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A NOTE ON IMPLICIT ITERATION PROCESSES

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Abstract

Let C be a closed, bounded, convex subset of a uniformly convex Banach space, and let $\{T_s\}$ be an asymptotic nonexpansive semigroup of nonlinear mappings acting within C. Consider the implicit iteration process defined by the sequence of equations:

$$x_{k+1} = c_k T_{s_{k+1}}(x_{k+1}) + (1 - c_k)x_k,$$

where each $c_k \in (0, 1)$ and the initial point $x_0 \in C$ is arbitrarily chosen. In this context, we investigate the conditions under which the sequence $\{x_k\}$ converges, either weakly or strongly, to a common fixed point of the semigroup $\{T_s\}$. We also touch upon the question of the stability of such processes.

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1. Introduction

In this paper, we will consistently refer to X as a uniformly convex real Banach space, while C will represent a nonempty, bounded, closed and convex subset of X.

An implicit iteration process for the fixed-point construction is based on the observation that for a nonexpansive mapping $T : C \to C$, any c with $0 < c < 1$ and an $x_0 \in C$, the equation $x = cT(x) + (1 - c)x_0$ has a unique solution $x_c \in C$. This uniqueness is assured by the Banach contraction principle. Note that $x_c \in C$ can be derived as the strong limit of the Picard iterates. The method is appealing from both theoretical and practical perspectives, as the convergence of Picard iterates has been extensively studied over many years, resulting in a plethora of practical algorithms that can be applied.

There are known results on weak or strong convergence of implicit iteration processes for nonexpansive semigroups in Hilbert spaces and uniformly convex Banach spaces with the Opial property (see [3, 13, 15, 17, 18]). The question of weak convergence when X is simultaneously uniformly convex and uniformly smooth was resolved in [7]. The strong convergence under a compactness assumption on C

was examined in [6, 9]. However, the case in which the semigroup is asymptotic nonexpansive has not yet been studied. Since the seminal paper [2], the concept of asymptotic nonexpansive mappings and its extension to semigroups have been firmly established in mainstream fixed point theory and its applications. Therefore, it is important to address this gap, which is the aim of this paper.

DEFINITION 1.1. A one-parameter family $\mathcal{T} = \{T_t : t \in J = [0, +\infty)\}$ of mappings from C into itself is called a continuous semigroup on C if it satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;
(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in J$;
(iii) for each $x \in C$, the mapping $t \mapsto T_t(x)$ is strong continuous at every $t \in J$.

Denote $F(T_t) = \{x \in C : T_t(x) = x\}$ and define the set of all common fixed points for the mappings in $\mathcal{T}$ as $F(\mathcal{T}) = \bigcap_{t\in J} F(T_t)$. Common fixed points are often interpreted as the stationary points of the system defined by the semigroup $\mathcal{T}$.

DEFINITION 1.2. A continuous semigroup $\mathcal{T}$ is called asymptotic nonexpansive if for each $t \in J$, there exists a number $a_t \geq 1$ such that $\lim_{t\to\infty} a_t = 1$, and

$$\|T_t(x) - T_t(y)\| \leq a_t\|x - y\|$$

holds for all $x, y \in C$. We will use the notation $b_t = a_t - 1$. The class of all asymptotic nonexpansive semigroups on C will be denoted by $\mathcal{ANS}(C)$.

The issue of whether asymptotic nonexpansive semigroups have common fixed points has been addressed in a broader context in [4, Theorem 3.4]. Below, we present this result as adapted to accommodate our specific setting.

THEOREM 1.1. *Let X be a uniformly convex Banach space and C be a nonempty, closed, bounded and convex subset of X. Let $\mathcal{T}$ be an asymptotic nonexpansive semigroup on C. Then, $\mathcal{T}$ has a common fixed point, and the set $F(\mathcal{T})$ of all common fixed points is closed and convex.*

DEFINITION 1.3. We will say that a semigroup $\mathcal{T} \in \mathcal{ANS}(C)$ is equicontinuous if the family of mappings $\{t \mapsto T_t(x) : x \in C\}$ is equicontinuous at $t = 0$.

Let us present a precise definition of the implicit iteration process and its related concepts.

