X atomic and then apply the fact has been amply shown, beginning with [3].

This notion is related to two others. Unique factorisation results for spaces and spectra have been considered in [6, 9, 14]. Here one needs the notion of an "irreducible" or "indecomposable" object X, and a slightly stronger notion of "prime".

We first show that the case of "spaces" and the case of "spectra" can be considered together, by concentrating on the fact that the hom-set $[X, X]$ is (under suitable assumptions) a profinite monoid. In this case we show that the "weaker" condition implies the "stronger", as follows.

(a) If X is indecomposable then its hom-set $[X, X]$ is "good", and

(b) if $[X, X]$ is "good" then X is both "atomic" and "prime".

We give some illustrative examples, including some which arise "in nature" as stable summands of classifying spaces BG. We conclude with the proofs.

Related results have been obtained by M. C. Crabb and J. R. Hubbuck; we are grateful to them for letters, and also to F. R. Cohen and F. P. Peterson.

2. First we unify the two cases to be considered.

Proposition 2.1. *The hom-set $[X, X]$ is a profinite monoid with zero in both the following cases.*

(2.2) *X is a p-complete CW-complex of finite type and $[X, X]$ means homotopy classes of pointed maps.*

(2.3) *X is a p-complete spectrum of finite type and $[X, X]$ means maps in the homotopy category of spectra.*

We will comment in Section 3.

*The Society is saddened by the sudden death on 7 January 1989 of Professor J. F. Adams, F.R.S.

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Proposition 2.4. *Suppose M is a profinite monoid with zero. Then either*

(*a*) *M contains a non-trivial idempotent, or*

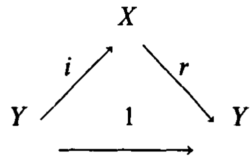
(*b*) *M is "good" in that the sense that each $f \in M$ is either invertible or topologically nilpotent.*

In (a), an idempotent is "non-trivial" if it is neither 0 nor 1.

In (b), "f is topologically nilpotent" means that as $n \to \infty$, so $f^n \to 0$ in the profinite topology on M.

The proof of (2.4), which is elementary, will be given in Section 3.

When $[X, X]$ contains a non-trivial idempotent, X is "reducible" or "decomposable". For spaces this means that X has a non-trivial retract; that is, there is a diagram



in which Y is not contractible and i, r are not equivalences. For spectra we can go on to infer a non-trivial decomposition as a wedge-sum, $X \simeq Y \vee Z$.

If X is "irreducible" or "indecomposable" then the possibility of a non-trivial idempotent is excluded, and we conclude that $[X, X]$ is "good".

Thus (2.1) plus (2.4) is an analogue, for homotopy-theorists, of a well-known algebraic result: under suitable finiteness conditions, if an R-module X is indecomposable, then every map $f: X \to X$ is either invertible or nilpotent. We were surprised to find that such a result survives in a context with no addition.

We will sketch the argument that if $[X, X]$ is good, then X is atomic. In the applications, a simple computable test will dismiss the possibility that f is topologically nilpotent. For example, suppose we choose any dimension n where $H_n(X; F_p) \neq 0$. If f is an equivalence, then $f_*: H_n(X; F_p) \to H_n(X; F_p)$ must be iso. But conversely, if $f_*: H_n(X; F_p) \to H_n(X; F_p)$ is iso, then it cannot be nilpotent, so f cannot be topologically nilpotent, and f must be an equivalence (assuming $[X, X]$ is good). We conclude that X is atomic.

Of course, many functors other than $H_n(-; F_p)$ would serve as well.

We will sketch the argument that if $[X, X]$ is good, then X is "prime".

If f and g are both topologically nilpotent, then the equation

$$f + g = 1 \neq 0$$

cannot hold even after passing to a finite quotient M_α of M and embedding M_α in a ring R (where we can add). In fact, in the finite quotient M_α we would have $f^m = 0$, $g^n = 0$; in R, f would commute with $1 - f = g$; so we would have $(f + g)^{m+n-1} = 0$.

