much as possible over the future premiums; are propositions to which I readily subscribe. But that the entire expense of obtaining an assurance —including, of course, under that head the medical fees, together with a due proportion of the current general expenses of the Office,—cannot consistently with prudence exceed the loading of the first year's premium, is an assertion which cannot be accepted without proof. To those who are in the habit of making their calculations upon true data the determination of the limit of a wise and profitable expenditure for the acquisition of business offers no difficulty whatever. When, therefore, those who tell us that it is unwise to exceed the loading in question can prove by calculation that the Office is better without the assurance than with it if obtained upon such terms, they will not indeed have answered Mr. Sprague's third objection, but they will at all events have convicted that gentleman of having rashly sanctioned an imprudent rate of expenditure.

The assertion that the expenditure is unjustifiable,—that there is in fact an "implied contract" with the assured that the loading only on the premiums received shall be available for expenditure—is easily disposed of. If it mean that there is an implied contract to this effect because it is the proper course, it merely amounts to begging the question at issue—while if it mean more than this it is simply and obviously untrue. It is generally known that the Deeds of Settlement of many Societies lay down rules of procedure quite incompatible with the net-premium mode of valuation. But indeed it is evident that the only possible "implied contract" is, that a sufficient fund shall at all times be maintained for the safety of the Office, and that the profits shall be equitably distributed among those entitled to them. In what manner these important conditions are to be fulfilled the public wisely leaves to the decision of the responsible actuary,—being sufficiently alive to the fact that the most obvious view of a scientific question is frequently the very reverse of the true one, to distrust its own opinion upon a matter in reference to which the most eminent authorities are by no means unanimous. If at the same time it were as careful to ascertain the title that officer has to its confidence, it would seldom have reason to regret having left him unfettered in his judgment.

I hope at some future time to submit to the Institute a description of the method of valuation which seems to me to be the best adapted to accomplish the end in view; and I shall then endeavour to show that, quite independently of the fatal objections urged by Mr. Sprague, the net-premium method, so far from possessing that title to preference which its admirers so unreasonably claim for it, is one which must inevitably disappear with that advance of true actuarial science, which the foundation of the Institute of Actuaries was designed to aid.

I remain, Sir,

Your very obedient servant,

*London, 1st June,* 1870. W. M. MAKEHAM.

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## ON HERR WILHELM LAZARUS'S PAPER "ON SOME PROBLEMS IN THE THEORY OF PROBABILITIES."

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—In the paper by Herr Wilhelm Lazarus, in the January number of the *Journal*, which treats of an important branch of the theory of proba-

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bilities in a very lucid manner, there is one passage which seems to me capable of being expressed a little more clearly. The obscurity has no doubt arisen from the great difficulty attendant upon the translation of a paper in German, upon a very abstruse mathematical subject, into a form intelligible to English readers. I hope, therefore, that an attempt to render the paper more easily intelligible will be of some service to your English readers.

The passage to which I allude is at the bottom of page 256, and is as follows:—"Making $z=u$, expanding $\Omega_1$ and $\Omega_2$ under this supposition, " and adding $\Omega_0$, which may be done in the simplest manner as we expand " equation (28) or (29) while calculating the most probable case—making " also $\Omega_0+\Omega_1+\Omega_2=\Omega$, we find in place of equation (8),

$$\text{“}\ \Omega=\frac{1}{\sqrt{\pi}}\int_0^{k_2}e^{-t^2}dt+\frac{1}{\sqrt{\pi}}\int_0^{k_3}e^{-t^2}dt+\frac{B_2}{A_2\sqrt{\pi}}e^{-k_2^2}-\frac{B_3}{A_3\sqrt{\pi}}e^{-k_3^2}\text{”}$$

Now I imagine that this passage will be to most readers by no means self-evident, and that some amplification of it may be acceptable to them.

Herr Lazarus shows that

$$\Omega_1=\frac{\int_0^p x^m(1-x)^{n-1}dx}{\int_0^1 x^m(1-x)^{n-1}dx}-\frac{\int_0^p x^{m+z}(1-x)^{n-z-1}dx}{\int_0^1 x^{m+z}(1-x)^{n-z-1}dx}$$

$$\Omega_2=\frac{\int_0^p x^{m-z-1}(1-x)^{n+z}dx}{\int_0^1 x^{m-z-1}(1-x)^{n+z}dx}-\frac{\int_0^p x^{m-1}(1-x)^{n}dx}{\int_0^1 x^{m-1}(1-x)^{n}dx}$$

These are simply equations (28) and (29) after making $u=z$; adding,

$$\Omega_1+\Omega_2=\frac{\int_0^p x^{m-z-1}(1-x)^{n+z}dx}{\int_0^1 x^{m-z-1}(1-x)^{n-z}dx}-\frac{\int_0^p x^{m+z}(1-x)^{n-z-1}dx}{\int_0^1 x^{m+z}(1-x)^{n-z-1}dx}$$
$$-\left\{\frac{\int_0^p x^{m-1}(1-x)^{n}dx}{\int_0^1 x^{m-1}(1-x)^{n}dx}-\frac{\int_0^p x^{m}(1-x)^{n-1}dx}{\int_0^1 x^{m}(1-x)^{n-1}dx}\right\}$$

