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EXTENSION AND AVERAGING OPERATORS FOR FINITE FIELDS

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Abstract In this paper we study L^p–L^r estimates of both the extension operator and the averaging operator associated with the algebraic variety $S = \{x \in \mathbb{F}_q^d \colon Q(x) = 0\}$, where $Q(x)$ is a non-degenerate quadratic form over the finite field $\mathbb{F}_q$ with q elements. We show that the Fourier decay estimate on S is good enough to establish the sharp averaging estimates in odd dimensions. In addition, the Fourier decay estimate enables us to simply extend the sharp L^2–L^4 conical extension result in $\mathbb{F}_q^3$, due to Mockenhaupt and Tao, to the L^2–$L^{2(d+1)/(d-1)}$ estimate in all odd dimensions $d \geqslant 3$. We also establish a sharp estimate of the mapping properties of the average operators in the case when the variety S in even dimensions $d \geqslant 4$ contains a $d/2$-dimensional subspace.

Keywords: extension problem; averaging operator; finite field

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1. Introduction

In the Euclidean setting the extension problem asks us to determine the optimal range of exponents $1 \leqslant p, r \leqslant \infty$ such that the following estimate holds:

$$\|(g\mathrm{d}\sigma)^\vee\|_{L^r(\mathbb{R}^d)} \leqslant C(p,r,d)\|g\|_{L^p(S,\mathrm{d}\sigma)} \quad \text{for all } g \in L^p(S,\mathrm{d}\sigma),$$

where $\mathrm{d}\sigma$ is a measure on the set S in $\mathbb{R}^d$. This problem was addressed in 1967 by Stein and it has been extensively studied. In particular, much attention has been given to the case in which the set S is related to a hypersurface. However, this problem has not been completely solved in higher dimensions. For a comprehensive survey of the extension problem, see [**1**, **3**, **10**, **12**] and the references therein.

Another interesting problem in classical harmonic analysis is the averaging problem: to determine the optimal range of exponents $1 \leqslant p, r \leqslant \infty$ such that the averaging estimate

$$\|f * \mathrm{d}\sigma\|_{L^r(\mathbb{R}^d)} \leqslant C(p,r,d)\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d), \tag{1.1}$$

holds, where $\mathrm{d}\sigma$ is a measure on a surface S in $\mathbb{R}^d$. This problem comes originally from investigating the regularity of the fundamental solution of a wave equation at a fixed time, and many interesting results relating to the problem have been obtained (see, for example, [**4**, **7**, **9**, **11**]).

In the finite field setting, the extension problem and the averaging problem were recently introduced by Mockenhaupt and Tao [**8**] and Carbery *et al.* [**2**], respectively. In this paper we aim to develop their work by studying those topics related to an algebraic variety

$$S = \{x \in \mathbb{F}_q^d \colon Q(x) = 0\},$$

where Q denotes a non-degenerate quadratic polynomial and $\mathbb{F}_q^d$ denotes the d-dimensional vector space over a finite field $\mathbb{F}_q$ with q elements. In [**8**], Mockenhaupt and Tao defined the cone for finite fields as

$$C_d = \{x \in \mathbb{F}_q^d \colon x_d x_{d-1} = x_1^2 + \cdots + x_{d-2}^2\},$$

which is a specific form of the variety S. Using combinatorial arguments, they proved that the L^2–L^4 extension estimate holds and it actually implies the complete answer to the extension problem for the cone C_3 in $\mathbb{F}_q^3$ (see [**8**]). In this paper we shall observe that the extension operator for the variety S yields the L^2–$L^{(2d+2)/(d-1)}$ extension estimate for all odd dimensions $d \geqslant 3$, but it is not necessarily true for even dimensions $d \geqslant 4$. Note that this result recovers the sharp extension result on the cone $C_3 \subset \mathbb{F}_q^3$, and gives non-trivial results in higher odd dimensions. We shall also investigate L^p–L^r estimates of the averaging operator over the variety S in even dimensions.

1.1. Notation and definitions

In order to clearly state our main results we begin by recalling some notation and definitions. We denote by $\mathbb{F}_q$ a finite field with q elements and assume that the characteristic of $\mathbb{F}_q$ is greater than two, namely q is a power of an odd prime. As usual, $\mathbb{F}_q^d$ refers to the d-dimensional vector space over a finite field $\mathbb{F}_q$. Let $g \colon \mathbb{F}_q^d \to \mathbb{C}$ be a complex-valued function on $\mathbb{F}_q^d$. We endow the space $\mathbb{F}_q^d$ with a counting measure $\mathrm{d}m$. Thus, the integral of the function g over $(\mathbb{F}_q^d, \mathrm{d}m)$ is given by

$$\int_{\mathbb{F}_q^d} g(m)\,\mathrm{d}m = \sum_{m \in \mathbb{F}_q^d} g(m).$$

For a fixed non-trivial additive character $\chi \colon \mathbb{F}_q \to \mathbb{C}$ and a complex-valued function g on $(\mathbb{F}_q^d, \mathrm{d}m)$, we define the Fourier transform of g by the following formula:

$$\hat{g}(x) = \int_{\mathbb{F}_q^d} \chi(-m \cdot x) g(m)\,\mathrm{d}m = \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) g(m), \tag{1.2}$$

where x is an element in the dual space of $(\mathbb{F}_q^d, \mathrm{d}m)$. Recall that the Fourier transform of the function g on $(\mathbb{F}_q^d, \mathrm{d}m)$ is actually defined on the dual space $(\mathbb{F}_q^d, \mathrm{d}x)$. Here, we endow the dual space $(\mathbb{F}_q^d, \mathrm{d}x)$ with a normalized counting measure $\mathrm{d}x$. We therefore see that if

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$f\colon (\mathbb{F}_q^d, \mathrm{d}x) \to \mathbb{C}$, then its integral over $(\mathbb{F}_q^d, \mathrm{d}x)$ is given by

$$\int_{\mathbb{F}_q^d} f(x)\,\mathrm{d}x = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(x)$$

and the Fourier transform of the function f defined on $(\mathbb{F}_q^d, \mathrm{d}x)$ is given by the formula

$$\hat{f}(m) = \int_{\mathbb{F}_q^d} \chi(-x \cdot m) f(x)\,\mathrm{d}x = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} \chi(x \cdot m) f(x), \tag{1.3}$$

where we recall that m is any element in $(\mathbb{F}_q^d, \mathrm{d}m)$ with the counting measure $\mathrm{d}m$, and we denote by $\mathrm{d}x$ the normalized counting measure on $(\mathbb{F}_q^d, \mathrm{d}x)$. We also recall that the Fourier inversion theorem holds: for $x \in (\mathbb{F}_q^d, \mathrm{d}x)$,

$$f(x) = \int_{m \in \mathbb{F}_q^d} \chi(m \cdot x) \hat{f}(m)\,\mathrm{d}m = \sum_{m \in \mathbb{F}_q^d} \chi(m \cdot x) \hat{f}(m).$$