DEFINITION 1.4. Let $\mathcal{T} \in \mathcal{ANS}(C)$. Assume that there exists $\gamma \geq 1$ such that $a_t \leq \gamma$ for all $t \in J$. Let $0 < \beta < 1$ be chosen so that $\beta\gamma < 1$ and let $0 < \alpha < \beta$. Assume that $\{c_k\}$ is a sequence of numbers such that $0 < \alpha \leq c_k \leq \beta < 1$ for every $k \in \mathbb{N}$. Let $\{t_k\}$ be a sequence of real numbers from $(0, +\infty)$ such that

$$\sum_{k=1}^{\infty} b_{t_k} < \infty$$

holds for every $x \in C$. Recall that $b_t = a_t - 1$. The implicit iteration process $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is defined by

$$\begin{cases} x_0 \in C, \\ x_{k+1} = c_k T_{t_{k+1}}(x_{k+1}) + (1 - c_k) x_k \quad \text{for } k \geq 0. \end{cases} \tag{1.1}$$

Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We say that the sequence $\{x_k\}_{k \in \mathbb{N}_0}$ is generated by the process $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ and write $\{x_k\} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$.

REMARK 1.5. Since the sequence $\{x_k\}$ generated by an implicit iteration process $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is constructed 'implicitly' by a set of equations (1.1), there is a legitimate question about the existence and uniqueness of such a construction process. Let us now address this question. For $k \in \mathbb{N}_0$, $u \in C$, $w \in C$, let us introduce the following notation:

$$P_{k,w}(u) = c_k T_{t_{k+1}}(u) + (1 - c_k) w.$$

Observe that for every $k \in \mathbb{N}$ and all $u, v \in C$,

$$\|P_{k,w}(u) - P_{k,w}(v)\| \leq c_k \|T_{t_{k+1}}(u) - T_{t_{k+1}}(v)\| \leq c_k a_{t_{k+1}} \|u - v\| \leq \beta\gamma \|u - v\|.$$

This means that each $P_{k,w}(u) : C \to C$ is a contraction. Therefore, it follows from the Banach contraction principle that each x_{k+1} in (1.1) is uniquely defined.

2. Auxiliary results

The following technical lemmas will be used in this paper.

LEMMA 2.1 [1]. *Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{d_{k,n}\}$ is a doubly indexed sequence of real numbers which satisfy*

$$\limsup_{k \to \infty} \limsup_{n \to \infty} d_{k,n} \leq 0 \quad \text{and} \quad r_{k+n} \leq r_k + d_{k,n}$$

for each $k, n \geq 1$. Then, $\{r_k\}$ converges to an $r \in \mathbb{R}$.

LEMMA 2.2 [14, 21]. *Let X be a uniformly convex Banach space, and let a sequence $\{c_n\}$ within the interval $(0, 1)$ be bounded away from 0 and 1. Assume that the two sequences $\{u_n\}, \{v_n\}$ of elements of X satisfy the inequalities*

$$\limsup_{n \to \infty} \|u_n\| \leq a, \quad \limsup_{n \to \infty} \|v_n\| \leq a, \quad \lim_{n \to \infty} \|c_n u_n + (1 - c_n) v_n\| = a.$$

Then, $\lim_{n \to \infty} \|u_n - v_n\| = 0$.

LEMMA 2.3 [16, Lemma 1]. *Let $\{t_n\}$ be a sequence of real numbers, and let $\tau \in \mathbb{R}$ be such that $\lim_{n \to \infty}(t_{n+1} - t_n) = 0$ and*

$$\liminf_{n \to \infty} t_n \leq \tau \leq \limsup_{n \to \infty} t_n.$$

Then, τ is a cluster point of the sequence $\{t_n\}$.

LEMMA 2.4. *Let X be a uniformly convex Banach space and C be a nonempty, closed, bounded and convex subset of X. Let $w \in F(\mathcal{T})$ and $\{x_k\} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$. Then, there exists $r \geq 0$ such that $\lim_{k\to\infty} \|x_k - w\| = r$.*

PROOF. It follows from the inequality

$$\begin{aligned} \|x_{k+1} - w\| &= \|c_k T_{t_{k+1}}(x_{k+1}) + (1 - c_k)x_k - w\| \\ &\leq c_k \|T_{t_{k+1}}(x_{k+1}) - T_{t_{k+1}}(w)\| + (1 - c_k)\|x_k - w\| \\ &\leq c_k(1 + b_{t_k})\|x_{k+1} - w\| + (1 - c_k)\|x_k - w\| \end{aligned}$$

that

$$\|x_{k+1} - w\| \leq \frac{c_k}{1 - c_k} b_{t_k} \|x_{k+1} - w\| + \|x_k - w\| \leq \frac{\beta}{1 - \beta} b_{t_k} \mathrm{diam}(C) + \|x_k - w\|.$$

