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# Algebraic Values of Entire Functions with Extremal Growth Orders: An Extension of a Theorem of Boxall and Jones

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*Abstract.* Given an entire function $f$ of finite order $\rho$ and positive lower order $\lambda$, Boxall and Jones proved a bound of the form $C(\log H)^{\eta(\lambda,\rho)}$ for the density of algebraic points of bounded degree and height at most $H$ on the restrictions to compact sets of the graph of $f$. The constant $C$ and exponent $\eta$ are effectively computable from certain data associated with the function. In this followup note, using different measures of the growth of entire functions, we obtain similar bounds for other classes of functions to which the original theorem does not apply.

## 1 Introduction

In this paper we are concerned with the study of the density of algebraic points of bounded height and degree on graphs of transcendental functions. A trivial upper bound of the form $C(d)H^{2d}$ follows immediately from Northcott's theorem, where $H$ and $d$ are the height and degree bounds, respectively. As such, one often seeks to obtain bounds of the form $C(\log H)^{\eta}$, where $C$ and $\eta$ depend on some parameters associated with the function. As one would expect, such bounds are usually nontrivial to prove.

The notion of height of an algebraic number that we will be using throughout this paper is the *absolute multiplicative height*, which, for the convenience of the reader, we recall below. After this, in order to place our results within the context of what is known in the general literature, we will briefly discuss a few related results, leading us to the Boxall–Jones theorem. In Section 2, we give the definitions of the measures of growth of the entire functions whose arithmetic properties we are going to study, as well as two key lemmas needed for our proofs. Section 3 consists of the statements and proofs of the main results.

We begin with the definition of the height of an algebraic number. Let $P(z) = a\prod_{j=1}^{d}(z-\alpha_j)$ be a polynomial with complex coefficients. The *Mahler measure* $\mathcal{M}(P)$ of $P$ is the quantity

$$\mathcal{M}(P) = |a|\prod_{j=1}^{d}\max\{1,|\alpha_j|\}.$$

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Let $\alpha$ be an algebraic number of degree $d$, with minimal polynomial $P$; then the *absolute multiplicative height* of $\alpha$, $H(\alpha)$ is defined as

$$H(\alpha) = \mathcal{M}(P)^{\frac{1}{d}}.$$

If $\alpha$ and $\beta$ are algebraic numbers, we use the notation $H(\alpha,\beta)$ to represent the quantity

$$\max\{H(\alpha), H(\beta)\}.$$

## 1.1 Some Known Results

The current line of research can be traced back to the Bombieri–Pila theorem for counting lattice points on graphs of real analytic functions.

Given a set $\Gamma \subset \mathbb{R}^2$ and a positive number $t \geq 1$, the homothetic dilation of $\Gamma$ by $t$, denoted by $t\Gamma$ is the set

$$t\Gamma := \{(tx_1, tx_2) : (x_1, x_2) \in \Gamma\}.$$

In [3] Bombieri and Pila considered, amongst several other variants, the following question. Let $f\colon [0,1] \to \mathbb{R}$ be an analytic function, and denote by $X_f \subset \mathbb{R}^2$ the graph of $f$. Given $t \geq 1$, how does the quantity $|tX_f \cap \mathbb{Z}^2|$ depend on $t$? When $f$ is a transcendental function, and for any $\epsilon > 0$, it was shown that there exists constant $c(f,\epsilon)$ such that $|tX_f \cap \mathbb{Z}^2| \leq c(f,\epsilon)t^\epsilon$ for all $t \geq 1$.

In [9], Pila extended and refined some of the results obtained in [3] to counting rational points of bounded height. Given that $f$ is a transcendental real analytic function on a closed and bounded interval $I$, and $\epsilon > 0$, it was shown that there is a constant $c(f,\epsilon)$ such that for any positive integer $H$, the number of rational points of height at most $H$ on $X_f$ is at most $c(f,\epsilon)H^\epsilon$.

