12

Measure-Theoretic Methods

The main goal of this chapter is to prove the existence of minimizers for the causal action principle in the case that $\mathcal{H}$ is *finite dimensional* and ρ is *normalized*, that is,

$$\dim \mathcal{H} =: f < \infty \quad \text{and} \qquad \rho(\mathcal{F}) = 1 \,. \tag{12.1}$$

After introducing the necessary methods (Sections 12.1 and 12.2), we first apply them to prove the existence of minimizers for causal variational principles in the compact setting (Section 12.3). In preparation for the proof for the causal action principle, we illustrate the constraints by a few examples (Section 12.4). The difficulties revealed by these examples can be resolved by working with the so-called moment measures. After introducing the needed mathematical methods (Section 12.5), the moment measures are introduced (Section 12.6). Then, the existence proof is completed (Section 12.7). In order to give a first idea for how to deal with an infinite total volume, we finally prove the existence of minimizers for causal variational principles in the non-compact setting (Section 12.8).

Our general strategy is to apply the *direct method in the calculus of variations*, which can be summarized as follows:

(a) Choose a *minimizing sequence*, that is, a sequence of measures (ρ_k), which satisfy the constraints such that

$$\mathcal{S}(\rho_k) \to \inf_\rho \mathcal{S}(\rho) \,. \tag{12.2}$$

Such a minimizing sequence always exists by definition of the infimum (note that the action, and therefore also its infimum, are nonnegative).

(b) Show that a subsequence of the measures converges in a suitable sense,

$$\rho_{k_l} \text{``}\!\longrightarrow\!\text{''}\, \rho \,. \tag{12.3}$$

Here, the quotation marks indicate that we still need to specify in which sense the sequence should converge (convergence in which space, strong or weak convergence, etc.).

(c) Finally, one must show that the action is lower semi-continuous, that is,

$$\mathcal{S}(\rho) \le \liminf_{l\to\infty} \mathcal{S}(\rho_{k_l}) \,. \tag{12.4}$$

Also, one must prove that the limit measure ρ satisfies the constraints.

Once these three steps have been carried out, the measure ρ is a desired minimizer. We point out that this procedure does *not* give a *unique* minimizer, simply because there may be different minimizing sequences, and because the choice of the subsequences may involve an arbitrariness. Indeed, for the causal action principle, we do not expect uniqueness. There should be many different minimizers, which describe different physical systems (like the vacuum, a system involving particles and fields, etc.). This intuitive picture is confirmed by the numerical studies in [74, 83], which show that, even if the dimension of $\mathcal{H}$ is small, there are many different minimizers.

12.1 The Banach–Alaoglu Theorem

For our purposes, it suffices to consider the case that the Banach space is separable, in which case the theorem was first proved by Banach (Alaoglu proved the generalization to non-separable Banach spaces; this makes use of Tychonoff's theorem and goes beyond what we need here). Indeed, the idea of proof of the theorem can be traced back to Eduard Helly's doctoral thesis in 1912, where the closely related "Helly's selection theorem" is proved (of course without reference to Banach spaces, which were introduced later). We closely follow the presentation in [116, Section 10.3].

Let $(E, \|.\|_E)$ be a separable (real or complex) Banach space and $(E^*, \|.\|_{E^*})$ its dual space with the usual sup-norm, that is,

$$\|\phi\|_{E^*} = \sup_{u \in E, \|u\|=1} \big|\phi(u)\big| \,. \tag{12.5}$$

A sequence $(\phi_n)_{n\in\mathbb{N}}$ in E^* is said to be *weak*-convergent* to $\phi \in E^*$ if

$$\lim_{n\to\infty} \phi_n(u) = \phi(u) \qquad \text{for all } u \in E\,. \tag{12.6}$$

Theorem 12.1.1 (Banach–Alaoglu theorem in the separable case) *Let E be a separable Banach space. Then every bounded sequence in E^* has a weak*-convergent subsequence.*

Proof Let ϕ_n be a bounded sequence in E^*, meaning that there is a constant $c > 0$ with

$$\|\phi_n\|_{E^*} \leq c \qquad \text{for all } n \in \mathbb{N}\,. \tag{12.7}$$

We let $(u_\ell)_{\ell\in\mathbb{N}}$ be a sequence in E which is dense in E. Combining (12.7) with (12.5), the estimate

$$|\phi_n(u_1)| \leq \|\phi_n\|_{E^*}\, \|u_1\|_E \leq c\, \|u_1\|_E, \tag{12.8}$$

shows that $(\phi_n(u_1))_{n\in\mathbb{N}}$ is a bounded sequence. Thus we can choose a convergent subsequence. By inductively choosing subsequences and taking the diagonal sequence, we obtain a subsequence (ϕ_{n_j}) such that the limit

$$\lim_{j\to\infty} \phi_{n_j}(u_\ell), \tag{12.9}$$

exists for all $\ell \in \mathbb{N}$. Hence setting

$$\phi(u_\ell) := \lim_{j\to\infty} \phi_{n_j}(u_\ell) \,, \tag{12.10}$$

we obtain a densely defined functional. Taking the limit in (12.8) (and the similar inequalities for $u_2, u_3, \ldots$), one sees that this functional is again continuous. Therefore, it has a unique continuous extension to E. By continuity, the resulting functional $\phi \in E^*$ satisfies the relations

$$\phi(u) = \lim_{j\to\infty} \phi_{n_j}(u) \qquad \text{for all } u \in \mathcal{H} \,. \tag{12.11}$$

In particular, it is again a linear. This concludes the proof. □

12.2 The Riesz Representation Theorem

In this section and Section 12.5, we shall introduce the methods from measure theory needed for the existence proofs. Apart from the books already mentioned in the preliminaries (Section 2.3), we also recommend the book [33] (this book is only concerned with measures in $\mathbb{R}^n$, but otherwise goes far beyond what we need here).

For our purposes, it suffices to restrict attention to the case that the base space $\mathcal{K}$ is a *compact* topological space. We always consider *bounded regular Borel measures* on $\mathcal{K}$ (for the preliminaries, see Section 2.3). In order to avoid confusion, we note that by a measure we always mean a *positive* measure (signed measures will not be considered in this book). A *bounded* measure is also referred to as a measure of *finite total volume*. Often, we *normalize* the measure such that $\mu(\mathcal{K}) = 1$.

In words, the Riesz representation theorem makes it possible to represent a linear functional on the Banach space of continuous functions of a topological space by a regular Borel measure on this topological space. We remark that we already came across the Riesz representation theorem in Section 3.2, where it was needed for the construction of spectral measures. However, in this context, we only needed the special case that the topological space was an interval of the real line. We now state the general theorem and outline its proof, mainly following the presentation in [101, §56].

As a simple example, one can choose $\mathcal{K}$ as the closed unit ball in $\mathbb{R}^n$. Restricting the Lebesgue measure to the Borel subsets of $\mathcal{K}$ gives a Radon measure. The Lebesgue measure itself is a completion of this Radon measure obtained by extending the σ-algebra of measurable sets by all subsets of Borel sets of measure zero. Since this completion is a rather trivial extension, in what follows we prefer to work with Radon measures or, equivalently, with normalized regular Borel measures.

Theorem 12.2.1 (Riesz representation theorem) *Let $\mathcal{K}$ be a compact topological space, and $E = C^0(\mathcal{K}, \mathbb{R})$ the Banach space of continuous functions on $\mathcal{K}$ with the usual sup-norm,*

$$\|f\| = \sup_{x\in\mathcal{K}} |f(x)| \,. \tag{12.12}$$

Let $\Lambda \in E^*$ be a continuous linear functional which is positive in the sense that

$$\Lambda(f) \geq 0 \qquad \textit{for all nonnegative functions } f \in C^0(\mathcal{K}, \mathbb{R})\,. \tag{12.13}$$

Then, there is a unique regular Borel measure μ such that

$$\Lambda(f) = \int_{\mathcal{K}} f \, \mathrm{d}\mu \qquad \textit{for all } f \in C^0(\mathcal{K}, \mathbb{R})\,. \tag{12.14}$$

Outline of the Proof We follow the strategy in [101, §56]. Given a Borel set $A \subset \mathcal{K}$, we set

$$\lambda(A) = \inf \big\{ \Lambda(f) \,\big|\, f \in C^0(\mathcal{K}, \mathbb{R}) \text{ and } f \geq \chi_A \big\} \in \mathbb{R}^+_0\,. \tag{12.15}$$

Intuitively speaking, λ gives us the desired "volume" of the set A. But there is the technical problem that λ is in general not a regular Borel measure. Instead, it merely is a *content*, meaning that it has the following properties:

(i) *nonnegative* and *finite*: $0 \leq \lambda(A) < \infty$
(ii) *monotone*: C, D compact and $C \subset D \implies \lambda(C) \leq \lambda(D)$
(iii) *additive*: C, D compact and disjoint $\implies \lambda(C \cup D) = \lambda(C) + \lambda(D)$
(iv) *subadditive*: C, D compact $\implies \lambda(C \cup D) \leq \lambda(C) + \lambda(D)$

At this stage, we are in a similar situation as in the elementary measure theory course after saying that a cube of length ℓ in $\mathbb{R}^3$ should have volume ℓ^3. In order to get from this "volume measure" to a measure in the mathematical sense, one has to proceed in several steps invoking the subtle and clever constructions of measure theory (due to Lebesgue, Hahn, Carathéodory and others) in order to get a mapping from a σ-algebra to the nonnegative real numbers which is σ-additive. In simple terms, repeating these constructions starting from the content introduced in (12.15) gives the desired Borel measure μ. For brevity, we here merely outline the constructions and refer for details to textbooks on measure theory (like, e.g., [101, Chapter X]).

The first step is to approximate (or exhaust) from inside by compact sets. Thus one introduces the *inner content* λ_* by

$$\lambda_*(U) = \sup \big\{ \lambda(C) \,\big|\, C \subset U \text{ compact} \big\}\,. \tag{12.16}$$

This inner content is monotone and countably additive. The second step is to exhaust from outside by open sets. This gives the *outer measure* μ^*,

$$\mu^*(U) = \inf \big\{ \lambda_*(\Omega) \,\big|\, \Omega \supset U \text{ open} \big\}\,. \tag{12.17}$$

The outer measure is defined for any subset of $\mathcal{K}$. Therefore, it remains to distinguish the measurable sets. This is accomplished by Carathéodory's criterion, which defines a set $A \subset \mathcal{K}$ to be *measurable* if

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B), \tag{12.18}$$

for every subset $B \subset \mathcal{K}$. Then Carathéodory's lemma (for a concise proof see, e.g., [18, Lemma 2.8]) implies that the measurable sets form a σ-algebra and that the restriction of μ^* to the measurable sets is indeed a measure, denoted by μ.

In order to complete the proof, one still needs to verify that every Borel set is μ-measurable. Moreover, it remains to show that the resulting Borel measure is regular. To this end, one first needs to show that the content λ is regular in the following sense:

(v) *regular*: For every compact C,

$$\lambda(C) = \inf \Big\{ \lambda(D) \;\Big|\; D \text{ compact and } C \subset \overset{\circ}{D} \Big\} . \tag{12.19}$$

As the proofs of these remaining points are rather straightforward and not very instructive, we refer for the details to [101, §54–§56]. □

12.3 Existence of Minimizers for Causal Variational Principles in the Compact Setting

We now apply the above methods to prove the existence of minimizers for causal variational principles in the compact setting. Our strategy is to apply the Banach–Alaoglu theorem to a specific Banach space, namely the continuous functions on a compact metric space. We first verify that this Banach space is separable.