Using the orthogonality relation of the non-trivial additive character, that is

$$\sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x) = 0 \quad \text{for } m \in \mathbb{F}_q^d \setminus \{(0, \dots, 0)\},$$

we see that the Plancherel Theorem holds such that

$$\|\hat{f}\|_{L^2(\mathbb{F}_q^d, \mathrm{d}m)} = \|f\|_{L^2(\mathbb{F}_q^d, \mathrm{d}x)}.$$

In other words, the Plancherel Theorem yields the following formula:

$$\sum_{m \in \mathbb{F}_q^d} |\hat{f}(m)|^2 = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2. \tag{1.4}$$

Let f and h be complex-valued functions defined on $(\mathbb{F}_q^d, \mathrm{d}x)$. The convolution function $f * h$ is defined on the space $(\mathbb{F}_q^d, \mathrm{d}x)$ and it follows the rule

$$f * h(y) = \int_{x \in \mathbb{F}_q^d} f(y - x) h(x)\,\mathrm{d}x = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(y - x) h(x).$$

It is not hard to see that

$$\widehat{(f * h)}(m) = \hat{f}(m) \cdot \hat{h}(m) \quad \text{and} \quad \widehat{(f \cdot h)}(m) = (\hat{f} * \hat{h})(m).$$

Remark 1.1. Throughout the paper we always consider the variable m as an element of $(\mathbb{F}_q^d, \mathrm{d}m)$ with the counting measure $\mathrm{d}m$. On the other hand,we always use the variables x or y to indicate an element of $(\mathbb{F}_q^d, \mathrm{d}x)$ with the normalized counting measure $\mathrm{d}x$. Notice from (1.2) and (1.3) that the definitions of the Fourier transforms take two different forms that depend on the domain of the Fourier transforms.

We now introduce the algebraic variety S in $(\mathbb{F}_q^d, \mathrm{d}x)$ on which we shall work. Given a non-degenerate quadratic polynomial $Q(x) \in \mathbb{F}_q[x_1, \dots, x_d]$, we define an algebraic variety S in $(\mathbb{F}_q^d, \mathrm{d}x)$ by the set

$$S = \{x \in \mathbb{F}_q^d \colon Q(x) = 0\}. \tag{1.5}$$

By a non-singular linear substitution, any non-degenerate quadratic polynomial $Q(x)$ can be transformed into $a_1x_1^2 + \cdots + a_dx_d^2$ for some $a_j \in \mathbb{F}_q \setminus \{0\}$, $j = 1, \dots, d$ (see [**6**, p. 280]). Hence, we may express the set S as follows:

$$S = \{x \in \mathbb{F}_q^d \colon a_1x_1^2 + a_2x_2^2 + \cdots + a_dx_d^2 = 0\} \subset (\mathbb{F}_q^d, \mathrm{d}x). \tag{1.6}$$

We endow the set S with a normalized surface measure $\mathrm{d}\sigma$ which is given by the relation

$$\int_S f(x)\,\mathrm{d}\sigma(x) = \frac{1}{|S|}\sum_{x\in S} f(x),$$

where $|S|$ denotes the cardinality of S. Note that the total mass of S is 1 and the measure σ is just a function on $(\mathbb{F}_q^d, \mathrm{d}x)$ given by

$$\sigma(x) = \frac{q^d}{|S|}S(x). \tag{1.7}$$

Here, and throughout the paper, we identify the set S with the characteristic function on the set S. For example, we write $E(x)$ for $\chi_E(x)$ where E is a subset of $\mathbb{F}_q^d$.

1.2. Definition of extension and averaging problems for finite fields

We recall the definition of the extension problem related to the algebraic variety S in $(\mathbb{F}_q^d, \mathrm{d}x)$. For $1 \leqslant p, r \leqslant \infty$, we denote by $R^*(p \to r)$ the smallest constant such that the extension estimate

$$\|(f\,\mathrm{d}\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, \mathrm{d}m)} \leqslant R^*(p \to r)\|f\|_{L^p(S, \mathrm{d}\sigma)}$$

holds for every function f defined on S in $(\mathbb{F}_q^d, \mathrm{d}x)$. By duality, we see that the quantity $R^*(p \to r)$ is also the smallest constant such that the following restriction estimate holds: for every function g on $(\mathbb{F}_q^d, \mathrm{d}m)$,

$$\|\hat{g}\|_{L^{p'}(S, \mathrm{d}\sigma)} \leqslant R^*(p \to r)\|g\|_{L^{r'}(\mathbb{F}_q^d, \mathrm{d}m)}. \tag{1.8}$$

Here, and throughout the paper, p' and r' denote the dual exponents of p and r respectively. In other words, $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$. The constant $R^*(p \to r)$ may depend on q: the size of the underlying finite field $\mathbb{F}_q$. However, the extension problem asks us to determine the exponents $1 \leqslant p, r \leqslant \infty$ such that $R^*(p \to r) \lesssim 1$, where the constant in the notation $\lesssim$ is independent of q and f. We recall that for positive numbers A and B, the notation $A \lesssim B$ means that there exists a constant $C > 0$ independent of the parameters q and f such that $A \leqslant CB$. We also use the notation $A \sim B$ to illustrate that there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1A \leqslant B \leqslant C_2A$.

Remark 1.2. A direct calculation yields the trivial estimate, $R^*(1 \to \infty) \lesssim 1$. Using Hölder's inequality and the nesting properties of L^p-norms, we also see that

$$R^*(p_1 \to r) \leqslant R^*(p_2 \to r) \quad \text{for } 1 \leqslant p_2 \leqslant p_1 \leqslant \infty$$

and

$$R^*(p \to r_1) \leqslant R^*(p \to r_2) \quad \text{for } 1 \leqslant r_2 \leqslant r_1 \leqslant \infty.$$

Therefore, the optimal result can be obtained once we find the smallest r and the largest p such that $R^*(p \to r) \lesssim 1$.