Although this was shown to be the best possible bound in general, in certain instances, such as when additional hypotheses are imposed on $f$, or when $f$ is some concrete function, it is sometimes possible to improve the bound to one of the form $c(\log H)^\eta$ for some $c, \eta > 0$.

In [8], Masser showed that there are at most $c(\frac{\log H}{\log\log H})^2$ rational points on the graph of the Riemann $\zeta$-function restricted to the interval $(2,3)$.

In [2], adapting Masser's method, Besson studied the density of algebraic points of bounded degree and height on the graph of the $\Gamma$-function and obtained a similar bound for restrictions of $\Gamma$ to intervals of the form $[n-1, n]$.

In [10], assuming only that $f$ is complex analytic and transcendental, Surroca obtained a $Cd^3(\log H)^2$ bound for the number of algebraic points of degree at most $d$ and height at most $H$ on the restriction to a compact subset of the graph of $f$. However, the bound is valid only for infinitely many real $H \geq 1$. Unfortunately, one cannot replace the "infinitely many real $H \geq 1$" with "for all sufficiently large $H$".

Recall that the order and lower order of an entire function $f$ are defined as

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

respectively.

***Remark 1.1*** If $\rho$ is finite, then $\rho$ is the infimum of the set of all $\alpha$ such that $M(r,f) \leq e^{r^\alpha}$ for sufficiently large $r$ and $\lambda$ is the supremum of the set of all $\beta$ such that $e^{r^\beta} \leq M(r,f)$ for sufficiently large $r$.

In [4], motivated by earlier work of Masser in [8], Boxall and Jones studied the density of algebraic points of bounded height and degree on graphs of entire functions of finite order $\rho$ and positive lower order $\lambda$ restricted to compact subsets of $\mathbb{C}$. They attain a bound of the form $C(\log H)^\eta$ where the constant $C$ and the exponent $\eta$ are effective and $\eta$ depends only on $\rho$ and $\lambda$. More specifically, they prove the following theorem.

***Theorem 1.2*** (Boxall and Jones [4]) *Let $f$ be a nonconstant entire function of order $\rho$ and lower order $\lambda$. Suppose $0 < \lambda \leq \rho < \infty$ and let $d \geq 1$ and $r > 0$. There is a constant $C > 0$ such that for all $H > e$, there are at most $C(\log H)^{\eta(\lambda,\rho)}$ complex numbers $z$ such that $|z| \leq r$, $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$, and $H(z, f(z)) \leq H$.*

Given the Boxall–Jones theorem, one can ask several followup questions towards possible generalizations. Can one find an analogue for meromorphic functions? Using Nevanlinna theory, we address this question elsewhere. Still within the case of entire functions, can one extend this result to functions of order zero or infinite order? The latter question is the focus of this paper.

## 1.2 A Proposition of Masser

One of the key ingredients of our proof strategy is an auxiliary polynomial constructed by Masser in [8], which enables us to "convert" the question of counting algebraic points on the graph of the function $f$ to that of finding an upper bound for the number of zeroes of a related function $g$, say, a task that can then be handled by analytic methods.

This is essentially a non-zero polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ such that $P(z, f(z)) = 0$ whenever

$$(z, f(z)) \in \overline{\mathbb{Q}}^2, \quad \deg(z, f(z)) \leq d, \quad \text{and} \quad H(z, f(z)) \leq H.$$

We give the exact details of Masser's polynomial below.

***Lemma 1.3*** (Masser [8, Prop. 2]) *Let $d \geq 1$ and $T \geq \sqrt{8d}$ be positive integers and let $A, Z, M$, and $H$ be positive real numbers such that $H \geq 1$. Let $f_1, f_2$ be functions analytic on an open neighbourhood of $\overline{B(0,2Z)}$, with $\max\{|f_1(z)|, |f_2(z)|\} \leq M$ on this set. Suppose $\mathcal{Z} \subset \mathbb{C}$ is finite and satisfies the following for all $z, w \in \mathcal{Z}$:*

- $|z| \leq Z$,
- $|w - z| \leq \frac{1}{A}$,
- $[\mathbb{Q}(f_1(z), f_2(z)) : \mathbb{Q}] \leq d$,
- $H(f_1(z), f_2(z)) \leq H$.