Proposition 12.3.1 *Let $\mathcal{K}$ be a compact metric space. Then, $C^0(\mathcal{K}, \mathbb{R})$ is a separable Banach space.*

Proof The proposition is a consequence of the Stone–Weierstrass theorem, whose proof can be found, for example, in [26, 7.3.1] We closely follow the proof given in [26, 7.4.4].

Covering $\mathcal{K}$ by a finite number of open balls of radii 1, 1/2, 1/3, ..., one gets an enumerable basis of the open sets $(U_n)_{n \in \mathbb{N}}$. For any $n \in N$, we let g_n be the continuous function

$$g_n(x) = d\big(x, \mathcal{K} \setminus U_n\big) . \tag{12.20}$$

Clearly, the algebra generated by these functions (by taking finite products and finite linear combinations) is again separable. Therefore, it suffices to show that this algebra is dense in $C^0(\mathcal{K}, \mathbb{R})$. To this end, we need to verify the assumptions of the Stone–Weierstrass theorem. The only assumption which is not obvious is that the algebra separates the points. This can be seen as follows: Let x and y be two distinct points in $\mathcal{K}$. Since the (U_n) are a basis of the topology, there is U_n with $x \in U_n$ and $y \notin U_n$. As a consequence, $g_n(x) > 0$ but $g_n(y) = 0$. □

We proceed by proving a compactness result for Radon measures.

Theorem 12.3.2 *Let ρ_n be a series of regular Borel measures on $C^0(\mathcal{K}, \mathbb{R})$ which are bounded in the sense that there is a constant $c > 0$ with*

$$\rho_n(\mathcal{K}) \leq c \qquad \text{for all } n . \tag{12.21}$$

Then, there is a subsequence (ρ_{n_k}), which converges as a measure, that is,

$$\lim_{k \to \infty} \int_{\mathcal{K}} f \, d\rho_{n_k} = \int_{\mathcal{K}} f \, d\rho \qquad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R}) . \tag{12.22}$$

Moreover, the total volume converges, that is,

$$\rho(\mathcal{K}) = \lim_{k\to\infty} \rho_{n_k}(\mathcal{K}) . \tag{12.23}$$

Proof Via

$$\phi_n(f) := \int_{\mathcal{K}} f \, \mathrm{d}\rho_n , \tag{12.24}$$

every measure can be identified with a positive linear functional on $E := C^0(\mathcal{K}, \mathbb{R})$. Since E is separable (Proposition 12.3.1), we can apply the Banach–Alaoglu theorem in the separable case (Theorem 12.1.1) to conclude that there is a weak*-convergent subsequence, that is,

$$\lim_{k\to\infty} \phi_{n_k}(f) = \phi(f) \qquad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R}) . \tag{12.25}$$

Clearly, since all ϕ_{n_k} are positive, the same is true for the limit ϕ. Therefore, the Riesz representation theorem (Theorem 12.2.1) makes it possible to represent ϕ by a regular Borel measure ρ, that is,

$$\phi(f) = \int_{\mathcal{K}} f \, \mathrm{d}\rho \qquad \text{for all } f \in C^0(\mathcal{K}, \mathbb{R}) . \tag{12.26}$$

Choosing f as the constant function, one obtains (12.23). This concludes the proof. □

Theorem 12.3.3 *Assume that $\mathcal{F}$ is a compact topological space and the Lagrangian is continuous,*

$$\mathcal{L} \in C^0(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0) . \tag{12.27}$$

Then, the causal variational principle where the causal action (6.8) *is minimized in the class of regular Borel measures under the volume constraint* (6.9) *is well posed in the sense that every minimizing sequence $(\rho_n)_{n\in\mathbb{N}}$ has a subsequence which converges as a measure to a minimizer ρ.*

Proof The existence of a convergent subsequence $(\rho_{n_k})_{k\in\mathbb{N}}$ is proven in Theorem 12.3.2. It remains to show that the action is continuous, that is,

$$\lim_{k\to\infty} \mathcal{S}\big(\rho_{n_k}\big) = \mathcal{S}(\rho) . \tag{12.28}$$

This is verified in detail as follows. Using that the Lagrangian is continuous in its second argument, we know that

$$\lim_{k\to\infty} \int_{\mathcal{F}} \mathcal{L}(x,y) \, \mathrm{d}\rho_{n_k}(y) = \int_{\mathcal{F}} \mathcal{L}(x,y) \, \mathrm{d}\rho(y) \qquad \text{for all } x \in \mathcal{F} . \tag{12.29}$$

Next, since $\mathcal{F}$ is compact, the Lagrangian is even uniformly continuous on $\mathcal{F} \times \mathcal{F}$. Therefore, given $\varepsilon > 0$, every point $x \in \mathcal{F}$ has an open neighborhood $U(x) \subset \mathcal{F}$ such that

$$\big|\mathcal{L}(\hat{x}, y) - \mathcal{L}(x,y)\big| < \varepsilon \qquad \text{for all } \hat{x} \in U(x) \text{ and } y \in \mathcal{F} . \tag{12.30}$$

Integrating over y with respect to any normed regular Borel measure $\tilde{\rho}$, it follows that

$$\left| \int_{\mathcal{F}} \mathcal{L}(\hat{x}, y) \, \mathrm{d}\tilde{\rho}(y) \;-\; \int_{\mathcal{F}} \mathcal{L}(x, y) \, \mathrm{d}\tilde{\rho}(y) \right| \leq \varepsilon \qquad \text{for all } \hat{x} \in U(x) \,. \tag{12.31}$$

Covering $\mathcal{F}$ by a finite number of such neighborhoods $U(x_1), \ldots, U(x_N)$, one can combine the pointwise convergence (12.29) for $x = x_1, \ldots, x_N$ with the estimate (12.31) to conclude that for any $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that

$$\left| \int_{\mathcal{F}} \mathcal{L}(x, y) \mathrm{d}\rho_{n_k}(y) - \int_{\mathcal{F}} \mathcal{L}(x, y) \mathrm{d}\rho(y) \right| \leq 3\varepsilon \qquad \text{for all } x \in \mathcal{F} \text{ and } k \geq k_0 . \tag{12.32}$$

Integrating over x with respect to ρ_{n_k} and ρ gives for all $k \geq k_0$ the respective inequalities

$$\left| \mathcal{S}\big(\rho_{n_k}\big) \;-\; \int_{\mathcal{F}} \mathrm{d}\rho_{n_k}(x) \int_{\mathcal{F}} \mathrm{d}\rho(y) \, \mathcal{L}(x, y) \right| \leq 3\varepsilon \,, \tag{12.33}$$

$$\left| \int_{\mathcal{F}} \mathrm{d}\rho(x) \int_{\mathcal{F}} \mathrm{d}\rho_{n_k}(y) \, \mathcal{L}(x, y) \;-\; \mathcal{S}(\rho) \right| \leq 3\varepsilon \,. \tag{12.34}$$

Combining these inequalities and using that the Lagrangian is symmetric in its two arguments, we conclude that

$$\big| \mathcal{S}\big(\rho_{n_k}\big) - \mathcal{S}(\rho) \big| \leq 6\varepsilon \,. \tag{12.35}$$

This gives the result. $\square$

We finally remark that the statement of this theorem also holds if the Lagrangian merely is lower semi-continuous, as is worked out in [66, Section 3.2].

12.4 Examples Illustrating the Constraints

Compared to causal variational principles in the compact setting, the existence proof for the causal action principle is considerably harder because we need to handle the constraints (5.37)–(5.39) and face the difficulty that the set $\mathcal{F}$ is unbounded and therefore non-compact. We now explain the role of the constraints in a few examples. The necessity of the volume constraint is quite obvious: If we dropped the constraint of fixed total volume (5.37), the measure $\rho = 0$ would be a trivial minimizer. The role of the trace constraint is already less obvious. It is explained in the next two examples.

Example 12.4.1 (Necessity of the trace constraint) Let x be the operator with the matrix representation

$$x = \mathrm{diag}\big(\underbrace{1, \ldots, 1}_{n \text{ times}}, \underbrace{-1, \ldots, -1}_{n \text{ times}}, 0, 0, \ldots \big) \,. \tag{12.36}$$

Moreover, we choose ρ as a multiple of the Dirac measure supported at x. Then the action $\mathcal{S}$ vanishes (see (5.36)), whereas the constraint $\mathcal{T}$ is strictly positive (see (5.39)). $\Diamond$

Example 12.4.2 (Nontriviality of the action with trace constraint) Let ρ be a normalized measure which satisfies the trace constraint in a nontrivial way, that is,

$$\int_{\mathcal{F}} \operatorname{tr}(x)\, \mathrm{d}\rho(x) = \text{const} \neq 0\,. \tag{12.37}$$

Let us prove that the action is nonzero. This will show that the trace constraint really avoids trivial minimizers of the causal action principle.

(a) Since the integral over the trace is nonzero, there is a point x in the support of ρ with $\operatorname{tr}(x) \neq 0$. We denote the nontrivial eigenvalues of x by $\nu_1, \ldots, \nu_{2n}$ and order them according to

$$\nu_1 \leq \cdots \leq \nu_n \;\leq\; 0 \;\leq\; \nu_{n+1} \leq \cdots \leq \nu_{2n}\,. \tag{12.38}$$

The fact that the trace of x is nonzero clearly implies that the ν_i do not all have the same absolute value. As a consequence, the nontrivial eigenvalues of the operator product x^2 given by $\lambda^{xx}_j = \nu_j^2$ are all nonnegative and not all equal. Using the form of the Lagrangian in (5.35), we conclude that $\mathcal{L}(x,x) > 0$.

(b) Since the Lagrangian is continuous in both arguments, there is an open neighborhood $U \subset \mathcal{F}$ of x such that $\mathcal{L}(y,z) > 0$ for all $y, z \in U$. Since x lies in the support of ρ, we know that $\rho(U) > 0$. As a consequence,

$$\mathcal{S} \geq \int_U \mathrm{d}\rho(x) \int_U \mathrm{d}\rho(y)\, \mathcal{L}(x,y) > 0, \tag{12.39}$$

(because if the integrals vanished, then the integrand would have to be zero almost everywhere, a contradiction).

We remark that this argument is quantified in [42, Proposition 4.3]. ◊

We now come to the boundedness constraint. In order to explain how it comes about, we give an explicit example with (4×4)-matrices (for a similar example with (2×2)-matrices, see Exercise 6.2).

