On Eddington's higher order equations of the gravitational field.

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Einstein's fundamental equations of the gravitational field are

$$G^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}G + \lambda g^{\mu\nu} = -\kappa T^{\mu\nu}, \ (\mu, \nu = 1, \ldots, 4) \tag{1}$$

where $T^{\mu\nu}$ are the components of the energy tensor and λ is the cosmical constant. In empty space these equations become

$$G^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}G + \lambda g^{\mu\nu} = 0, \tag{2}$$

which may be reduced to

$$G^{\mu\nu} = \lambda g^{\mu\nu} \tag{3}$$

since $G = 4\lambda$, by contraction of (2).

Eddington[1] has shown (§ 60, p. 138) that when the cosmical term in (2) is neglected these equations may be derived by Hamiltonian differentiation with respect to $g_{\mu\nu}$ of G, *viz.*

$$\frac{\mathfrak{d}G}{\mathfrak{d}g^{\mu\nu}} = -(G^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}G) = 0 \tag{4}$$

Distinguishing Eddington's equations and paragraphs by the letter E we have, by (E 35.3)

$$\delta\sqrt{-g} = \tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}, \tag{5}$$

so that (2) may be obtained by variation of the integral

$$\int (G - 2\lambda)\sqrt{-g}\, d\tau, \tag{6}$$

i.e. by replacing G in (4) by

$$K = G - 2\lambda. \tag{6.1}$$

The spherically symmetrical solution of (3) may be written in the form (E 45.3)

$$ds^2 = -\gamma^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 + \gamma dt^2 \quad \text{with } \gamma = 1 - 2m/r - \tfrac{1}{3}\lambda r^2 \tag{7}$$

where m is a constant of integration. When $\lambda = 0$ (7) reduces to the Schwarzschild solution for which

$$\gamma = 1 - 2m/r. \tag{7.1}$$

Now Eddington has suggested (E § 62, p. 141) that instead of (4) the equations

$$\frac{\mathfrak{d}\bar{K}}{\mathfrak{d}g_{\mu\nu}}=0 \tag{8}$$

might have been used, $\bar{K}$ being some fundamental invariant other than G. In particular he considers

$$K' = G_{\mu\nu}G^{\mu\nu} = g^{\alpha\beta}g^{\sigma\rho}g^{\mu\epsilon}g^{\nu\pi}B_{\mu\nu\sigma\rho}B_{\epsilon\pi\alpha\beta}, \tag{9.1}$$

$$K'' = B_{\mu\nu\sigma\rho}B^{\mu\nu\sigma\rho} = g^{\mu\alpha}g^{\nu\beta}g^{\sigma\epsilon}g^{\rho\pi}B_{\mu\nu\sigma\rho}B_{\alpha\beta\epsilon\pi}, \tag{9.2}$$

as being the simplest alternative invariants; but on these grounds we should also consider the square of the scalar curvature

$$K''' = G^2 = g^{\mu\nu}g^{\sigma\rho}g^{\alpha\beta}g^{\epsilon\pi}B_{\mu\nu\sigma\rho}B_{\alpha\beta\epsilon\pi}. \tag{9.3}$$

Eddington shows that the Schwarzschild solution (7.1) is also a solution of the alternative equations

$$\frac{\mathfrak{d}K'}{\mathfrak{d}g_{\mu\nu}}=0, \tag{10.1}$$

$$\frac{\mathfrak{d}K''}{\mathfrak{d}g_{\mu\nu}}=0; \tag{10.2}$$

and it is easily seen also to be a solution of

$$\frac{\mathfrak{d}K'''}{\mathfrak{d}g_{\mu\nu}}=0. \tag{10.3}$$

The author now considers these and other alternative field laws and shows, amongst other results, that in 4-space *every* solution of (3) is also a solution of (10.1) and (10.3), and that those solutions of (3) which represent spaces of constant Riemannian curvature are also solutions of (10.2). In particular it is verified that (7) (with λ replaced by a constant of integration a) is not only a solution of (10.1) and (10.3) but also of (10.2), although the space is not of constant Riemannian curvature unless $m = 0$.