*Then there is a nonzero polynomial $P(X,Y)$ of total degree at most $T$ such that $P(f_1(z), f_2(z)) = 0$ for all $z \in \mathcal{Z}$ provided*

$$(AZ)^T > (4T)^{96d^2/T}(M+1)^{16d}H^{48d^2}.$$

*Moreover, if $|\mathcal{Z}| \geq T^2/8d$, then $P(X, Y)$ can be chosen such that all the coefficients are integers each with absolute value at most*

$$2^{1/d}(T+1)^2 H^T.$$

***Remark 1.4*** The "moreover" part in the conclusion of this lemma does not appear in the proposition as stated in [8], but it follows as a byproduct of his proof of the proposition. For our purposes, we do need this detailed information on the size of the coefficients of the polynomial.

In addition, we will be using the special case of the lemma where $f_1(z) = z$ and $f_2(z) = f(z)$.

## 2 Other Notions of Growth of Entire Functions and Auxiliary Lemmas

The following definition was introduced in the literature as a finer measure of growth for entire functions of order zero. For properties of these functions and related constructions thereof, one can consult [7].

Let $f$ be a nonconstant entire function. The logarithmic order $\rho_0$ of $f$ is defined as

$$\rho_0(f) := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r}.$$

The logarithmic lower order of $f$ is then defined to be

$$\lambda_0(f) := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r}.$$

***Remark 2.1*** In analogy with Remark 1.1, if $\rho_0$ is finite, then $\rho_0$ is the infimum of the set of all $\alpha$ such that $M(r,f) \leq r^{(\log r)^{\alpha-1}}$ for sufficiently large $r$ and $\lambda_0$ is the supremum of the set of all $\beta$ such that $r^{(\log r)^{\beta-1}} \leq M(r,f)$ for sufficiently large $r$.

The notion of the *hyper-order* of an entire function was introduced in [12] to study the growth properties of functions with infinite order.

Let $f$ be a nonconstant entire function. The *hyper-order* $\rho_2$ of $f$ is defined as

$$\rho_2(f) := \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

Analogously, the *lower hyper-order* of $f$ is defined to be

$$\lambda_2(f) := \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

***Remark 2.2*** If $\rho_2$ is finite, then $\rho_2$ is the infimum of the set of all $\alpha$ such that $M(r,f) \leq \exp(\exp(r^{\rho_2}))$ and $\lambda_2$ is the supremum of the set of all $\beta$ such that $\exp(\exp(r^{\lambda_2})) \leq M(r,f)$ for sufficiently large $r$.

The next lemma gives a quantitative way of covering the zeroes of a polynomial $P(z)$ with a collection of disks outside of which $|P(z)| > 1$.

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***Lemma 2.3*** (Boutroux–Cartan) *Let $P(z) \in \mathbb{C}[z]$ be a monic polynomial with degree $n \geq 1$. Then $|P(z)| > 1$ for all complex $z$ outside a collection of at most $n$ disks the sum of whose radii is $2e$.*

In the following lemma, we use the notation $n(r, \frac{1}{f})$ to denote the number of zeroes of $f$ in $\overline{B(0, r)}$. This is a standard Nevanlinna theoretic notation.