Example 12.4.3 (Necessity of the boundedness constraint) The following example explains why the *boundedness constraint* (5.39) is needed in order to ensure the existence of minimizers. It was first given in [43, Example 2.9]. Let $\mathcal{H} = \mathbb{C}^4$. We introduce the four 4×4-matrices acting on $\mathcal{H}$ by

$$\gamma^\alpha = \begin{pmatrix} \sigma^\alpha & 0 \\ 0 & -\sigma^\alpha \end{pmatrix}, \quad \alpha = 1,2,3 \qquad \text{and} \qquad \gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{12.40}$$

where the σ^α are again the Pauli matrices (1.27). Given a parameter $\tau > 1$, we consider the following mapping from the sphere $S^3 \subset \mathbb{R}^4$ to the linear operators on $\mathcal{H}$,

$$F \;:\; S^3 \to \mathrm{L}(\mathcal{H})\,, \qquad F(x) = \sum_{i=1}^4 \tau\, x^i \gamma^i + \mathbb{1}\,. \tag{12.41}$$

(a) *The matrices $F(x)$ have two positive and two negative eigenvalues:*
Since the computation of the eigenvalues of 4×4-matrices is tedious, it is

preferable to proceed as follows. The matrices γ^j are the Dirac matrices of Euclidean $\mathbb{R}^4$, satisfying the anti-commutation relations

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\, \mathbb{1} \qquad (i, j = 1, \ldots, 4)\,. \tag{12.42}$$

As a consequence,

$$F(x) - \mathbb{1} = \sum_{i=1}^{4} \tau\, x^i \gamma^i \tag{12.43}$$

$$\begin{aligned} \big(F(x) - \mathbb{1}\big)^2 &= \sum_{i,j=1}^{4} \tau^2\, x^i\, x^j \gamma^i \gamma^j = \frac{\tau^2}{2} \sum_{i,j=1}^{4} x^i\, x^j \big\{\gamma^i, \gamma^j\big\} \\ &= \frac{\tau^2}{2} \sum_{i,j=1}^{4} x^i\, x^j\, 2\, \delta_{ij}\, \mathbb{1} = \tau^2\, \mathbb{1}\,. \end{aligned} \tag{12.44}$$

Hence, the matrix $F(x)$ satisfies the polynomial equation

$$\big(F(x) - \mathbb{1}\big)^2 = \tau^2\, \mathbb{1}\,. \tag{12.45}$$

We conclude that $F(x)$ has the eigenvalues

$$\nu_\pm = 1 \pm \tau\,. \tag{12.46}$$

Since $F(x) - \mathbb{1}$ is trace-free, each eigenvalue must appear with multiplicity two. Using that $\tau > 1$, we conclude that $F(x)$ really has two positive and two negative eigenvalues.

(b) *Construction of a causal fermion system:*
Let μ be the normalized Lebesgue measure on $S^3 \subset \mathbb{R}^4$. Setting $\rho = F_* \mu$ defines a causal fermion system of spin dimension two and total volume one. Since the matrices $F(x)$ all have trace four, we also know that

$$\int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho(x) = \int_{S^3} \mathrm{tr}(F(x))\, \mathrm{d}\mu(x) = 4\,. \tag{12.47}$$

Therefore, the volume constraint (5.37) and the trace constraint (5.38) are satisfied, both with constants independent of τ.

(c) *Computation of the eigenvalues of $F(x)\, F(y)$:*
Again, this can be calculated most conveniently using the Clifford relations.

$$\begin{aligned} F(x)\, F(y) &= \Big(\sum_{i=1}^{4} \tau\, x^i \gamma^i + \mathbb{1}\Big)\Big(\sum_{j=1}^{4} \tau\, y^j \gamma^j + \mathbb{1}\Big) \\ &= \big(1 + \tau^2\, \langle x, y\rangle\big)\mathbb{1} + \tau \sum_{i=1}^{4} (x^i + y^i)\gamma^i + \frac{\tau^2}{2} \sum_{i,j=1}^{4} x^i y^j \left[\gamma^i, \gamma^j\right]. \end{aligned} \tag{12.48}$$

Using that

$$\gamma^i \left[\gamma^i, \gamma^j\right] = -\left[\gamma^i, \gamma^j\right] \gamma^i\,, \tag{12.49}$$

we conclude that

$$\begin{aligned}&\Big(F(x)\,F(y)-\big(1+\tau^2\,\langle x,y\rangle\big)\mathbb{1}\Big)^2\\&=\tau^2\sum_{i=1}^4(x^i+y^i)^2+\bigg(\frac{\tau^2}{2}\sum_{i,j=1}^4 x^iy^j\left[\gamma^i,\gamma^j\right]\bigg)^2 .\end{aligned}\tag{12.50}$$

This can be simplified with the help of the relations

$$\sum_{i=1}^4(x^i+y^i)^2=2+2\,\langle x,y\rangle \tag{12.51}$$

$$\bigg(\sum_{i,j=1}^4 x^iy^j\left[\gamma^i,\gamma^j\right]\bigg)^2=-4\sin^2\vartheta=-4\left(1-\langle x,y\rangle^2\right), \tag{12.52}$$

where ϑ is the angle between the vectors $x,y\in\mathbb{R}^4$. The relation (12.52) can be verified in detail as follows. The rotational symmetry of the Euclidean Dirac operator on $\mathbb{R}^4$ means that for every rotation $O\in\mathrm{SO}(4)$ there is a unitary operator $U\in\mathrm{SU}(4)$ such that

$$O^i_j\,\gamma^j=U\gamma^iU^{-1}\,. \tag{12.53}$$

Making use of this rotational symmetry, we can arrange that the vector x is the basis vector e_1 and that $y=\cos\vartheta\,e_1+\sin\vartheta\,e_2$. As a consequence,

$$\sum_{i,j=1}^4 x^iy^j\left[\gamma^i,\gamma^j\right]=\sin\vartheta\,[\gamma^1,\gamma^2]=2\,\sin\vartheta\,\gamma^1\gamma^2 \tag{12.54}$$

$$\bigg(\sum_{i,j=1}^4 x^iy^j\left[\gamma^i,\gamma^j\right]\bigg)^2=4\,\sin^2\vartheta\,\gamma^1\gamma^2\gamma^1\gamma^2\,, \tag{12.55}$$

and applying the anti-commutation relations gives (12.52).

Combining the above equations, we find that the product $F(x)F(y)$ satisfies the polynomial equation

$$\begin{aligned}\Big(F(x)\,F(y)-\big(1+\tau^2\,\langle x,y\rangle\big)\mathbb{1}\Big)^2&=2\,\tau^2\big(1+\langle x,y\rangle\big)-\tau^4\,\big(1-\langle x,y\rangle^2\big)\\&=\tau^2\,\Big(1+\langle x,y\rangle\Big)\Big(2-\tau^2\,\big(1-\langle x,y\rangle\big)\Big)\,.\end{aligned}\tag{12.56}$$

Taking the square root, the zeros of this polynomial are computed by

$$\lambda_{1/2}=1+\tau^2\,\langle x,y\rangle\pm\tau\,\sqrt{1+\langle x,y\rangle}\,\sqrt{2-\tau^2\,(1-\langle x,y\rangle)}\,. \tag{12.57}$$

Moreover, taking the trace of (12.48), one finds

$$\operatorname{tr}\big(F(x)\,F(y)\big)=4\,\big(1+\tau^2\,\langle x,y\rangle\big)\,. \tag{12.58}$$

This implies that each eigenvalue in (12.57) has the algebraic multiplicity of two.

(d) *Computation of the Lagrangian:*
We again denote the angle between the vectors $x, y \in \mathbb{R}^4$ by ϑ. If ϑ is sufficiently small, the term $(1 - \langle x, y \rangle)$ is close to zero, and thus the arguments of the square roots in (12.57) are all positive. However, if ϑ becomes so large that

$$\vartheta \;\geq\; \vartheta_{\max} := \arccos\bigg(1 - \frac{2}{\tau^2} \bigg) \,, \tag{12.59}$$

then the argument of the last square root in (12.57) becomes negative, so that the $\lambda_{1/2}$ form a complex conjugate pair. Moreover, a short calculation shows that

$$\lambda_1 \lambda_2 = (1+\tau)^2 (1-\tau)^2 > 0 \,, \tag{12.60}$$

implying that if the $\lambda_{1/2}$ are both real, then they have the same sign. Using this information, the Lagrangian simplifies to

$$\begin{aligned} \mathcal{L}\big(F(x), F(y)\big) &= \frac{1}{8} \sum_{i,j=1}^{4} \Big(\big|\lambda^{xy}_i\big| - \big|\lambda^{xy}_j\big| \Big)^2 = \frac{1}{2} \sum_{i,j=1}^{2} \Big(\big|\lambda_i\big| - \big|\lambda_j\big| \Big)^2 \\ &= \frac{1}{2}\, \Theta(\vartheta_{\max} - \vartheta) \sum_{i,j=1}^{2} \Big(\lambda_i - \lambda_j \Big)^2 = \Theta(\vartheta_{\max} - \vartheta) \big(\lambda_1 - \lambda_2\big)^2 \\ &= 4\tau^2 \,(1 + \cos\vartheta)\, \big(2 - \tau^2\,(1 - \cos\vartheta)\big)\; \Theta(\vartheta_{\max} - \vartheta)\,. \end{aligned} \tag{12.61}$$

(e) *Computation of the action:*
Inserting this Lagrangian in (5.36) and using the definition of the push-forward measure, we obtain

$$\begin{aligned} \mathcal{S} &= \int_{S^3} \mathrm{d}\mu(x) \int_{S^3} \mathrm{d}\mu(y)\, \mathcal{L}\big(F(x), F(y)\big) \\ &= \int_{S^3} \mathrm{d}\mu(y)\, \mathcal{L}\big(F(x), F(y)\big) = \frac{2}{\pi} \int_0^{\vartheta_{\max}} \mathcal{L}(\cos\vartheta)\, \sin^2\vartheta\; \mathrm{d}\vartheta \\ &= \frac{512}{15\pi}\, \frac{1}{\tau} + \mathcal{O}(\tau^{-2})\,. \end{aligned} \tag{12.62}$$

Thus setting $F_k = F|_{\tau=k}$, we have constructed a divergent minimizing sequence. However, the integral in the boundedness constraint (5.39) also diverges as $k \to \infty$. This example shows that, leaving out the boundedness constraint, there is no minimizer. ◊

We finally remark that this example is not as artificial or academic as it might appear at first sight. Indeed, as observed in the master thesis [109], when discretizing a Dirac system in $\mathbb{R} \times S^3$ (where the sphere can be thought of as a spatial compactification of Minkowski space), then in the simplest case of four occupied Dirac states (referred to as "one shell," i.e., $\dim \mathcal{H} = 4$), this system reduces precisely to the last example. In simple terms, this observation can be summarized by saying that Clifford structures tend to make the causal action small.

12.5 The Radon–Nikodym Theorem

As already mentioned at the beginning of the previous section, one difficulty in the existence proof for the causal action principle is the fact that the set $\mathcal{F}$ is unbounded and thus non-compact. In order to deal with this difficulty, we need one more mathematical tool: the Radon–Nikodym theorem. We now give the proof of the Radon–Nikodym theorem by von Neumann following the presentation in [136, Chapter 6]. An alternative method of proof is given in [101, 33]. As in Section 12.2, it again suffices to consider the case that the base space $\mathcal{K}$ is a *compact* topological space.

Definition 12.5.1 *A Radon measure λ is* **absolutely continuous** *with respect to another Radon measure ν, denoted by*

$$\lambda \ll \nu \,, \tag{12.63}$$

if the implication

$$\nu(E) = 0 \quad \Longrightarrow \quad \lambda(E) = 0 \tag{12.64}$$

holds for any Borel set E. The measure λ is **concentrated** *on the Borel set A if $\lambda(E) = \lambda(E \cap A)$ for all Borel sets E. The measures λ and μ are* **mutually singular***, denoted by*

$$\lambda \perp \nu \,, \tag{12.65}$$

if there are disjoint Borel sets A and B such that λ is concentrated in A and ν is concentrated in B.

In order to avoid confusion, we point out that the supports of two mutually singular measures are not necessarily disjoint, as one sees in the simple example of the Lebesgue measure on $(0,1)$ and the Dirac measure supported at the origin,

$$\lambda := \mathrm{d}x|_{(0,1)} \qquad \text{and} \qquad \mu = \delta_0 \,. \tag{12.66}$$

Since the support is by definition a closed set (see (2.69)), the support of $dx|_{(0,1)}$ contains the origin, which is precisely the support of the Dirac measure. But clearly, the two measures are concentrated on the sets $(0,1)$ and $\{0\}$, respectively, and are thus mutually singular.