We now assume that "X divides YZ". For spectra this means that we assume given a retraction

$$X \to Y \vee Z \to X.$$

We take f, g to be the composites

$$X \to Y \to X$$

$$X \to Z \to X.$$

We then have $f+g=1$ in $[X,X]$; assuming $[X,X]$ is good, we deduce that either f or g is an equivalence, and X is a retract either of Y or of Z. That is, "if X divides YZ, then X divides either Y or Z".

For spaces, $Y \vee Z$ should become $Y \times Z$. We cannot argue in quite the same way because we cannot add in $[X,X]$; but we can obtain the equation $f+g=1$ after passing to a suitable ring $R = \mathrm{End}\, h(X)$, where h is a suitable functor $\pi_r(-)\otimes F_p$ or $QH^r(-;F_p)$ with $h(X)\neq 0$.

We turn to the examples.

Example 2.5 There is a p-local spectrum X of finite type which is indecomposable (but becomes decomposable on completion) and for which $[X,X]$ is not good.

This justifies the assumption of p-completeness above. The construction will be given in Section 4.

In the case of spectra, $[X,X]$ is a profinite ring R. When R is good it is local: the topologically-nilpotent elements make up the unique maximal ideal $\mathrm{rad}(R)$. (Given the indications above, the proof may be left to the reader; the result is due to [9].)

Remark 2.6. In this case the quotient $R/\mathrm{rad}(R)$ is a finite field.

The proof is easy, but this too is postponed to Section 4.

This raises the question, which finite fields occur as $R/\mathrm{rad}(R)$. Here we present two examples; one involves infinite spectra which "arise in nature", and the other involves finite spectra constructed by hand.

Example 2.7. Each finite field arises as $R/\mathrm{rad}(R)$ for a suitable X which is an indecomposable stable summand of a classifying space BG. Indeed, if the field is of characteristic p, then G can be a p-group.

For simplicity we now take $p=2$.

Example 2.8. Each finite field of characteristic 2 arises as $R/\mathrm{rad}(R)$ for a suitable X which is a finite spectrum

The constructions will be given in Section 4.

3. We begin by commenting on (2.1).

The cases of spaces, (2.2), is presumably known; but we sketch a proof avoiding certain difficulties.

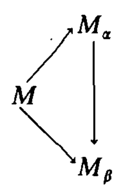
First we set up some finite quotients of the monoid $N=[X,X]$. Let X_α be a space whose homotopy groups $\pi_r(X_\alpha)$ are finite p-groups, and zero except for a finite number of r. Then $[X, X_\alpha]$ is a finite set and $M=[X,X]$ acts on it from the right; let M_α be the image of M in the monoid $\mathrm{End}([X,X_\alpha])$. Then M_α is a finite monoid; and the map $M\to M_\alpha$ is continuous because the map $m\mapsto i_\alpha m$: $[X,X]\to[X,X_\alpha]$ is continuous for each of the finitely many i_α in $[X,X_\alpha]$.

Secondly we note that the map $M\to\prod_\alpha M_\alpha$ is mono. In fact, according to Sullivan [13] we can arrange an isomorphism

$$[X,X]\xrightarrow{\{m\mapsto i_\alpha m\}}\lim_{\substack{\leftarrow \\ i_\alpha}}[X,X_\alpha].$$

Thus $i_\alpha m'=i_\alpha m''$ for all i_α implies $m'=m''$.

Thirdly we order the M_α by considering diagrams



(without requiring any relation between the spaces X^α, X_β). We show that the map.

$$M\to\lim_{\leftarrow} M_\alpha$$

is an epi by a standard compactness argument.

This completes the sketch proof that $[X,X]$ is a profinite monoid.

The case of spectra, (2.3), is due to [9, Proposition 4, p. 155}.

We turn to the proof of (2.4).