***Lemma 2.4*** (A corollary of Jensen's formula) *Let $G$ be a nonconstant entire function such that $G(0) \neq 0$. Let $0 < r < R < \infty$. Then*

$$n\Big(r, \frac{1}{G}\Big) \leq \frac{1}{\log \frac{R}{r}} \log\Big(\frac{M(R, G)}{|G(0)|}\Big).$$

## 3 Main Results

We now state and prove the main results of this paper, first for certain functions of order zero and then for functions with infinite order. The proofs of the two theorems are very similar, and indeed, are just (routine) adaptations of the Boxall–Jones argument. We shall, however, include both of them for the sake of completeness.

Before moving on to our first result, we briefly discuss the related recent work of Comte and Yomdin, with a view towards contrasting it with ours.

### 3.1 A Result of Comte and Yomdin

In [5], Comte and Yomdin studied polynomial zero estimates for certain transcendental functions analytic in a disk. One of their results gives a $C(\log H)^{\beta}$ bound for the number of rational points of bounded height on restrictions of the graphs of functions $f(z)$ defined by lacunary series whose coefficients satisfy some growth condition. When $f$ defined as such is an entire function, the restriction on the coefficients allows it to be a function of order zero. We give a precise statement of their result below.

***Theorem 3.1*** ([5, Theorem 5.2(3)]) *Let the sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ be such that there exists $q > 2$ such that for any $k \geq 1$, $n_k^2 < n_{k+1} < n_k^q$. Let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$, where $a_k \in \mathbb{Q}$ and suppose that there exists $p > 0$ such that $|a_k| \geq e^{-n_k^p}$ for all $k$. Let $r > 0$ and suppose $f$ is analytic on a neighborhood of $B(0, r)$. Let $H > e$. Then there exist $C, \beta > 0$ such that there are at most $C(\log H)^{\beta}$ rational points of height at most $H$ on the graph of $f$ restricted to $\overline{B(0, \frac{r}{4})}$.*

When $f(z) = \sum_{k=n}^{\infty} a_n z^n$ is an entire function, the order of $f$ can be found by the formula

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{-\log |a_n|}. \tag{3.1}$$

For the proof of this formula, the reader can consult [6, Theorem 14.1.1]. In [1], an analogous formula is proved for the logarithmic order. More precisely, when $f(z) = \sum_{k=n}^{\infty} a_n z^n$ is an entire function of order zero, the logarithmic order of $f$ can be found by the formula

$$\rho_0 = 1 + \limsup_{n\to\infty} \frac{\log n}{\log(-\frac{1}{n}\log|a_n|)}. \tag{3.2}$$

We note that if $f(z)$ as defined in Theorem 3.1 is an entire function, then by equation (3.1), $\rho(f) \geq 0$, and by equation (3.2), $\rho_0(f) \geq 1$. Furthermore, no assumption is made on the lower (logarithmic) order of $f$.

In this regard their result is slightly more general than ours, since the functions to which our theorem applies are the ones for which $1 < \rho_0(f) < \infty$. In fact, we proved our theorem for functions for which $\lambda_0(f) > 3$. This was just a technical assumption, and it may be possible to replace it with $\lambda_0(f) > 1$, ideally. However, we do not make any assumption on the growth of the coefficients of the functions, and therefore, our result applies to a wider class of functions.

## 3.2 Functions with Order Zero

**Theorem 3.2** *Let $f$ be a nonconstant entire function of log-order $\rho$ and lower log-order $\lambda$ such that $3 < \lambda \leq \rho < \infty$. Let $d$, $\alpha$, $\beta$, $\epsilon$, and $s$ be as follows: $s > 0, d \geq 1$, $0 < \epsilon < \frac{\lambda-1}{2}$, $\beta = \lambda - 1 - \epsilon$, and $\alpha = \rho + \beta$. Then there is a constant $C > 0$ such that for all $H > e^e$, there are at most $C(\log H)^{4\alpha^2}$ numbers $z \in \mathbb{C}$ such that $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$, $H(z, f(z) \leq H$ and $|z| \leq s$.*

**Proof** Let $H > e^e$. We shall denote by $C$ a positive constant independent of $H$. The constant $C$ cannot be the same at each occurrence. By $|P|$ we are referring to the modulus of the coefficient of a polynomial with largest absolute value.