Now the expression between brackets, on the right hand side of this equation, using the notation adopted by Herr Lazarus, is equal to $\overset{m}{\underset{\mu}{\Sigma}}(p^\mu)-\overset{m+1}{\underset{\mu}{\Sigma}}(p^\mu)$, which is nothing but $\Omega_0$, or the probability of the occurrence of the most probable combination, *i.e.*, exactly $m$ E's, and $\mu-m$ F's. Thus

$$\Omega_1+\Omega_2=\frac{\int_0^p x^{m-z-1}(1-x)^{n+z}dx}{\int_0^1 x^{m-z-1}(1-x)^{n-z}dx}-\frac{\int_0^p x^{m+z}(1-x)^{n-z-1}dx}{\int_0^1 x^{m+z}(1-x)^{n-z-1}dx}-\Omega_0$$



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or

$$\Omega_0+\Omega_1+\Omega_2=\frac{\int_0^p x^{m-z-1}(1-x)^{n+z}dx}{\int_0^1 x^{m-z-1}(1-x)^{n+z}dx}-\frac{\int_0^p x^{m+z}(1-x)^{n-z-1}dx}{\int_0^1 x^{m+z}(1-x)^{n-z-1}dx}\ ..(\text{I})$$

$$=\tfrac{1}{2}\mp\frac{1}{\sqrt{\pi}}\int_0^{k_3}e^{-t^2}dt\left(\mp\text{ as }\frac{m-z-1}{\mu-1}>\text{ or }<p\right)-\frac{\text{B}_3}{\text{A}_3\sqrt{\pi}}e^{-k_3^2}$$

$$-\left\{\tfrac{1}{2}\mp\int_0^{k_2}e^{-t^2}dt\left(\mp\text{ as }\frac{m+z}{\mu-1}>\text{ or }<p\right)-\frac{\text{B}_2}{\text{A}_2\sqrt{\pi}}e^{-k_2^2}\right\}$$

$$=\pm\int_0^{k_2}e^{-t^2}dt\left(\pm\text{ as }\frac{m+z}{\mu-1}>\text{ or }<p\right)$$

$$\mp\int_0^{k_3}e^{-t^2}dt\left(\mp\text{ as }\frac{m-z-1}{\mu-1}>\text{ or }<p\right)$$

$$+\frac{\text{B}_2}{\text{A}_2\sqrt{\pi}}e^{-k_2^2}-\frac{\text{B}_3}{\text{A}_3\sqrt{\pi}}e^{-k_3^2}$$

Herr Lazarus appears to have dropped the distinction with regard to the use of the signs preceding the first two terms of this expression; but I do not see his reason for so doing.

The expression for $\Omega$ might have been obtained in the same form directly; only then, as Herr Lazarus points out, the elements of which it is composed would have been lost sight of.

We have

$$\Omega=\sum_{\mu}^{m-z}(p^\mu)-\sum_{\mu}^{m+z+1}(p^\mu)$$

and using the equation (25), viz.,

$$\sum_{\mu}^{\rho}(p^\mu)=\frac{\int_0^p x^{\rho-1}(1-x)^\tau dx}{\int_0^1 x^{\rho-1}(1-x)^\tau dx}\ (\tau+\rho=\mu)$$

we have

$$\Omega=\frac{\int_0^p x^{m-z-1}(1-x)^{n+z}dx}{\int_0^1 x^{m-z-1}(1-x)^{n+z}dx}-\frac{\int_0^p x^{m+z}(1-x)^{n-z-1}dx}{\int_0^1 x^{m+z}(1-x)^{n-z-1}dx}$$

which is the equation (I) already obtained.

Herr Lazarus refers to Laplace and Poisson as his guides in the method by which he obtains the value of the integral $\int x^\alpha(1-x)^\beta dx$ between the limits 0 and $p$, and 0 and 1. Laplace's demonstration of this method will be found in his "*Théorie Analytique des Probabilités,*"

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Seconde Partie, Chapitre Premier; and an excellent English version of this is given in the article on Probabilities (sections 62–69) by Professor de Morgan, in the "*Encyclopædia Metropolitana.*"

I am, Sir,
Your obedient servant,

*June* 7*th*, 1870. WILLIAM SUTTON.
18, *Lincoln's Inn Fields.*

P.S. The last two terms of equation (49) in Herr Lazarus's paper are printed incorrectly.

They ought to be $+\frac{B_4}{A_4\sqrt{\pi}}e^{-k_4^2}-\frac{B_3}{A_3\sqrt{\pi}}e^{-k_3^2}$.

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