

Counting Rational Points on Ruled Varieties

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Abstract. In this paper, we prove a general result computing the number of rational points of bounded height on a projective variety V which is covered by lines. The main technical result used to achieve this is an upper bound on the number of rational points of bounded height on a line. This upper bound is such that it can be easily controlled as the line varies, and hence is used to sum the counting functions of the lines which cover the original variety V .

1 Introduction

In algebraic geometry, the general notion of studying an algebraic variety by studying families of curves which cover it is a very old and fruitful one. However, it has not been much used to study the density of rational points on algebraic varieties, because there has not been the necessary uniformity in the results for lower-dimensional varieties and their counting functions.

Heath-Brown suggests in [HB1] that the technique could be quite widely used, and gives an example of how to compute the counting function of a certain cubic surface by studying families of cubic curves lying on it. He has recently improved these results dramatically in an excellent paper [HB2], in which he derives a multitude of uniform bounds on the number of rational points on algebraic varieties. Broberger generalises these results still further in [Br].

In the spirit of Heath-Brown's idea, we will describe in this paper a method for obtaining upper bounds (and in many cases, asymptotic formulae) for the counting functions for rational points with respect to height on projective varieties which are covered by lines.

In particular, we consider an algebraic variety V defined over a number field K , embedded in projective space \mathbf{P}^n for some n , and consider the usual (multiplicative) height function, not normalised to be independent of the field K . We define the counting function thus:

$$N_L(B) = \text{card}\{P \in V(k) \mid H(P) \leq B\}$$

The counting function counts the number of K -rational points of V whose height is at most B . (That this is well defined is an immediate consequence of a theorem of Northcott—see for example [Vo], Proposition 1.2.9.(g).)

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If V is covered by a set \mathcal{L} of lines, then we might hope to compute the counting function of V by summing the respective counting functions for the lines in \mathcal{L} :

$$N_V(B) = \sum_{L \in \mathcal{L}} N_L(B).$$

There are several excellent estimates for $N_L(B)$ in the literature. The first of these was that of Schanuel [Sch], in which Schanuel calculates very precisely the asymptotics of the counting function for \mathbf{P}^1 over a number field K . The chief drawback of this is that the constant in the leading term depends on the specific embedding of the line into \mathbf{P}^n , so for our purposes we will need a more specific calculation.

This, too, has been done, by Thunder [Th], in which he calculates the asymptotics of the counting function for an arbitrary line in \mathbf{P}^n over an arbitrary number field. Thunder makes clear that the leading term in the counting function for a line L is

$$\frac{c_K}{H(L)} B^2,$$

where $H(L)$ denotes the height of the Plücker point corresponding to L and c_K is a constant depending only on the number field K . However, for our purposes, we will want strict control of how many points of small height lie on lines of large height, so the presence of an unbounded error term of any kind (which is definitely necessary in all theorems of the sort that Schanuel and Thunder sought) is fatal to our line of reasoning.

Thus, we must prove yet another result about heights of rational points on lines, one which allows such a strict control—this result is Theorem 2.1. To get this control, we sacrifice the quality of the constant c_K , and we obtain only an upper bound, rather than a lower one. However, Theorem 2.1 is still good enough to give exactly the right exponent on B in the counting function for many algebraic varieties (see Theorem 3.1), so our sacrifices are certainly outweighed by our gains.

Theorem 3.1 fits into the extensive literature which computes the counting functions of algebraic varieties, which is too large to summarise in a satisfactory fashion here. Suffice it to say that it is compatible with the conjectures of Batyrev and Manin, and that a more comprehensive overview of the history of the subject can be found in [Si].

2 Rational Points on Lines

Let K be an algebraic number field with ring of integers \mathcal{O}_K , and let L be a line in \mathbf{P}_K^n . We wish to compute an upper bound for the counting function

$$N_L(B) = \text{card}\{P \in L \mid H(P) \leq B\},$$

where $H(P)$ denotes the standard height in projective space:

$$H([x_0 : \cdots : x_n]) = \prod_v \max_i \{|x_i|_v\},$$

where v ranges over all (isomorphism classes of) valuations on K . Note that we do not normalise the height to be independent of the field K .