(a) Consider a small variation of K'.

$$\delta K' = \delta(G_{\mu\nu}G^{\mu\nu}) = G_{\mu\nu}\delta G^{\mu\nu} + G^{\mu\nu}\delta G_{\mu\nu}.$$

If

$$G_{\mu\nu} = a g_{\mu\nu}, \tag{11}$$

where a is a constant, then

$$\begin{aligned}\delta K' &= a(g_{\mu\nu}\delta G^{\mu\nu} + g^{\mu\nu}\delta G_{\mu\nu})\\ &= a(\delta G - G^{\mu\nu}\delta g_{\mu\nu} + \delta G - G_{\mu\nu}\delta g^{\mu\nu})\\ &= 2a\delta G - a^2(g^{\mu\nu}\delta g_{\mu\nu} + g_{\mu\nu}\delta g^{\mu\nu})\\ &= 2a\delta G.\end{aligned}$$

Hence $\delta\mathfrak{K}' = \delta(\sqrt{-g}K') = \sqrt{-g}\delta K' + K'\delta\sqrt{-g}$

$$= 2a\sqrt{-g}\,\delta G + 4a^2\delta\sqrt{-g}$$
$$= 2a(\delta\mathfrak{G} - G\delta\sqrt{-g}) + 4a^2\delta\sqrt{-g}$$
$$= 2a\delta(\mathfrak{G} - 2a\sqrt{-g}),$$

i.e.
$$\delta\mathfrak{K}' = 2a\delta\mathfrak{K}. \tag{11.1}$$

Hence the condition $\int \delta\mathfrak{K}'d\tau = 0$ leads to the condition

$$\int \delta\mathfrak{K}d\tau = 0, \tag{11.2}$$

which is satisfied in virtue of the fact that equation (11) is equivalent to equation (11.2). Consequently
every solution of (11) *is a solution of the alternative equations* (10.1), where a is to be regarded as a constant of integration.

It is of interest to note that when $a \neq 0$ this result does not hold when the dimensional number is other than four. Thus in a space of $n(\neq 4)$ dimensions we have as before $\delta K' = 2a\delta G$;

but now $\delta\mathfrak{K}' = \sqrt{-g}\,\delta K' + K'\delta\sqrt{-g}$

$$= 2a\sqrt{-g}\,\delta G + na^2\delta\sqrt{-g}$$
$$= 2a(\delta\mathfrak{G} - G\delta\sqrt{-g}) + na^2\delta\sqrt{-g}$$
$$= 2a\delta\mathfrak{K} - (n-4)a^2\,\delta\sqrt{-g},$$

where the K in (6.1) is now replaced by $K = G - (n-2)a$.

Hence, if $\int \delta\mathfrak{K}\,d\tau = 0$ it follows that

$$\int \delta\mathfrak{K}'d\tau = (n-4)\,a^2\int \delta\sqrt{-g}\,d\tau, \tag{11.3}$$

which will in general fail to vanish.

As a particular case of the general result above we shall verify that (7), with
$$\gamma = 1 - 2m/r - ar^2/3, \tag{12}$$
(both m and a being constants of integration) is a solution of (10.1). To conform with Eddington's notation we write

$$e^{-\lambda} = e^{\nu} = \gamma = 1 - 2m/r - ar^2/3 \tag{12.1}$$

where now, of course, a does not denote the cosmical constant.

Using the form of the $G_{\mu\nu}$ given by (E 38.61-5) we find

$$\mathfrak{K}' = e^{-\frac{3}{2}\lambda + \frac{1}{2}\nu}\{2r^2\Omega^2 + 2r(\nu' - \lambda')\,\Omega + \tfrac{1}{2}(3\nu'^2 - 2\nu'\lambda' + 3\lambda'^2)$$
$$+ 2(e^{\lambda} - 1)(\lambda' - \nu')/r + 2(e^{\lambda} - 1)^2/r^2\}. \tag{13}$$

Here $\Omega = \frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 \,(= g^{44}B_{1144})$, a dash denotes differentiation with respect to r, and we have set $\theta = \pi/2$ since it is not necessary to consider variations from the symmetrical condition.

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Using (12.1) after the various partial differentiations have been performed, we find, following some straightforward but tedious algebraic work, that