We now introduce the averaging problem over the algebraic variety S in $(\mathbb{F}_q^d, \mathrm{d}x)$. We denote by $A(p \to r)$ the smallest constant such that the following averaging estimate holds: for every f defined on $(\mathbb{F}_q^d, \mathrm{d}x)$, we have

$$\|f * \mathrm{d}\sigma\|_{L^r(\mathbb{F}_q^d, \mathrm{d}x)} \leqslant A(p \to r) \|f\|_{L^p(\mathbb{F}_q^d, \mathrm{d}x)},$$

where $\mathrm{d}\sigma$ is the normalized surface measure on S defined as in (1.7). Like the extension problem, the averaging problem asks one to determine the exponents $1 \leqslant p, r \leqslant \infty$ such that $A(p \to r) \lesssim 1$.

2. Statement of main results

2.1. Results on extension problems

As mentioned before, Mockenhaupt and Tao [**8**] proved that the L^2–L^4 estimate implies the complete solution to the extension problem related to the cone in $\mathbb{F}_q^3$. Using simple arguments, we modestly extend their result to higher dimensions.

Theorem 2.1. *Let S be the variety defined as in (1.5) or (1.6). If $d \geqslant 3$ is odd, then we have*

$$R^*\left(2 \to \frac{2d+2}{d-1}\right) \lesssim 1, \tag{2.1}$$

and if $d \geqslant 4$ is even, then

$$R^*\left(2 \to \frac{2d}{d-2}\right) \lesssim 1. \tag{2.2}$$

In addition, there exist specific varieties S for which each result of (2.1) and (2.2) gives a sharp L^2–L^r extension estimate.

Remark 2.2. We shall see that Theorem 2.1 is, in fact, a direct result from the well-known standard Tomas–Stein-type argument. However, the conclusions of Theorem 2.1 are very interesting, in part because they are inconsistent with the facts in the Euclidean case. For example, if $S \subset \mathbb{R}^d$ is a compact subset of the cone, then it is well known that the L^2–$L^{2d/(d-2)}$ estimate gives the sharp L^2–L^r extension estimate for all dimensions $d \geqslant 3$ (see [**12**] or [**13**]). Note that the conclusion (2.1) is much more accurate than

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that in the Euclidean case, although the conclusion (2.2) in even dimensions is exactly the same. In the Euclidean setting, the curvature on the surface plays an important role in determining the extension estimates. On the other hand, the extension estimates for finite fields can be determined in accordance with the maximal size of affine subspaces in the surface S. This explains why the result (2.1) for odd dimensions is much better than the result (2.2) for even dimensions. In fact, the surface S in even dimensions may contain a $d/2$-dimensional subspace but this never happens in odd dimensions, because $d/2$ is not an integer for odd d. The conclusion (2.1) shows that if $d \geqslant 3$ is odd, then $q^{(d-1)/2}$ is the maximal cardinality of subspaces contained in the variety S.

2.2. Results on averaging problems

Theorem 2.3. *Let S be the algebraic variety in $(\mathbb{F}_q^d, \mathrm{d}x)$ defined as in (1.5) or (1.6). If $d \geqslant 3$ is odd, then we have*

$$A(p \to r) \lesssim 1 \iff \left(\frac{1}{p}, \frac{1}{r}\right) \in \mathbb{T}, \tag{2.3}$$

where $\mathbb{T}$ denotes the convex hull of points $(0,0)$, $(0,1)$, $(1,1)$ and $(d/(d+1), 1/(d+1))$. On the other hand, if $d \geqslant 4$ is even, then

$$A(p \to r) \lesssim 1 \quad \text{for } \left(\frac{1}{p}, \frac{1}{r}\right) \in \Omega \setminus \{P_1, P_2\} \tag{2.4}$$

where Ω denotes the convex hull of points

$$(0,0), \quad (0,1), \quad (1,1), \quad P_1 = \left(\frac{d^2-2d+2}{d(d-1)}, \frac{1}{(d-1)}\right) \quad \text{and} \quad P_2 = \left(\frac{d-2}{d-1}, \frac{d-2}{d(d-1)}\right).$$

In addition, if $d \geqslant 4$ is even and $P_1 = (1/p, 1/r)$ then the restricted strong-type estimate

$$\|f * \mathrm{d}\sigma\|_{L^r(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim \|f\|_{L^{p,1}(\mathbb{F}_q^d, \mathrm{d}x)} \tag{2.5}$$

holds, and if $d \geqslant 4$ is even and $P_2 = (1/r', 1/p')$ then the weak-type estimate

$$\|f * \mathrm{d}\sigma\|_{L^{p',\infty}(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim \|f\|_{L^{r'}(\mathbb{F}_q^d, \mathrm{d}x)} \tag{2.6}$$

holds. Finally, the averaging results in even dimensions are sharp in the sense that if $(1/p, 1/r) \notin \Omega$ and S contains a $d/2$-dimensional subspace, then the L^p–L^r averaging estimate is impossible.

The results in Theorem 2.3 are also interesting since they contradict well-known facts in the Euclidean case. In the Euclidean space it is well known that if a hypersurface has everywhere non-vanishing Gaussian curvature, then the L^p–L^r averaging estimate holds if and only if $(1/p, 1/r)$ lies in the triangle with vertices $(0,0)$, $(1,1)$ and $(d/(d+1), 1/(d+1))$.* However, if the Gaussian curvature is allowed to vanish, then the

* If $1 \leqslant r < p \leqslant \infty$, then the L^p–L^r averaging estimate is impossible in the Euclidean case, but it always holds in the finite field setting. Therefore, it would be interesting to find the difference only in the case when $1 \leqslant p \leqslant r \leqslant \infty$.

averaging estimates get worse (see [**7**, **11**] and [**9**]). For example, since it is clear that $S = \{x \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots + x_d^2 = 0\}$ has everywhere vanishing Gaussian curvature away from the origin, the averaging estimates in the Euclidean case must be less accurate than our result (2.3) in the finite field case. The other interesting point from Theorem 2.3 is that the sharp averaging estimates (2.3) in odd dimensions are better than those in even dimensions. The main reason for the difference is that, given Remark 2.2, the variety S in odd dimensions $d \geqslant 3$ can only contain a subspace H with cardinality at most $q^{(d-1)/2}$. We shall see that the Fourier transform of the surface measure $\mathrm{d}\sigma$ yields a good decay estimate such that the sharp averaging estimates (2.3) can be obtained directly by the well-established Euclidean arguments. On the other hand, if the dimension $d \geqslant 4$ is even, then a relatively large subspace H with the cardinality $q^{d/2}$ may lie in S. In this case, the averaging problem becomes much harder, but we can still obtain relatively good results by applying our extension result (2.2).