Set $d_{k,n} = (\beta/(1-\beta))\mathrm{diam}(C) \sum_{i=k+1}^{k+n} b_{t_i}$ and note that $\|x_{k+n} - w\| \leq \|x_k - w\| + d_{k,n}$ for every $n \in \mathbb{N}$. Since $\sum_{i=1}^{+\infty} b_{t_i} < \infty$, it follows that $\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} = 0$. Consequently, by using Lemma 2.1 with $r_k = \|x_k - w\|$, we infer that there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$. The proof is complete. □

To streamline our discussion, we will introduce the concept of asymptotic nonexpansive sequences.

DEFINITION 2.5. We say that a sequence $\{T_k\}$ of mappings that act within C is an asymptotic nonexpansive sequence if there exists a sequence $\{A_k\}$ of numbers such that $A_k \geq 1$, $\lim_{k\to\infty} A_k = 1$ and $\|T_k(x) - T_k(y)\| \leq A_k \|x - y\|$ for all $x, y \in C$ and all $k \in \mathbb{N}$. By $\mathcal{A}(C)$, we will denote the class of all asymptotic nonexpansive sequences of mappings $\{T_k\}$ acting within C. We say that a sequence $\{x_k\}$ of elements of C is generated by $\{T_k\}$ if $x_1 \in C$ and $x_{k+1} = T_k(x_n)$ for every $k \in \mathbb{N}$. Denoting $B_k = A_k - 1$, let us define $\mathcal{A}_c(C)$ as the class of all $\{T_k\} \in \mathcal{A}(C)$ such that $\sum_{k=1}^{\infty} B_k < \infty$.

The following lemma introduces the key technique for proving the weak convergence of the iteration processes generated by asymptotic nonexpansive sequences of operators acting in uniformly smooth Banach spaces.

LEMMA 2.6 [5, Lemma 4.6]. *Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X. Let $\{T_k\} \in \mathcal{A}_c(C)$ and let $\{x_k\}$ be a sequence generated by $\{T_k\}$. Assume that $w_1, w_2 \in \bigcap_{k=1}^{\infty} F(T_k)$. Then,*

$$\langle y - z, J(w_1 - w_2)\rangle = 0$$

for any two weak cluster points y, z of the sequence $\{x_k\}$.

Let us recall the definition of the Opial property.

DEFINITION 2.7 [12]. A Banach space X is said to have the Opial property if, for each sequence $\{x_n\}$ of elements from X that weakly converges to a point $x \in X$ (denoted as $x_n \rightharpoonup x$) and for any $y \in X$ such that $y \neq x$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Recall that Hilbert spaces possess the Opial property. Additionally, the sequence spaces l^p for $1 \leq p < \infty$ exhibit the Opial property. However, many significant uniformly convex Banach spaces, such as L^p for $1 < p \neq 2$, do not possess the Opial property. The following version of the demiclosedness principle will address both the Opial property and the uniformly smooth situations. Notable examples of spaces that are uniformly convex and uniformly smooth include all L^p for $p > 1$. For the case of X possessing the Opial property, this result is based on [11, Theorem 22.17]. In the uniformly smooth case, the reader is referred to [5, Theorem 5.1].

THEOREM 2.1 (The demiclosedness principle). *Let X be a uniformly convex Banach space. Assume that either X has the Opial property or that X is uniformly smooth. Let C be a nonempty, bounded, closed and convex subset of X, and let $\mathcal{T} \in \mathcal{ANS}(C)$. Assume that there exists $w \in X$ and a sequence $\{x_n\}$ of elements of C such that $x_n \rightharpoonup w$. If $\{x_n\}$ is an approximate fixed point sequence for each $T_s \in \mathcal{T}$ (that is, $\|T_s(x_n) - x_n\| \to 0$ as $n \to \infty$), then $w \in F(\mathcal{T})$.*

3. Strong convergence

The following strong convergence result generalises and extends [6, Theorem 2.6] from the context of nonexpansive semigroups to the asymptotic nonexpansive setting.