$$\left.\begin{aligned}
\frac{\partial K'}{\partial \lambda} &= -\tfrac{3}{2}K' + 2e^{-\frac{1}{2}\lambda+\frac{1}{2}\nu}\left\{(\lambda'-\nu')/r + 2(e^{\lambda}-1)/r^2\right\} = 4a - 6a^2r^2,\\
\frac{\partial K'}{\partial \lambda'} &= e^{-\frac{3}{2}\lambda+\frac{1}{2}\nu}\Big\{-\Omega(r^2\nu' + 2r) + \tfrac{1}{2}r\nu'(\lambda'-\nu') + (3\lambda'-\nu')\\
&\qquad + 2(e^{\lambda}-1)/r\Big\} = 4ar - 6am - 2a^2r^3,\\
\frac{\partial K'}{\partial \lambda''} &= 0,\\
\frac{\partial K'}{\partial \nu} &= \tfrac{1}{2}K' = 2a^2r^2,\\
\frac{\partial K'}{\partial \nu'} &= e^{-\frac{3}{2}\lambda+\frac{1}{2}\nu}\Big\{\Omega\left[r^2(2\nu'-\lambda') + 2r\right] + r(\nu'-\lambda')(\nu'-\tfrac{1}{2}\lambda')\\
&\qquad + (3\nu'-\lambda') - 2(e^{\lambda}-1)/r\Big\} = -4ar + 2am + \tfrac{10}{3}a^2r^3,\\
\frac{\partial K'}{\partial \nu''} &= e^{-\frac{3}{2}\lambda+\frac{1}{2}\nu}\left\{2r^2\Omega + r(\nu'-\lambda')\right\} = -2ar^2 + 4amr + \tfrac{2}{3}a^2r^4.
\end{aligned}\right\} \tag{13.1}$$

If these values are substituted in the expressions for the Lagrange derivatives

$$\left.\begin{aligned}
[K']_{\lambda} &= \frac{\partial K'}{\partial \lambda} - \frac{d}{dr}\frac{\partial K'}{\partial \lambda'} + \frac{d^2}{dr^2}\frac{\partial K'}{\partial \lambda''},\\
[K']_{\nu} &= \frac{\partial K'}{\partial \nu} - \frac{d}{dr}\frac{\partial K'}{\partial \nu'} + \frac{d^2}{dr^2}\frac{\partial K'}{\partial \nu''},
\end{aligned}\right\} \tag{13.2}$$

they will be found to vanish identically.

(b) Unfortunately it does not appear possible to deal with K'' in the manner above by using equation (11). But a somewhat more specialised result may be obtained by considering spaces of constant Riemannian curvature[2]. We therefore replace (11) by the equation

$$B_{\mu\nu\sigma\rho} = \tfrac{1}{3}a\,(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho}). \tag{14}$$

Contracting for σ and ρ, we see that (11) is satisfied. Then

$$\begin{aligned}
\delta K'' &= B_{\mu\nu\sigma\rho}\delta B^{\mu\nu\sigma\rho} + B^{\mu\nu\sigma\rho}\delta B_{\mu\nu\sigma\rho}\\
&= \frac{a}{3}\left\{(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})\,\delta B^{\mu\nu\sigma\rho} + (g^{\mu\nu}g^{\sigma\rho} - g^{\mu\sigma}g^{\nu\rho})\,\delta B_{\mu\nu\sigma\rho}\right\}\\
&= \frac{a}{3}\left\{4\delta G - B^{\mu\nu\sigma\rho}\delta(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho}) - B_{\mu\nu\sigma\rho}\delta(g^{\mu\nu}g^{\sigma\rho} - g^{\mu\sigma}g^{\nu\rho})\right\}
\end{aligned}$$

since $(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})B^{\mu\nu\sigma\rho} = g_{\mu\nu}g_{\sigma\rho}B^{\mu\nu\sigma\rho} + g_{\mu\sigma}g_{\nu\rho}B^{\mu\sigma\nu\rho} = 2G$,

$B^{\mu\nu\sigma\rho}$ being anti-symmetrical in ν and σ. As it is also anti-symmetrical in μ and ρ we get

$$\delta K'' = \frac{a}{3}(4\delta G - 4G^{\mu\nu}\delta g_{\mu\nu} - 4G_{\mu\nu}\delta G^{\mu\nu})$$

$$= \tfrac{4}{3}a\delta G - \tfrac{4}{3}a^2(g^{\mu\nu}\delta g_{\mu\nu} + g_{\mu\nu}\delta g^{\mu\nu}) = \tfrac{4}{3}a\delta G.$$

Hence $\delta \mathfrak{K}'' = \sqrt{-g}\delta K'' + K''\delta\sqrt{-g} = \tfrac{4}{3}a(\sqrt{-g}\delta G + 2a\delta\sqrt{-g})$

$$= \tfrac{4}{3}a(\delta\mathfrak{G} - 2a\delta\sqrt{-g});$$

Hence
$$\delta\mathfrak{K}'' = \tfrac{4}{3}a\delta\mathfrak{K}. \tag{15}$$

Therefore, as in the case of (11.1) we conclude that *any solution of* (11) *defining the line-element of a space of constant Riemannian curvature is also a solution of the alternative equations* (10.2).

