RESIDUE RINGS OF SEMI-PRIMARY HEREDITARY RINGS*

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Introduction: Throughout this paper we assume that all rings contain an identity. We say that R is a semi-primary ring if its (Jacobson) radical N is nilpotent, and R/N is an Artinian ring. We say that R admits a splitting, and we write $R=A+B$ if A is a subring of R, if B is a two-sided ideal in R, and if $A\cap B=0$.

It has been shown in [1] that for a semi-primary ring R $l.gl.\dim R = r.gl.\dim R = 1+l.\text{proj. dim } N$. This common value is denoted by $gl.\dim R$.

It has been shown in [2] that if R is a semi-primary hereditary ring, and I is a two-sided ideal in R, then $gl.\dim R/I<\infty$.

We prove that if R is a semi-primary ring and $gl.\dim R/N^2<\infty$, then R is a residue ring of a semi-primary hereditary ring. This is a generalization of a similar result in [3]. The crucial step is a splitting theorem that we prove for a semi-primary ring R, for which $eNe=0$ for any primitive idempotent $e\in R$. This splitting theorem seems also useful in studying certain types of semi-primary subrings of a simple ring.

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§1. A Splitting Theorem.

For the rest of this section, let $R=\sum_{u=1}^{t} Re_u$ be a complete decomposition for the semi-primary ring R, i.e. $e_1 \ldots\ldots e_t$ are primitive orthogonal idempotents (e.g. [4, pp 53–57]). Furthermore, assume $e_vNe_v=0$ for $v=1, \ldots\ldots, t$. When writing $e_i, e_j \ldots\ldots$ we always assume $1\le i, j, \ldots\ldots \le t$, unless otherwise stated.

Since for any e_i, e_iNe_i is the radical of e_iRe_i, and e_iRe_i/e_iNe_i is a division ring, we have:

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LEMMA 1: *e_iRe_i is a division ring for $i=1, \ldots\ldots, t$. Every element $e_ire_j \in R$ induces a homomorphism (by right multiplication) of Re_i into Re_j, and vice versa. In particular, if Re_i is isomorphic to Re_j, then $e_kNe_i \neq 0$ iff $e_kNe_j \neq 0$ for any k, $1 \leq k < t$.*
Thus:

LEMMA 2: *Let Re_i be isomorphic to Re_j, then $e_iNe_j=0$.*
One easily verifies that this is equivalent to:

LEMMA 2*: *Every non-zero homomorphism between isomorphic components is an isomorphism.*

Let Γ_0 be $\sum_{i,j} e_iRe_j$ where (i, j) ranges over all pairs such that Re_i and Re_j are isomorphic to Re_{i_0} for some fixed i_0. Let $R_0 = Hom_R(\sum_k Re_k, \sum_k Re_k)$, where k ranges over all indices such that Re_k is isomorphic to Re_{i_0}. Let $R_1 = Hom_R(\sum_{i=1}^{s} B_i, \sum_{i=1}^{s} B_i)$, where $B_i = Re_{i_0}$ for $i=1, \ldots\ldots, s$, and s is the number of components in the complete decomposition for R which are isomorphic to Re_{i_0}. Finally, let $\Gamma_1 = (e_{i_0}Re_{i_0})_s$-the $s \times s$ matrix algebra over the division ring $e_{i_0}Re_{i_0}$. With these notations we have:

LEMMA 3: *The subring Γ_0 of R is a simple ring.*

Proof: It is clear that R_1 and R_0 are isomorphic. It is also clear that $\Gamma_0(\Gamma_1)$ is anti-isomorphic to $R_0(R_1)$. Thus Γ_0 and Γ_1 are isomorphic.

Let Γ be $\sum_{i,j} e_iRe_j$, where (i, j) ranges over all pairs such that Re_i is isomorphic to Re_j. Since on Γ we have a natural splitting, into subsums taken over any fixed isomorphism class of components, it follows from Lemma 3 that:

PROPOSITION 1: The subring Γ of R is a semi-simple ring.

