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SUCCESSIVE ITERATIONS FOR POSITIVE EXTREMAL SOLUTIONS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS ON A HALF-LINE

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Abstract

In this paper, positive solutions of fractional differential equations with nonlinear terms depending on lower-order derivatives on a half-line are investigated. The positive extremal solutions and iterative schemes for approximating them are obtained by applying a monotone iterative method. An example is presented to illustrate the main results.

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1. Introduction

Fractional calculus has gained considerable attention from both theoretical and applied points of view in recent years. There are numerous applications in a variety of fields such as electrical networks, chemical physics, fluid flow, economics, signal and image processing, viscoelasticity, porous media, aerodynamics, modelling for physical phenomena exhibiting anomalous diffusion, and so on. In contrast to integer-order differential and integral operators, fractional-order differential operators are nonlocal in nature and provide the means to look into hereditary properties of several materials and processes. This aspect of fractional-order operators has helped to improve the mathematical modelling of many real-world problems in the physical and technical sciences. A detailed description of theory and applications of the subject can be found in the texts [3, 8, 9, 17, 19].

Another important contribution of fractional calculus has been observed in the investigation of backward problems. It is well known that the backward problem in time is *severely* ill-posed for the parabolic problem (involving a first-order derivative

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with respect to time: that is, $\alpha = 1$). This severe ill-posedness means that the stability in the backward problem cannot be restored even by strengthening the norm within Sobolev norms for estimating the initial value in L^2 spaces. However, for a fractional order $0 < \alpha < 1$, the backward problem is only *moderately* ill-posed [16]. For the application of fractional calculus in inverse problems concerning the determination of the fractional order, we refer the reader to the paper [5].

In this paper, we consider a nonlinear fractional boundary value problem on a half-line given by

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), D^{\alpha-1}u(t)) = 0, & 1 < \alpha \le 2, \\ u(0) = 0, \quad D^{\alpha-1}u(\infty) = \beta u(\xi), \end{cases} \tag{1.1}$$

where $t \in J = [0, +\infty)$, $f \in C[J \times \mathbb{R} \times \mathbb{R}, J]$ and D^{α} is the standard Riemann–Liouville fractional derivative of order α. Here we emphasise that the nonlinearity in problem (1.1) depends on the unknown function and its lower-order fractional derivative.

It is imperative to note that the available literature on fractional differential equations is mainly concerned with a finite domain rather than with the infinite domain. For work dealing with the existence of solutions (or positive solutions) of nonlinear fractional differential equations on *infinite intervals* (unbounded domains), see [1, 2, 4, 10, 11, 20, 21, 24, 26, 30, 33]. In particular, Zhao and Ge [33] applied the idea of the Leray–Schauder nonlinear alternative theorem to study the existence of positive solutions for the following nonlocal fractional boundary value problem:

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t)) = 0, & 1 < \alpha \le 2, \\ u(0) = 0, \quad \lim_{t \to +\infty} D^{\alpha-1}u(t) = \beta u(\xi), \end{cases}$$

where $t \in J = [0, +\infty)$, $f \in C(J \times \mathbb{R}, [0, +\infty))$, $0 \le \xi < \infty$ and D^{α} is the standard Riemann–Liouville fractional derivative. Su and Zhang [21] used Schauder's fixed point theorem to find the sufficient conditions for the existence of solutions for a problem involving fractional differential equations with nonlinear term depending on a lower-order derivative on the unbounded interval:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), D^{\alpha-1}u(t)) = 0, & 1 < \alpha \le 2, \\ u(0) = 0, \quad \lim_{t \to +\infty} D^{\alpha-1}u(t) = u_{\infty}, \quad u_{\infty} \in \mathbb{R}, \end{cases}$$

where $t \in J = [0, +\infty)$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and D^{α} and $D^{\alpha-1}$ are the standard Riemann–Liouville fractional derivatives.