Theorem 12.5.2 **(Radon–Nikodym)** *Let μ and λ be Radon measures on a compact topological space $\mathcal{K}$.*

(a) *There is a unique pair of Borel measures λ_a and λ_s such that*

$$\lambda = \lambda_a + \lambda_s \qquad \textit{and} \qquad \lambda_a \ll \mu \,, \quad \lambda_s \perp \mu \,. \tag{12.67}$$

(b) *There is a unique function $h \in L^1(\mathcal{K}, \mathrm{d}\mu)$ such that*

$$\lambda_a(E) = \int_E h \, \mathrm{d}\mu \qquad \textit{for every Borel set } E \,. \tag{12.68}$$

The pair (λ_a, λ_s) is also referred to as the **Lebesgue decomposition** of λ with respect to μ.

Proof of Theorem 12.5.2. The uniqueness of the decomposition is easily seen as follows: Suppose that (λ'_a, λ'_s) is another Lebesgue decomposition. Then

$$\lambda'_a - \lambda_a = \lambda_s - \lambda'_s \,. \tag{12.69}$$

Since $\lambda_s \perp \mu$ and $\lambda'_s \perp \mu$, the measures λ_s and $\lambda_{s'}$ are concentrated in a Borel set A with $\mu(A) = 0$. Evaluating (12.69) on Borel subsets of A, the left-hand side vanishes because λ_a and λ'_a are both absolutely continuous with respect to μ. Hence $\lambda'_s - \lambda_s = 0$. Using this relation in (12.69), we also conclude that $\lambda'_a - \lambda_a = 0$. This proves uniqueness.

For the existence proof, we let ρ be the measure $\rho = \lambda + \mu$. Then

$$\int_{\mathcal{K}} f \, \mathrm{d}\rho = \int_{\mathcal{K}} f \; \mathrm{d}\lambda + \int_{\mathcal{K}} f \, \mathrm{d}\mu \tag{12.70}$$

for any nonnegative Borel function f. If $f \in L^2(\mathcal{K}, \mathrm{d}\rho)$, the Schwarz inequality gives

$$\left| \int_{\mathcal{K}} f \; \mathrm{d}\lambda \right| \le \int_{\mathcal{K}} |f| \, \mathrm{d}\rho \le \sqrt{\rho(\mathcal{K})} \; \|f\|_{L^2(\mathcal{K}, \mathrm{d}\rho)} \,. \tag{12.71}$$

Thus the mapping $f \mapsto \int_{\mathcal{K}} f \, d\lambda$ is a bounded linear functional on $L^2(\mathcal{K}, \mathrm{d}\rho)$. By the Fréchet–Riesz theorem, we can represent this linear functional by a function $g \in L^2(\mathcal{K}, \mathrm{d}\rho)$, that is,

$$\int_{\mathcal{K}} f \; \mathrm{d}\lambda = \int_{\mathcal{K}} g \, f \, \mathrm{d}\rho \qquad \text{for all } f \in L^2(\mathcal{K}, \mathrm{d}\rho) \,. \tag{12.72}$$

We next want to show that, by modifying g on a set of ρ-measure zero, we can arrange that g takes values in the interval $[0, 1]$. To this end, we let E be any Borel set with $\rho(E) > 0$. Evaluating (12.72) for $f = \chi_E$, we obtain

$$0 \le \frac{1}{\rho(E)} \int_E g \, \mathrm{d}\rho = \frac{\lambda(E)}{\rho(E)} \le 1 \,. \tag{12.73}$$

Now the claim follows from elementary measure theory (see, e.g., [136, Theorem 1.40]).

Using (12.70), we can rewrite (12.72) as

$$\int_{\mathcal{K}} (1 - g) \, f \; \mathrm{d}\lambda = \int_{\mathcal{K}} g \, f \, \mathrm{d}\mu \qquad \text{for all nonnegative } f \in L^2(\mathcal{K}, \mathrm{d}\rho) \,. \tag{12.74}$$

We introduce the Borel sets

$$A = \{x \in \mathcal{K} \,|\, 0 \le g(x) < 1\} \qquad \text{and} \qquad B = \{x \in \mathcal{K} \,|\, g(x) = 1\} \,. \tag{12.75}$$

and define the measures λ_a and λ_s by

$$\mathrm{d}\lambda_a = \chi_A \; \mathrm{d}\lambda \qquad \text{and} \qquad \mathrm{d}\lambda_s = \chi_B \; \mathrm{d}\lambda \,. \tag{12.76}$$

Choosing $f = \chi_B$ in (12.74), one sees that $\mu(B) = 0$, implying that $\lambda_s \perp \mu$.

Moreover, since g is bounded, we can evaluate (12.74) for

$$f = \big(1 + g + \cdots + g^n\big) \, \chi_E \tag{12.77}$$

for any $n \in \mathbb{N}$ and any Borel set E. Using the same transformation with "telescopic sums" as in the evaluation of the geometric or Neumann series, we obtain

$$\int_E \left(1 - g^{n+1}\right) \,\mathrm{d}\lambda = \int_E g\left(1 + g + \cdots + g^n\right) \mathrm{d}\mu \,. \tag{12.78}$$

At every point of B, the factor $(1 - g^{n+1})$ in the integrand on the left vanishes. At every point of A, on the other hand, the factor $(1 - g^{n+1})$ is monotone increasing in n and converges to one. Hence, Lebesgue's monotone convergence theorem implies that the left-hand side of (12.78) converges to

$$\lim_{n\to\infty} \int_E \left(1 - g^{n+1}\right) \,\mathrm{d}\lambda = \lambda\big(E \cap A\big) \,. \tag{12.79}$$

The integrand on the right-hand side of (12.78), on the other hand, is monotone increasing in n, so that the limit

$$h(x) := \lim_{n\to\infty} g(x)\left(1 + g(x) + \cdots + g^n(x)\right) \quad \text{exists in } \mathbb{R}^+_0 \cup \{\infty\} \,. \tag{12.80}$$

Moreover, the monotone convergence theorem implies that

$$\lim_{n\to\infty} \int_E g\left(1 + g + \cdots + g^n\right) \mathrm{d}\mu = \int_E h \,\mathrm{d}\mu \;\in\; \mathbb{R}^+_0 \cup \{\infty\} \,. \tag{12.81}$$

We conclude that, in the limit $n \to \infty$, the relation (12.78) yields

$$\lambda_a(E) = \lambda\big(E \cap A\big) = \int_E h \,\mathrm{d}\mu \qquad \text{for any Borel set } E \,. \tag{12.82}$$

Choosing $E = \mathcal{K}$, one sees that $h \in L^1(\mathcal{K}, \mathrm{d}\mu)$. This concludes the proof of (12.68). Finally, the representation (12.68) implies that $\lambda_a \ll \mu$. $\square$

12.6 Moment Measures

We now introduce an important concept needed for the existence proof: the moment measures. We again assume that the Hilbert space is finite-dimensional and that the measure ρ is normalized (12.1). We consider $\mathcal{F}$ with the metric induced by the sup-norm on $\mathrm{L}(\mathcal{H})$, that is,

$$d(p, q) = \|p - q\|, \tag{12.83}$$

(and $\|.\|$ as in (5.40)). The basic difficulty in applying the abstract theorems is that $\mathcal{F}$ is *not compact* (indeed, it is star-shaped in the sense that $p \in \mathcal{F}$ implies $\lambda p \in \mathcal{F}$ for all $\lambda \in \mathbb{R}$). If the metric space is non-compact, our abstract results no longer apply, as becomes clear in the following simple example.

Example 12.6.1 Consider the Banach space $C^0_0(\mathbb{R}, \mathbb{R})$ of compactly supported continuous functions. Let $\rho_n = \delta_n$ be the sequence of Dirac measures supported at $n \in \mathbb{N}$. Then for any $f \in C^0_0(\mathbb{R}, \mathbb{R})$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f \,\mathrm{d}\rho_n = \lim_{n\to\infty} f(n) = 0 \,. \tag{12.84}$$

Hence, the sequence $(\rho_n)_{n\in\mathbb{R}}$ converges as a measure to zero. Thus the limiting measure is no longer normalized. This shows that Theorem 12.3.2 fails to hold if the base space is non-compact. ◊

The way out is to make use of the fact that the causal action as well as the constraints are formed of functionals that are homogeneous under the scaling $p \to \lambda p$ of degree zero, one or two. This makes it possible to restrict attention to a compact subset of $\mathcal{F}$ and to consider three measures on this compact set. We now give the needed definitions.

Definition 12.6.2 *Let $\mathcal{K}$ be the compact metric space*

$$\mathcal{K} = \{p \in \mathcal{F} \textit{ with } \|p\| = 1\} \cup \{0\} \,. \tag{12.85}$$

For a given measure ρ on $\mathcal{F}$, we define the measurable sets of $\mathcal{K}$ by the requirement that the sets $\mathbb{R}^+\Omega = \{\lambda p \,|\, \lambda \in \mathbb{R}^+, p \in \Omega\}$ and $\mathbb{R}^-\Omega$ should be ρ-measurable in $\mathcal{F}$. We introduce the measures $\mathfrak{m}^{(0)}$, $\mathfrak{m}^{(1)}_\pm$ and $\mathfrak{m}^{(2)}$ by

$$\mathfrak{m}^{(0)}(\Omega) = \frac{1}{2}\,\rho\big(\mathbb{R}^+\Omega \setminus \{0\}\big) + \frac{1}{2}\,\rho\big(\mathbb{R}^-\Omega \setminus \{0\}\big) + \rho\big(\Omega \cap \{0\}\big) \tag{12.86}$$

$$\mathfrak{m}^{(1)}_+(\Omega) = \frac{1}{2} \int_{\mathbb{R}^+\Omega} \|p\|\, \mathrm{d}\rho(p) \tag{12.87}$$

$$\mathfrak{m}^{(1)}_-(\Omega) = \frac{1}{2} \int_{\mathbb{R}^-\Omega} \|p\|\, \mathrm{d}\rho(p) \tag{12.88}$$

$$\mathfrak{m}^{(2)}(\Omega) = \frac{1}{2} \int_{\mathbb{R}^+\Omega} \|p\|^2\, \mathrm{d}\rho(p) \;+\; \frac{1}{2} \int_{\mathbb{R}^-\Omega} \|p\|^2\, \mathrm{d}\rho(p)\,. \tag{12.89}$$

The measures $\mathfrak{m}^{(l)}$ and $\mathfrak{m}^{(l)}_\pm$ are referred to as the l^{th} **moment measure**.

As a short notation, it is convenient to abbreviate the difference of the first moment measures by

$$\mathfrak{m}^{(1)}(\Omega) := \mathfrak{m}^{(1)}_+(\Omega) - \mathfrak{m}^{(1)}_-(\Omega)\,. \tag{12.90}$$

We remark that $\mathfrak{m}^{(1)}$ can be regarded as a *signed measure* (see, e.g., [101, §28] or [136, Chapter 6]). For simplicity, we here avoid the concept of signed measures by working instead with the two (positive) measures $\mathfrak{m}^{(1)}_\pm$. Nevertheless, we introduce an $\mathfrak{m}^{(1)}$-integral as a short notation for the difference of the integrals with respect to $\mathfrak{m}^{(1)}_+$ and $\mathfrak{m}^{(1)}_-$, that is,

$$\int_{\mathcal{K}} h \; \mathrm{d}\mathfrak{m}^{(1)} := \int_{\mathcal{K}} h \; \mathrm{d}\mathfrak{m}^{(1)}_+ - \int_{\mathcal{K}} h \; \mathrm{d}\mathfrak{m}^{(1)}_- \,, \tag{12.91}$$

where for simplicity we always assume that h is continuous.