THEOREM 3.1. *Let C be a convex compact subset of a uniformly convex Banach space X. Let $\mathcal{T}$ be an asymptotic pointwise nonexpansive semigroup. Assume that there exists $\gamma \geq 1$ such that the inequality $a_t \leq \gamma$ holds for all $t \in J$. Assume that $\{x_k\} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is an implicit iteration process and that the sequence $\{t_k\}$ satisfies the conditions*

$$0 = \liminf_{n\to\infty} t_n < \limsup_{n\to\infty} t_n,$$

$$\lim_{n\to\infty} (t_{n+1} - t_n) = 0.$$

Then, there exists a common fixed point $x \in F(\mathcal{T})$ such that $\|x_k - x\| \to 0$.

PROOF. Fix any t with $0 < t < \limsup_{n\to\infty} t_n$. According to Lemma 2.3, there exists a subsequence $\{t_{k_n}\}$ of $\{t_k\}$ such that

$$\lim_{n\to\infty} t_{k_n} = t. \tag{3.1}$$

We will prove that

$$\lim_{n\to\infty} \|T_{t_{k_n}}(x_{k_n}) - x_{k_n}\| = 0. \tag{3.2}$$

Fix temporarily a $w \in F(\mathcal{T})$, which exists according to Theorem 1.1. By Lemma 2.4, there exists $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$. Since $a_{t_k} \to 1$, it follows that

$$\limsup_{k\to\infty} \|T_{t_k}(x_k) - w\| = \limsup_{k\to\infty} \|T_{t_k}(x_k) - T_{t_k}(w)\| \leq \limsup_{k\to\infty} a_{t_k}\|x_k - w\| = r.$$

Denote $v_k = x_{k-1} - w$, $u_k = T_{t_k}(x_k) - w$, and observe that $\lim_{k\to\infty} \|v_k\| = r$, $\limsup_{k\to\infty} \|u_k\| \le r$ and $\lim_{k\to\infty} \|c_k u_k + (1 - c_k)v_k\| = \lim_{k\to\infty} \|x_k - w\| = r$. We can, therefore, apply Lemma 2.2 to obtain $\lim_{k\to\infty} \|T_{t_k}(x_k) - x_{k-1}\| = \lim_{k\to\infty} \|u_k - v_k\| = 0$, which implies that $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0$ because of the equality

$$\|x_{k+1} - x_k\| = \|c_k T_{t_{k+1}}(x_{k+1}) + (1 - c_k)x_k - x_k\| = c_k\|T_{t_{k+1}}(x_{k+1}) - x_k\|.$$

Finally,

$$\lim_{k\to\infty} \|T_{t_k}(x_k) - x_k\| \le \lim_{k\to\infty} \|T_{t_k}(x_k) - x_{k-1}\| + \lim_{k\to\infty} \|x_{k-1} - x_k\| = 0,$$

proving (3.2). Since C is compact, there exists a subsequence $\{x_{k_{n_i}}\}$ of $\{x_{k_n}\}$ and an element of $x \in C$ such that

$$\lim_{i\to\infty} \|T_{t_{k_{n_i}}}(x_{k_{n_i}}) - x\| = 0. \tag{3.3}$$

Denote $s_i = t_{k_{n_i}}$, $w_i = x_{k_{n_i}}$. From (3.2), (3.3) and the inequality

$$\|w_i - x\| \le \|w_i - T_{s_i}(w_i)\| + \|T_{s_i}(w_i) - x\|,$$

we derive $\|w_i - x\| \to 0$. Therefore,

$$\begin{aligned}\|T_{s_i}(x) - x\| &\le \|T_{s_i}(x) - T_{s_i}(w_i)\| + \|T_{s_i}(w_i) - w_i\| + \|w_i - x\| \\ &\le a_{s_i}(x)\|x - w_i\| + \|T_{s_i}(w_i) - w_i\| + \|w_i - x\| \\ &\le \gamma\|x - w_i\| + \|T_{s_i}(w_i) - w_i\| + \|w_i - x\| \to 0, \end{aligned} \tag{3.4}$$

as $i \to \infty$. From (3.4), (3.1) and the continuity of the semigroup $\mathcal{T}$, we conclude that