In the above referenced work [21, 33], only the existence of solutions for the given problems was discussed by using the standard tools of fixed point theory. Applying a similar procedure, one can easily show the existence of solutions for problem (1.1). However, it is more interesting and useful to devise a strategy that not only ensures the existence of solutions for the problem at hand but also provides means for finding solutions. With this in mind, we seek the minimal and maximal positive solutions for problem (1.1) by using the monotone iterative method, which is different from the

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approach employed in [1, 2, 4, 10, 11, 20, 21, 24, 26, 30, 33]. To approximate the minimal and maximal positive solutions, we give two explicit computable iterative sequences. For more details of the application of this method in fractional differential equations, see [6, 7, 9, 12–15, 18, 22, 23, 25, 27–29, 31, 32].

2. Preliminaries and lemmas

First of all, we recall definitions of the Riemann–Liouville fractional derivative and integral.

Definition 2.1 [8]. The Riemann–Liouville fractional derivative of order δ for a continuous function f is defined by

$$D^{\delta} f(t) = \frac{1}{\Gamma(n-\delta)}\Big(\frac{d}{dt}\Big)^{n} \int_0^t (t-s)^{n-\delta-1} f(s)\,ds, \quad n = [\delta]+1,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 [8]. The Riemann–Liouville fractional integral of order δ for a function f is defined as

$$I^{\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\,ds, \quad \delta > 0,$$

provided that the integral exists.

For the analysis below, we define two Banach spaces,

$$X = \Big\{u \in C(J,\mathbb{R}) : \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty\Big\},$$
$$Y = \Big\{u \in X : D^{\alpha-1}u(t) \in C(J,\mathbb{R}), \sup_{t\in J} |D^{\alpha-1}u(t)| < +\infty\Big\},$$

equipped with the respective norms $\|u\|_X = \sup_{t\in J}(|u(t)|/(1+t^{\alpha-1}))$ and $\|u\|_Y = \max\{\|u\|_X, \sup_{t\in J} |D^{\alpha-1}u(t)|\}$.

Define a cone $P \subset Y$ by

$$P = \{u \in Y : u(t) \geq 0, t \in J\}.$$

We now introduce the assumptions that we need in the sequel.

(H_1): $\beta, \xi > 0, \Gamma(\alpha) > \beta\xi^{\alpha-1}$.

(H_2): There exist nonnegative functions $a(t), b(t), c(t)$ defined on $[0,\infty)$ and constants $p, q \geq 0$, such that

$$f(t,u,v) \leq a(t) + b(t)|u|^p + c(t)|v|^q$$

and

$$\int_0^{+\infty} a(t)\,dt = a^* < +\infty, \quad \int_0^{+\infty} b(t)(1+t^{\alpha-1})^p\,dt = b^* < +\infty, \quad \int_0^{+\infty} c(t)\,dt = c^* < +\infty.$$

(H_3): f is nondecreasing with respect to the second and last variables.

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LEMMA 2.3 [21]. *Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following conditions hold:*

(i) *for any $u(t) \in U$, $u(t)/(1 + t^{\alpha-1})$ and $D^{\alpha-1}u(t)$ are equicontinuous on any compact interval of J;*

(ii) *for any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that*

$$\left|\frac{u(t_1)}{1 + t_1^{\alpha-1}} - \frac{u(t_2)}{1 + t_2^{\alpha-1}}\right| < \varepsilon$$

and $|D^{\alpha-1}u(t_1) - D^{\alpha-1}u(t_2)| < \varepsilon$ for any $t_1, t_2 \geq T$ and $u \in U$.

LEMMA 2.4 [33]. *Let $h \in C([0, +\infty))$ with $\int_0^\infty h(s)\,ds < \infty$. If $\Gamma(\alpha) \neq \beta\xi^{\alpha-1}$, then the boundary value problem*

$$\begin{cases} D^\alpha u(t) + h(t) = 0, \\ u(0) = 0, \quad D^{\alpha-1}u(\infty) = \beta u(\xi), \quad \beta, \xi > 0, \end{cases}$$

