J. Aust. Math. Soc. 74 (2003), 1–3

A SIMPLE REMARK ON DIRICHLET SERIES

WINFRIED KOHNEN

(Received 4 July 2001; revised 11 October 2001)

Communicated by W. W. L. Chen

Abstract

It is shown that a twist of a Dirichlet series by a Dirichlet character may lead to very different analytic properties.

2000 *Mathematics subject classification*: primary 11M06.

1. Introduction

Let $L(s)$ be an ordinary Dirichlet series absolutely convergent in some right half-plane which has meromorphic continuation to the entire complex plane. Let χ be a Dirichlet character. Then one might often expect that the twist $L(s, \chi)$ of $L(s)$ by χ has the same analytic properties. For instance, as is well known this in general can be proved if $L(s)$ is attached to an automorphic form and so in addition one knows that $L(s)$ is of finite order and when completed with appropriate Γ-factors satisfies a functional equation, see for example [2] in the case of GL_2 and [1] in the context of Koecher-Maass series on Sp_n.

In the present paper we would like to give explicit evidence that the above expectation is not always satisfied automatically. In fact, we will give an example of a Dirichlet series with rational coefficients which is absolutely convergent for $\mathrm{Re}(s) > 1$, extends to an entire function and such that there is a character twist which has analytic continuation to the entire plane except for an essential singularity at $s = 1$. To exclude any misunderstandings, let us point out that our function neither is of finite order nor satisfies a functional equation of appropriate type and so in particular does not satisfy the conditions of any converse theorem.

Our example is based on the following simple observation, roughly being stated as follows: if $L(s)$ is a Dirichlet series, then formally (ignoring any convergence) $e^{L(s)}$ can be written as a Dirichlet series, and (under good circumstances) the association $L(s) \mapsto e^{L(s)}$ preserves analyticity, sends poles to essential singularities and is compatible with twists.

2. Statement of result and proof

Let r and m be positive integers. We define $e_r(m)$ to be the number of r-tuples $(m_1, \ldots, m_r)$ of integers greater than 1 such that $m_1 \cdots m_r = m$.

In addition we set

$$e_0(m) := \begin{cases} 1 & \text{if m=1;} \\ 0 & \text{if m>1.} \end{cases}$$

Put

$$e(m) := \sum_{r\geq 0} \frac{e_r(m)}{r!}. \tag{1}$$

Note that the sum in (1) actually is finite, hence $e(m)$ is a rational number.

Let χ be a fixed Dirichlet character modulo N with $N > 1$ which is not the principal character χ_0 modulo N. Define a formal Dirichlet series by

$$D(s) := \sum_{m\geq 1} \chi(m)e(m)m^{-s}$$

and denote by

$$D(s, \bar{\chi}) = \sum_{m\geq 1} \chi_0(m)e(m)m^{-s}$$

its twist by $\bar{\chi}$.

For a complex number s, we put $\sigma := \mathrm{Re}(s)$ as usual.

THEOREM. (i) *The series $D(s)$ and $D(s, \bar{\chi})$ are absolutely convergent for $\sigma > 1$.*

(ii) *The function $D(s)$ extends to an entire function, while $D(s, \bar{\chi})$ has analytic continuation to the complex plane except for an essential singularity at $s = 1$.*

PROOF. (i) Since $e(m) \geq 0$ it is sufficient to prove that $D_0(s) := \sum_{m\geq 1} e(m)m^{-s}$ is convergent for $\sigma > 1$. This, however, is obvious. Indeed, for any $r \geq 0$ the formula

$$\sum_{m\geq 1} e_r(m)m^{-s} = (\zeta(s) - 1)^r \qquad (\sigma > 1)$$

https://doi.org/10.1017/S1446788700003062 Published online by Cambridge University Press

holds, by definition. Hence the desired convergence follows from the convergence of the Taylor series of the exponential function around zero, and in addition we obtain the equality

$$D_0(s) = \sum_{r \geq 0} \frac{1}{r!}(\zeta(s) - 1)^r = e^{\zeta(s)-1} \qquad (\sigma > 1) \tag{2}$$

(note that the interchange of orders of summations is justified because all coefficients are non-negative).

(ii) Twisting (2) with χ we obtain

$$D(s) = \sum_{r \geq 0} \frac{1}{r!}(L(s, \chi) - 1)^r = e^{L(s,\chi)-1} \quad (\sigma > 1). \tag{3}$$

Since $L(s, \chi)$ has analytic continuation to an entire function, we see that the same is true for $D(s)$.

Twisting the expressions in (3) by $\bar{\chi}$, we finally obtain

$$D(s, \bar{\chi}) = \sum_{r \geq 0} \frac{1}{r!}(L(s, \chi_0) - 1)^r = e^{L(s,\chi_0)-1} \quad (\sigma > 1).$$

Since

$$L(s, \chi_0) = \prod_{p|N} \left(1 - \frac{1}{p}\right) \frac{1}{s-1} + g(s) \quad (s \neq 1)$$

with $g(s)$ entire, this proves the second part of the assertion of (ii). □

References

[1] H. Maass, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Math. 216 (Springer, Berlin, 1970).

[2] A. Ogg, *Modular forms and Dirichlet series* (Benjamin, Amsterdam, 1969).

Universität Heidelberg
Mathematisches Institut
INF 288
D-69120 Heidelberg
Germany
e-mail: winfried@mathi.uni-heidelberg.de