We would first like to obtain a nonzero polynomial $P(X, Y) \in \mathbb{Z}[X, Y]$ of degree at most $T = (\log H)^\alpha$ such that $|P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$ and $P(z, f(z)) = 0$ whenever $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$, $H(z, f(z) \leq H$, and $|z| \leq s$. To this end, let

$$A = \frac{1}{2s}, \qquad Z = 3s, \qquad T = (\log H)^\alpha,$$
$$S = M(6s, f), \qquad M = \max\{6s, S\}.$$

We then have that $\max\{|z|, |f(z)|\} \leq M$ for all $z \in \overline{B(0, 2s)}$ (in particular).

From the above choices of $A$, $Z$, $T$, and $M$, we have, on the one hand, that

$$(AZ)^T = (3/2)^T, \quad \text{and hence} \quad \log(AZ)^T = C(\log H)^\alpha.$$

On the other hand,

$$\begin{aligned}\log[(4T)^{96d^2/T}(M+1)^{16d}H^{48d^2}] &= 96d^2/T\log(4T) + 16d\log(M+1) + 48d^2\log H \\ &\leq C(\log H).\end{aligned}$$

Therefore,

$$(AZ)^T > (4T)^{96d^2/T}(M+1)^{16d}H^{48d^2}.$$

We note that the bound we are trying to prove is worse than $C(\log H)^{2\alpha}$. We can thus assume that there are at least $\frac{T^2}{8d}$ complex numbers such that $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$

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and $H(z, f(z) \leq H$. By Lemma 1.3, Masser's proposition, there is a polynomial $P(X, Y)$ satisfying all our requirements.

Let $G(z) = P(z, f(z))$. We would like to bound the number of zeroes of $G$ in $\overline{B(0,s)}$. To do this, first let $k$ be the highest power of $Y$ in $P(X, Y)$. We can assume $k \geq 1$. Let $\widetilde{P}(X, Y) = Y^k P(X, \frac{1}{Y})$, $R(X) = \widetilde{P}(X, 0)$, and $Q(X, Y) = \widetilde{P}(X, Y) - R(X)$. We note that $R(X)$ is not identically zero. Let $\widetilde{Q}(X, Y) = \frac{1}{Y} Q(X, Y)$. The highest power of $X$ in $\widetilde{Q}$ is at most $T$ and $|\widetilde{Q}| \leq |P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$. Finally, $\widetilde{Q}$ has at most $(T+1)^2$ terms.

We would now like to find some $z_i \in \mathbb{C}$ such that $|G(z_i)| = |P(z_i, f(z_i))| \geq 1$.

Let $z = re^{i\theta} \in \mathbb{C}$ be such that $|f(z)| = M(r, f) \geq 1$. Then

$$\left|\widetilde{Q}\Big(z, \frac{1}{f(z)}\Big)\right| \leq 2^{\frac{1}{d}}(T+1)^4 H^T r^T.$$

Therefore,

$$\left|Q\Big(z, \frac{1}{f(z)}\Big)\right| \leq \frac{1}{2},$$

provided

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2} M(r, f). \tag{3.3}$$

We would like to estimate the size of $r$ (in terms of $H$) that would make the above inequality hold.

Let $0 < C_1 < C_2$ be such that

$$C_1(\log H)^{(\log H)^\alpha} \leq r \leq C_2(\log H)^{(\log H)^\alpha};$$

then (bearing in mind that $T = C(\log H)^\alpha$),

$$C(\log T) + T(\log H) + T\log r \leq C_2'(\log H)^{2\alpha}(\log\log H). \tag{3.4}$$

Abusing notation slightly, inequality (3.4) above implies that, for a carefully chosen $C > 0$, the logarithm of the left hand side of inequality (3.3) is dominated by $C(\log H)^{2\alpha}(\log\log H)$.