has a unique solution

$$u(t) = \int_0^{+\infty} G(t, s)h(s)\,ds, \tag{2.1}$$

where

$$G(t, s) = \frac{1}{\Delta}\begin{cases} [\Gamma(\alpha) - \beta(\xi - s)^{\alpha-1}]t^{\alpha-1} - [\Gamma(\alpha) - \beta\xi^{\alpha-1}](t - s)^{\alpha-1}, & s \leq t, s \leq \xi, \\ [\Gamma(\alpha) - \beta(\xi - s)^{\alpha-1}]t^{\alpha-1}, & 0 \leq t \leq s \leq \xi, \\ \Gamma(\alpha)t^{\alpha-1} - [\Gamma(\alpha) - \beta\xi^{\alpha-1}](t - s)^{\alpha-1}, & 0 \leq \xi \leq s \leq t, \\ \Gamma(\alpha)t^{\alpha-1}, & s \geq t, s \geq \xi, \end{cases} \tag{2.2}$$

and $\Delta = \Gamma(\alpha)[\Gamma(\alpha) - \beta\xi^{\alpha-1}]$.

From (2.1),

$$D^{\alpha-1}u(t) = \int_0^{+\infty} G^*(t, s)h(s)\,ds,$$

where

$$G^*(t, s) = \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}}\begin{cases} \beta\xi^{\alpha-1} - \beta(\xi - s)^{\alpha-1}, & s \leq t, s \leq \xi, \\ \Gamma(\alpha) - \beta(\xi - s)^{\alpha-1}, & 0 \leq t \leq s \leq \xi, \\ \beta\xi^{\alpha-1}, & 0 \leq \xi \leq s \leq t, \\ \Gamma(\alpha), & s \geq t, s \geq \xi. \end{cases} \tag{2.3}$$

LEMMA 2.5. *For $(s, t) \in J \times J$, if condition (H_1) holds, then*

$$0 \leq \frac{G(t, s)}{1 + t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}}, \quad 0 \leq G(t, s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha) - \beta\xi^{\alpha-1}}$$

and

$$0 \leq G^*(t, s) \leq \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}}.$$

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PROOF. The assertion is obvious from (2.2) and (2.3). □

LEMMA 2.6. *If condition* (H_2) *is satisfied, then*

$$\int_0^{+\infty} f(s, u(s), D^{\alpha-1}u(s))\, ds \le a^* + b^*\|u\|_Y^p + c^*\|u\|_Y^q, \quad \forall u \in Y.$$

PROOF. For $u \in Y$, by condition (H_2),

$$\begin{aligned}
&\int_0^{+\infty} f(s, u(s), D^{\alpha-1}u(s))\, ds \\
&\quad \le \int_0^{+\infty} [a(s) + b(s)|u(s)|^p + c(s)|D^{\alpha-1}u(s)|^q]\, ds \\
&\quad \le a^* + \int_0^{+\infty} b(s)(1 + s^{\alpha-1})^p \frac{|u(s)|^p}{(1 + s^{\alpha-1})^p}\, ds + \int_0^{+\infty} c(s)|D^{\alpha-1}u(s)|^q\, ds \\
&\quad \le a^* + b^*\|u\|_Y^p + c^*\|u\|_Y^q.
\end{aligned}$$
□

3. Main results

Using Lemma 2.4 with $h(t) = f(t, u(t), Tu(t))$, we define an integral operator Q associated with problem (1.1) by

$$Qu(t) = \int_0^{+\infty} G(t, s) f(s, u(s), D^{\alpha-1}u(s))\, ds. \tag{3.1}$$

Notice that problem (1.1) has a solution if and only if the operator equation $u = Qu$ has a fixed point, where Q is given by (3.1).

LEMMA 3.1. *Assume that conditions* (H_1) *and* (H_2) *hold. Then the operator* $Q : Y \to Y$ *is completely continuous.*

PROOF. The proof consists of two steps.

(a) The operator $Q : Y \to Y$ is relatively compact.

Let Ω be any bounded subset of Y. Then for any $u \in \Omega$, there exists a constant $M > 0$ such that $\|u\|_Y \le M$. By Lemmas 2.5 and 2.6,

$$\begin{aligned}
\|Qu\|_X &= \sup_{t \in J} \int_0^{\infty} \frac{G(t, s)}{1 + t^{\alpha-1}} |f(s, u(s), D^{\alpha-1}u(s))|\, ds \\
&\le \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} \int_0^{+\infty} |f(s, u(s), D^{\alpha-1}u(s))|\, ds \\
&\le \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} [a^* + b^*\|u\|_Y^p + c^*\|u\|_Y^q] \\
&\le \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} [a^* + M^p b^* + M^q c^*],
\end{aligned}$$

which implies that $Q\Omega$ is uniformly bounded.