Suppose given $f\in M$. Let M_α be a finite quotient of M. We show first that some power f^n of f (with $n\geqq 1$) becomes idempotent in M_α.

In fact, if infinitely many powers f^n lie in the same finite set M_α, then two of them must be equal, say $f^a=f^{a+b}$ for some $a\geqq 1, b\geqq 1$. Applying f and iterating we get $f^c=f^{c+bd}$ for $c\geqq a$. Taking $bd\geqq a$ we get $f^{bd}=f^{2bd}$ is idempotent in M_α.

Next let F be $\{f^n | n\geqq 1\}$, the set of powers of f, and let $\bar{F}$ be the closure of F in M. We show that $\bar{F}$ contains an idempotent (possibly 0 or 1). In fact, for each finite quotient M_α of M, let E_α be the set of elements in $\bar{F}$ which map to idempotents in M_α. Then E_α is closed and non-empty (for by the last paragraph it contains some power of f). Indeed, the sets E_α have the finite intersection property, for any finite intersection $E_\alpha\cap E_\beta\cap\ldots\cap E_\delta$ contains another E_ε (consider the pull-back of $M_\alpha, M_\beta, \ldots, M_\delta$). M is compact, so there is an element common to all the E_α, i.e. an idempotent in $\bar{F}$.

It is possible that the idempotent in $\bar{F}$ is 1; in this case we will show that f is invertible. In fact, assume $1\in\bar{F}$; then in each finite quotient M_α of M we have $1=f^n$ for

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some $n \geqq 1$, so f has an inverse $1 = f^{n-1}$ in M_α. This inverse is unique, and these inverse elements give an element of $\lim_{\leftarrow} M_\alpha$, providing an inverse for f in M.

It is possible that the idempotent in $\bar{F}$ is 0; in this case we will show that f is topologically nilpotent. In fact, assume $0 \in \bar{F}$, and let M_α be a finite quotient of M. Then some f^n maps to 0 in M_α; hence f^r maps to 0 in M_α for $r \geqq n = n(\alpha)$. Thus $f^r \to 0$ in the profinite topology.

This completes the proof of (2.4).

4. We begin with (2.5).

Our spectrum X will have $H_0(X) = Z_{(p)} \oplus Z_{(p)}$, so that $\mathrm{End}(H_0(X))$ is a ring of 2×2 matrices. We will construct X so that the image of

$$[X, X] \to \mathrm{End}(H_0(X))$$

is a ring

$$\frac{Z_{(p)}[A]}{(A^2 - A + p)}.$$

The proposed minimum polynomial $x^2 - x + p$ has no real roots (because $b^2 - 4ac < 0$), and *a fortiori* no roots in $Z_{(p)}$. It does have p-adic roots α_0, α_1 congruent to $0, 1 \bmod p$ (e.g. by Hensel's Lemma). A convenient matrix with this minimum polynomial is

$$A = \begin{bmatrix} 0 & 1 \\ -p & 1 \end{bmatrix};$$

this has $(1, \alpha_0)^T$, $(1, \alpha_1)^T$ as eigenvectors with eigenvalues α_0, α_1.

Next we need a sequence of elements $g_i \in \pi^S_{n_i - 1}(S^0)$, $i = 1, 2, \ldots$, such that g_i has order p^i in $\pi^S_*(S^0)$ and still has order p^i in $\pi^S_*(S^0)^*/I_i$, where I_i is the ideal generated by the g_j with $j \neq i$. These conditions can easily be satisfied by suitable elements in the image of the J-homomorphism.

