5

A Brief Introduction to Causal Fermion Systems

In this chapter, we define and explain the basic objects and structures of a causal fermion system. Since causal fermion systems introduce a new language to describe our physical world, we begin with preliminary considerations that explain how the basic objects of the theory come about and how to think about them. In order to provide different perspectives, the preliminary considerations motivate causal fermion systems in two somewhat different ways. In Section 5.1, the motivating question is whether and how spacetime structures can be encoded in quantum mechanical wave functions. In Section 5.2, on the other hand, we begin with the example of a two-dimensional lattice system and ask the question how one can formulate physical equations in this discrete spacetime without making use of specific lattice structures like the nearest neighbor relations and the lattice spacing. By extending the setting from the motivating examples (Section 5.3), we are led to the general definition of a causal fermion system (Section 5.4). Next, as a further example, we explain how the Minkowski vacuum can be described by a causal fermion system (Section 5.5). In order to formulate equations describing the dynamics of a causal fermion system, we introduce a variational principle, the so-called causal action principle (Section 5.6). We proceed by explaining how to obtain a spacetime as well as structures therein (Section 5.7). We conclude by discussing the form of the causal action principle (Section 5.8) and by explaining the underlying physical concepts (Section 5.9).

5.1 Motivation: Encoding Spacetime Structures in Wave Functions

For the introductory considerations, following [86, Section 2.1.1], we begin with a quantum particle described by a quantum mechanical wave function ψ satisfying the Klein–Gordon equation (1.22) in Minkowski space or in a curved spacetime. Suppose that we have access only to the information contained in the absolute square $|\psi(x)|^2$ of this wave function. We ask the question: Given this information, what can we infer on the structure of spacetime? First, let the wave function ψ be a solution evolved from compactly supported initial data ψ_0 as illustrated in Figure 5.1. Then, finite speed of propagation guarantees that the absolute square $|\psi(x)|^2$ vanishes outside the causal future of the support of the initial data. In this way, the support of $|\psi(x)|^2$ gives us some information on the causal structure of our spacetime. But, of course, there is only a limited amount of information that can be extracted from a single wave function. However, if instead we

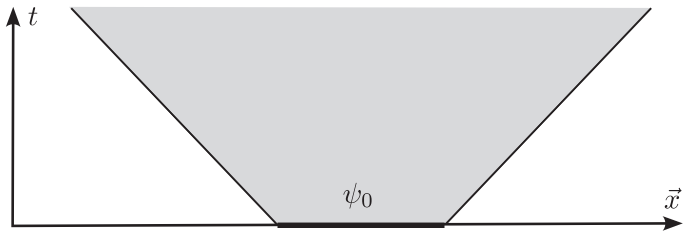

Figure 5.1 Causal propagation of a wave function. From [86], Creative Commons Attribution 4.0 license.

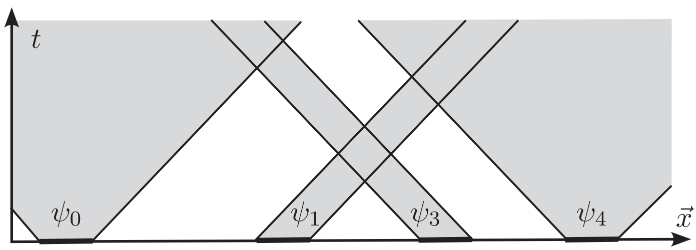

Figure 5.2 Probing with many wave functions. From [86], Creative Commons Attribution 4.0 license.

probe with many wave functions, as illustrated in Figure 5.2, we gain more information. If we aggregate the information contained in all wave functions evolved from compactly supported initial data, then we can extract the complete causal structure of our spacetime. We remark that this determines the metric up to a conformal factor [103, 119].

We next consider the situation if an electromagnetic background field is present. The coupling of the scalar field to the electromagnetic field is described by the Klein–Gordon equation (1.23). Now the wave functions are deflected by the electromagnetic force. Therefore, their absolute square also encodes information on the electromagnetic field. In order to retrieve this information, one can use the following procedure. Suppose that we have access to two wave functions ψ and ϕ and that we can also measure the absolute value of superpositions, that is,

$$\big|\alpha\psi(x) + \beta\phi(x)\big|^2 = \big|\alpha\psi(x)\big|^2 + 2\,\mathrm{Re}\Big(\overline{\alpha}\,\beta\,\overline{\psi(x)}\,\phi(x)\Big) + \big|\beta\phi(x)\big|^2, \tag{5.1}$$

for arbitrary complex coefficients α and β. By varying these coefficients, we can determine the quantity

$$\overline{\psi(x)}\phi(x)\,, \tag{5.2}$$

which tells us about the correlation of the two wave function ψ and ϕ at the spacetime point x. This allows us to probe the electromagnetic field, as shown schematically in Figure 5.3. Here, we do not need to be specific on what “probing” exactly means (e.g., one could determine deflection angles, recover the Aharanov–Bohm phase shifts of the wave function, etc.). All that counts is that we can get information also on the electromagnetic field. Generally speaking, the more wave functions we have to our disposal, the more information on the electromagnetic