Next, we show that the operator $Q : Y \to Y$ is equicontinuous.

Let $I \subset J$ be any compact interval and let Ω be any bounded subset of Y. Then, for all $t_1, t_2 \in I, t_2 > t_1$ and $u \in \Omega$,

$$\begin{aligned} \left|\frac{Qu(t_2)}{1+t_2^{\alpha-1}} - \frac{Qu(t_1)}{1+t_1^{\alpha-1}}\right| &= \left|\int_0^\infty \left(\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right) f(s,u(s),D^{\alpha-1}u(s))\,ds\right| \\ &\le \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| |f(s,u(s),D^{\alpha-1}u(s))|\,ds. \end{aligned} \tag{3.2}$$

Since $G(t,s) \in C(J \times J)$, for any compact set $I \times I$, $G(t,s)/(1+t^{\alpha-1})$ is uniformly continuous. Note that this function only depends on t for $s \ge t$. So it is uniformly continuous on $I \times (J \backslash I)$. Thus, for all $s \in J$ and $t_1,\ t_2 \in I$,

$$\forall \varepsilon > 0,\ \exists\, \delta(\varepsilon) \text{ such that if } |t_1 - t_2| < \delta,\ \text{ then } \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| < \varepsilon. \tag{3.3}$$

By Lemma 2.6, for all $u \in \Omega$, we get

$$\int_0^\infty |f(s,u(s),D^{\alpha-1}u(s))|\,ds < \infty, \qquad \forall u \in \Omega.$$

This, together with (3.2) and (3.3), implies that $Qu(t)/(1+t^{\alpha-1})$ is equicontinuous on I.

Observe that

$$D^{\alpha-1}Qu(t) = \int_0^{+\infty} G^*(t,s) f(s,u(s),D^{\alpha-1}u(s))\,ds$$

and the function $G^*(t,s) \in C(J \times J)$ does not depend on t. Thus it is obvious that $D^{\alpha-1}Qu(t)$ is equicontinuous on I. Furthermore,

$$\lim_{t\to\infty} \frac{G(t,s)}{1+t^{\alpha-1}} = \frac{1}{\Gamma(\alpha)[\Gamma(\alpha) - \beta\xi^{\alpha-1}]} \begin{cases} \beta\xi^{\alpha-1} - \beta(\xi - s)^{\alpha-1}, & 0 \le s \le \xi, \\ \beta\xi^{\alpha-1}, & \xi \le s, \end{cases}$$

In view of the above argument, it is easy to verify that for any given $\varepsilon > 0$, there exists a constant $T' = T'(\varepsilon) > 0$ such that

$$\left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| < \varepsilon$$

for any $t_1, t_2 \ge T'$ and $s \in J$. Thus, by Lemma 2.6 and (3.2), we infer that the same property holds for $Q\Omega$, uniformly on Ω. Hence, the operator Q is equiconvergent at ∞. As the function $G^*(t,s)$ does not depend on t, we easily obtain that $D^{\alpha-1}Qu(t)$ is equiconvergent at ∞. Therefore, it follows by Lemma 2.3 that $Q\Omega$ is relatively compact on J.

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(b) The operator $Q : Y \to Y$ is continuous.

Let $u_n, u \in X$ such that $u_n \to u$ $(n \to \infty)$. Then $\|u_n\|_Y < \infty, \|u\|_Y < \infty$. By Lemmas 2.5 and 2.6,

$$\begin{aligned} Qu_n(t) &= \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s, u_n(s), D^{\alpha-1}u_n(s))\,ds \\ &\le \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} \int_0^\infty |f(s, u_n(s), D^{\alpha-1}u_n(s))|\,ds \\ &\le \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} [a^* + b^*\|u_n\|_Y^p + c^*\|u_n\|_Y^q] < \infty \end{aligned}$$

and

$$\begin{aligned} D^{\alpha-1}Qu_n(t) &= \int_0^{+\infty} G^*(t,s) f(s, u_n(s), D^{\alpha-1}u_n(s))\,ds \\ &\le \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} \int_0^\infty |f(s, u_n(s), D^{\alpha-1}u_n(s))|\,ds \\ &\le \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}} [a^* + b^*\|u_n\|_Y^p + c^*\|u_n\|_Y^q] < \infty. \end{aligned}$$