We now take

$$X = (S^0 \vee S^0) \cup \left(\bigcup_i e^{n_i} \right),$$

where we work localised at p but omit the notation for it, and where the attaching map for e^{n_i} has components $(g_i, \alpha_i g_i)$. (Here α_i means α_0 or α_1 according as i is even or odd.) This has the following effect. A self-map f of $S^0 \vee S^0$, given by a matrix B, extends over e^{n_i}, with degree d_i on e^{n_i}, if and only if $(1, \alpha_i)^T$ is an eigenvector for $B \bmod p^i$, with eigenvalue $d_i \bmod p^i$; it extends over X if and only if this condition holds for all i, that is, if and only if $(1, \alpha_0)^T$ and $(1, \alpha_1)^T$ are p-adic eigenvectors for B. By construction A satisfies this condition, so A comes from a map $X \to X$. Conversely, suppose B satisfies it; then B is a p-adic linear combination of I and A; here the coefficients of I, A are the entries B_{11}, B_{12} in B, so they lie in $Z_{(p)}$. This proves that the image of

$$[X,X] \to \mathrm{End}\,(H_0(X))$$

is

$$\frac{Z_{(p)}[A]}{(A^2-A+p)}.$$

We show that X is indecomposable (over $Z_{(p)}$). In fact,

$$\frac{Z_{(p)}[A]}{(A^2-A+p)}$$

is in integral domain; so for any idempotent $e \in [X,X]$, either e or $1-e$ maps to 0 in $\mathrm{End}\,H_0(X)$, and the other maps to 1; the one which maps to 1 maps each cell e^{n_i} with degree congruent to $1 \bmod p^i$, and must be an equivalence.

We show that $[X,X]$ is not good. In fact, the map which realises A is neither an equivalence nor topologically nilpotent, for on $H_0(X;F_p)$ it induces an idempotent of rank 1.

Proof of (2.6). Consider the quotient map q from R to a finite quotient ring $R_\alpha \neq 0$. Under q invertible elements map to invertible elements, and topologically nilpotent elements map to nilpotent ones; thus $q^{-1}(\mathrm{rad}\,(R_\alpha)) = \mathrm{rad}\,(R)$ and

$$R/\mathrm{rad}\,(R) \cong R_\alpha/\mathrm{rad}\,(R_\alpha).$$

So $R/\mathrm{rad}\,R$ is finite; being a finite division algebra, it must be a finite field.

We turn to (2.7). Here we need some hold on the ring of stable maps $\{BG_+, BG_+\}$.

Lemma 4.1. *Let G be a finite p-group. Then the group ring $F_p[\mathrm{Out}\,(G)]$ is a quotient of the ring $\{BG_+, BG_+\}$.*

The obvious map is in the direction

$$Z[\mathrm{Out}\,(G)] \to \{BG_+, BG_+\};$$

but we definitely need a quotient of $\{BG_+, BG_+\}$.

Sketch proof. The ring $\{BG_+, BG_+\}$ is known [**11**, p. 397, Corollary 2.3; **10**, p. 128, Corollary 15] as a consequence of the Segal conjecture. In particular, we have

$$F_p \otimes \{BG_+, BG_+\} \cong F_p \otimes A(G,G)$$

where $A(G,G)$ is a ring that plays the same role here that the Burnside ring $A(G)$ does in studying $\{BG_+, S^0\}$ [**1, 10**]. (In [**11**] the present $A(G,G)$ is written $F(G,G)$.) As a Z-module, $A(G,G)$ is free, with a base of elements which may be written $\theta_* i^*$. Here i runs over the inclusions of subgroups $i: H \to G$, and i^* corresponds to the transfer map $Tr: BG_+ \to BH_+$; θ runs over homomorphisms $\theta: H \to G$, and θ_* corresponds to the induced map $B\theta_+: BH_+ \to BG_+$ [**1**, Section 9; **7**, p. 433]. If θ is epi, then we must have $H = G$ and $i = 1$, and θ must be iso. Let I be the Z-submodule of $A(G,G)$ generated by the remaining elements $\theta_* i^*$, in which θ is not epi. We claim I is an ideal.

Consider a product $\theta_* i^* \phi_* j^*$, and assume first that θ is not epi. The product $i^* \phi_* j^*$ can be reduced to a sum of terms $\sum_\alpha (\psi_\alpha)_* k^*$, so we obtain a sum of terms $(\theta\psi_\alpha)_* k^*$ in which $\theta\psi_\alpha$ is not epi.