$$\|T_t(x) - x\| \le \|T_t(x) - T_{s_i}(x)\| + \|T_{s_i}(x) - x\| \to 0.$$

Thus, $T_t(x) = x$. We now need to prove this equality for any $s > 0$. Observe that there exist t, u with $0 < t < \limsup_{n\to\infty} t_n$, $0 < u < \limsup_{n\to\infty} t_n$ and $k \in \mathbb{N}_0$ such that $s = t + ku$. Therefore, $T_s(x) = x$, because

$$T_s(x) = T_{ku}(T_t(x)) = T_{ku}(x) = T_{u+\ldots+u}(x) = x.$$

We conclude that $x \in F(\mathcal{T})$. It only remains to show that $\lim_{k\to\infty} \|x_k - x\| = 0$. To achieve this, note that, according to Lemma 2.4, there exists a nonnegative number r such that $\lim_{k\to\infty} \|x_k - x\| = r$. We previously established that $\lim_{i\to\infty} \|x_{k_{n_i}} - x\| = \lim_{i\to\infty} \|w_i - x\| = 0$, which requires $r = 0$. Thus, the proof is complete. □

4. Weak convergence

Observe that the properties of the implicit iteration processes, discussed so far, did not depend on our choice of the sequence $\{t_k\}$. Not surprisingly, to be able to prove the weak convergence of such processes, we will have to impose some restrictions on $\{t_k\}$. This leads us to a notion of a normalised implicit iteration process introduced in [7, Definition 3.2].

DEFINITION 4.1. We will say that an implicit iteration process $\{x_k\} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is normalised if the following two conditions are satisfied:

$$\lim_{k\to\infty} t_k = 0 \tag{4.1}$$

and

$$\lim_{k\to\infty} \frac{1}{t_k}\|T_{t_k}(x_k) - x_k\| = 0. \tag{4.2}$$

REMARK 4.2. It is important to note that a sequence $\{t_k\}$ can always be constructed to satisfy (4.1) and (4.2). For one such method of construction, refer to [7, Remark 3.1]. This method had been studied in relation to the convergence of processes within Hilbert spaces and uniformly convex Banach spaces with the Opial property. Additional references include [13, 16, 17].

The role of sequences defining the normalised process becomes clear in view of the following corollary to the demiclosedness principle.

LEMMA 4.3. *Let C be a nonempty, closed, bounded and convex subset of a uniformly convex Banach space X. Assume that either X has the Opial property or that X is uniformly smooth. Let $\mathcal{T} \in \mathcal{ANS}(C)$ be equicontinuous on C. Assume that there exists $\gamma \geq 1$ such that $a_t \leq \gamma$ holds for all $t \in J$. Let $\{s_i\}$ denote a sequence of strictly positive real numbers converging to zero as $i \to \infty$. Additionally, let $\{x_i\}$ be a sequence of elements from C such that $\{x_i\}$ converges weakly to some element $w \in C$. If*

$$\lim_{i\to\infty} \frac{1}{s_i}\|T_{s_i}(x_i) - x_i\| = 0, \tag{4.3}$$

then $w \in F(\mathcal{T})$.

PROOF. Let us fix an arbitrary $t > 0$. Without any loss of generality, we can assume that $s_i < t$ for every $i \in \mathbb{N}$. Denoting for simplicity $p_i = [t/s_i]$,

$$\begin{aligned}
\|x_i - T_t(x_i)\| &\leq \sum_{k=0}^{p_i-1} \|T_{(k+1)s_i}(x_i) - T_{ks_i}(x_i)\| + \|T_{p_i s_i}(x_i) - T_t(x_i)\| \\
&\leq \|T_{s_i}(x_i) - x_i\| \sum_{k=0}^{p_i-1} a_{ks_i} + \|T_{p_i s_i}(x_i) - T_t(x_i)\| \\
&\leq \frac{t}{s_i}\gamma\|T_{s_i}(x_i) - x_i\| + \|T_{t-p_i s_i}(x_i) - x_i\|.
\end{aligned}$$

Since $(t/s_i)\|T_{s_i}(x_i) - x_i\| \to 0$ by (4.3) and $t - p_i s_i \to 0$, it follows from the equicontinuity of $\mathcal{T}$ that $\|x_i - T_t(x_i)\| \to 0$. Therefore, the sequence $\{x_i\}$ is an approximate fixed point sequence for every $t \in J$. By assumption, $x_i \rightharpoonup w$. Consequently, it follows from the demiclosedness principle (Theorem 2.1) that $w \in F(\mathcal{T})$, as claimed. □

At this point, we have gathered all the necessary preparatory information to prove our weak convergence theorem for normalised implicit iteration processes in uniformly convex Banach spaces that possess the Opial property.