2.3. Outline of the remainder of paper

In § 3, we summarize the necessary conditions for $R^*(p \to r) \lesssim 1$ and $A(p \to r) \lesssim 1$. In § 4, we compute the explicit form of the Fourier transform on the variety S, which plays a crucial role in obtaining our results. The proof of Theorem 2.1 is given in § 5. In the last section, we complete the proof of Theorem 2.3.

3. Necessary conditions for the L^p–L^r extension and averaging estimates

In this section, we review the necessary conditions for $R^*(p \to r) \lesssim 1$ and $A(p \to r) \lesssim 1$. Mockenhaupt and Tao [**8**] introduced the necessary conditions for the L^p–L^r extension estimates related to the cone $C_3 = \{x \in \mathbb{F}_q^3 : x_2 x_3 = x_1^2\}$ and proved that the necessary conditions are in fact sufficient. Based on arguments similar to those in [**8**], it is not hard to find the necessary conditions for the case of higher dimensions. Here, we state the necessary conditions for the L^p–L^r extension estimates related to the variety $S = \{x \in \mathbb{F}_q^d : a_1 x_1^2 + \cdots + a_d x_d^2 = 0\}$, $d \geqslant 3$, and we leave the proof to the readers.

Lemma 3.1. *If $d \geqslant 4$ is even and S contains a $d/2$-dimensional subspace, then the necessary conditions for $R^*(p \to r) \lesssim 1$ give*

$$r \geqslant \frac{2d-2}{d-2} \quad \text{and} \quad r \geqslant \frac{dp}{(d-2)(p-1)}.$$

On the other hand, if $d \geqslant 3$ is odd, S contains a $(d-1)/2$-dimensional subspace and $-a_i a_j^{-1}$ is a square number for some $i, j = 1, 2, \ldots, d$ with $i \neq j$, then the necessary conditions for $R^*(p \to r) \lesssim 1$ are given by the relations

$$r \geqslant \frac{2d-2}{d-2} \quad \text{and} \quad r \geqslant \frac{(d+1)p}{(d-1)(p-1)}.$$

The necessary conditions for the L^p–L^r averaging estimates are given by Carbery *et al*. [**2**]. In our case, the necessary conditions can be stated as follows.

Lemma 3.2. *For $a_j \neq 0$, $j = 1, \dots, d$, let $S = \{x \in \mathbb{F}_q^d \colon a_1 x_1^2 + \cdots + a_d x_d^2 = 0\}$. Then $A(p \to r) \lesssim 1$ only if $(1/p, 1/r)$ lies in the convex hull of the points*

$$(0,0), \quad (0,1), \quad (1,1) \quad \text{and} \quad \left(\frac{d}{d+1}, \frac{1}{d+1}\right). \tag{3.1}$$

Moreover, if $d \geqslant 4$ is even and S contains a $d/2$-dimensional affine subspace H, then $A(p \to r) \lesssim 1$ only if $(1/p, 1/r)$ lies in the convex hull of the points

$$(0,0), \quad (0,1), \quad (1,1), \quad \left(\frac{d^2-2d+2}{d(d-1)}, \frac{1}{d-1}\right) \quad \text{and} \quad \left(\frac{d-2}{d-1}, \frac{d-2}{d(d-1)}\right). \tag{3.2}$$

4. The Fourier transform of the surface measure $d\sigma$

In this section we obtain an explicit formula for the Fourier transform of the surface measure $d\sigma$ on the surface S defined as in (1.6). We shall see that the Fourier transform is closely related to the classical Gauss sums. Moreover, it plays a key role in proving our main results on both the extension problem and the averaging problem. It is useful to review classical Gauss sums in the finite field setting. In the remainder of this paper, we fix the additive character χ as a canonical additive character of $\mathbb{F}_q$ and η always denotes the quadratic character of $\mathbb{F}_q$. Recall that $\eta(t) = 1$ if s is a square number in $\mathbb{F}_q \setminus \{0\}$, and $\eta(t) = -1$ if t is not a square number in $\mathbb{F}_q \setminus \{0\}$. We also recall that $\eta(0) = 0$, $\eta^2 \equiv 1$, $\eta(ab) = \eta(a)\eta(b)$ for $a, b \in \mathbb{F}_q$ and $\eta(t) = \eta(t^{-1})$ for $t \neq 0$. For each $t \in \mathbb{F}_q$, the Gauss sum $G_t(\eta, \chi)$ is defined by

$$G_t(\eta, \chi) = \sum_{s \in \mathbb{F}_q \setminus \{0\}} \eta(s)\chi(ts).$$

The absolute value of the Gauss sum is given by the relation

$$|G_t(\eta, \chi)| = \begin{cases} q^{1/2} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

In addition, we have the following formula:

$$\sum_{s \in \mathbb{F}_q} \chi(ts^2) = \eta(t) G_1(\eta, \chi) \quad \text{for any } t \neq 0. \tag{4.1}$$

For the nice proofs of the properties related to the Gauss sums, see [**6**, Chapter 5] and [**5**, Chapter 11]. When we complete the square and apply a change of variable, the formula (4.1) yields the following equation: for each $a \in \mathbb{F}_q \setminus \{0\}$, $b \in \mathbb{F}_q$,

$$\sum_{s \in \mathbb{F}_q} \chi(as^2 + bs) = G_1(\eta, \chi)\eta(a)\chi\left(\frac{b^2}{-4a}\right). \tag{4.2}$$

We shall name the method used to obtain the formula (4.2) the 'complete square method'. Relating the inverse Fourier transform of $d\sigma$ to the Gauss sum, we shall obtain an explicit form of $(d\sigma)^\vee$: the inverse Fourier transform of the surface measure on S. We have the following lemma.