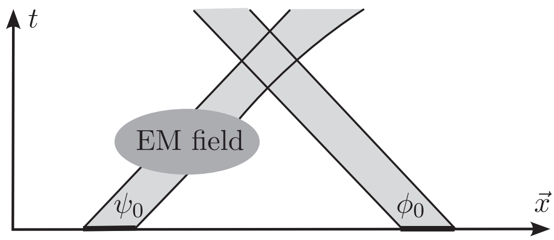

Figure 5.3 Probing an electromagnetic field. From [86], Creative Commons Attribution 4.0 license.

field can be retrieved. It seems sensible to expect that, after suitably increasing the number of wave functions, we can recover both the spacetime structures and the matter fields therein from the knowledge of the absolute squares of all these wave functions alone.

Now we go one step further and formulate the idea of encoding spacetime structures in a family of wave functions in mathematical terms. To this end, we consider a (for simplicity finite) number f of linearly independent wave functions $\psi_1, \ldots, \psi_f : \mathscr{M} \to \mathbb{C}$, mapping from a classical spacetime $\mathscr{M}$ to the complex numbers. On the complex vector space $\mathcal{H}$ spanned by these wave functions we introduce a scalar product $\langle .|. \rangle_{\mathcal{H}}$ by demanding that the wave functions $\psi_1, \ldots, \psi_f$ are orthonormal, that is,

$$\langle \psi_k | \psi_l \rangle_{\mathcal{H}} = \delta_{kl} \,. \tag{5.3}$$

We thus obtain an f-dimensional Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. At any spacetime point $x \in \mathscr{M}$ we can now introduce the *local correlation operator* $F(x) : \mathcal{H} \to \mathcal{H}$ as the linear operator whose matrix representation in the basis $\psi_1, \ldots, \psi_f$ is given by

$$\big(F(x)\big)^j_{\;k} = \overline{\psi_j(x)} \psi_k(x) \,. \tag{5.4}$$

The diagonal entries of this matrix are the absolute squares of the wave functions, whereas the off-diagonal entries tell us about the correlation of two different wave functions at the spacetime point x. This is why we refer to $F(x)$ as the local correlation operator. Alternatively, the local correlation operator can be characterized in a basis-invariant form by the identity

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} = \overline{\psi(x)} \phi(x) \qquad \text{for all } \psi, \phi \in \mathcal{H} \,. \tag{5.5}$$

By construction, the operator $F(x)$ is positive semi-definite and has rank at most one (in order not to distract from the main construction, this will be explained in more detail after (5.12) in Section 5.2). By varying the point x, we obtain a map $F : \mathscr{M} \to \mathcal{F}$ from the classical spacetime $\mathscr{M}$ to the set $\mathcal{F}$ of positive semi-definite linear operators of rank at most one,

$$\mathcal{F} := \big\{ y \in \mathrm{L}(\mathcal{H}) \,\big|\, y \text{ positive semi-definite of rank at most one} \big\} \,. \tag{5.6}$$

This map encodes all the physical information[1] contained in the wave functions of $\mathcal{H}$.

Next, we need to formalize the idea that we want to restrict attention to the information encoded in the wave functions. This entails that we want to disregard all the information contained in the usual structures of Minkowski space or a curved spacetime (like the causal structure, the metric, the spinor bundle, and all that). In order to do so mathematically, we focus on the family of all local correlation operators. Thus, instead of considering F as a mapping from our classical spacetime to $\mathcal{F}$, we restrict attention to its image $M := F(\mathcal{F})$ as a subset of $\mathcal{F}$,

$$M \subset \mathcal{F} \,. \tag{5.7}$$

In this way, Minkowski space and the corresponding classical spacetime structures no longer enter our description. Instead, spacetime and all structures therein are encoded in and must be recovered from the information contained in the family of wave functions. This point of view of recovering all spacetime structures from the wave functions will be taken seriously in this book, and we will unravel its consequences step by step.