Assume secondly that ϕ is not epi. By the last paragraph it is sufficient to consider the case in which θ is iso and $i = 1$; but then we get $(\theta\phi)_* j^*$, in which $\theta\phi$ is not epi. Thus I is an ideal.

We now see that the quotient ring $A(G,G)/I$ is $Z[\mathrm{Out}(G)]$. (Two automorphisms θ of G give the same basis element in $A(G,G)$ if and only if they differ by conjugation in G). Thus

$$A(G,G)/(I + (p)) \cong F_p[\mathrm{Out}(G)],$$

and this proves the lemma.

Lemma 4.2. *The finite field F_q, where $q = p^n$, may be obtained as a quotient of the ring $\{BG_+, BG_+\}$ for a suitable p-group G.*

Proof. By a theorem of Bryant and Kovacs [**2, 8**, p. 403, Theorem 13.5] there is a p-group G whose abelianisation $G/[G,G]$ is the additive group $(Z/p)^n$ of F_q and whose automorphism group $\mathrm{Aut}\, G$ acts on $G/[G,G]$ as the multiplicative group $F_q^\times$ of F_q. Clearly this action factors through $\mathrm{Out}(G)$, so we get epimorphisms

$$F_p[\mathrm{Out}(G)] \to F_p[F_q^\times] \to F_q.$$

Using (4.1), we get a map of rings from $\{BG_+, BG_+\}$ onto F_q.

(2.7) now follows. If we take a complete decomposition of 1 into orthogonal idempotents in $\{BG_+, BG_+\}$, then just one of the idempotents maps to 1 in F_q; if X is the corresponding summand of BG_+, then $\{X, X\}$ maps onto F_q and $R/\mathrm{rad}(R) \cong F_q$, as in the proof of (2.6).

We turn to (2.8). In order to realise the finite field F_{2^q}, we begin with $W = \bigvee_1^q S^0$. (We work completed, but for simplicity we omit the notation for it.) We next form

$$X = W \cup_f CS^8 W$$

where the attaching map f has to be described. For any $w \in \pi_0(W)$, f is to carry $S^8 w$ to

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$$w\cdot\bar{v}+\phi(w)\cdot\varepsilon;$$

here $\bar{v},\varepsilon$ are the two generators for $\pi_8^S(S^0)=Z_2\oplus Z_2$, and ϕ has to be described. Since the result depends only on $\phi(w)\bmod 2$, we may interpret ϕ as an endomorphism of

$$V=\pi_0(W)\otimes F_2=H_0(W;F_2)=H_0(X;F_2).$$

We take $\phi\colon V\to V$ to be some linear map whose minimum polynomial is an irreducible polynomial P of degree q over F_2.

An endomorphism of $H_*(X;F_2)$ is now given by a linear map $\lambda\colon V\to V$ in degree 0 and a linear map $\mu\colon V\to V$ in degree 9. Such a pair (λ,μ) is induced by a map $g\colon X\to X$ if and only if it commutes with the boundary map, that is

$$\lambda w\cdot\bar{v}+\lambda\phi w\cdot\varepsilon=\mu w\cdot\bar{v}+\phi\mu w\cdot\varepsilon.$$

Equivalently, $\lambda=\mu$ and $\lambda\phi=\phi\lambda$, that is, λ commutes with ϕ.

Multiplication by ϕ gives V the structure of a module over $F_2[\phi]/P\cong F_{2^q}$; this structure is of course a 1-dimensional vector space over F_{2^q}. The possible maps λ are the endomorphisms of this structure, i.e. multiplication by the elements of F_{2^q}. This shows that the image of

$$R=[X,X]\to\mathrm{End}(V)$$

is F_{2^q}.

It is now clear that X is indecomposable; and $R/\mathrm{rad}(R)\cong F_{2^q}$, as in the proof of (2.6).

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