THEOREM 4.1. *Let C be a nonempty, closed, bounded and convex subset of a uniformly convex Banach space X with the Opial property. Let $\mathcal{T} \in \mathcal{ANS}(C)$ be equicontinuous on C. Assume that there exists $\gamma \geq 1$ such that $a_t(x) \leq \gamma$ holds for every $x \in C$ and for all $t \in J = [0, +\infty)$. Let $\{x_k\} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ be a normalised implicit iteration process. Then, there exists a common fixed point $w \in F(\mathcal{T})$ such that $x_k \rightharpoonup w$.*

PROOF. Since $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is normalised, it follows that $t_k \to 0$ and that

$$\lim_{k\to\infty} \frac{1}{t_k} \|T_{t_k}(x_k) - x_k\| = 0. \tag{4.4}$$

This implies that

$$\lim_{k\to\infty} \|T_{t_k}(x_k) - x_k\| = 0.$$

Consider $y, z \in C$, which are two weak cluster points of the sequence $\{x_k\}$. Then, there exist two subsequences $\{y_k\}$ and $\{z_k\}$ of $\{x_k\}$ such that $y_k \rightharpoonup y$ and $z_k \rightharpoonup z$. It follows from Lemma 4.3 that $y, z \in F(\mathcal{T})$. According to Lemma 2.4, the limits

$$r_1 = \lim_{k\to\infty} \|x_k - y\|, \quad r_2 = \lim_{k\to\infty} \|x_k - z\|$$

exist. We assert that $y = z$. For a contradiction, assume $y \neq z$. Due to the Opial property,

$$\begin{aligned} r_1 = \liminf_{k\to\infty} \|y_k - y\| &< \liminf_{k\to\infty} \|y_k - z\| = r_2 \\ &= \liminf_{k\to\infty} \|z_k - z\| < \liminf_{k\to\infty} \|z_k - y\| = r_1. \end{aligned}$$

This contradiction indicates that $y = z$, showing that the sequence $\{x_k\}$ has no more than one weak cluster point. Since C is weakly sequentially compact, it follows that $\{x_k\}$ indeed has exactly one weak cluster point $w \in C$, meaning that $x_k \rightharpoonup w$. Since the process is normalised, we also have (4.4). Consequently, by applying Lemma 4.3, we can conclude that $w \in \mathcal{T}$. This concludes the proof. □

The scenario becomes more intricate when the assumption of the Opial property is substituted with the uniform smoothness of X. To start, we need to demonstrate how to represent an implicit iteration process as an explicit iteration process generated by a suitably constructed asymptotic nonexpansive sequence of mappings.

LEMMA 4.4. *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Let $\mathcal{T} \in \mathcal{ANS}(C)$. Assume that there exists $\gamma \geq 1$ such that $a_t \leq \gamma$ for all $t \in J$. Let $\{u_k\} = P(C, \mathcal{T}, u_1, \{c_k\}, \{t_k\})$ be an implicit iteration process. Then, there exists a sequence $\{Z_k\} \in \mathcal{A}_c(C)$ such that $u_{k+1} = Z_k(u_k)$ for all k.*

PROOF. For each $k \in \mathbb{N}$, define the mapping $Z_k : C \to C$ by $Z_k(w) = z_{k,w}$, where $z_{k,w}$ is the unique fixed point of the contractive mapping

$$P_{k,w}(u) = c_k T_{t_{k+1}}(u) + (1 - c_k)w.$$

By straightforward calculation, we obtain $u_{k+1} = Z_k(u_k)$ for $k \in \mathbb{N}$. It is easy to prove that $F(\mathcal{T}) \subset \bigcap_{k=1}^{\infty} F(Z_k)$. We will prove that the sequence $\{Z_k\} \in \mathcal{A}_c(C)$. Let us take any $v_1, v_2 \in C$. Observe that, for $i = 1, 2$, the equation

$$Z_k(v_i) = u_i \tag{4.5}$$

is equivalent to

$$u_i = c_k T_{t_{k+1}}(u_i) + (1 - c_k)v_i. \tag{4.6}$$

Using the notation introduced in (4.5) and (4.6),

$$\begin{aligned} \|Z_k(v_1) - Z_k(v_2)\| &= \|u_1 - u_2\| \\ &\le c_k\|T_{t_{k+1}}(u_1) - T_{t_{k+1}}(u_2)\| + (1 - c_k)\|v_1 - v_2\| \\ &\le c_k a_{t_k}\|u_1 - u_2\| + (1 - c_k)\|v_1 - v_2\|. \end{aligned} \tag{4.7}$$