It turns out that working as in (5.7) merely with a subset of $\mathcal{F}$ is not quite sufficient. In order to get into the position to formulate physical equations, we need one more structure: a measure ρ on spacetime. Here, by a "measure on spacetime," we mean a mapping which to a subset $\Omega \subset M$ associates a nonnegative number, which can be thought of as the "volume" of the spacetime region corresponding to Ω. In nontechnical terms, this measure can be obtained by combining the volume measure of Minkowski space with the map F. More precisely, we take the preimage $F^{-1}(\Omega) \subset \mathcal{F}$ and integrate over it,

$$\rho(\Omega) := \int_{F^{-1}(\Omega)} \mathrm{d}\mu \,, \tag{5.8}$$

where $d\mu = \mathrm{d}^4x$ is the volume measure in Minkowski space $\mathscr{M}$ (and similarly $\mathrm{d}\mu = \sqrt{|\det g|}\, \mathrm{d}^4x$ in curved spacetime). In more mathematical terms, the measure ρ is the *push-forward* of μ under F (for basics on measure theory and the push-forward measure, see Section 2.3).

This construction leads us to consider a measure ρ on a set of linear operators on a Hilbert space as the basic structure describing a physical system in spacetime. These are indeed all the basic ingredients to define a causal fermion system. The only modification to be made later is that, instead of complex wave functions, we will work with sections of a spinor bundle. One consequence of that is that the local correlation operators will no longer be positive semi-definite. Instead, they will be of finite rank with a fixed upper bound on the number of positive and negative eigenvalues.

[1] Here, by "physical," we mean the information up to local gauge phases, which drop out in (5.4). Local gauge freedom and local gauge transformations will be discussed in Section 5.9.

Before coming to these generalizations (Section 5.3), we next explain why encoding information in the wave functions also has benefits if one wants to formulate physical equations in a setting that goes beyond a classical continuous spacetime.

5.2 Motivating Example: Formulating Equations in Discrete Spacetimes

It is generally believed that for distances as small as the Planck length, spacetime can no longer be described by Minkowski space or a Lorentzian manifold, but that it should have a different, possibly discrete structure. There are different approaches to modeling such spacetimes. The simplest approach is to replace Minkowski space with a discrete lattice. Indeed, causal fermion systems provide another, more general approach. In any such approach, one faces the challenge of how to formulate physical equations if one gives up the continuous structure of spacetime and thus can no longer work with partial differential equations like the Klein–Gordon equation or the Dirac equation.

In order to explain the underlying problem more concretely, we now have a closer look at the simple example of a spacetime lattice (this example was first given in [48, Section 1]). For simplicity, we consider a two-dimensional lattice (one space and one time dimension), but higher-dimensional lattices could be described similarly. Thus let $\mathscr{M} \subset \mathbb{R}^{1,1}$ be a rectangular lattice in two-dimensional Minkowski space. We denote the spacing in time direction by Δt and in spatial direction by Δx (see Figure 5.4). The usual procedure for setting up equations on a lattice is to replace derivatives with difference quotients, giving rise to an evolution equation that can be solved time step by time step according to deterministic rules. A simple example is the discretization of the two-dimensional wave equation for a function $\phi : \mathscr{M} \to \mathbb{C}$ on the lattice,

$$\begin{aligned} 0 = \Box\phi(t,x) := \frac{1}{(\Delta t)^2}\Big(\phi(t+\Delta t,x) - 2\phi(t,x) + \phi(t-\Delta t,x)\Big) \\ - \frac{1}{(\Delta x)^2}\Big(\phi(t,x+\Delta x) - 2\phi(t,x) + \phi(t,x-\Delta x)\Big)\,. \end{aligned} \tag{5.9}$$

Solving this equation for $\phi(t+\Delta t,x)$ gives a deterministic rule for computing $\phi(t+\Delta t,x)$ from the values of ϕ at earlier times t and $t-\Delta t$ (see again Figure 5.4).

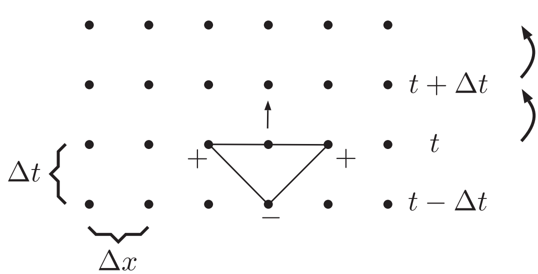

Figure 5.4 Time evolution of a lattice system $\mathscr{M} \subset \mathbb{R}^{1,1}$. From [48], Creative Commons Atribution 3.0 license.

While this method for setting up equations in a discrete spacetime is very simple and yields well-defined evolution equations, it also has several drawbacks:

- The method in (5.9) for discretizing the continuum equations is very *ad hoc*. Why do we choose a regular lattice, why do we work with difference quotients? There are many other ways of discretizing the wave equation.
- The method is *not background-free*. In order to speak of the "lattice spacing," the lattice must be thought of as being embedded in a two-dimensional ambient spacetime.
- The concept of a spacetime lattice is not invariant under general coordinate transformations. In other words, the assumption of a spacetime lattice is *not compatible with the equivalence principle*.