The ρ-integrals of homogeneous functions can be rewritten as integrals over $\mathcal{K}$ using the moment measures, as we now make precise.

Definition 12.6.3 *A function $h \in C^0(\mathcal{F})$ is called* **homogeneous of degree ℓ** *with $\ell \in \{0, 1, 2\}$ if*

$$h(\nu x) = \nu^\ell\, h(x) \qquad \textit{for all } \nu \in \mathbb{R} \textit{ and } x \in \mathcal{F}\,. \tag{12.92}$$

Lemma 12.6.4 *Let $h \in C^0(\mathcal{F})$ be a function which is homogeneous of degree $\ell \in \{0,1,2\}$. Then*

$$\int_{\mathcal{F}} h \, \mathrm{d}\rho = \int_{\mathcal{K}} h \; \mathrm{d}\mathfrak{m}^{(l)} \,. \tag{12.93}$$

Proof We first note that, using the homogeneity (12.92), the function h is uniquely determined by its restriction to $\mathcal{K}$. Moreover, using an approximation argument with Lebesgue's dominated convergence theorem, it suffices to consider a function h which is homogeneous of degree ℓ and *simple* in the sense that its restriction to $\mathcal{K}$ takes a finite number of values, that is,

$$h\big|_{\mathcal{K}} = \sum_{i=1}^{N} c_i \, \chi_{\Omega_i} , \tag{12.94}$$

with suitable Borel sets $\Omega_1, \ldots, \Omega_N \subset \mathcal{K}$ and real coefficients $c_1, \ldots, c_N$. For such simple functions, the integrals go over to finite sums, and we obtain

$$\int_{\mathcal{F}} h \, \mathrm{d}\rho = \sum_{i=1}^{N} c_i \int_{\mathbb{R}^+ \Omega_i} \|p\|^\ell \, \mathrm{d}\rho(p) = \sum_{i=1}^{N} c_i \, \mathfrak{m}^{(\ell)}(\Omega_i) = \int_{\mathcal{K}} h \; \mathrm{d}\mathfrak{m}^{(l)} \,, \tag{12.95}$$

as desired. This concludes the proof. □

Applying this lemma, the normalization $\rho(\mathcal{F}) = 1$ can be expressed in terms of the moment measures as

$$\mathfrak{m}^{(0)}(\mathcal{K}) = 1 \,, \tag{12.96}$$

whereas the action (5.36) as well as the functionals in the constraints (5.39) and (5.38) can be written as

$$\mathcal{S}(\rho) = \iint_{\mathcal{K}\times\mathcal{K}} \mathcal{L}(p,q) \; \mathrm{d}\mathfrak{m}^{(2)}(p) \; \mathrm{d}\mathfrak{m}^{(2)}(q) \tag{12.97}$$

$$\mathcal{T}(\rho) = \iint_{\mathcal{K}\times\mathcal{K}} |p\,q|^2 \; \mathrm{d}\mathfrak{m}^{(2)}(p) \; \mathrm{d}\mathfrak{m}^{(2)}(q) \tag{12.98}$$

$$\int_{\mathcal{F}} \mathrm{tr}(x) \, \mathrm{d}\rho(x) = \int_{\mathcal{K}} \mathrm{tr}(p) \; \mathrm{d}\mathfrak{m}^{(1)}_+(p) - \int_{\mathcal{K}} \mathrm{tr}(p) \; \mathrm{d}\mathfrak{m}^{(1)}_-(p) \,. \tag{12.99}$$

Working with the moment measures has the advantage that they are measures on the *compact* space $\mathcal{K}$. We also learn that two measures ρ and $\tilde{\rho}$ whose moment measures coincide yield the same values for the functionals $\mathcal{S}$ and $\mathcal{T}$ as well as for the integral (12.99) entering the trace constraint. It is most convenient to work exclusively with the moment measures. At the end, we shall construct a suitable representative ρ of the limiting moment measures. A key step for making this method work is the following a priori estimate.

Lemma 12.6.5 *There is a constant $\varepsilon = \varepsilon(f,n) > 0$ such that for every measure ρ on $\mathcal{F}$ the corresponding moment measures (see Definition 12.6.2) satisfy for all measurable $\Omega \subset \mathcal{K}$ the following inequalities:*

$$\Big(\mathfrak{m}^{(1)}_+(\Omega) + \mathfrak{m}^{(1)}_-(\Omega)\Big)^2 \leq \mathfrak{m}^{(0)}(\Omega) \, \mathfrak{m}^{(2)}(\Omega) \tag{12.100}$$

$$\mathfrak{m}^{(2)}(\mathcal{K}) \leq \frac{\sqrt{\mathcal{T}(\rho)}}{\varepsilon} \,. \tag{12.101}$$

Proof The inequality (12.100) follows immediately from Hölder's inequality,

$$\big|2\big(\mathfrak{m}_+^{(1)}(\Omega)+\mathfrak{m}_-^{(1)}(\Omega)\big)\big|^2 \le \Big(\int_{\mathbb{R}\Omega} \|p\|\, \mathrm{d}\rho(p)\Big)^2$$
$$\le \rho(\mathbb{R}\Omega) \int_{\mathbb{R}\Omega} \|p\|^2\, \mathrm{d}\rho(p) \le 4\mathfrak{m}^{(0)}(\Omega)\, \mathfrak{m}^{(2)}(\Omega)\,. \tag{12.102}$$

In order to prove (12.101), we introduce the mapping

$$\phi \,:\, \mathcal{K} \times \mathcal{K} \to \mathbb{R} \,:\, (p,q) \mapsto |p\, q|\,. \tag{12.103}$$

Clearly, ϕ is continuous and

$$\phi(p,p) = |p^2| = \mathrm{Tr}(p^2) \ge \|p\|^2 = 1, \tag{12.104}$$

here we used that the Hilbert–Schmidt norm is larger than the absolute square of each eigenvalue. Therefore, every point $r \in \mathcal{K}$ has a neighborhood $U(r) \subset \mathcal{K}$ with

$$\phi(p,q) \ge \frac{1}{2} \qquad \text{for all } p,q \in U(r)\,. \tag{12.105}$$

Since $\mathcal{K}$ is compact, there is a finite number of points $r_1, \ldots, r_N$ such that the corresponding sets $U_i := U(r_i)$ cover $\mathcal{K}$. Due to the additivity property of measures, there is an index $i \in \{1, \ldots, N\}$ such that

$$\mathfrak{m}^{(2)}(U_i) \ge \frac{\mathfrak{m}^{(2)}(\mathcal{K})}{N}\,. \tag{12.106}$$

We write $\mathcal{T}$ in the form (12.98) and apply (12.105) as well as (12.106) to obtain

$$\mathcal{T}(\rho) \ge \iint_{U_i \times U_i} |p\, q|^2\; \mathrm{d}\mathfrak{m}^{(2)}(p)\; \mathrm{d}\mathfrak{m}^{(2)}(q) \ge \frac{1}{2}\, \mathfrak{m}^{(2)}(U_i)^2 \ge \frac{\mathfrak{m}^{(2)}(\mathcal{K})^2}{2N^2}\,. \tag{12.107}$$

Setting $\varepsilon = 1/(\sqrt{2}N)$, the result follows. □

12.7 Existence of Minimizers for the Causal Action Principle

After the above preparations, we can follow the strategy of the direct method in the calculus of variations described at the beginning of Chapter 12 to obtain the following result.

Theorem 12.7.1 *Let $\mathcal{H}$ be a finite-dimensional Hilbert space and $n \in \mathbb{N}$. Let ρ_k be a minimizing sequence of regular Borel measures on $\mathcal{F}$ satisfying our constraints* (5.37), (5.38) *and* (5.39) *(for fixed and finite constants). Then there is a regular Borel measure ρ which also satisfies the constraints (again with the same constants) and*

$$\mathcal{S}(\rho) = \liminf_{n\to\infty} \mathcal{S}(\rho_n)\,. \tag{12.108}$$

In short, the method for constructing ρ is to take a limit of the moment measures of the ρ_k and to realize this limit by the measure ρ. In more detail, we proceed as follows. In view of Lemma 12.6.5, we know that the moment measures are uniformly bounded measures on the compact metric space $\mathcal{K}$. Applying the compactness result of Theorem 12.3.2 (based on the Banach–Alaoglu theorem and the Riesz representation theorem), we conclude that for a suitable subsequence (which we again denote by (ρ_k)), the moment measures converge in the $C^0(\mathcal{K})^*$-topology to regular Borel measures,

$$\mathfrak{m}_k^{(0)} \to \mathfrak{m}^{(0)}\,, \qquad \mathfrak{m}_{k,\pm}^{(1)} \to \mathfrak{m}_\pm^{(1)} \qquad \text{and} \qquad \mathfrak{m}_k^{(2)} \to \mathfrak{m}^{(2)}\,, \tag{12.109}$$

which again have the properties (12.96), (12.100) and (12.101).

We next form the Radon–Nikodym decompositions of $\mathfrak{m}_\pm^{(1)}$ and $\mathfrak{m}^{(2)}$ with respect to $\mathfrak{m}^{(0)}$. The inequality (12.100) shows that every set of $\mathfrak{m}^{(0)}$-measure zero is also a set of measure zero with respect to $\mathfrak{m}_+^{(1)}$ and $\mathfrak{m}_-^{(1)}$. In other words, the measures $\mathfrak{m}_\pm^{(1)}$ are absolutely continuous with respect to $\mathfrak{m}^{(0)}$. Hence, applying Theorem 12.5.2, we obtain the Radon–Nikodym decompositions

$$d\mathfrak{m}_\pm^{(1)} = f_\pm \, d\mathfrak{m}^{(0)} \qquad \text{with} \qquad f_\pm \in L^1(\mathcal{K}, d\mathfrak{m}^{(0)})\,. \tag{12.110}$$

As a consequence, the difference of measures $\mathfrak{m}^{(1)}$ in (12.90) has the decomposition

$$d\mathfrak{m}^{(1)} = f^{(1)} \, d\mathfrak{m}^{(0)} \qquad \text{with} \qquad f^{(1)} := f_+ - f_- \in L^1(\mathcal{K}, d\mathfrak{m}^{(0)})\,. \tag{12.111}$$

As we do not know whether also $\mathfrak{m}^{(2)}$ is absolutely continuous with respect to $m^{(0)}$, Theorem 12.5.2 gives the decomposition

$$d\mathfrak{m}^{(2)} = f^{(2)} \, d\mathfrak{m}^{(0)} + d\mathfrak{m}^{(2)}_{\text{sing}} \qquad \text{with} \qquad f^{(2)} \in L^1(\mathcal{K}, d\mathfrak{m}^{(0)})\,, \tag{12.112}$$

where the measure $\mathfrak{m}^{(2)}_{\text{sing}}$ is singular with respect to $\mathfrak{m}^{(0)}$.