It follows from (4.7) that

$$(1 - c_k a_{t_k})\|u_1 - u_2\| \le (1 - c_k)\|v_1 - v_2\|,$$

which implies

$$\|Z_k(v_1) - Z_k(v_2)\| = \|u_1 - u_2\| \le \frac{1 - c_k}{1 - c_k a_{t_k}}\|v_1 - v_2\|.$$

Since each $a_{t_k} \ge 1$ and $a_{t_k} \to 1$, it follows that each A_k, defined by

$$A_k = \frac{1 - c_k}{1 - c_k a_{t_k}},$$

possesses the same properties. Using the constants β and γ from Definition 1.4, we obtain

$$B_k := 1 - A_k = c_k \frac{b_{t_k}}{1 - c_k a_{t_k}} \le \frac{\beta}{1 - \beta\gamma} b_{t_k}. \tag{4.8}$$

Since $\beta\gamma < 1$ and $\sum_{k=1}^{\infty} b_{t_k} < \infty$, it follows from (4.8) that $\sum_{k=1}^{\infty} B_k < \infty$. Finally, we conclude that the sequence $\{Z_k\}$ belongs to $\mathcal{A}_c(C)$, as claimed. □

LEMMA 4.5. *Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X. Let $\mathcal{T}$ be an asymptotic nonexpansive semigroup ($\mathcal{T} \in \mathcal{ANS}(C)$), which is equicontinuous on C. Assume that there exists $\gamma \ge 1$ such that $a_t(x) \le \gamma$ for every $x \in C$ and for all $t \in J = [0, +\infty)$. Also assume that $\{x_k\} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is a normalised implicit iteration process and that $w_1, w_2 \in F(\mathcal{T})$. Then,*

$$\langle y - z, J(w_1 - w_2)\rangle = 0 \tag{4.9}$$

for any two weak cluster points y, z of the sequence $\{x_k\}$.

PROOF. Let $Z_k : C \to C$ be the mappings defined in Lemma 4.4. Recall that we established there that $x_{k+1} = Z_k(x_k)$, $F(\mathcal{T}) \subset \bigcap_{k=1}^{\infty} F(Z_k)$ and that the sequence $\{Z_k\}$ belongs to $\mathcal{A}_c(C)$. Therefore, by applying Lemma 2.6, we conclude that the equality (4.9) holds for any two weak cluster points $y,\ z$ of the sequence $\{x_k\}$. □

The lemma presented above plays a vital role in establishing the primary weak convergence result for the case when X is uniformly smooth.

THEOREM 4.2. *Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X. Let $\mathcal{T} \in \mathcal{ANS}(C)$ be equicontinuous on C. Assume that there exists $\gamma \geq 1$ such that $a_t(x) \leq \gamma$ holds for every $x \in C$ and for all $t \in J = [0, +\infty)$. Also assume that $\{x_k\} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is a normalised implicit iteration process. Then, there exists a common fixed point $w \in F(\mathcal{T})$ such that $x_k \rightharpoonup w$.*

PROOF. Since $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is normalised, it follows that $t_k \to 0$ and that

$$\lim_{k\to\infty} \frac{1}{t_k} \|T_{t_k}(x_k) - x_k\| = 0. \tag{4.10}$$

Consider $y, z \in C$, two weak cluster points of the sequence $\{x_k\}$. There exist two subsequences $\{x_{\alpha_n}\}$ and $\{x_{\beta_n}\}$ of the sequence $\{x_k\}$ such that $x_{\alpha_n} \rightharpoonup y,\ \ x_{\beta_n} \rightharpoonup z$. From (4.10), we conclude that

$$\lim_{n\to\infty} \frac{1}{t_{\alpha_n}} \|T_{t_{\alpha_n}}(x_{\alpha_n}) - x_{\alpha_n}\| = 0, \quad \lim_{n\to\infty} \frac{1}{t_{\beta_n}} \|T_{t_{\beta_n}}(x_{\beta_n}) - x_{\beta_n}\| = 0.$$