In view of these shortcomings, the following basic question arises:

Can one formulate equations without referring to the nearest neighbor relation and the lattice spacing?

The answer to this question is yes, and we will now see how this can be done in the example of our two-dimensional lattice system. Although our example is somewhat oversimplified, this consideration will lead us quite naturally to the setting of causal fermion systems.

In order to formulate the equations, we consider on our lattice a family of complex-valued wave functions $\psi_1, \ldots, \psi_f : \mathscr{M} \to \mathbb{C}$ (for simplicity a finite number, i.e., $f < \infty$). At this stage, these wave functions do not need to satisfy any wave equation. On the complex vector space $\mathcal{H}$ spanned by these wave functions we introduce a scalar product $\langle .|. \rangle_{\mathcal{H}}$ by demanding that the wave functions $\psi_1, \ldots, \psi_f$ are orthonormal, that is,

$$\langle \psi_k | \psi_l \rangle_{\mathcal{H}} = \delta_{kl} \,. \tag{5.10}$$

We thus obtain an f-dimensional Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. Note that the scalar product is given abstractly (meaning that it has no representation in terms of the wave functions as a sum over lattice points). Next, for any lattice point $(t, x) \in \mathscr{M}$ we introduce the so-called *local correlation operator* $F(t,x) : \mathcal{H} \to \mathcal{H}$ as the linear operator whose matrix representation in the basis $\psi_1, \ldots, \psi_f$ is given by

$$(F(t,x))^j_k = \overline{\psi_j(t,x)} \psi_k(t,x) \,. \tag{5.11}$$

The diagonal elements of this matrix are the absolute squares $|\psi_k(t,x)|^2$ of the corresponding wave functions. The off-diagonal elements, on the other hand, tell us about the correlation of the j^{th} and k^{th} wave function at the lattice point (t, x). This is the reason for the name "local correlation operator." This operator can also be characterized in a basis-invariant way by the relations

$$\langle \psi \,|\, F(t,x)\, \phi \rangle_{\mathcal{H}} = \overline{\psi(t,x)} \phi(t,x) \,, \tag{5.12}$$

to be satisfied for all $\psi, \phi \in \mathcal{H}$.

We now analyze some properties of the local correlation operators. Taking the complex conjugate, one sees immediately that the matrix defined by (5.11) is Hermitian. Stated equivalently independent of bases, the local correlation operator is a *symmetric* linear operator on $\mathcal{H}$ (see Definition 2.2.5 in the preliminaries). Moreover, a local correlation operator has *rank at most one* and is *positive semi-definite*. This can be seen in detail by expressing it in terms of the operator

$$e(t,x) : \mathcal{H} \to \mathbb{C} \,, \qquad \psi \mapsto \psi(t,x) \,, \tag{5.13}$$

which to every vector associates the corresponding wave function evaluated at the spacetime point (t,x) (this mapping is sometimes referred to as the *evaluation map*). Indeed, rewriting the right-hand side of (5.12) as

$$\overline{\psi(t,x)}\phi(t,x) = \overline{\big(e(t,x)\psi\big)}\big(e(t,x)\phi\big) = \langle \psi \,|\, e(t,x)^* \, e(t,x)\, \phi \rangle_{\mathcal{H}} \,, \tag{5.14}$$

where $e(t,x)^* : \mathbb{C} \to \mathcal{H}$ is the adjoint of the operator $e(t,x)$ as defined by (2.46), we can compare with the left-hand side of (5.12) to conclude that

$$F(t,x) = e(t,x)^* \, e(t,x) \,. \tag{5.15}$$

This shows that $F(t,x)$ is positive semi-definite. Moreover, being a mapping to $\mathbb{C}$, the operator $e(t,x)$ has rank at most one. As a consequence, also $F(t,x)$ has rank at most one.

It is useful to denote the set of all operators with the above properties by

$$\begin{aligned} \mathcal{F} := \big\{ F \in \mathrm{L}(\mathcal{H}) \,\big|\, & F \text{ is symmetric,} \\ & \text{positive semi-definite and has rank at most one} \big\} \,. \end{aligned} \tag{5.16}$$

Varying the lattice point, we obtain a mapping (see Figure 5.5)

$$F : \mathscr{M} \to \mathcal{F} \,, \qquad (t,x) \mapsto F(t,x) \,. \tag{5.17}$$

For clarity, we note that the set $\mathcal{F}$ is *not* a vector space, because a linear combination of operators in $\mathcal{F}$ in general has a rank greater than one. But it is a *conical* set in the sense that a positive multiple of any operator in $\mathcal{F}$ is again in $\mathcal{F}$ (this is why in Figure 5.5 the set $\mathcal{F}$ is depicted as a cone).