The underlying additive group of R admits a decomposition $R = \sum_{i,j=1}^{t} e_iRe_j$. Let $R_1 = \sum_{i,j} e_iRe_j$ where (i, j) ranges over all pairs such that Re_i is not isomorphic to Re_j. We have $R = \Gamma + R_1$, and it is clear that $R_1 \subset N$. Our next step is to show that $R_1 = N$. We will be done once we show that R_1 is a two-sided ideal in R. Since $r = \sum_{i,j=1}^{t} e_ire_j$ for any $r \in R$, and since R_1 is closed under addition, it suffices to show that $e_ire_j \in R_1$, implies $e_ire_jse_k \in R_1$ and $e_lve_ire_j \in R_1$ for all $1 \leq i, j, k, l \leq t$ and $r, s, v \in R$. But $e_ire_jse_k \notin R_1$ only if Re_i is isomorphic to Re_k, whence by Lemma 2* this element induces an isomorphism of Re_i onto Re_k, and this is impossible since $e_ire_j \in R_1 \subset N$. A similar argument shows that $e_lve_ire_j \in R_1$.

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This proves:

THEOREM 1. THE SPLITTING THEOREM: *Let R be a semi-primary ring, and let $R=\sum_{u=1}^{t} Re_u$ be a complete decomposition for R. If $e_iNe_i=0$ for $i=1,\dots,t$, then R admits a splitting $R=\Gamma+N$. $\Gamma=\sum_{i,j} e_iRe_j$ where (i, j) ranges over all pairs such that Re_i is isomorphic to Re_j. $N=\sum_{i,j} e_iRe_j$ where (i, j) ranges over all pairs such that Re_i is not isomorphic to Re_j.*

With the assumptions and notations of Theorem 1, using Lemma 1 one can easily prove that the center of R is a direct product of fields. The center of R is a field only if 0 and 1 are the unique central idempotents in R. One can also show that if $R=\Gamma_1+N$ is another splitting for R, then there exists an invertible element s in R such that the automorphism $r\rightarrow srs^{-1}$ takes Γ onto Γ_1.

The splitting theorem enables us to view N as a $\Gamma-\Gamma$ bimodule. Define $\Omega(\Gamma, N)=\sum_{i=0}^{\infty} N^{(i)}$, where $N^{(0)}=\Gamma$ and $N^{(i)}=N^{(i-1)}\otimes_{\Gamma} N$. Letting $n_1\otimes\dots\dots\otimes n_i\otimes n^1\otimes\dots\dots\otimes n^j=n_1\otimes\dots\dots\otimes n_i\otimes n^1\otimes\dots\dots\otimes n^j$ and extending $\otimes$ distributively, $\Omega(\Gamma, N)$ becomes a ring (identifying $N^{(i)}\otimes_{\Gamma}\Gamma$, $\Gamma\otimes_{\Gamma}N^{(i)}$ and $N^{(i)}$ for $i\geq 0$). Letting $f(n_1\otimes\dots\dots\otimes n_k)=n_1\dots\dots\cdot n_k$, and extending f linearly, f is a ring epimorphism from $\Omega(\Gamma, N)$ onto R. If for some m, $N^{(m)}=0$ then $M=\sum_{i=1}^{m-1} N^{(i)}$ is a nilpotent two-sided ideal and $\Omega(\Gamma, N)/M$ is semi-simple. Thus $\Omega(\Gamma, N)$ is a semi-primary ring with radical M. Furthermore, $M=\Omega(\Gamma, N)\otimes_{\Gamma}N$, and since N is Γ-projective, M is $\Omega(\Gamma, N)$-projective. By [1], this implies that $\Omega(\Gamma, N)$ is an hereditary ring.

If $E_0,\dots\dots,E_k$ are primitive idempotents in R, then $(E_0,\dots\dots,E_k)$ is an R-connected sequence of length k if $E_iNE_{i+1}\neq 0$ for $i=0,\dots\dots,k-1$. It is obvious that $N^{(m)}=0$ if there are no R-connected sequences of length m.