By the Lebesgue dominated convergence theorem and continuity of f, we obtain

$$\lim_{n\to\infty} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s, u_n(s), D^{\alpha-1}u_n(s))\,ds = \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s, u(s), D^{\alpha-1}u(s))\,ds$$

and

$$\lim_{n\to\infty} \int_0^\infty G^*(t,s) f(s, u_n(s), D^{\alpha-1}u_n(s))\,ds = \int_0^\infty G^*(t,s) f(s, u(s), D^{\alpha-1}u(s))\,ds.$$

In consequence,

$$\begin{aligned} \|Qu_n - Qu\|_X = \sup_{t\in J} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} &|f(s, u_n(s), D^{\alpha-1}u_n(s)) \\ &- f(s, u(s), D^{\alpha-1}u(s))|\,ds \to 0 \quad (n \to \infty), \end{aligned}$$

and

$$\sup_{t\in J} |D^{\alpha-1}Qu_n - D^{\alpha-1}Qu| \to 0 \quad (n \to \infty).$$

This shows that the operator Q is continuous.

From the above steps, we conclude that the operator $Q : Y \to Y$ is completely continuous. This completes the proof. □

THEOREM 3.2. *Let the conditions* $(H_1), (H_2)$ *and* (H_3) *be satisfied. Then there exists a positive constant* R *such that problem (1.1) has minimal and maximal positive*

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solutions v^*, u^* respectively in $(0, Rt^{\alpha-1}]$, which can be obtained by means of the following two explicit monotone iterative sequences:

$$v_{n+1} = \int_0^{+\infty} G(t,s)a(s)f(s, v_n(s), D^{\alpha-1}v_n(s))\,ds, \quad \textit{with initial value } v_0(t) = 0,$$
$$u_{n+1} = \int_0^{+\infty} G(t,s)a(s)f(s, u_n(s), D^{\alpha-1}u_n(s))\,ds, \quad \textit{with initial value } u_0(t) = Rt^{\alpha-1}. \tag{3.4}$$

Moreover,

$$v_0 \le v_1 \le \cdots \le v_n \le \cdots \le v^* \le \cdots \le u^* \le \cdots \le u_n \le \cdots \le u_1 \le u_0.$$

Proof. For computational convenience, we set

$$L = \frac{1}{\Gamma(\alpha) - \beta\xi^{\alpha-1}}.$$

In view of $\Gamma(\alpha) > \beta\xi^{\alpha-1}$, Lemma 2.5 leads to the fact that $(Qu)(t) \ge 0$ for any $u \in P, t \in J$. Thus, $Q(P) \subset P$.

For $0 \le p, q < 1$, choose

$$R \ge \max\{3La^*, (3Lb^*)^{1/(1-p)}, (3Lc^*)^{1/(1-q)}\}$$

and define $B = \{u \in Y, \|u\|_Y \le R\}$. In what follows, we first show that $Q(B) \subset B$.

For any $u \in B$, by Lemmas 2.5 and 2.6,

$$\begin{aligned}\|Qu\|_X &= \sup_{t\in J} \int_0^{\infty} \frac{G(t,s)}{1+t^{\alpha-1}} f(s, u(s), D^{\alpha-1}u(s))\,ds \\ &\le L\int_0^{+\infty} f(s, u(s), D^{\alpha-1}u(s))\,ds \\ &\le L[a^* + b^*\|u\|_Y^p + c^*\|u\|_Y^q] \\ &\le L[a^* + R^p b^* + R^q c^*] \\ &\le R\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in J} |D^{\alpha-1}Qu(t)| &= \sup_{t\in J} \int_0^{\infty} G^*(t,s) f(s, u(s), D^{\alpha-1}u(s))\,ds \\ &\le L\int_0^{+\infty} f(s, u(s), D^{\alpha-1}u(s))\,ds \\ &\le R.\end{aligned}$$

This implies that $\|Qu\|_Y \le R$, for all $u \in B$. Thus $Q(B) \subset B$.