From Lemma 4.3, it follows that $y \in F(\mathcal{T})$ and $z \in F(\mathcal{T})$. According to Lemma 4.5, this implies $\|y - z\|^2 = \langle y - z, J(y - z)\rangle = 0$, leading us to conclude that $y = z$. Therefore, the sequence $\{x_k\}$ has at most one weak cluster point. Since C is weakly sequentially compact, it follows that $\{x_k\}$ has exactly one weak cluster point $w \in C$. This means that $x_k \rightharpoonup w$. It follows from Lemma 4.3 that $w \in F(\mathcal{T})$. The proof is now complete. □

The above proof follows the pattern of [7, Theorem 4.2] for nonexpansive semigroups. The asymptotic aspects of the proof are hidden in Lemmas 4.3 and 4.5 that form its backbone.

5. Stability considerations

In Theorems 4.1 and 4.2, we demonstrated that, under certain conditions, a sequence generated by a normalised implicit iteration process $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$, starting from an arbitrary point $x_0 \in C$, converges weakly to a common fixed point of the asymptotic nonexpansive semigroup $\mathcal{T}$. Thus, this process serves as a method for approximating these common fixed points.

In practical implementations, each implicit iteration step may introduce computational errors. Therefore, it is crucial to assess the stability of this process: specifically, whether it continues to converge weakly to a common fixed point when each iteration k produces a point x_{k+1} that is close to, but not exactly equal to, $Z_k(x_k)$, where Z_k is defined as in Lemma 4.4.

Many stability schemas have been studied and employed over the years. In this short note, we focus on stability under summable errors, meaning that for any sequence $\{x_k\}$ within C where $\sum_{k=1}^{\infty} \|x_{k+1} - Z_k(x_k)\| < \infty$, the sequence converges in the weak topology to a common fixed point of $\mathcal{T}$, assuming that the sequence $u_{k+1} = Z_k(u_k)$ is weak-convergent to a (possibly different) common fixed point of $\mathcal{T}$ for any initial element $u_1 \in C$. The ideas of stability under summable errors for iteration processes can be traced back to the 1960s and have flourished in recent years due to their direct computational applications (see [8, 10, 19, 20]). Specifically, the papers [8, 10] operate within a framework similar to that used in the current paper. Below, we provide a sketch of a stability result relevant to the study presented in the previous section.

THEOREM 5.1. *Let X be a uniformly convex Banach space. Additionally, assume that X is either uniformly smooth or it possesses the Opial property. Let C be a bounded, closed and convex subset of X. Let $\mathcal{T}$ be an equicontinuous asymptotic nonexpansive semigroup. Assume that there exists $\gamma \geq 1$ such that $a_t(x) \leq \gamma$ for every $x \in C$ and for all $t \in J = [0, +\infty)$. Also assume that $\{u_k\} = P(C, \mathcal{T}, u_1, \{c_k\}, \{t_k\})$ is a normalised implicit iteration process. Then, the iteration process $\{Z_k\}$ associated with $\{u_k\} = P(C, \mathcal{T}, u_1, \{c_k\}, \{t_k\})$ is stable under summable errors.*

PROOF. Recall that $u_{k+1} = Z_k(u_k)$, where each u_k is generated by the implicit iteration process $\{u_k\} = P(C, \mathcal{T}, u_1, \{c_k\}, \{t_k\})$, and $u_1 \in C$ was chosen arbitrarily. It can be concluded from Theorem 4.1 regarding the Opial property case and from Theorem 4.2 for uniformly smooth X that, regardless of the choice of the starting point, the sequence generated by the implicit iteration process converges in the weak topology to a common fixed point of the semigroup $\mathcal{T}$. By employing [10, Theorem 4.2], we conclude that the iteration process $\{Z_k\}$ is stable under summable errors, as claimed. □

By using our Theorem 3.1 in conjunction with [8, Theorem 3.3], one can derive a similar result regarding the stability of implicit iteration processes that strongly converge to a common fixed point. It must be emphasised that a further, more detailed examination will be required to obtain a comprehensive understanding of the stability issues related to the implicit iteration processes.

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