On the other hand, since $r \geq C_1(\log H)^{(\log H)^\alpha}$, we have that

$$(\log r)^{\beta+1} \geq C(\log H)^{\alpha(\beta+1)}(\log\log H)^{\beta+1}.$$

By assumption, $\lambda > 3$ and therefore by definition $\beta > 1$. Hence,

$$C(\log H)^{2\alpha}(\log\log H) \leq C(\log H)^{\alpha(\beta+1)}(\log\log H)^{\beta+1}. \tag{3.5}$$

By combining inequalities (3.4) and (3.5), we have that

$$\log[2^{\frac{1}{d}}(T+1)^4 H^T r^T] \leq C(\log H)^{\alpha(\beta+1)}(\log\log H)^{\beta+1}.$$

It follows that for large enough $H$ and possibly a different choice of $C > 0$,

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2} r^{(\log r)^\beta}.$$

Since $\beta < \lambda - 1$ and $H$ is sufficiently large, by Remark 2.1,

$$r^{(\log r)^\beta} \leq M(r, f).$$

We thus get that:

$$\left|Q\Big(z, \frac{1}{f(z)}\Big)\right| \le \frac{1}{2},$$

as required.

Note that the degree of $R(X)$ is also at most $T$. For $i = 1, \dots, [T] + 14$, say, let $r_i$ be the $i$th integer after $C(\log H)^{(\log H)^\alpha}$. Let $z_i$ be such that $|z_i| = r_i$ and $|f(z_i)| = M(r_i, f)$. By the Boutroux–Cartan lemma, there will be at least one $i$ such that $|R(z_i)| > 1$. For such $i$, we have:

$$\left|\widetilde{P}\Big(z_i, \frac{1}{f(z_i)}\Big)\right| \ge \frac{1}{2}.$$

We can (again by Remark 2.1) conclude that

$$|G(z_i)| = |P(z_i, f(z_i))| = \left|f(z_i)^k \widetilde{P}\Big(z_i, \frac{1}{f(z_i)}\Big)\right| \ge \frac{1}{2} {r_i}^{k(\log r_i)^\beta},$$

and therefore $|G(z_i)| \ge 1$. We note that $\overline{B(0,s)} \subset \overline{B(z_i, \mu)}$ where $\mu = C(\log H)^{(\log H)^\alpha}$.

By the maximum modulus principle and Lemma 2.4, the corollary of Jensen's formula, we have that

$$n\Big(s, \frac{1}{G}\Big) \le \frac{1}{\log 2} \log\Big(\frac{M(3\mu, G)}{|G(z_i)|}\Big) \le \frac{\log M(3\mu, G)}{\log 2}.$$

Since $\alpha > \rho - 1$ and $H$ is sufficiently large, by Remark 2.1, we have that

$$M(3\mu, G) \le |P|(T+1)^2 (3\mu)^T (3\mu)^{T(\log 3\mu)^\alpha}.$$

Recalling the definitions (or bounds) of $|P|$, $T$, and $\mu$ in terms of $H$, we can thus deduce that

$$\log M(3\mu, G) \le C(\log H)^{4\alpha^2}.$$

Therefore,

$$n\Big(s, \frac{1}{G}\Big) \le C(\log H)^{4\alpha^2},$$

as required. ∎

We note that the constant $C$ can be computed from $d, s, \lambda, \rho$, and some constant $r_0 > 0$ such that $r^{(\log r)^{\beta-1}} \le M(r, f) \le r^{(\log r)^{\alpha-1}}$ for all $r \ge r_0$.

The exponent of $\log H$ could be potentially improved through more careful considerations.