Lemma 12.7.2 *The two functions $f^{(1)}$ and $f^{(2)}$ in the Radon–Nikodym decompositions* (12.111) *and* (12.112) *can be chosen such that*

$$\big|f^{(1)}\big|^2 \leq f^{(2)}\,. \tag{12.113}$$

Proof Since $\mathfrak{m}^{(2)}_{\text{sing}} \perp \mathfrak{m}^{(0)}$, there is a Borel set V such that

$$\chi_V \, d\mathfrak{m}^{(0)} = d\mathfrak{m}^{(0)} \qquad \text{and} \qquad \chi_V \, d\mathfrak{m}^{(2)}_{\text{sing}} = 0\,. \tag{12.114}$$

Using the Radon–Nikodym decompositions (12.111) and (12.112) in (12.100), we obtain for any Borel set $U \subset V$ the inequality

$$\left| \int_U f^{(1)} \, d\mathfrak{m}^{(0)} \right|^2 \leq \mathfrak{m}^{(0)}(U) \int_U f^{(2)} \, d\mathfrak{m}^{(0)}\,. \tag{12.115}$$

If the function $f^{(1)}$ does not change signs on U, we conclude that

$$\inf_U \big|f^{(1)}\big|^2 \leq \sup_U f^{(2)}\,. \tag{12.116}$$

By decomposing U into the two sets where $f^{(1)}$ is positive and negative, respectively, one readily sees that this inequality even holds for any Borel set $U \subset V$. As a consequence, the inequality $|f^{(1)}|^2 \leq f^{(2)}$ holds almost everywhere (with respect to the measure $\mathfrak{m}^{(0)}$), concluding the proof. □

In particular, we conclude that $f^{(1)}$ even lies in $L^2(\mathcal{K}, d\mathfrak{m}^{(0)})$. Setting $f = f^{(1)}$ and $d\mathfrak{n} = (f^{(2)} - |f|^2)\, d\mathfrak{m}^{(0)} + d\mathfrak{m}^{(2)}_{\text{sing}}$, we obtain the decomposition

$$d\mathfrak{m}^{(1)} = f\, d\mathfrak{m}^{(0)}\,, \qquad d\mathfrak{m}^{(2)} = |f|^2\, d\mathfrak{m}^{(0)} + d\mathfrak{n}\,, \tag{12.117}$$

where $f \in L^2(\mathcal{K}, d\mathfrak{m}^{(0)})$, and $\mathfrak{n}$ is a positive measure that need not be absolutely continuous with respect to $\mathfrak{m}^{(0)}$. From the definition (12.90), it is clear that f is odd, that is,

$$f(-p) = -f(p) \quad \text{for all } p \in \mathcal{K}\,. \tag{12.118}$$

The remaining task is to represent the limiting moment measures $\mathfrak{m}^{(l)}$ in (12.117) by a measure ρ. Unfortunately, there is the basic problem that such a measure can exist only if $\mathfrak{m}^{(2)}$ is absolutely continuous with respect to $\mathfrak{m}^{(0)}$, as the following consideration shows: Assume conversely that $\mathfrak{m}^{(2)}$ is not absolutely continuous with respect to $\mathfrak{m}^{(0)}$. Then, there is a measurable set $\Omega \subset \mathcal{K}$ with $\mathfrak{m}^{(0)}(\Omega) = 0$ and $\mathfrak{m}^{(2)}(\Omega) \neq 0$. Assume furthermore that there is a measure ρ on $\mathcal{F}$ which represents the limiting moment measures in the sense that (12.86)–(12.89) hold. From (12.86), we conclude that the set $\mathbb{R}\Omega \subset \mathcal{F}$ has ρ-measure zero. But then the integral (12.89) also vanishes, a contradiction.

This problem can also be understood in terms of the limiting sequence ρ_k. We cannot exclude that there is a star-shaped region $\mathbb{R}\Omega \subset \mathcal{F}$ such that the measures $\rho_k(\mathbb{R}\Omega)$ tend to zero, but the corresponding moment integrals (12.89) have a nonzero limit. Using a notion from the calculus of variations for curvature functionals, we refer to this phenomenon as the possibility of *bubbling*. This bubbling effect is illustrated by the following example.

Example 12.7.3 *(Bubbling)* We choose $f = 2$ and $n = 1$. Furthermore, we let $\mathscr{M} = [0, 1)$ with μ the Lebesgue measure. For any parameters $\kappa \geq 0$ and $\varepsilon \in (0, \frac{1}{2})$, we introduce the mapping $F_\varepsilon : \mathscr{M} \to \mathcal{F}$ by

$$F_\varepsilon(x) = \frac{1}{1-2\varepsilon} \times \begin{cases} -\kappa\, \varepsilon^{-\frac{1}{2}}\, \sigma^3 & \text{if } x \leq \varepsilon \\ \mathbb{1} + \sigma^1 \cos(\nu x) + \sigma^2 \sin(\nu x) & \text{if } \varepsilon < x \leq 1-\varepsilon \\ \kappa\, \varepsilon^{-\frac{1}{2}}\, \sigma^3 & \text{if } x > 1-\varepsilon\,, \end{cases} \tag{12.119}$$

where we set $\nu = 2\pi/(1-2\varepsilon)$ (and σ^j are the Pauli matrices). The corresponding measure ρ_ε on $\mathcal{F}$ has the following properties. On the set

$$S := \{\mathbb{1} + v^1\sigma^1 + v^2\sigma^2 \text{ with } (v^1)^2 + (v^2)^2 = 1\}\,, \tag{12.120}$$

which can be identified with a circle S^1, ρ_ε is a multiple of the Lebesgue measure. Moreover, ρ_ε is supported at the two points

$$p_\pm := \pm \frac{\kappa\, \varepsilon^{-\frac{1}{2}}}{1-2\varepsilon}\, \sigma^3 \qquad \text{with} \qquad \rho_\varepsilon(\{p_+\}) = \rho_\varepsilon(\{p_-\}) = \varepsilon\,. \tag{12.121}$$

A short calculation shows that the trace constraint is satisfied. Furthermore, the separations of the points p_+ and p_- from each other and from S are either spacelike or just in the boundary case between spacelike and timelike. Thus for computing the action, we only need to take into account the pairs (p_+, p_+), (p_-, p_-) as well as pairs (x, y) with $x, y \in S$. A straightforward computation yields

$$\mathcal{S}(\rho_\varepsilon) = \frac{3}{(1-2\varepsilon)^2}\,, \qquad \mathcal{T}(\rho_\varepsilon) = \frac{6}{(1-2\varepsilon)^2} + \frac{16\kappa^2}{(1-2\varepsilon)^3} + \frac{16\kappa^4}{(1-2\varepsilon)^4}\,. \tag{12.122}$$

Let us consider the limit $\varepsilon \searrow 0$. From (12.122) we see that the functionals $\mathcal{S}$ and $\mathcal{T}$ converge,

$$\lim_{\varepsilon\searrow 0} \mathcal{S} = 3\,, \qquad \lim_{\varepsilon\searrow 0} \mathcal{T} = 6 + 16\left(\kappa^2 + \kappa^4\right). \tag{12.123}$$

Moreover, there are clearly no convergence problems on the set S. Thus it remains to consider the situation at the two points $p_\pm$, (12.121), which move to infinity as ε tends to zero. These two points enter the moment measures only at the corresponding normalized points $\hat{p}_\pm = p_\pm/\|p_\pm\| \in \mathcal{K}$. A short calculation shows that the limiting moment measures $\mathfrak{m}^{(l)} = \lim_{\varepsilon\searrow 0} \mathfrak{m}^{(l)}_\varepsilon$ satisfy the relations

$$\mathfrak{m}^{(0)}(\{\hat{p}_\pm\}) = \mathfrak{m}^{(1)}(\{\hat{p}_\pm\}) = 0 \qquad \text{but} \qquad \mathfrak{m}^{(2)}(\{\hat{p}_\pm\}) = \kappa^2 > 0\,. \tag{12.124}$$

Hence, $\mathfrak{m}^{(2)}$ is indeed *not* absolutely continuous with respect to $\mathfrak{m}^{(0)}$.

In order to avoid misunderstandings, we point out that this example does *not* show that bubbling really occurs for minimizing sequences because we do not know whether the family $(\rho_\varepsilon)_{0<\varepsilon<1/2}$ is minimizing. But at least, our example shows that bubbling makes it possible to arrange arbitrary large values of $\mathcal{T}$ without increasing the action $\mathcal{S}$ (see (12.123) for large κ). ♢

In order to handle possible bubbling phenomena, it is important to observe that the second moment measure does not enter the trace constraint. Therefore, by taking out the term $d\mathfrak{n}$ in (12.117) we *decrease* the functionals $\mathcal{S}$ and $\mathcal{T}$ (see (12.97) and (12.98)), without affecting the trace constraint. It remains to show that the resulting moment measure can indeed be realized by a measure ρ. This is proven in the next lemma.

Lemma 12.7.4 *For any normalized regular Borel measure $\mathfrak{m}^{(0)}$ on $\mathcal{K}$ and any function $f \in L^2(\mathcal{K}, \mathbb{R})$, there is a normalized regular Borel measure $\tilde{\rho}$ whose moment measures $\tilde{\mathfrak{m}}^{(l)}$ are given by*

$$\tilde{\mathfrak{m}}^{(0)} = \mathfrak{m}^{(0)}\,, \qquad \mathrm{d}\tilde{\mathfrak{m}}^{(1)} = f\,\mathrm{d}\mathfrak{m}^{(0)}\,, \qquad \mathrm{d}\tilde{\mathfrak{m}}^{(2)} = |f|^2\,\mathrm{d}\mathfrak{m}^{(0)}\,. \tag{12.125}$$

Proof We introduce the mapping

$$F : \mathcal{K} \to \mathcal{F}\,, \qquad F(x) = f(x)\,x\,. \tag{12.126}$$

Choosing $\tilde{\rho} := F_*\mathfrak{m}^{(0)}$, a direct computation shows that the corresponding moment measures indeed satisfy (12.125). □

This concludes the proof of Theorem 12.7.1. We finally remark that the fact that we dropped the measure $d\mathfrak{n}$ in (12.117) implies that the value of $\mathcal{T}$ might decrease in the limit. This is the only reason why the boundedness constraint (5.39) is formulated as an inequality, rather than an equality. It is not clear if the causal action principle also admits minimizers if the inequality in (5.39) is replaced by an equality. We conjecture that the answer is yes. But at present, there is no proof. We note that, for the physical applications, it makes no difference if (5.39) is an equality or an inequality because in this case one works with the corresponding Euler–Lagrange equations, where the constraints enter only via Lagrange multiplier terms (for details, see [13]).

12.8 Existence of Minimizers for Causal Variational Principles in the Non-compact Setting

In Theorem 12.7.1, the existence of minimizers was established in the case that the Hilbert space $\mathcal{H}$ is finite-dimensional and the total volume $\rho(\mathcal{F})$ of spacetime is finite. From the physical point of view, this existence result is quite satisfying because one can take the point of view that it should be possible to describe our universe at least approximately by a causal fermion system with $\dim \mathcal{H} < \infty$ and $\rho(\mathcal{F}) < \infty$. From the mathematical point of view, however, it is interesting and important to also study the cases of an infinite-dimensional Hilbert space and/or infinite total volume. The case $\dim \mathcal{H} < \infty$ and $\rho(\mathcal{F}) = \infty$ is not sensible (see Exercise 12.5). In the case $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) < \infty$, on the other hand, there are minimizing sequences that converge to zero. Therefore, it remains to study the *infinite-dimensional setting* $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) = \infty$ already mentioned in Section 5.6. In this setting, the existence theory is difficult and has not yet been developed. Therefore, our strategy is to approach the problem in two steps. The first step is to deal with infinite total volume; this has been carried out in [66]. The second step, which involves the difficulty of dealing with nonlocally compact spaces, is currently under investigation (for the first results, see [114]).