Lemma 4.1. *Let* $\mathrm{d}\sigma$ *be the surface measure on* S *defined as in (1.6). If* $d \geqslant 3$ *is odd, then we have*

$$(\mathrm{d}\sigma)^{\vee}(m) = \begin{cases} q^{d-1}|S|^{-1} & \text{if } m = (0,\dots,0), \\ 0 & \text{if } m \neq (0,\dots,0), \\ & \quad \dfrac{m_1^2}{a_1} + \cdots + \dfrac{m_d^2}{a_d} = 0, \\ \dfrac{G_1^{d+1}}{q|S|}\eta(-a_1\cdots a_d)\eta\biggl(\dfrac{m_1^2}{a_1} + \cdots + \dfrac{m_d^2}{a_d}\biggr) & \text{if } \dfrac{m_1^2}{a_1} + \cdots + \dfrac{m_d^2}{a_d} \neq 0. \end{cases}$$

If $d \geqslant 2$ *is even, then we have*

$$(\mathrm{d}\sigma)^{\vee}(m) = \begin{cases} q^{d-1}|S|^{-1} + \dfrac{G_1^d}{|S|}(1-q^{-1})\eta(a_1\cdots a_d) & \text{if } m = (0,\dots,0), \\ \dfrac{G_1^d}{|S|}(1-q^{-1})\eta(a_1\cdots a_d) & \text{if } m \neq (0,\dots,0), \\ & \quad \dfrac{m_1^2}{a_1} + \cdots + \dfrac{m_d^2}{a_d} = 0, \\ -\dfrac{G_1^d}{q|S|}\eta(a_1\cdots a_d) & \text{if } \dfrac{m_1^2}{a_1} + \cdots + \dfrac{m_d^2}{a_d} \neq 0. \end{cases}$$

Here, and throughout this paper, we write G_1 *for the Gauss sum* $G_1(\eta,\xi)$*, where* η *denotes the quadratic character of* $\mathbb{F}_q$*.*

Proof. Using the definition of the inverse Fourier transform and the orthogonality relations of the non-trivial additive character χ of $\mathbb{F}_q$, we see

$$\begin{aligned} (\mathrm{d}\sigma)^{\vee}(m) &= |S|^{-1}\sum_{x\in S}\chi(x\cdot m) \\ &= |S|^{-1}q^{-1}\sum_{x\in\mathbb{F}_q^d}\sum_{s\in\mathbb{F}_q}\chi(s(a_1x_1^2+\cdots+a_dx_d^2))\,\chi(x\cdot m) \\ &= q^{d-1}|S|^{-1}\delta_0(m) + |S|^{-1}q^{-1}\sum_{x\in\mathbb{F}_q^d}\sum_{s\neq 0}\chi(s(a_1x_1^2+\cdots+a_dx_d^2))\,\chi(x\cdot m) \\ &= q^{d-1}|S|^{-1}\delta_0(m) + |S|^{-1}q^{-1}\sum_{s\neq 0}\prod_{j=1}^{d}\sum_{x_j\in\mathbb{F}_q}\chi(sa_jx_j^2+m_jx_j). \end{aligned}$$

Using the complete square method (4.2), we compute the sums over $x_j \in \mathbb{F}_q$ and obtain that

$$\begin{aligned} &(\mathrm{d}\sigma)^{\vee}(m) \\ &\quad = q^{d-1}|S|^{-1}\delta_0(m) + G_1^d|S|^{-1}q^{-1}\eta(a_1\cdots a_d)\sum_{s\neq 0}\eta^d(s)\chi\biggl(-\frac{1}{4s}\biggl(\frac{m_1^2}{a_1}+\cdots+\frac{m_d^2}{a_d}\biggr)\biggr). \end{aligned}$$

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Case 1. Suppose that $d \geqslant 3$ is odd. Then $\eta^d \equiv \eta$, because η is the multiplicative character of order two. Therefore, if

$$\frac{m_1^2}{a_1} + \cdots + \frac{m_d^2}{a_d} = 0,$$

then the proof is complete, because $\sum_{s\in\mathbb{F}_q\setminus\{0\}} \eta(s) = 0$. On the other hand, if

$$\frac{m_1^2}{a_1} + \cdots + \frac{m_d^2}{a_d} \neq 0,$$

then the statement follows from using a change of variable,

$$-\frac{1}{4s}\left(\frac{m_1^2}{a_1} + \cdots + \frac{m_d^2}{a_d}\right) \to s,$$

and the fact that $\eta(4) = 1$, $\eta(s) = \eta(s^{-1})$ for $s \neq 0$ and $G_1 = \sum_{s\neq 0} \eta(s)\chi(s)$.

Case 2. Suppose that $d \geqslant 2$ is even. Then $\eta^d \equiv 1$. The proof is complete, because $\sum_{s\neq 0} \chi(as) = -1$ for all $a \neq 0$, and $\sum_{s\neq 0} \chi(as) = (q-1)$ if $a = 0$. □

Lemma 4.1 yields the following corollary.

Corollary 4.2. *If $d \geqslant 3$ is odd, then it follows that*

$$\left.\begin{aligned} (\mathrm{d}\sigma)^\vee(0,\dots,0) &= 1, \\ |(\mathrm{d}\sigma)^\vee(m)| &\lesssim q^{-(d-1)/2} \quad \text{if } m \neq (0,\dots,0), \end{aligned}\right\} \tag{4.3}$$

and if $d \geqslant 4$ is even, then we have

$$\left.\begin{aligned} (\mathrm{d}\sigma)^\vee(0,\dots,0) &= 1, \\ |(\mathrm{d}\sigma)^\vee(m)| &\lesssim q^{-(d-2)/2} \quad \text{if } m \neq (0,\dots,0). \end{aligned}\right\} \tag{4.4}$$

Proof. Recall that the inverse Fourier transform of the surface measure $\mathrm{d}\sigma$ is given by the relation

$$(\mathrm{d}\sigma)^\vee(m) = \int_S \chi(x\cdot m)\,\mathrm{d}\sigma = \frac{1}{|S|}\sum_{x\in S} \chi(x\cdot m),$$

where $m \in (\mathbb{F}_q^d, \mathrm{d}m)$. Therefore, it is clear that $(\mathrm{d}\sigma)^\vee(0,\dots,0) = 1$ for all $d \geqslant 2$. If we compare this with the values $(\mathrm{d}\sigma)^\vee(0,\dots,0)$ given by Lemma 4.1, then we see that $|S| \sim q^{d-1}$ for $d \geqslant 3$. Since the absolute of the Gauss sum G_1 is exactly $q^{1/2}$, the statements in Corollary 4.2 follow immediately from Lemma 4.1. □

5. Proof of Theorem 2.1 (extension theorems)

We begin by proving the last statement in Theorem 2.1. We choose a variety S with $a_j = 1$ for j odd and $a_j = -1$ otherwise. It follows that if $d \geqslant 3$ is odd, then the variety S contains the $(d-1)/2$-dimensional subspace

$$H = \{(t_1, t_1, \dots, t_j, t_j, \dots, t_{(d-1)/2}, t_{(d-1)/2}, 0) \colon t_k \in \mathbb{F}_q^d,\ k = 1, 2, \dots, (d-1)/2\},$$

and if $d \geqslant 4$ is even, then it contains the $d/2$-dimensional subspace

$$W = \{(t_1, t_1, \dots, t_j, t_j, \dots, t_{d/2}, t_{d/2})\colon t_k \in \mathbb{F}_q^d,\ k = 1, 2, \dots, d/2\}.$$

Thus, the last statement in Theorem 2.1 follows immediately from the necessary conditions in Lemma 3.1. Next, observe that the statements of (2.1) and (2.2) follow from Corollary 4.2 and the following lemma, which can be proved by a routine modification of the Euclidean Tomas–Stein-type argument.