## 3.3 Functions with Infinite Order

***Theorem 3.3*** *Let $f$ be a nonconstant entire function of hyper-order $\rho$ and lower hyper-order $\lambda$ such that $4 < \lambda \le \rho < \infty$. Let $d, \alpha, \beta$, and $s$ be as follows: $s > 0, d \ge 1, \beta = \frac{\lambda}{2}$ and $\alpha = \max\{1, \rho\} + \beta$. Then there is a constant $C > 0$ such that for all $H > e^{e^e}$, there are at most $C(\log H)^{\frac{2\alpha^2}{\beta}}$ numbers $z \in \mathbb{C}$ such that $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \le d$, $H(z, f(z) \le H$, and $|z| \le s$.*

**Proof** Throughout the proof we assume that $H$ is sufficiently large. We shall denote by $C$ a positive constant independent of $H$. The constant $C$ may not be the same at each occurrence. By $|P|$ we are referring to the modulus of the coefficient of a polynomial with largest absolute value.

We would first like to obtain a nonzero polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ of degree at most $T = (\log H)^{\frac{\alpha}{\beta}}$ such that $|P| \leq 2^{\frac{1}{d}}(T+1)^2 H^T$ and $P(z, f(z)) = 0$ whenever $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$, $H(z, f(z) \leq H$, and $|z| \leq s$. To this end, let

$$A = \frac{1}{2s}, \qquad Z = 3s, \qquad T = (\log H)^{\frac{\alpha}{\beta}},$$
$$S = M(6s, f), \qquad M = \max\{6s, S\}.$$

We then have that $\max\{|z|, |f(z)|\} \leq M$ for all $z \in \overline{B(0,2s)}$ (in particular).

From the above choices of $A$, $Z$, $T$, and $M$, we have, on the one hand, that

$$(AZ)^T = (3/2)^T, \quad \text{and hence} \quad \log(AZ)^T = C(\log H)^{\frac{\alpha}{\beta}}.$$

On the other hand,

$$\begin{aligned}\log\left[(4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}\right] &= \frac{96d^2}{T}\log(4T) + 16d\log(M+1) + 48d^2\log H \\ &= C(\log H).\end{aligned}$$

Therefore,

$$(AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}.$$

We note that the bound we are trying to prove is worse than $C(\log H)^{\frac{2\alpha}{\beta}}$. We can thus assume that there are at least $\frac{T^2}{8d}$ complex numbers such that $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$ and $H(z, f(z) \leq H$. By Lemma 1.3, there is a polynomial $P(X,Y)$ satisfying all our requirements.

Let $G(z) = P(z, f(z))$. We would like to bound the number of zeroes of $G$ in $\overline{B(0,s)}$. To do this, first let $k$ be the highest power of $Y$ in $P(X,Y)$. We can assume that $k \geq 1$. Let $\widetilde{P}(X,Y) = Y^k P(X, \frac{1}{Y})$, $R(X) = \widetilde{P}(X,0)$, and $Q(X,Y) = \widetilde{P}(X,Y) - R(X)$. We note that $R(X)$ is not identically zero. Let $\widetilde{Q}(X,Y) = \frac{1}{Y}Q(X,Y)$. The highest power of $X$ in $\widetilde{Q}$ is at most $T$ and $|\widetilde{Q}| \leq |P| \leq 2^{\frac{1}{d}}(T+1)^2H^T$. Finally, $\widetilde{Q}$ has at most $(T+1)^2$ terms.

We would now like to find some $z_i \in \mathbb{C}$ such that $|G(z_i)| = |P(z_i, f(z_i))| \geq 1$.