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The definition of the operator Q and condition (H_3) imply that the operator Q is nondecreasing.

Denote $v_0(t) = 0, v_1 = Q0 = Qv_0, v_2 = Q^2 0 = Qv_1$, for all $t \in J$. Since $v_0(t) = 0 \in B$ and $Q : B \to B$, we have $v_1 \in Q(B) \subset B$ and $v_2 \in Q(B) \subset B$. So,

$$v_1(t) = (Q0)(t) \geq 0 = v_0(t), \quad \forall t \in J.$$

By the nondecreasing nature of the operator Q, we get

$$v_2(t) = (Qv_1)(t) \geq (Qv_0)(t) = v_1(t), \quad \forall t \in J.$$

By induction, we can now define a sequence $v_{n+1} = Qv_n,\ n = 0, 1, 2, \ldots$. Clearly the sequence $\{v_n\}_{n=1}^{\infty} \subset Q(B) \subset B$ and satisfies

$$v_{n+1}(t) \geq v_n(t), \quad \forall t \in J,\ n = 0, 1, 2, \ldots. \tag{3.5}$$

By the complete continuity of the operator Q, we have that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists a $v^* \in B$ such that $v_{n_k} \to v^*$ as $k \to \infty$. This, together with (3.5), implies that $\lim_{n\to\infty} v_n = v^*$.

Since Q is continuous and $v_{n+1} = Qv_n$, we have $Qv^* = v^*$, that is, v^* is a fixed point of the operator Q.

Denote $u_0(t) = Rt^{\alpha-1}, u_1 = Qu_0, u_2 = Q^2 u_0 = Qu_1$, for all $t \in J$. Since $u_0(t) \in B$ and $Q : B \to B$, we get $u_1 \in Q(B) \subset B$ and $u_2 \in Q(B) \subset B$. By Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}
u_1(t) &= \int_0^{+\infty} G(t, s) f(s, u_0(s), D^{\alpha-1} u_0(s))\, ds \\
&\leq \int_0^{+\infty} Lt^{\alpha-1} f(s, u_0(s), D^{\alpha-1} u_0(s))\, ds \\
&\leq Lt^{\alpha-1}(a^* + b^* \|u_0\|_Y^p + c^* \|u_0\|_Y^q) \\
&\leq Rt^{\alpha-1} = u_0(t), \quad \forall t \in J.
\end{aligned}$$

Noting that Q is nondecreasing, we get

$$u_2(t) = (Qu_1)(t) \leq (Qu_0)(t) = u_1(t), \quad \forall t \in J.$$

As before, by induction, we define $u_{n+1} = Qu_n, n = 0, 1, 2, \ldots$. Then the sequence $\{u_n\}_{n=1}^{\infty} \subset Q(B) \subset B$ and satisfies the relation

$$u_{n+1}(t) \leq u_n(t), \quad \forall t \in J,\ n = 0, 1, 2, \ldots. \tag{3.6}$$

Similarly to earlier arguments, it can be shown that there exists a $u^* \in B$ such that $\lim_{n\to\infty} u_n = u^*$.

Since Q is continuous and $u_{n+1} = Qu_n$, we have $Qu^* = u^*$ which, in turn, implies that u^* is a fixed point of the operator Q.

We are now in a position to show that u^* and v^* are the maximal and minimal positive solutions of (1.1) respectively in $(0, Rt^{\alpha-1}]$. We first establish that if

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$w \in [0, bt^{\alpha-1}]$ is any solution of (1.1), then $v_0(t) = 0 \le w(t) \le Rt^{\alpha-1} = u_0(t)$ and $Qw = w$. Using the monotone nature of Q, we have that $v_1(t) = Qv_0(t) \le w(t) \le Qu_0(t) = u_1(t)$, for all $t \in J$.