We now outline the basic strategy in the simplest possible case (more details and a more general treatment can be found in [66]). We consider *causal variational principles in the non-compact setting* as introduced in Section 6.3. Moreover, we consider the *smooth setting* by assuming that the Lagrangian is smooth,

$$\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0)\,, \tag{12.127}$$

and has the properties (i) and (ii) in Definition 6.2.1. Moreover, we again assume that the Lagrangian has *compact range* (see Definition 8.1.1). The goal of this section is to prove the following theorem.

Theorem 12.8.1 *Under the above assumptions, there is a regular Borel measure ρ on $\mathcal{F}$ (not necessarily bounded) which satisfies the EL equations*

$$\ell\big|_{\operatorname{supp}\rho} \equiv \inf_M \ell = 0 \qquad \text{with} \qquad \ell(x) := \int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - 1\,. \tag{12.128}$$

For the proof, we first exhaust $\mathcal{F}$ by compact sets $(K_j)_{j\in\mathbb{N}}$, that is,

$$K_1 \subset K_2 \subset \cdots \subset \mathcal{F} \qquad \text{and} \qquad \bigcup_{j=1}^\infty K_j = \mathcal{F}\,. \tag{12.129}$$

On each K_j, we consider the restricted variational principle where we minimize the action

$$\mathcal{S}_{K_j}(\rho) = \int_{K_j} \mathrm{d}\rho(x) \int_{K_j} \mathrm{d}\rho(y)\, \mathcal{L}(x,y), \tag{12.130}$$

under variations of ρ within the class of normalized regular Borel measure on K_j. Using the existence theory in the compact setting (see Theorem 12.3.3), each of these restricted variational principles has a minimizer, which we denote by ρ_j. Each of these measures satisfies the EL equations stated in Theorem 7.1.1. Thus, introducing the functions

$$\ell_j \in C^0(K_j, \mathbb{R}) , \qquad \ell(x) := \int_{K_j} \mathcal{L}(x,y)\, \mathrm{d}\rho_j(y) - \mathfrak{s}_j , \tag{12.131}$$

one can choose the parameters $s_j > 0$ such that

$$\ell_j\big|_{\operatorname{supp}\rho_j} \equiv \inf_{K_j} \ell_j = 0 . \tag{12.132}$$

Typically, the support of the measures ρ_j will be "spread out" over larger and larger subsets of $\mathcal{F}$. This also means that, working with normalized measures, the measures ρ_j typically converge to the trivial measure $\rho = 0$. In order to ensure a nontrivial measure, we must perform a suitable *rescaling*. To this end, we introduce the measures

$$\tilde{\rho}_j = \frac{\rho_j}{\mathfrak{s}_j} . \tag{12.133}$$

These new measures are no longer normalized, but they satisfy the EL equations with $\tilde{\mathfrak{s}}_j = 1$, that is,

$$\tilde{\ell}_j\big|_{\operatorname{supp}\tilde{\rho}_j} \equiv \inf_{K_j} \tilde{\ell}_j = 0 \qquad \text{with} \qquad \tilde{\ell}_j(x) := \int_{K_j} \mathcal{L}(x,y)\, d\tilde{\rho}_j(y) - 1 . \tag{12.134}$$

Our next task is to construct a limit measure ρ of the measures $\tilde{\rho}_j$. We first extend the measures $\tilde{\rho}_j$ by zero to all of $\mathcal{F}$ and denote them by $\rho^{[j]}$,

$$\rho^{[j]}(U) := \rho_j(U \cap K_j) \qquad \text{for any Borel subset } U \subset \mathcal{F} . \tag{12.135}$$

In the next lemma, we show that these measures are bounded on every compact set.

Lemma 12.8.2 *For every compact subset $K \subset \mathcal{F}$, there is a constant $C_K > 0$ such that*

$$\rho^{[j]}(K) \leq C_K \qquad \textit{for all } j \in \mathbb{N} . \tag{12.136}$$

Proof Since $\mathcal{L}(x,.)$ is continuous and strictly positive at x, there is an open neighborhood $U(x)$ of x with

$$\mathcal{L}(y,z) \geq \frac{\mathcal{L}(x,x)}{2} > 0 \qquad \text{for all } y,z \in U(x) . \tag{12.137}$$

Covering K by a finite number of such neighborhoods $U(x_1), \ldots, U(x_L)$, it suffices to show the inequality (12.136) for the sets $K \cap U(x_\ell)$ for any $\ell \in \{1, \ldots, L\}$. Moreover, we choose N so large that $K_N \supset K$ and fix $k \geq N$. If $K \cap \operatorname{supp}\rho^{[k]} = \varnothing$, there is nothing to prove. Otherwise, there is a point $z \in K \cap \operatorname{supp}\rho^{[k]}$. Using the EL equations (12.134) at z, it follows that

$$1 = \int_{\mathcal{F}} \mathcal{L}(z,y)\mathrm{d}\rho^{[k]}(y) \geq \int_{U(x_\ell)} \mathcal{L}(z,y)\mathrm{d}\rho^{[k]}(y) \geq \frac{\mathcal{L}(x_\ell,x_\ell)}{2}\rho^{[k]}(U(x_\ell)). \tag{12.138}$$

Hence,

$$\rho^{[k]}(U(x_\ell)) \le \frac{2}{\mathcal{L}(x_\ell, x_\ell)} \,. \tag{12.139}$$

This inequality holds for any $k \ge N$. We introduce the constants $c(x_\ell)$ as the maximum of $2/\mathcal{L}(x_\ell, x_\ell)$ and $\rho^{[1]}(U(x_\ell)), \ldots, \rho^{[N-1]}(U(x_\ell))$. Since the open sets $U(x_1), \ldots, U(x_L)$ cover K, we finally introduce the constant C_K as the sum of the constants $c(x_1), \ldots, c(x_L)$. □

Given a compact set K, combining the result of the previous lemma with the compactness of measures on compact topological spaces (see Theorem 12.3.2), we conclude that there is a subsequence $(\rho^{[j_n]})$ whose restrictions to K converge as a measure (i.e., in the sense (12.22)). Proceeding inductively for the compact sets $K_1, K_2, \ldots$ and choosing a diagonal sequence, one gets a subsequence of measures on $\mathcal{F}$, denoted by $\rho^{(k)}$, whose restriction to any compact set K_j converges, that is,

$$\rho^{(k)}\big|_{K_j} \text{ converges as } k \to \infty \text{ to } \rho|_{K_j} \text{ for all } j \in \mathbb{N} \,, \tag{12.140}$$

where ρ is a regular Borel measure on $\mathcal{F}$ (typically of infinite total volume). The convergence of measures in (12.140) is referred to as *vague convergence* (for more details, see [8, Definition 30.1] or [66, Section 4.1]).

It remains to show that the obtained measure ρ is a nontrivial minimizer. In order to show that it is nontrivial, we make use of the EL equations (12.134). Let $x \in \mathcal{F}$. Then (12.134) implies that

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho^{(k)}(y) \ge 1 \,. \tag{12.141}$$

Since $\mathcal{L}$ has compact range, we may pass to the limit to conclude that

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) \ge 1 \,. \tag{12.142}$$

This shows (in a quantitative way) that the measure ρ is nonzero.

Our final step for proving the EL equations (12.128) is to show that the EL equations (12.134) are preserved in the limit. In view of the lower bound in (12.142), it remains to show that ℓ vanishes on the support of ρ. Thus let $x \in \operatorname{supp} \rho$. We choose a compact subset $K \subset \mathcal{F}$ such that x lies in its interior. Again using that the Lagrangian has compact range, there is another compact subset $K' \subset \mathcal{F}$ such that (8.1.1) holds. The fact that x lies in the support and that the measures $\rho^{(k)}$ converge vaguely to ρ implies that there is a sequence $x_k \in \operatorname{supp} \rho^{(k)}$ which converges to x. The EL equations for each $\rho^{(k)}$ imply that, for sufficiently large k,

$$\int_{K'} \mathcal{L}(x_k,y)\, \mathrm{d}\rho^{(k)}(y) = 1 \,. \tag{12.143}$$

Taking the limit is a bit subtle because both the argument x_k of the Lagrangian and the integration measure depend on k. Therefore, we begin with the estimate

$$\begin{aligned} &\bigg| \int_{K'} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \int_{K'} \mathcal{L}(x_k,y)\, \mathrm{d}\rho^{(k)}(y) \bigg| \\ &\qquad \le \big|\ell(x) - \ell^{(k)}(x)\big| - \sup_j \big|\ell^{(j)}(x) - \ell^{(j)}(x_k)\big| \,, \end{aligned} \tag{12.144}$$

where we set

$$\ell^{(k)}(z) := \int_{K'} \mathcal{L}(z,y)\, \mathrm{d}\rho^{(k)}(y) - 1 \,. \tag{12.145}$$

The first summand on the right-hand side of (12.144) tends to zero because the measures $\rho^{(k)}$ converge vaguely to ρ. The second summand, on the other hand, tends to zero because the functions $\ell^{(j)}$ are equicontinuous (for more details on this argument, see [66, Section 4.2]). This concludes the proof of Theorem 12.8.1.

We finally note that, starting from the EL equations (12.128), one can also show that ρ is a *minimizer under variations of finite volume*, meaning that for every regular Borel measure $\tilde{\rho}$ satisfying (6.13), the difference of actions (6.14) is nonnegative (6.15). The proof can be found in [66, Section 4.3].

12.9 Tangent Cones and Tangent Cone Measures

In the previous sections of this chapter measure-theoretic methods have been used in order to prove the existence of minimizers. But methods of measure theory are also useful for analyzing the structure of the minimizing measure. Since these methods might be important for the future development of the theory, we now briefly explain the concept of a tangent cone measure (more details and applications can be found in [56, Section 6]). We have the situation in mind that spacetime M does not have a smooth manifold structure, so that the powerful methods of differential topology and geometry (like the tangent space, the exponential map, etc.) cannot be used in spacetime. Nevertheless, the structure of spacetime can be analyzed locally as follows. Let $x \in M$ be a spacetime point. We want to analyze a neighborhood of x in M. To this end, it is useful to consider a continuous mapping $\mathcal{A}$ from M to the symmetric operators on the spin space at x. We always assume that this mapping vanishes at x, that is,

$$\mathcal{A} : M \to \mathrm{Symm}(S_x) \qquad \text{with} \qquad \mathcal{A}(x) = 0 \,. \tag{12.146}$$

There are different possible choices for $\mathcal{A}$. The simplest choice is

$$\mathcal{A} \;:\; y \mapsto \pi_x\,(y-x)\,x|_{S_x} \,. \tag{12.147}$$

Here the factor x on the right is needed for the operator to be symmetric because

$$\begin{aligned} \prec\!\psi|\mathcal{A}\phi\!\succ_x &\overset{(5.54)}{=} -\langle \psi \,|\, x\,(\pi_x\,(y-x)\,x)\,\phi\rangle_{\mathcal{H}} = -\langle \psi \,|\, x\;(y-x)\,x\,\phi\rangle_{\mathcal{H}} \\ &= -\langle \pi_x\,(y-x)\,x\,\psi \,|\, x\,\phi\rangle_{\mathcal{H}} = \prec\!\mathcal{A}\psi|\phi\!\succ_x \,. \end{aligned} \tag{12.148}$$

Alternatively, one can consider mappings involving the operators s_y or π_y, like for example

$$\mathcal{A} \;:\; y \mapsto \pi_x\,(s_y - s_x)\,x|_{S_x} \tag{12.149}$$

$$\mathcal{A} \;:\; y \mapsto \pi_x\,(\pi_y - \pi_x)\,x|_{S_x}, \tag{12.150}$$

where π_x again denotes the orthogonal projection in $\mathcal{H}$ on S_x. But, of course, many other choices of $\mathcal{A}$ are possible. The detailed choice of $\mathcal{A}$ depends on the application in mind.