We point out that the local correlation operators do not involve the lattice spacing or the nearest neighbor relation (as a matter of fact, we did not even

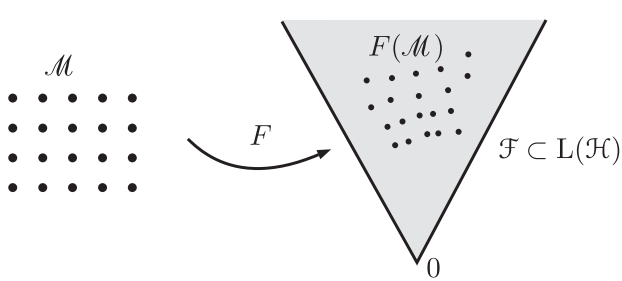

Figure 5.5 Embedding in $\mathcal{F}$. From [48], Creative Commons Atribution 3.0 license.

use that $\mathscr{M}$ is a lattice); instead, they contain information only on the local correlations of the wave functions at each lattice point. With this in mind, our strategy for formulating equations which do not involve the specific structures of the lattice is to work exclusively with the local correlation operators, that is, with the subset $F(\mathscr{M}) \subset \mathcal{F}$. In other words, in Figure 5.5, we want to disregard the lattice on the left and work exclusively with the objects on the right.

How can one set up equations purely in terms of the local correlation operators? In order to explain the general procedure, we consider a finite number of operators $F_1, \ldots, F_L \in \mathcal{F}$. Each of these operators can be thought of as encoding information on the local correlations of the wave functions at a corresponding spacetime point. However, this "spacetime point" is no longer a lattice point because the notions of lattice spacing and nearest lattice point have been dropped. At this stage, spacetime is merely a point set, where each point is an operator on the Hilbert space. In order to obtain a "spacetime" in the usual sense (like Minkowski space, a Lorentzian manifold or a generalization thereof), one needs additional structures and relations between the spacetime points. Such relations can be obtained by multiplying the operators. Indeed, the operator product $F_i\, F_j$ tells us about correlations of the wave functions at different spacetime points. Taking the trace of this operator product gives a real number. Our method for formulating physical equations is to use the operators F_i and their products to set up a variational principle. This variational formulation has the advantage that symmetries give rise to conservation laws by Noether's theorem (as will be explained in Chapter 9). Therefore, we want to minimize an action $\mathcal{S}$ defined in terms of the operators $F_1, \ldots, F_L$. A simple example is to

$$\text{minimize} \qquad \mathcal{S}(F_1, \ldots, F_L) := \sum_{i,j=1}^{L} \operatorname{Tr}(F_i\, F_j)^2, \tag{5.18}$$

under variations of the points $F_1, \ldots, F_L \in \mathcal{F}$. In order to obtain a mathematically sensible variational principle, one needs to impose certain constraints. Here, we do not enter the details because the present example is a bit too simple (see, however, Exercise 5.1). Instead, we merely use it as a motivation for the general setting of causal fermion systems, which we now introduce.

5.3 Toward the General Definition of a Causal Fermion System

In order to get from the previous motivating examples to the general setting of causal fermion systems, we extend the above constructions in several steps:

(a) The previous example works similarly in higher dimensions, in particular for a lattice $\mathscr{M} \subset \mathbb{R}^{1,3}$ in four-dimensional Minkowski space. This has no effect on the resulting structure of a finite number of distinguished operators $F_1, \ldots, F_L \in \mathcal{F}$.

(b) Suppose that we consider multicomponent wave functions $\psi : \mathscr{M} \to \mathbb{C}^N$. Then, clearly, we cannot directly multiply two such wave functions pointwise as was done on the right-hand side of (5.11). However, assuming that we are given an inner product on $\mathbb{C}^N$, which we denote by $\prec .|.\succ$ (in mathematical terms, this inner product is a nondegenerate sesquilinear form; we always use the convention that the wave function in the first argument is complex conjugated), we can adapt the definition of the local correlation operator (5.11) to

$$(F(t,x))^j_k = -\prec \psi_j(t,x) | \psi_k(t,x) \succ, \tag{5.19}$$

(the minus sign compared to (5.11) merely is a useful convention). The resulting local correlation operator is no longer an operator of rank at most one, but it has rank at most N (as can be seen, e.g. by writing it similar to (5.15) in the form $F(t,x) = -e(t,x)^* \, e(t,x)$ with the evaluation map $e(t,x) : \mathcal{H} \to \mathbb{C}^N, \psi \mapsto \psi(t,x)$). If the inner product $\prec .|.\succ$ on $\mathbb{C}^N$ is positive definite, then the operator $F(t,x)$ is negative semi-definite. However, in the physical applications in mind, this inner product will *not* be positive definite. Indeed, a typical example in mind is that of four-component Dirac spinors. The Lorentz invariant inner product $\overline{\psi}\phi$ on Dirac spinors in Minkowski space (with the usual adjoint spinor $\overline{\psi} := \psi^\dagger \gamma^0$) is indefinite of signature $(2,2)$. In order to describe systems involving leptons and quarks, one must take direct sums of Dirac spinors, giving the signature (n,n) with $n \in 2\mathbb{N}$. With this in mind, we assume more generally that

$$\prec .|.\succ \quad \text{has signature } (n,n) \text{ with } n \in \mathbb{N}\,. \tag{5.20}$$