Let $z = re^{i\theta} \in \mathbb{C}$ be such that $|f(z)| = M(r,f) \geq 1$. Then

$$\left|\widetilde{Q}\Big(z, \frac{1}{f(z)}\Big)\right| \leq 2^{\frac{1}{d}}(T+1)^4 H^T r^T.$$

Therefore,

$$\left|Q\Big(z, \frac{1}{f(z)}\Big)\right| \leq \frac{1}{2}$$

provided

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2}M(r,f).$$

We would like to find an explicit bound for $r$ in terms of $H$ for which the above desired inequalities will hold. We note that if for some $0 < C_1 < C_2$,

$$C_1(\log\log H)^{\frac{\alpha}{\beta}} \leq r \leq C_2(\log H)^{\frac{\alpha}{\beta}}, \quad \text{say;}$$

then, on the one hand,

$$\log[2^{\frac{1}{d}}(T+1)^4 H^T r^T] = C(\log T) + T(\log H) + T\log r, \tag{3.6}$$
$$\leq C_2(\log H)^{\frac{2\alpha}{\beta}},$$

whilst on the other hand

$$\exp(r^\beta) \geq C_1(\log H)^\alpha.$$

Recall that $\alpha := \max\{1,\rho\} + \beta$. Since $\beta := \frac{\lambda}{2}$ where $\lambda > 4$, we have that

$$(\log H)^\alpha \geq (\log H)^{\frac{2\alpha}{\beta}}.$$

Therefore, it follows from inequality (3.6) that

$$\log[2^{\frac{1}{d}}(T+1)^4 H^T r^T] \leq \exp(r^\beta).$$

In conclusion, for large enough $H$ (and possibly a different choice of the constant $C$), we have that

$$2^{\frac{1}{d}}(T+1)^4 H^T r^T \leq \frac{1}{2}\exp(\exp(r^\beta)).$$

Since $\beta < \lambda$ and $H$ is sufficiently large, by Remark 2.2,

$$\exp(\exp(r^\beta)) \leq M(r,f).$$

We thus get that

$$\Big|Q\Big(z, \frac{1}{f(z)}\Big)\Big| \leq \frac{1}{2},$$

as required.

Note that the degree of $R(X)$ is also at most $T$. For $i = 1,\dots,[T]+14$, say, let $r_i$ be the $i$-th integer after $C(\log\log H)^{\frac{\alpha}{\beta}}$. Let $z_i$ be such that $|z_i| = r_i$ and $|f(z_i)| = M(r_i,f)$. By the Boutroux–Cartan lemma, there will be at least one $i$ such that $|R(z_i)| > 1$. For such $i$, we have

$$\Big|\widetilde{P}\Big(z_i, \frac{1}{f(z_i)}\Big)\Big| \geq \frac{1}{2}.$$

We can (again by Remark 2.2) conclude that

$$|G(z_i)| = |P(z_i, f(z_i))| = \Big|f(z_i)^k\widetilde{P}\Big(z_i, \frac{1}{f(z_i)}\Big)\Big| \geq \frac{1}{2}\exp(\exp(kr_i^\beta)),$$

and therefore $|G(z_i)| \geq 1$. We note that $\overline{B(0,s)} \subset \overline{B(z_i,\mu)}$ where $\mu = C(\log H)^{\frac{\alpha}{\beta}}$.

By the maximum modulus principle and Lemma 2.4, we have that

$$n\Big(s, \frac{1}{G}\Big) \leq \frac{1}{\log 2}\log\Big(\frac{M(3\mu,G)}{|G(z_i)|}\Big) \leq \frac{\log M(3\mu,G)}{\log 2}.$$

Since $\alpha > \rho$,

$$M(3\mu,G) \leq |P|(T+1)^2(3\mu)^T(3\mu)^{T(\log 3\mu)^\alpha}.$$

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We can thus deduce that

$$\log M(3\mu, G) \leq C(\log H)^{\frac{2\alpha^2}{\beta}}.$$

Therefore,

$$n(s, \frac{1}{G}) \leq C(\log H)^{\frac{2\alpha^2}{\beta}},$$

as required. ∎

We note that the constant $C$ can be computed from $d, s, \lambda, \rho$ and some constant $r_0 > 0$ such that $\exp(\exp(r^\beta)) \leq M(r, f) \leq \exp(\exp(r^\alpha))$ for all $r \geq r_0$.

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