Repeating the above process several times, we obtain

$$v_n(t) \le w(t) \le u_n(t), \quad \forall t \in J,\ n = 0, 1, 2, \ldots. \tag{3.7}$$

In view of $u^* = \lim_{n\to\infty} u_n$ and $v^* = \lim_{n\to\infty} v_n$, it follows from (3.5)–(3.7) that

$$v_0 \le v_1 \le \cdots \le v_n \cdots \le v^* \le w \le u^* \cdots \le u_n \le \cdots \le u_1 \le u_0. \tag{3.8}$$

Since $f(t, 0) \not\equiv 0$, for all $t \in J$, it follows that 0 is not a solution of problem (1.1). It follows from (3.8) that u^* and v^* are the maximal and minimal positive solutions of (1.1) respectively in $(0, Rt^{\alpha-1}]$, which can be obtained by the corresponding iterative sequences in (3.4).

With regard to the range of p and q, the method is similar, so we omit the details.

This completes the proof. □

4. Example

Consider the following nonlocal fractional boundary value problem for a nonlinear fractional differential equation on a half-line:

$$\begin{cases} D^{1.25}u(t) + \dfrac{2}{(5+t)^2} + \dfrac{e^{-3t}|u(t)|^p}{(1+\sqrt[4]{t})^p} + \dfrac{2\ln(1+t)|D^{0.25}u(t)|^q}{(2+t^2)^2} = 0, \quad t \in [0, +\infty), \\ u(0) = 0, \quad D^{0.25}u(\infty) = 0.5u(1), \end{cases} \tag{4.1}$$

where $\alpha = 1.25, \beta = 0.5, \xi = 1$ and

$$f(t, u, v) = \frac{2}{(5+t)^2} + \frac{e^{-3t}|u|^p}{(1+\sqrt[4]{t})^p} + \frac{2\ln(1+t)|v|^q}{(2+t^2)^2}, \quad 0 \le p, q < 1.$$

Obviously $\Gamma(1.25) \approx 0.913\,15, \beta\xi^{\alpha-1} = 0.5$. Thus (H_1) holds.

Next, taking

$$a(t) = \frac{2}{(5+t)^2}, \quad b(t) = \frac{e^{-3t}}{(1+\sqrt[4]{t})^p}, \quad c(t) = \frac{2t}{(2+t^2)^2},$$

we have

$$\begin{aligned} |f(t, u, v)| &= \frac{2}{(5+t)^2} + \frac{e^{-3t}|u|^p}{(1+\sqrt[4]{t})^p} + \frac{2\ln(1+t)|v|^q}{(2+t^2)^2} \\ &\le \frac{2}{(5+t)^2} + \frac{e^{-3t}|u|^p}{(1+\sqrt[4]{t})^p} + \frac{2t|v|^q}{(2+t^2)^2} \\ &\triangleq a(t) + b(t)|u|^p + c(t)|v|^q, \end{aligned}$$

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and

$$a^* = \int_0^{+\infty} a(t)\,dt = \int_0^{+\infty} \frac{2}{(5+t)^2}\,dt = \frac{2}{5} < +\infty,$$
$$b^* = \int_0^{+\infty} b(t)(1+t^{\alpha-1})^p\,dt = \int_0^{+\infty} \frac{e^{-3t}}{(1+\sqrt[4]{t})^p}(1+\sqrt[4]{t})^p\,dt = \int_0^{+\infty} e^{-3t}\,dt = \frac{1}{3} < +\infty,$$
$$c^* = \int_0^{+\infty} c(t)\,dt = \int_0^{+\infty} \frac{2t}{(2+t^2)^2}\,dt = \frac{1}{2} < +\infty,$$

implying that (H_2) holds.

From the expression for f, it is easy to see that f is nondecreasing with respect to the second and last variables. This means that (H_3) holds.

Hence, by Theorem 3.2, it follows that there exists a positive constant R such that the fractional boundary value problem (4.1) has the minimal and maximal positive solutions v^*, u^* respectively in $(0, Rt^{\alpha-1}]$, which can be approximated by the following iterative sequences:

$$v_{n+1} = \int_0^{+\infty} G(t,s)a(s)f(s, v_n(s), D^{\alpha-1}v_n(s))\,ds, \quad \text{with initial value } v_0(t) = 0,$$
$$u_{n+1} = \int_0^{+\infty} G(t,s)a(s)f(s, u_n(s), D^{\alpha-1}u_n(s))\,ds, \quad \text{with initial value } u_0(t) = Rt^{\alpha-1}.$$

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