Then, the resulting local correlation operators are symmetric operators of rank at most $2n$, which (counting multiplicities) have at most n positive and at most n negative eigenvalues.

(c) Finally, it is useful to generalize the setting such as to allow for continuous spacetimes and for spacetimes which may have both continuous and discrete components. In preparation, we note that the sums over the operators $F_1, \ldots, F_L$ in (5.18) can be written as integrals,

$$\mathcal{S}(\rho) = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \;\, \mathrm{Tr}(xy)^2\,, \tag{5.21}$$

if the measure ρ on $\mathcal{F}$ is chosen as the sum of Dirac measures supported at these operators,

$$\rho = \sum_{i=1}^{L} \delta_{F_i}\,. \tag{5.22}$$

Note that, in this formulation, the measure plays a double role: First, it distinguishes the points $F_1, \ldots, F_L$ as those points where the measure is nonzero, as is made mathematically precise by the notion of the *support* of the measure (for details, see Definition 2.3.4), that is,

$$\operatorname{supp} \rho = \{F_1, \ldots, F_L\}\,. \tag{5.23}$$

Second, a measure makes it possible to integrate over its support, an operation which for the measure (5.22) reduces to the sum over $F_1, \ldots, F_L$.

Now one can extend the setting simply by considering (5.21) for more general measures on $\mathcal{F}$ (like, for example, regular Borel measures). The main advantage of working with measures is that we get into a mathematical framework in which variational principles like (5.18) can be studied with powerful analytic methods.

5.4 Basic Definition of a Causal Fermion System

Motivated by the previous considerations we now give the basic definition of a causal fermion system. This definition evolved over several years. Based on preparations in [41], the present formulation was first given in [80].

Definition 5.4.1 (Causal fermion system) *Given a separable complex Hilbert space $\mathcal{H}$ with scalar product $\langle .|. \rangle_{\mathcal{H}}$ and a parameter $n \in \mathbb{N}$ (the* **spin dimension***), we let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all symmetric operators on $\mathcal{H}$ of finite rank, which (counting multiplicities) have at most n positive and at most n negative eigenvalues. Moreover, let ρ be a positive measure on $\mathcal{F}$ (defined on a σ-algebra of subsets of $\mathcal{F}$). We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a* **causal fermion system**.

The definition of a causal fermion system is illustrated in Figure 5.6.

The set $\mathcal{F}$ is invariant under the transformation where an operator is multiplied by a real number, as is indicated in the figure by the double cones. The support of the measure, denoted by

$$M := \operatorname{supp} \rho \,, \tag{5.24}$$

is referred to as *spacetime* (intuitively speaking, the support of a measure consists of all points where the measure is nonzero; for mathematical details, see Definition 2.3.4). In contrast to the example of the lattice system, where spacetime consisted of discrete points (5.23), in general, the measure ρ can also have continuous components. For example, M could be a subset of $\mathcal{F}$ having the additional structure of being a four-dimensional manifold. The space $\mathcal{F}$ should be thought of as a space of very large dimension,[2] so that M typically is a low-dimensional subset of $\mathcal{F}$. The measure $\rho(\Omega)$ of a measurable subset $\Omega \subset M$ can be regarded as the volume of the spacetime region Ω. In the example of the lattice system, this volume is simply the number of spacetime points in Ω, whereas for a continuous spacetime, it is the four-dimensional Lebesgue measure of Ω. It is a specific feature of a causal fermion system that a spacetime point $x \in M$ is a linear operator on the Hilbert space $\mathcal{H}$. This endows spacetime with a lot of additional structure. In particular, as will be explained in Section 5.7, the spacetime point operators give rise to a family of spinorial wave functions and to causal and geometric structures. The general idea is that a causal fermion system describes a spacetime together with all structures therein. Before entering these structures in more detail, we

[2] This statement is made precise in [60, 67] as follows. The operators of $\mathcal{F}$ of maximal rank $2n$ form a Banach manifold. If the Hilbert space $\mathcal{H}$ is finite-dimensional, then this manifold also has a finite dimension given by $4n \dim \mathcal{H} - 4n^2$; see also Proposition 3.1.3 in the preliminaries.