Lemma 5.1. *Let $\mathrm{d}\sigma$ be the surface measure on the algebraic variety $S \subset (\mathbb{F}_q^d, \mathrm{d}x)$ defined as in (1.6). If $|(\mathrm{d}\sigma)^\vee(m)| \lesssim q^{-\alpha/2}$ for some $\alpha > 0$ and for all $m \in \mathbb{F}_q^d \setminus (0, \dots, 0)$, then we have*

$$R^*\left(2 \to \frac{2(\alpha+2)}{\alpha}\right) \lesssim 1.$$

Proof. By duality, it suffices to prove that the following restriction estimate holds: for every function g defined on $(\mathbb{F}_q^d, \mathrm{d}m)$, we have

$$\|\hat{g}\|^2_{L^2(S,\mathrm{d}\sigma)} \lesssim \|g\|^2_{L^{(2(\alpha+2))/(\alpha+4)}(\mathbb{F}_q^d,\mathrm{d}m)}.$$

By the orthogonality principle and Hölder's inequality, we see that

$$\|\hat{g}\|^2_{L^2(S,\mathrm{d}\sigma)} \leqslant \|g * (\mathrm{d}\sigma)^\vee\|_{L^{(2(\alpha+2))/\alpha}(\mathbb{F}_q^d,\mathrm{d}m)} \|g\|_{L^{(2(\alpha+2))/(\alpha+4)}(\mathbb{F}_q^d,\mathrm{d}m)}.$$

It therefore suffices to show that, for every function g on $(\mathbb{F}_q^d, \mathrm{d}m)$,

$$\|g * (\mathrm{d}\sigma)^\vee\|_{L^{(2(\alpha+2))/\alpha}(\mathbb{F}_q^d,\mathrm{d}m)} \lesssim \|g\|_{L^{(2(\alpha+2))/(\alpha+4)}(\mathbb{F}_q^d,\mathrm{d}m)}.$$

Define $K = (\mathrm{d}\sigma)^\vee - \delta_0$. Since $(\mathrm{d}\sigma)^\vee(0, \dots, 0) = 1$, we see that $K(m) = 0$ if $m = (0, \dots, 0)$, and $K(m) = (\mathrm{d}\sigma)^\vee(m)$ if $m \in \mathbb{F}_q^d \setminus \{(0, \dots, 0)\}$. It follows that

$$\begin{aligned} \|g * \delta_0\|_{L^{(2(\alpha+2))/\alpha}(\mathbb{F}_q^d,\mathrm{d}m)} &= \|g\|_{L^{(2(\alpha+2))/\alpha}(\mathbb{F}_q^d,\mathrm{d}m)} \\ &\leqslant \|g\|_{L^{(2(\alpha+2))/(\alpha+4)}(\mathbb{F}_q^d,\mathrm{d}m)}, \end{aligned}$$

where the inequality follows from the fact that $\mathrm{d}m$ is the counting measure and that $2(\alpha+2)/\alpha \geqslant 2(\alpha+2)/(\alpha+4)$. Thus, it is enough to show that, for every g on $(\mathbb{F}_q^d, \mathrm{d}m)$,

$$\|g * K\|_{L^{(2(\alpha+2))/\alpha}(\mathbb{F}_q^d,\mathrm{d}m)} \lesssim \|g\|_{L^{(2(\alpha+2))/(\alpha+4)}(\mathbb{F}_q^d,\mathrm{d}m)}. \tag{5.1}$$

We now claim that the following two estimates hold: for every function g on $(\mathbb{F}_q^d, \mathrm{d}m)$,

$$\|g * K\|_{L^2(\mathbb{F}_q^d,\mathrm{d}m)} \lesssim q \|g\|_{L^2(\mathbb{F}_q^d,\mathrm{d}m)}, \tag{5.2}$$

$$\|g * K\|_{L^\infty(\mathbb{F}_q^d,\mathrm{d}m)} \lesssim q^{-\alpha/2} \|g\|_{L^1(\mathbb{F}_q^d,\mathrm{d}m)}. \tag{5.3}$$

Note that the estimate (5.1) follows by interpolating (5.2) and (5.3). It therefore remains to show that both (5.2) and (5.3) hold. Using Plancherel, the inequality (5.2) follows

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from the following observation:

$$\begin{aligned}\|g * K\|_{L^2(\mathbb{F}_q^d,\mathrm{d}m)} &= \|\hat{g}\hat{K}\|_{L^2(\mathbb{F}_q^d,\mathrm{d}x)}\\ &\leqslant \|\hat{K}\|_{L^\infty(\mathbb{F}_q^d,\mathrm{d}x)}\|\hat{g}\|_{L^2(\mathbb{F}_q^d,\mathrm{d}x)}\\ &\lesssim q\|g\|_{L^2(\mathbb{F}_q^d,\mathrm{d}m)}, \end{aligned} \tag{5.4}$$

where the last line is due to the observation that for each $x \in (\mathbb{F}_q^d, \mathrm{d}x)$,

$$\hat{K}(x) = \mathrm{d}\sigma(x) - \hat{\delta}_0(x) = q^d|S|^{-1}S(x) - 1 \lesssim q.$$

On the other hand, the estimate (5.3) follows from Young's inequality and the assumption on the Fourier decay estimates away from the origin. Thus, the proof is complete. □

6. Proof of Theorem 2.3 (averaging theorems)

6.1. Proof of (2.3)

Because of the necessary condition (3.1) in Theorem 3.2, it suffices to prove that if $(1/p, 1/r) \in \mathbb{T}$, then $A(p \to r) \lesssim 1$, where $\mathbb{T}$ is the convex hull of points $(0,0)$, $(0,1)$, $(1,1)$ and $(d/(d+1), 1/(d+1))$. Since both $\mathrm{d}\sigma$ and $(\mathbb{F}_q^d, \mathrm{d}x)$ have total mass 1 it is clear that if $1 \leqslant r \leqslant p \leqslant \infty$, then

$$\|f * \mathrm{d}\sigma\|_{L^r(\mathbb{F}_q^d,\mathrm{d}x)} \leqslant \|f\|_{L^p(\mathbb{F}_q^d,\mathrm{d}x)}. \tag{6.1}$$

Using the Interpolation Theorem, it is enough to prove that

$$A((d+1)/d \to d+1) \lesssim 1.$$

Since the dimension $d \geqslant 3$ is odd, it follows from the first part of Corollary 4.2 that

$$|(\mathrm{d}\sigma)^\vee(m)| \lesssim q^{-(d-1)/2} \quad \text{if } m \neq (0,\dots,0),$$

and we complete the proof by using the lemma below due to the authors in [**2**].