Schanuel [Sch] derived some very precise asymptotics for $N_L(B)$:

$$N_L(B) = cB^2 + E(B)$$

where c is a specific constant and $E(B)$ is an error term which is $o(B^2)$. Schanuel computes both of these quite precisely in the case that $L = \mathbf{P}_K^1$.

However, this estimate will not suffice for our present purpose, since we wish to control the set of points of small height on our lines as well. Furthermore, since our lines will not generally be identical to \mathbf{P}^1 , we wish to explore the dependence of c on the height $H(L)$ of the line L , which we define to be the height of the corresponding Plücker point in the Grassmannian $G(1, n)$. An asymptotic version of this has been derived by Thunder [Th], but like Schanuel’s result, does not control the behaviour of the points of small height.

Thus, say L corresponds to a 2-dimensional subspace of K^{n+1} , spanned by the vectors (a_0, \dots, a_n) and (b_0, \dots, b_n) . We define:

$$H(L) = H((a_0dx_0 + \dots + a_n dx_n) \wedge (b_0dx_0 + \dots + b_n dx_n))$$

where the result of the wedge product is interpreted as a point in $\mathbf{P}_K^{(n^2+n)/2}$ with homogeneous coordinates $\{dx_i \wedge dx_j\}$ for $i \neq j$.

We can now state the main result of this section:

Theorem 2.1 *The counting function for L satisfies the following inequalities:*

$$N_L(B) \leq \frac{c_K}{H(L)} B^2 + 1,$$

where c_K is a positive real constant depending only on the field K .

Proof Our first step will be to identify the K -rational points of L with a set of lattice points in a finite-dimensional Euclidean space. Let $[x_0 : \dots : x_n]$ be a K -rational point on L . By clearing denominators, we can ensure that $x_i \in \mathcal{O}_K$ for all i . In fact, by choosing a fixed set \mathcal{J} of representatives for the class group of K , we can ensure that the coordinates x_i generate an ideal in \mathcal{J} . This representation for $[x_0 : \dots : x_n]$ is unique up to multiplication by a unit of \mathcal{O}_K .

Let M be the rank two \mathcal{O}_K -module M in K^{n+1} consisting of all the vectors in L whose coordinates all lie in \mathcal{O}_K . Let $d = [K : \mathbf{Q}]$, and denote by $\sigma_1, \dots, \sigma_{r_1}$ the embeddings of K into \mathbf{R} , and by $\tau_1, \dots, \tau_{r_2}$ the embeddings of K into \mathbf{C} , where $d = r_1 + 2r_2$.

Using these embeddings, we can embed M as a lattice of rank $2d$ in $V = (\mathbf{R}^{n+1})^{r_1} \oplus (\mathbf{C}^{n+1})^{r_2}$. We will abuse notation by hereafter identifying M with its image in V . Define:

$$|(a_0, \dots, a_n)|_i = \begin{cases} \max_j (|\sigma_i(a_j)|) & \text{if } i \leq r_1 \\ \max_j (|\tau_{i-r_1}(a_j)|^2) & \text{if } i > r_1 \end{cases}$$

Thus, we can view $[x_0 : \dots : x_n]$ as a point in M whose coordinates generate an element of the fixed set \mathcal{J} of ideals. Thus, there exists a positive constant c_1 depending only on the field K such that:

$$(1) \quad \frac{1}{c_1} \prod_i |(x_0, \dots, x_n)_i| \leq H([x_0 : \dots : x_n]) \leq c_1 \prod_i |(x_0, \dots, x_n)_i|.$$

We also define:

$$(2) \quad \|(x_0, \dots, x_n)\| = \max_i \{|(x_0, \dots, x_n)_i|^{d_i}\},$$

where d_i is 1 or 1/2, depending on whether i corresponds to a real or complex embedding, respectively.