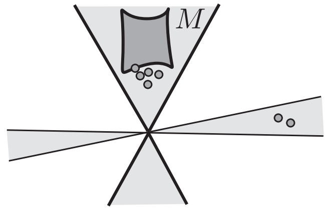

Figure 5.6 A causal fermion system. From [48], Creative Commons Atribution 3.0 license.

illustrate the general definition by the simple and concrete example of Dirac wave functions in Minkowski space.

5.5 Example: Dirac Wave Functions in Minkowski Space

As a further example, we now explain how to construct a causal fermion system in Minkowski space. Recall that in Section 1.4 (and similarly in curved spacetime in Section 4.5), for a given parameter $m \in \mathbb{R}$ we introduced the Hilbert space $(\mathcal{H}_m, (.|.))$ of all solutions of the Dirac equation with mass m (recall that the scalar product is defined as the spatial integral (1.37)). We now choose a closed subspace $\mathcal{H}$ of this Hilbert space and denote the scalar product $(.|.)$ restricted to this subspace by $\langle .|. \rangle_{\mathcal{H}}$ (changing the notation from round to pointed brackets clarifies that we consider $\langle .|. \rangle_{\mathcal{H}}$ as an abstract scalar product, without referring to its representation as a spatial integral (1.37)). We thus obtain the

$$\text{Hilbert space} \qquad \big(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}}\big) \,. \tag{5.25}$$

By construction, the vectors in this Hilbert space are solutions of the Dirac equation. They can be thought of as the "occupied states" of the system. We prefer the notion of *physical wave functions*, where "physical" means intuitively that these wave functions are realized in our physical system (whatever this means; we shall not enter philosophical issues here). The choice of the subspace $\mathcal{H} \subset \mathcal{H}_m$ is part of the input which characterizes the physical system. For example, in order to describe the vacuum, one chooses $\mathcal{H}$ as the subspace of all negative-energy solutions of the Dirac equation (see Section 1.5). In order to model a system involving electrons, however, the subspace $\mathcal{H}$ must be chosen to include the electronic wave functions of positive frequency. At this stage, we do not need to specify $\mathcal{H}$, and in order to clarify the concepts, it seems preferable to keep our considerations on a general abstract level. Specific choices and explicit computations can be found in [45, Section 1.2] and in later chapters of this book (Chapters 15–19).

We point out that the functions in $\mathcal{H}$ do not need to be continuous (instead, as mentioned at the end of Section 1.4, they are weak solutions whose restriction to any Cauchy surface merely is an L^2-function). Therefore, we cannot evaluate the wave functions pointwise at a spacetime point $x \in \mathscr{M}$. However, for the following constructions, it is crucial to do so. The way out is to introduce so-called *regularization operators* $(\mathfrak{R}_\varepsilon)$ with $0 < \varepsilon < \varepsilon_{\max}$ as linear operators that map $\mathcal{H}$ to the continuous wave functions,

$$\mathfrak{R}_\varepsilon \,:\, \mathcal{H} \to C^0(\mathscr{M}, S\mathscr{M}) \qquad \text{linear}\,. \tag{5.26}$$

In the limit $\varepsilon \searrow 0$, these operators should go over to the identity (in a suitable sense which we do not specify here as it will not be needed). The physical picture is that on a small length scale, which can be thought of as the Planck length scale $\varepsilon \approx 10^{-35}$ meters, the structure of spacetime must be modified. The regularization operators specify this microscopic structure of spacetime. Different choices of regularization operators are possible. A simple example of a regularization operator is obtained by mollifying with a test function. Thus, we let $h \in C^\infty_0(\mathscr{M}, \mathbb{R})$ be a nonnegative test function with

$$\int_{\mathscr{M}} h(x)\, \mathrm{d}^4x = 1\,. \tag{5.27}$$

We define the operators $\mathfrak{R}_\varepsilon$ for $\varepsilon > 0$ as the convolution operators (for basics on the convolution, see the paragraph after (2.111) in Section 2.4)

$$(\mathfrak{R}_\varepsilon u)(x) := \frac{1}{\varepsilon^4} \int_{\mathscr{M}} h\Big(\frac{x-y}{\varepsilon}\Big)\, u(y)\; \mathrm{d}^4y\,. \tag{5.28}$$

Another method is to work in Fourier space (for preliminaries, see Sections 1.5 and 2.4) by setting

$$u(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \hat{u}(k)\, \mathrm{e}^{-\mathrm{i}kx}\,, \tag{5.29}$$

and to regularize by multiplication with an exponentially decaying cutoff function, that is,

$$\big(\mathfrak{R}_\varepsilon u\big)(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \hat{u}(k)\, \mathrm{e}^{-\varepsilon\,|\omega|}\, \mathrm{e}^{-\mathrm{i}kx} \qquad \text{with} \qquad \omega = k^0\,. \tag{5.30}$$

This so-called *iε-regularization* is most convenient for explicit computations (for more details, see [45, §2.4.1]). Clearly, these methods of regularizing Dirac solutions are very special and should be thought of merely as a mathematical tool for constructing simple and explicit examples of causal fermion systems.