Lemma 6.1. *Let* $\mathrm{d}\sigma$ *be the surface measure on the algebraic variety* $S \subset (\mathbb{F}_q^d, \mathrm{d}x)$ *defined as in (1.5). If* $|(\mathrm{d}\sigma)^\vee(m)| \lesssim q^{-\alpha/2}$ *for all* $m \in \mathbb{F}_q^d \setminus (0,\dots,0)$ *and for some* $\alpha > 0$, *then we have*

$$A\left(\frac{\alpha+2}{\alpha+1} \to \alpha+2\right) \lesssim 1.$$

Proof. Consider a function K on $(\mathbb{F}_q^d, \mathrm{d}m)$ defined as $K = (\mathrm{d}\sigma)^\vee - \delta_0$. We want to prove that for every function f on $(\mathbb{F}_q^d, \mathrm{d}x)$,

$$\|f * \mathrm{d}\sigma\|_{L^{\alpha+2}(\mathbb{F}_q^d,\mathrm{d}x)} \lesssim \|f\|_{L^{(\alpha+2)/(\alpha+1)}(\mathbb{F}_q^d,\mathrm{d}x)}.$$

Since $\mathrm{d}\sigma = \hat{K} + \hat{\delta}_0 = \hat{K} + 1$ and $\|f * 1\|_{L^{\alpha+2}(\mathbb{F}_q^d,\mathrm{d}x)} \lesssim \|f\|_{L^{(\alpha+2)/(\alpha+1)}(\mathbb{F}_q^d,\mathrm{d}x)}$, it suffices to show that for every f on $(\mathbb{F}_q^d, \mathrm{d}x)$,

$$\|f * \hat{K}\|_{L^{\alpha+2}(\mathbb{F}_q^d,\mathrm{d}x)} \lesssim \|f\|_{L^{(\alpha+2)/(\alpha+1)}(\mathbb{F}_q^d,\mathrm{d}x)}. \tag{6.2}$$

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Note that this can be done by interpolating the following two estimates:

$$\|f * \hat{K}\|_{L^2(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim q^{-\alpha/2} \|f\|_{L^2(\mathbb{F}_q^d, \mathrm{d}x)} \tag{6.3}$$

$$\|f * \hat{K}\|_{L^\infty(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim q \|f\|_{L^1(\mathbb{F}_q^d, \mathrm{d}x)}. \tag{6.4}$$

The inequality (6.3) follows from the Plancherel Theorem, the size assumption of $|(\mathrm{d}\sigma)^\vee|$ and the definition of K. On the other hand, the inequality (6.4) follows from Young's inequality and the observation that $\|\hat{K}\|_{L^\infty(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim q$. Thus, the proof of Lemma 6.1 is complete. □

6.2. Proof of Theorem 2.3 in the case of even dimensions

First, observe that the statement for the sharpness follows from the necessary condition (3.2). Also recall from (6.1) that $A(p \to r) \lesssim 1$ for $1 \leqslant q \leqslant p \leqslant \infty$.

It is clear by duality that the statement (2.5) implies the statement (2.6). By the Interpolation Theorem, we also see that the statements of (2.5) and (2.6) imply the statement (2.4). Therefore, it suffices to prove the restricted strong-type estimate (2.5). More precisely, it amounts to showing

$$\|E * \mathrm{d}\sigma\|_{L^{d-1}(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim \|E\|_{L^{d(d-1)/(d^2-2d+2)}(\mathbb{F}_q^d, \mathrm{d}x)} \quad \text{for all } E \subset \mathbb{F}_q^d. \tag{6.5}$$

We now consider the Bochner–Riesz kernel K on $(\mathbb{F}_q^d, \mathrm{d}m)$ defined by $K = (\mathrm{d}\sigma)^\vee - \delta_0$, where $\delta_0(m) = 1$ if $m = (0, \ldots, 0)$ and 0 otherwise. Our task is to establish the following two inequalities: for all $E \subset \mathbb{F}_q^d$,

$$\|E * \hat{\delta}_0\|_{L^{d-1}(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim \|E\|_{L^{d(d-1)/(d^2-2d+2)}(\mathbb{F}_q^d, \mathrm{d}x)} \quad \text{for all } E \subset \mathbb{F}_q^d, \tag{6.6}$$

$$\|E * \hat{K}\|_{L^{d-1}(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim \|E\|_{L^{d(d-1)/(d^2-2d+2)}(\mathbb{F}_q^d, \mathrm{d}x)} \quad \text{for all } E \subset \mathbb{F}_q^d. \tag{6.7}$$

Since $\hat{\delta}_0 = 1$ and the total mass of $\mathbb{F}_q^d$ is 1, the inequality (6.6) follows immediately from Young's inequality for convolution. On the other hand, the inequality (6.7) can be obtained by interpolating the following two inequalities: for all $E \subset \mathbb{F}_q^d$,

$$\|E * \hat{K}\|_{L^\infty(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim q \|E\|_{L^1(\mathbb{F}_q^d, \mathrm{d}x)}, \tag{6.8}$$

$$\|E * \hat{K}\|_{L^2(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim q^{(-d+3)/2} \|E\|_{L^{2d/(d+2)}(\mathbb{F}_q^d, \mathrm{d}x)}. \tag{6.9}$$

Since the inequality (6.8) follows immediately from Young's inequality and the observation that $\|\hat{K}\|_{L^\infty(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim q$, it remains to prove that (6.9) holds. Namely, we must show that

$$\|E * \hat{K}\|_{L^2(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim q^{-d+1/2} |E|^{(d+2)/2d} \quad \text{for all } E \subset \mathbb{F}_q^d.$$