2. Applications. We first deal with the case $gl.\dim R/N^2<\infty$. Thus let R be a semi-primary ring and $gl.\dim R/N^2<\infty$. Let $\tilde{R}=R/N^2$, $\tilde{N}=N/N^2$. With the notations of section 1 we have that $\tilde{R}=\sum_{u=1}^{t}\tilde{R}\tilde{e}_u$ is a complete decomposition for $\tilde{R}$, where $\tilde{e}_i$ is the canonical image of e_i in $\tilde{R}$ for $i=1,\dots\dots,t$. By a result in [3] concerning semi-primary rings for which the square of the radical is zero, we conclude that $\tilde{R}$-connected sequences are bounded in length. This implies:

LEMMA 4: *R-connected sequences are bounded in length.*

Proof: We show that if $e_iNe_j\neq 0$ then there exists an $\tilde{R}$-connected sequence

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of the form $(\tilde{e}_i, \ldots\ldots, \tilde{e}_j)$. If $(\tilde{e}_i, \tilde{e}_j)$ is $\tilde{R}$-connected we are done. Otherwise, $e_iNe_j \in N^2$ and there readily follows the existence of a primitive idempotent e_k such that $e_iNe_kNe_j \neq 0$. If $(\tilde{e}_i, \tilde{e}_k, \tilde{e}_j)$ is $\tilde{R}$-connected we are done. Otherwise, either $e_iNe_k \in N^2$ or $e_kNe_j \in N^2$. Let $e_iNe_k \in N^2$, then we can find a primitive idempotent e_l such that $0 \neq e_iNe_lNe_kNe_j \in N^3$. Since N is nilpotent, this procedure must end and the result follows.

In particular, we must have $e_iNe_i = 0$ for $i = 1, \ldots\ldots, t$, thus by Theorem 1, $R = \Gamma + N$. The ring $\Omega(\Gamma, N)$ as constructed at the end of § 1 is a semi-primary hereditary ring in this case. Combining this with the result in [2] concerning residue rings of semi-primary hereditary rings we have:

THEOREM 2. *Let R be a semi-primary ring, then the following are equivalent:*

(a) *R is a residue ring of a semi-primary hereditary ring.*

(b) *All residue rings of R have finite global dimension.*

(c) $gl \cdot dim\, R/N^2 < \infty$.

Remark that under each of these equivalent conditions $eNe = 0$ for any primitive idempotent $e \in R$.

In particular, if R is a semi-primary hereditary ring, its center is a direct product of fields. The center of R is a field only if 0 and 1 are the unique central idempotents in R.

For the rest, let D be a division ring and let D_n denote the $n \times n$ matrix algebra over D. Let R be a semi-primary subring of D_n, such that $R = \sum_{i=1}^{n} Re_i$ is a complete decomposition for R. Without loss of generality we may assume that e_i is the matrix whose $(\alpha, \beta)^{th}$ component is $(e_i)_{\alpha\beta} = \delta_{i\alpha}\delta_{i\beta}$ for all $i, \alpha, \beta = 1, \ldots \ldots, n$. We can (naturally) identify $e_iD_ne_i$ with D, and e_iRe_i with a subring of D, for $i = 1, \ldots\ldots, n$. In particular $e_iNe_i = 0$ for $i = 1, \ldots\ldots, n$, and by Theorem 1 $R = \Gamma + N$. We want to show now that $\Omega(\Gamma, N)$ is a semi-primary hereditary ring. This follows from the fact that any element $e_ire_j \in R$ induces an isomorphism from D_ne_i onto D_ne_j. Thus in particular $e_ire_j \neq 0$ and $e_kse_i \neq 0$ imply $e_kse_ire_j \neq 0$, or $e_iNe_j \neq 0$ and $e_jNe_k \neq 0$ imply $e_iNe_jNe_k \neq 0$. Since N is nilpotent this implies that R-connected sequences are bounded in length. Thus we proved:

THEOREM 3. *Let R be a semi-primary subring of D_n, containing n orthogonal idempotents, then $gl \cdot dim\, R/I < \infty$ for any two-sided ideal I in R.*

Let R be a semi-primary subring of D_n. Let $C(R)$ be the subset of D_n

consisting of elements $V \in D_n$ for which $Vr = rV$ for all $r \in R$. Set $C(D_n)$ to be the center of D_n. One can show that $C(R) = C(D_n)$ implies that (a) 0 and 1 are the unique central idempotents in R and (b) R contains n orthogonal idempotents. If D is a field one easily verifies that (a) and (b) imply $C(R) = C(D_n)$.

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