If $\epsilon \in \mathcal{O}_K^*$ is a unit, then we have the relation:

$$|\epsilon(x_0, \dots, x_n)_i| = |\epsilon|_i |(x_0, \dots, x_n)_i|,$$

where $|\epsilon|_i$ represents the absolute value of ϵ with respect to the embedding σ_i (if i is at most r_1) or $\tau_{i-r_1}^2$ (if i is greater than r_1). Thus, by the Dirichlet Unit Theorem (see for example [Ne], Theorem 7.3), there is a positive real constant c_2 depending only on K such that for any element $\mathbf{v} = (v_1, \dots, v_{r_1+r_2}) \in \mathbf{R}^{r_1} \oplus \mathbf{C}^{r_2}$, there exists a unit $\epsilon \in \mathcal{O}_K^*$ such that $|\epsilon v_i|^{d_i} \leq c_2 |\epsilon v_j|^{d_j}$ for all i and j , where d_i and d_j are as in (2).

Applying this to $\mathbf{v} = (|(x_0, \dots, x_n)_1|, \dots, |(x_0, \dots, x_n)_{r_1+r_2}|)$ reveals that there is a positive real constant c_3 depending only on K such that through multiplication by a suitable unit, we may assume that for all i and j :

$$|(x_0, \dots, x_n)_i|^{d_i} \leq c_3 |(x_0, \dots, x_n)_j|^{d_j}.$$

Thus, by equations (1) and (2), we may find a positive real constant c_4 depending only on K such that for all K -rational points $P \in L$, we can find a representation $P = [x_0 : \dots : x_n]$ as above such that:

$$\frac{1}{c_4} H(P) \leq \|(x_0, \dots, x_n)\|^d \leq c_K H(P)$$

Thus, when calculating an upper bound for $N_L(B)$, it will suffice to compute an upper bound for the following function:

$$N'_L(B) = \{ \mathbf{v} = (x_0, \dots, x_n) \in M' \mid \|\mathbf{v}\| \leq B \}$$

where M' denotes the set of vectors in M , counted modulo the action of K^* . In particular, we have:

$$N'_L(B) \geq N_L(B)^d$$

We will use the following well known result (it follows, for example, from work of Thunder [Th]):

Lemma 2.2 *There is a positive real constant α depending only on K such that the determinant of M is equal to $\alpha H(L)$.*

We therefore have reduced to showing that there is a positive real constant c , depending only on the field K , such that:

$$N'_L(B) \leq \frac{c}{\det(M)} B^{2d} + 1.$$

Thus, fix a positive real number B . If $N_L(B) \leq 1$, then the result is clear, so assume that $N_L(B) \geq 2$. Then we can find two K -linearly independent lattice points P_1 and P_2 in M with $H(P_i) \leq B$, since K -linearly dependent points in M contribute only one point to $N_L(B)$.

Choose a basis $\{\alpha_1, \dots, \alpha_d\}$ for \mathcal{O}_K over \mathbf{Z} , and consider the set:

$$\{\alpha_1 P_1, \dots, \alpha_d P_1, \alpha_1 P_2, \dots, \alpha_d P_2\}.$$

There is a positive real constant α depending only on K such that the height of $\alpha_j P_i$ is at most $\alpha H(P_i)$.

For any real number H , consider the set

$$V_M(H) = \{ \mathbf{v} \in M \otimes \mathbf{R} \mid \|\mathbf{v}\| \leq H \}.$$

This set is convex and centrally symmetric, so by the previous arguments, it follows that the real simplex spanned by the vectors $\alpha_j P_i$ is contained in $V_M(\alpha B)$. In particular, we conclude that every element of L with height at most B corresponds to a point of M which is a vertex of a $2d$ -dimensional real simplex which is entirely contained in $V_M(\alpha B)$ and whose vertices are all elements of M .

The number of such simplexes is at most $\frac{V \alpha^{2d}}{\det(M)} B^{2d}$, where V is the volume of the standard $2d$ -dimensional real simplex. Each such simplex has $2d + 1$ vertices, so we conclude that:

$$N'_L(B) \leq \frac{V(2d+1)\alpha^{2d}}{\det(M)} B^{2d} + 1$$

and hence the theorem follows. ■

3 Ruled Varieties

In the spirit of Heath-Brown's remark in [HB1], Theorem 2.1 enables us to give easy upper bounds for the counting functions for rational points on ruled varieties. For instance, consider the following situation.