Now (7) reduces to the de Sitter line-element (E 70.1) when $m = 0$, and the latter defines a space of constant Riemannian curvature. Hence we know now that (10.2) is satisfied by the two particular solutions

$$\gamma = 1 - 2m/r \qquad \text{and} \qquad \gamma = 1 - \tfrac{1}{3}ar^2.$$

It is natural to enquire therefore whether (12) as it stands may not perhaps also be a solution of (10.2). By direct substitution in the equations following (E 62.6) this may be verified to be actually the case. Thus using (12.1) again we find

$$\left.\begin{aligned}
\frac{\partial\mathfrak{K}''}{\partial\lambda} &= \frac{16m}{r^3} + \frac{8a}{3} - \frac{72m^2}{r^4} - 4a^2r^2,\\
\frac{\partial\mathfrak{K}''}{\partial\lambda'} &= -\frac{8m}{r^2} + \frac{8ar}{3} + \frac{24m^2}{r^3} - 4am - \frac{4a^2r^3}{3},\\
\frac{\partial\mathfrak{K}''}{\partial\lambda''} &= 0,\\
\frac{\partial\mathfrak{K}''}{\partial\nu} &= \frac{24m^2}{r^4} + \frac{4a^2r^2}{3},\\
\frac{\partial\mathfrak{K}''}{\partial\nu'} &= \frac{8m}{r^2} - \frac{8ar}{3} - \frac{40m^2}{r^3} + \frac{20am}{3} + \frac{20a^2r^3}{9},\\
\frac{\partial\mathfrak{K}''}{\partial\nu''} &= -\frac{8m}{r} - \frac{4ar^2}{3} + \frac{16m^2}{r^2} + \frac{16amr}{3} + \frac{4a^2r^4}{9}.
\end{aligned}\right\} \tag{16}$$

With these values we find that the Lagrange derivatives $[\mathfrak{K}'']_\lambda$ and $[\mathfrak{K}'']_\nu$ vanish identically. Hence

(12) *is also a solution of the alternative equations* (10.2).

Comparing (13.1) and (16) we notice that when $m = 0$

$$\frac{[\mathfrak{K}'']_\lambda}{[\mathfrak{K}']_\lambda} = \frac{[\mathfrak{K}'']_\nu}{[\mathfrak{K}']_\nu} = \tfrac{2}{3},$$

as is required by (11.1) and (15), *viz.*

$$\frac{\delta \mathfrak{K}''}{\delta \mathfrak{K}'} = \tfrac{2}{3} \tag{16.1}$$

As in the case of K' the result established above (for $a \neq 0$) again requires the space to be four-dimensional. The equation analogous to (11.3) is

$$\int \delta \mathfrak{K}'' d\tau = \frac{2a^2(n-4)}{n-1} \int \delta \sqrt{-g}\, d\tau. \tag{17}$$

(c) It is scarcely necessary to consider the case (10.3) in detail. It will be sufficient to quote the result that, assuming $G = na$,

$$\int \delta \mathfrak{K}''' d\tau = na^2(n-4) \int \delta \sqrt{-g}\, d\tau. \tag{17.1}$$

Hence *when* $n = 4$ *the results stated for* K' *apply here also.*

It may be remarked that $\mathfrak{K}'''$ is identical with Weyl's action density (E § 90. p. 209) in the absence of electromagnetic fields.

(d) The three alternative fundamental invariants which have been considered so far have in common the property of being quadratic in the second derivatives of the $g_{\mu\nu}$. It is of interest to examine whether the results established would also follow for more complex invariants. Consider for example the m^{th} power of the scalar curvature in 4-dimensional space. Let

$$K^{(m)} = G^m. \tag{18}$$

Then

$$\begin{aligned} \delta K^{(m)} &= mG^{m-1}\delta G = m(4a)^{m-1}\delta G, \\ \delta \mathfrak{K}^{(m)} &= \sqrt{-g}\,\delta K^{(m)} + K^{(m)}\delta\sqrt{-g} \\ &= m(4a)^{m-1}(\delta \mathfrak{G} - G\delta\sqrt{-g}) + (4a)^m \delta\sqrt{-g} \\ &= m(4a)^{m-1}\delta\left(\mathfrak{G} - \frac{4(m-1)}{m} a\sqrt{-g}\right). \end{aligned}$$

Hence $\int \delta \mathfrak{K}^{(m)} d\tau = 0$ is consistent with (11.2) only if $m = 2$, *i.e.* in the case dealt with above. In particular we conclude that (12), ($a \neq 0$), is not a solution of $[K^{(m)}]_\lambda = [K^{(m)}]_\nu = 0$, except in that case. The author has considered various other invariants involving 3rd, 4th,... powers of the second derivatives of the $g_{\mu\nu}$, and these yielded similar results.

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