It suffices to prove the following inequality, which gives a better estimate in the case when $1 \leqslant |E| \leqslant q^{1/2d}$:

$$\|E * \hat{K}\|_{L^2(\mathbb{F}_q^d, \mathrm{d}x)} \lesssim \begin{cases} q^{-d+1/2} |E|^{(d+2)/2d} & \text{if } 1 \leqslant |E| \leqslant q^{d/2}, \\ q^{-d+1} |E|^{1/2} & \text{if } q^{1/2d} \leqslant |E| \leqslant q^d. \end{cases} \tag{6.10}$$

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Using the Plancherel Theorem, we have

$$\begin{aligned}\|E * \hat{K}\|^2_{L^2(\mathbb{F}_q^d, \mathrm{d}x)} &= \|\hat{E}K\|^2_{L^2(\mathbb{F}_q^d, \mathrm{d}m)} \\ &= \sum_{m\in\mathbb{F}_q^d} |\hat{E}(m)|^2 |K(m)|^2 \\ &= \sum_{m\neq(0,\dots,0)} |\hat{E}(m)|^2 |(\mathrm{d}\sigma)^\vee(m)|^2,\end{aligned}$$

where the last line follows from the definition of K and the fact that $(\mathrm{d}\sigma)^\vee(0,\dots,0) = 1$. Since $|S| \sim q^{d-1}$, $|\eta| \equiv 1$, and the absolute value of the Gauss sum G_1 is $q^{1/2}$, using the explicit formula for $(\mathrm{d}\sigma)^\vee$ in the second part of Lemma 4.1 shows that

$$\begin{aligned}\|E * \hat{K}\|^2_{L^2(\mathbb{F}_q^d, \mathrm{d}x)} &\sim \frac{1}{q^{d-2}} \sum_{\substack{m\neq(0,\dots,0):\\ m_1^2/a_1+\cdots+m_d^2/a_d=0}} |\hat{E}(m)|^2 + \frac{1}{q^d} \sum_{\substack{m\neq(0,\dots,0):\\ m_1^2/a_1+\cdots+m_d^2/a_d\neq0}} |\hat{E}(m)|^2 \\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

From the Plancherel Theorem (1.4), we see that

$$\mathrm{II} \leqslant \frac{1}{q^d} \sum_{m\in\mathbb{F}_q^d} |\hat{E}(m)|^2 = q^{-2d}|E|.$$

We claim that the upper bound of I is given by

$$\mathrm{I} \lesssim \min\{q^{-2d+1}|E|^{(d+2)/d}, q^{-2d+2}|E|\}, \tag{6.11}$$

which shall be proved later. It follows that

$$\begin{aligned}\|E * \hat{K}\|^2_{L^2(\mathbb{F}_q^d, \mathrm{d}x)} &\lesssim \min\{q^{-2d+1}|E|^{(d+2)/d}, q^{-2d+2}|E|\} + q^{-2d}|E| \\ &\sim \min\{q^{-2d+1}|E|^{(d+2)/d}, q^{-2d+2}|E|\}.\end{aligned}$$

By a direct calculation, we see that this estimate implies (6.10). Thus, our last work is to prove the claim (6.11). Note that (6.11) can be obtained by using the following lemma based on the dual extension theorem.

Lemma 6.2. *For any subset E of $(\mathbb{F}_q^d, \mathrm{d}x)$ and $b_j \neq 0$ for $j = 1, \dots, d$, if $d \geqslant 4$ is even, then we have*

$$\sum_{m\in S} |\hat{E}(m)|^2 := \sum_{m\in S} \Big|q^{-d} \sum_{x\in E} \chi(-m\cdot x)\Big|^2 \lesssim \min\{q^{-(d+1)}|E|^{(d+2)/d}, q^{-d}|E|\},$$

where $S = \{m \in \mathbb{F}_q^d : b_1m_1^2 + \cdots + b_dm_d^2 = 0\} \subset (\mathbb{F}_q^d, \mathrm{d}m)$

Proof. It is clear from the Plancherel Theorem that

$$\sum_{m\in S} |\hat{E}(m)|^2 \leqslant \sum_{m\in\mathbb{F}_q^d} |\hat{E}(m)|^2 = q^{-d}|E|.$$

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It therefore remains to show that

$$\sum_{m\in S}|\hat{E}(m)|^2 := \sum_{m\in S}|q^{-d}\sum_{x\in E}\chi(-m\cdot x)|^2 \lesssim q^{-(d+1)}|E|^{(d+2)/d}. \tag{6.12}$$

Since the space $(\mathbb{F}_q^d, \mathrm{d}x)$ is isomorphic to its dual space $(\mathbb{F}_q^d, \mathrm{d}m)$ as an abstract group, we may identify the space $(\mathbb{F}_q^d, \mathrm{d}x)$ with the dual space $(\mathbb{F}_q^d, \mathrm{d}m)$. Thus, they possess the same algebraic structures. Recall that we have endowed them with different measures: the counting measure $\mathrm{d}m$ for $(\mathbb{F}_q^d, \mathrm{d}m)$ and the normalized counting measure $\mathrm{d}x$ for $(\mathbb{F}_q^d, \mathrm{d}x)$. For these reasons, the inequality (6.12) is essentially the same as the following: for every subset E of $(\mathbb{F}_q^d, \mathrm{d}m)$,

$$\sum_{x\in S} q^{-2d}|\hat{E}(x)|^2 \lesssim q^{-(d+1)}|E|^{(d+2)/d}, \tag{6.13}$$

where S is considered as

$$S = \{x \in \mathbb{F}_q^d \colon b_1x_1^2 + \cdots + b_dx_d^2 = 0\} \subset (\mathbb{F}_q^d, \mathrm{d}x)$$

and

$$\hat{E}(x) = \sum_{m\in\mathbb{F}_q^d}\chi(-m\cdot x)E(m).$$

By duality (1.8), the statement (2.2) in Theorem 2.1 implies that the following restriction estimate holds: for every function g on $(\mathbb{F}_q^d, \mathrm{d}m)$,

$$\|\hat{g}\|^2_{L^2(S,\mathrm{d}\sigma)} \lesssim \|g\|^2_{L^{2d/(d+2)}(\mathbb{F}_q^d,\mathrm{d}m)}.$$

If we take $g(m) = E(m)$, then we have

$$\frac{1}{|S|}\sum_{x\in S}|\hat{E}(x)|^2 \lesssim |E|^{(d+2)/d}.$$

Since $|S| \sim q^{d-1}$, (6.13) holds and the proof is complete. □

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