Let $V \subset \mathbf{P}^n$ be a projective variety defined over a number field K . Assume that V admits a K -rational morphism $\phi: V \rightarrow X$ to a projective variety X over K such that the fibres of ϕ are lines. Then we can define a morphism $\psi: X \rightarrow G(1, n)$ by $\psi(P) = [\phi^{-1}(P)]$, where $G(1, n)$ denotes the Grassmannian of lines in \mathbf{P}^n . Let D be the Plücker divisor on $G(1, n)$ —that is, the pullback of $\mathcal{O}(1)$ via the Plücker embedding of $G(1, n)$. Note that ψ is injective, since ϕ is a morphism. We now have the following result:

Theorem 3.1 Using the notation of the previous paragraph, assume that the counting function of X with respect to the divisor $A = \psi^*(D)$ satisfies

$$N_X(B) = \text{card}\{P \in X(K) \mid H_A(P) \leq B\} = O(B^\epsilon)$$

for some $\epsilon < \frac{1}{m}$, where m satisfies that $mL - \phi^*A$ is ample for a hyperplane section L . Then we have

$$(1/c)B^2 \leq N_V(B) \leq cB^2$$

for some positive constant c .

Proof The first inequality is clear by Schanuel’s Theorem [Sch], since V contains at least one K -rational line. Thus, we turn our attention to the second inequality. Write H for the usual height function in \mathbf{P}^n , and let $F = \phi^*A$. Via the height machine (see for example [Vo], Proposition 1.2.9), we obtain a constant α such that for all K -rational points P of V

$$(3) \quad H_F(P) \leq \alpha H(P)^m.$$

We can now calculate as follows:

$$\begin{aligned} N_V(B) &\leq \sum_{P \in X(K), H_A(P) \leq B^m} \alpha N_{\phi^{-1}(P)}(B) \\ &\leq \sum_{P \in X(K), H_A(P) \leq B^m} \left(\frac{c_K}{H_A(P)} B^2 + 1 \right), \end{aligned}$$

where this last inequality is from Theorem 2.1 and the fact that $H(\phi^{-1}(P)) = H_A(P)$. The hypothesis of the theorem now easily implies that this sum is asymptotically less than cB^2 for a positive constant c , and the theorem is proven. ■

Remarks In particular, Theorem 3.1 applies to all (relatively) minimal ruled surfaces (see section V.2 of [Ha] for a discussion of such surfaces). (This is not quite true, since the two cases of $\mathbf{P}^1 \times \mathbf{P}^1$ and \mathbf{P}^2 blown up at a single point do not satisfy the hypotheses of Theorem 3.1, but they can be handled in a similar manner, or indeed by any number of elementary approaches as well.)

The arithmetic of relatively minimal ruled surfaces over a rational base curve has been dealt with admirably in several places, including most notably in the very general treatment of Batyrev and Tschinkel [BT] in the context of toric varieties, and in a more specific and explicit way by Billard [Bi]. Note that in both these works, the authors not only obtain the exponent in the leading term of the counting function, but they also compute the constant in the leading term and compute error terms, neither of which we are able to do here.

Finally, we remark that the results of Theorem 3.1 are consistent with the conjectures of Batyrev and Manin [BM].

Theorem 2.1 can in principle be applied to any variety which is a union of lines in \mathbf{P}^n , by the simple expedient of summing the counting functions of the individual

lines, and controlling the point of smallest height on each line. Such an analysis proceeds trivially for \mathbf{P}^n , for example, which is the union of a pencil of lines through a fixed point, and the exponent on the upper bound thereby obtained is sharp $(n + 1)$. Similar analyses can be done for cones—in both cases, the point of smallest height on (almost all) lines is the basepoint of the linear system which sweeps out the variety.

One might hope to obtain analogues of Theorem 2.1 for curves other than lines. Indeed, Heath-Brown accomplishes this in [HB1] with Theorems 2 and 3 (the latter is conditional on a certain hypothesis on the ranks of elliptic curves) and especially Theorem 5 in [HB2], and Broberg with Corollary 1 in [Br]. However, the chief advantage of Theorem 2.1 is that the asymptotic growth of the counting function shrinks as the height of the line grows, making it much easier to sum counting functions over infinitely many lines. It would be interesting to try to obtain analogues of Theorem 2.1 for curves of higher degree.

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