A *conical set* is a set of the form $\mathbb{R}^+ A$ with $A \subset \mathrm{Symm}(S_x)$. We denote the conical sets whose pre-images under $\mathcal{A}$ are both ρ–measurable by $\mathfrak{M}$. For a conical set $A \subset \mathrm{Symm}(S_x)$, we consider countable coverings by measurable conical sets,

$$A \subset \bigcup_{k=1}^{\infty} A_k \qquad \text{with} \qquad A_k \in \mathfrak{M}\,. \tag{12.151}$$

We denote the set of such coverings by $\mathcal{P}$. We define

$$\mu^*_{\mathrm{con}}(A) = \inf_{\mathcal{P}} \sum_{k=1}^{\infty} \liminf_{\delta \searrow 0} \frac{1}{\rho\big(B_\delta(x)\big)}\, \rho\Big(\mathcal{A}^{-1}\big(A_k\big) \cap B_\delta(x)\Big), \tag{12.152}$$

where $B_\delta(x) \subset \mathrm{L}(\mathcal{H})$ is the Banach space ball. We remark for clarity that, since $x \in M := \operatorname{supp} \rho$, it follows that the measure $\rho(B_\delta(x))$ is nonzero for all $\delta > 0$.

The above mapping μ^*_{con} defines an outer measure on the conical sets in $\mathrm{Symm}(S_x)$. By applying the Carathéodory extension lemma (see, e.g., [8, 15]), one can construct a corresponding measure denoted by μ_{con}. By restriction one obtains a Borel measure (for details, see [56, Section 6.1]).

Definition 12.9.1 *The conical Borel sets of* $\mathrm{Symm}(S_x)$ *are denoted by* $\mathfrak{B}_{\mathrm{con}}(x)$*. We denote the measure obtained by applying the above construction by*

$$\mu_x \;:\; \mathfrak{B}_{\mathrm{con}}(x) \to [0, \infty]\,. \tag{12.153}$$

It is referred to as the **tangent cone measure** *corresponding to* $\mathcal{A}$*. The* **tangent cone** $\mathcal{C}_x$ *is defined as the support of the tangent cone measure,*

$$\mathcal{C}_x := \operatorname{supp} \mu_x \subset \mathrm{Symm}(S_x)\,. \tag{12.154}$$

In simple terms, the tangent cone $\mathcal{C}_x$ distinguishes directions in which the measure ρ is nonzero. The tangent cone measure, on the other hand, is a measure supported on the tangent cone. By integrating functionals on conical subsets of $\mathrm{Symm}(S_x)$ with respect to this measure, one can get fine information on the structure of the measure ρ in different directions. For example, one can set up a variational principle by maximizing a suitable integral of this type under variations of the Clifford section at x. As a concrete application, this method is used in [56, Section 6.2] in order to choose a distinguished Clifford section at x.

12.10 Exercises

Exercise 12.1 Let Λ be the functional

$$\Lambda \;:\; C^0\big([0,1], \mathbb{R}\big) \to \mathbb{R}\,, \quad \Lambda(f) = \sup_{x \in [0,1]} f(x)\,. \tag{12.155}$$

Can this functional be represented by a measure? Analyze how your findings are compatible with the Riesz representation theorem.

Exercise 12.2 Let ρ be the Borel measure on $[0, \pi]$ given by

$$\rho(\Omega) = \int_\Omega \sin x \;\mathrm{d}x \;+\; \sum_{n=1}^{\infty} \frac{1}{n^2}\, \chi_\Omega\Big(\frac{1}{n}\Big)\,. \tag{12.156}$$

Compute the Lebesgue decomposition of ρ with respect to the Lebesgue measure.

Exercise 12.3 *(Normalized regular Borel measures: compactness results)*

(a) Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of normalized regular Borel measures on $\mathbb{R}$ with the property that there is a constant $c > 0$ such that

$$\int_{-\infty}^{\infty} x^2 \,\mathrm{d}\rho_n(x) \leq c \qquad \text{for all } n\,. \tag{12.157}$$

Show that a subsequence converges again to a normalized Borel measure on $\mathbb{R}$. *Hint:* Apply the compactness result in Theorem 12.3.2 to the measures restricted to the interval $[-L, L]$ and analyze the behavior as $L \to \infty$.

(b) More generally, assume that for a given nonnegative function $f(x)$,

$$\int_{-\infty}^{\infty} f(x) \,\mathrm{d}\rho_n(x) \leq c \qquad \text{for all } n\,. \tag{12.158}$$

Which condition on f ensures that a subsequence of the measures converges to a normalized Borel measure? Justify your answer by a counter example.

Exercise 12.4 Let $\mathscr{M} \subset \mathbb{R}$ be a closed embedded submanifold of $\mathbb{R}^3$. We choose a compact set $K \subset \mathbb{R}^3$ which contains $\mathscr{M}$. On $C^0(K,\mathbb{R})$ we introduce the functional

$$\Lambda \,:\, C^0(K,\mathbb{R}) \to \mathbb{R}\,, \quad \Lambda(f) = \int_{\mathscr{M}} f(x)\,\mathrm{d}\mu_{\mathscr{M}}(x), \tag{12.159}$$

where $\mathrm{d}\mu_{\mathscr{M}}$ is the volume measure corresponding to the induced Riemannian metric on $\mathscr{M}$. Show that this functional is linear, bounded and positive. Apply the Riesz representation theorem to represent this functional by a Borel measure on K. What is the support of this measure?

Exercise 12.5 This exercise explains why the causal action principle is ill-posed in the case $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) < \infty$. The underlying estimates were first given in the setting of discrete spacetimes in [42, Lemma 5.1].

(a) We let $\mathcal{H}_0$ be a finite-dimensional Hilbert space of dimension n and let $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$ be a causal fermion system of finite total volume $\rho_0(\mathcal{F}_0)$. Let $\iota : \mathcal{H}_0 \to \mathcal{H}$ be an isometric embedding. Construct a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$, which has the same action, the same total volume and the same values for the trace and boundedness constraints as the causal fermion system $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$.

(b) Let $\mathcal{H}_1 = \mathcal{H}_0 \oplus \mathcal{H}_0$. Construct a causal fermion system $(\mathcal{H}_1, \mathcal{F}_1, \rho_1)$ which has the same total volume and the same value of the trace constraint as $(\mathcal{H}_0, \mathcal{F}_0, \rho_0)$ but a smaller action and a smaller value of the boundedness constraint. *Hint:* Let $F_{1/2} : \mathrm{L}(\mathcal{H}_0) \to \mathrm{L}(\mathcal{H}_1)$ be the linear mappings

$$\big(F_1(A)\big)(u \oplus v) = (Au) \oplus 0\,, \qquad \big(F_2(A)\big)(u \oplus v) = 0 \oplus (Av)\,. \tag{12.160}$$

Show that $F_{1/2}$ map $\mathcal{F}_0$ to $\mathcal{F}_1$. Define ρ_1 by

$$\rho_1 = \frac{1}{2}\Big((F_1)_*\rho + (F_2)_*\rho\Big)\,. \tag{12.161}$$

(c) Iterate the construction in (b) and apply (a) to obtain a series of measures on $\mathfrak{F}$ of fixed total volume and with a fixed value of the trace constraint, for which the action and the values of the boundedness constraint tend to zero. Do these measures converge? If yes, what is the limit?

Exercise 12.6 *(Riesz representation theorem - part 1)* Let Λ be the functional

$$\Lambda : C^0([0,1], \mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) := \sup_{x\in[0,1]} f(x). \tag{12.162}$$

Can this functional be represented by a measure? Analyze how your findings are compatible with the Riesz representation theorem.

Exercise 12.7 *(Riesz representation theorem - part 2)* Let $\mathbb{M}$ be a closed embedded submanifold of $\mathbb{R}^3$. We choose a compact set $K \subset \mathbb{R}^3$ which contain $\mathbb{M}$. On $C^0(K, \mathbb{R})$ we introduce the functional

$$\Lambda : C^0(K, \mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) = \int_{\mathbb{M}} f(x)\, \mathrm{d}\mu_{\mathbb{M}}(x), \tag{12.163}$$

where $\mathrm{d}\mu_{\mathbb{M}}$ is the volume measure corresponding to the induced Riemannian metric on $\mathbb{M}$. Show that this functional is linear, bounded and positive. Apply the Riesz representation theorem to represent this functional by a Borel measure on K. What is the support of this measure?

Exercise 12.8 *(Radon–Nikodym decomposition)* Let ρ be the Borel measure on $[0, \pi]$ given by

$$\rho(\Omega) := \int_\Omega \sin x \; \mathrm{d}x + \sum_{n=1}^\infty \frac{1}{n^2}\, \chi_\Omega\left(\frac{1}{n}\right). \tag{12.164}$$

Compute the Radon–Nikodym decomposition of ρ with respect to the Lebesgue measure.

Exercise 12.9 *(Derivative of measures)* Let μ be the counting measure on the σ-algebra $\mathcal{P}(\mathbb{N})$. Consider the measure

$$\lambda(\varnothing) := 0, \quad \lambda(E) := \sum_{n\in E} (1+n)^2, \quad E \in \mathcal{P}(\mathbb{N}). \tag{12.165}$$

Show that μ and λ are equivalent (one is absolutely continuous with respect to the other) and determine the Radon–Nikodym derivative $\frac{\mathrm{d}\mu}{d\lambda}$.

Exercise 12.10 *(Minimizers)* Let M denote the 2-sphere $S^2 \subset \mathbb{R}^3$ and let $\mathrm{d}\mu_M$ be the normalized canonical surface measure. Consider a Lagrangian on $M \times M$ defined by

$$\mathcal{L}(x, y) := \frac{1}{1 + \|x - y\|_{\mathbb{R}^3}} \quad \text{for all } x, y \in M. \tag{12.166}$$

Show that the action $\mathcal{S}(\mu_M)$ is minimal under variations of the form

$$\mathrm{d}\rho_{x_0,t} := (1-t)\mathrm{d}\mu_M + t\; \mathrm{d}\delta_{x_0}, \quad \text{with } t \in [0, 1), \tag{12.167}$$

where δ_{x_0} is the Dirac measure centered at $x_0 \in M$.

Exercise 12.11 *(Moment measures)* Let $\mathcal{F} = \mathbb{R}^2$ and $K = S^1 \cup \{0\}$ be a compact subset of $\mathcal{F}$. Moreover, let ρ be a Borel measure ρ on $\mathcal{F}$. Compute the moment measures $\mathfrak{m}^{(0)}$, $\mathfrak{m}^{(1)}$ and $\mathfrak{m}^{(2)}$ for the following choices of ρ:

(a) $\rho = F_*(\mu_{S^1})$, where $F : S^1 \hookrightarrow \mathbb{R}^2$ is the natural injection and μ_{S^1} is the normalized Lebesgue measure on S^1.

(b) $\rho = \delta_{(0,0)} + \delta_{(1,1)} + \delta_{(5,0)}$ (where $\delta_{(x,y)}$ denote the Dirac measure supported at $(x, y) \in \mathbb{R}^2$).

(c) $\rho = F_*(\mu_{\mathbb{R}})$, where $\mu_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$ and

$$F : \mathbb{R} \to \mathbb{R}^2, \quad F(x) = (x, 2). \tag{12.168}$$