Before going on, we briefly remark for the reader familiar with quantum field theory (QFT) how the above regularization is related to the ultraviolet regularization procedures used in relativistic QFT. Both in QFT and the setting of causal fermion systems, regularizations are needed in order to make the theory mathematically well defined. In the renormalization program in QFT, one shows that the UV regularization can be taken out if other parameters of the theory (like masses and coupling constants) are suitably rescaled. Then, the regularization can be understood merely as a computational tool. In the causal fermion systems, however, the physical picture behind the regularization is quite different. Namely, in our setting, the *regularized* objects are to be considered as the fundamental physical objects. The regularization models the microscopic structure of spacetime and has therefore a physical significance.

Next, for any $x \in \mathscr{M}$, we consider the bilinear form

$$b^\varepsilon_x \,:\, \mathcal{H} \times \mathcal{H} \to \mathbb{C}\,, \quad b^\varepsilon_x(u,v) = -\prec\!(\mathfrak{R}_\varepsilon\, u)(x) \,|\, (\mathfrak{R}_\varepsilon\, v)(x)\!\succ . \tag{5.31}$$

This bilinear form is well defined and bounded because $\mathfrak{R}_\varepsilon$ is defined pointwise and because evaluation at x gives a linear operator of finite rank (see Exercise 5.3). Thus for any $v \in \mathcal{H}$, the anti-linear form $b^\varepsilon_x(.,v) : \mathcal{H} \to \mathbb{C}$ is continuous. By the Fréchet–Riesz theorem (Theorem 2.2.4), there is a unique vector $w^\varepsilon \in \mathcal{H}$ such that $b^\varepsilon_x(u,v) = \langle u | w^\varepsilon \rangle_{\mathcal{H}}$ for all $u \in \mathcal{H}$. The mapping $v \mapsto w^\varepsilon$ is linear and bounded. We thus obtain a bounded linear operator $F^\varepsilon(x)$ on $\mathcal{H}$ such that

$$b^\varepsilon_x(u,v) = \langle u \,|\, F^\varepsilon(x)\, v \rangle_{\mathcal{H}} \qquad \text{for all } u, v \in \mathcal{H}\,, \tag{5.32}$$

referred to as the *local correlation operator*. Taking into account that the inner product on the Dirac spinors at x has signature $(2,2)$, the local correlation operator $F^\varepsilon(x)$ is a symmetric operator on $\mathcal{H}$ of rank at most four, which has at most two positive and at most two negative eigenvalues.

Varying the point $x \in \mathscr{M}$, for any ε we obtain a mapping

$$F^\varepsilon \,:\, \mathscr{M} \to \mathcal{F}\,, \tag{5.33}$$

where $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ is the set of all symmetric operators on $\mathcal{H}$ of finite rank which (counting multiplicities) have at most two positive and at most two negative eigenvalues. We sometimes refer to F^ε as the *local correlation map*. The last step is to drop all other structures (like the metric and causal structures of Minkowski space, the spinorial structures, etc.). As mentioned earlier, the basic concept behind causal fermion systems is to work exclusively with the local correlation operators corresponding to the physical wave functions. In order to formalize this concept, we introduce the measure ρ^ε on $\mathcal{F}$ as the push-forward of the volume measure on $\mathscr{M}$ (for details, see Section 2.3 or Exercise 2.18),

$$\rho^\varepsilon := F^\varepsilon_* \mu\,. \tag{5.34}$$

We thus obtain a causal fermion system of spin dimension $n = 2$ (see Definition 5.4.1). The local correlation operators are encoded in ρ as the support M of this measure. Working exclusively with the structures of a causal fermion system, we no longer have the usual spacetime structures (particles, fields, causal structure, geometry, ...). The underlying idea is that all these spacetime structures are encoded in the local correlation operators. At this point, it is not obvious that this concept is sensible. But, as we shall see in the later sections in this book, it is indeed possible to reconstruct all spacetime structures from the local correlation operators. In this sense, the structures of a causal fermion system give a complete description of the physical system.

5.6 The Causal Action Principle

Having given the general definition of a causal fermion system (see Definition 5.4.1), the question arises how physical equations can be formulated in this setting. To this end, we now introduce a variational principle, the so-called *causal* *action principle*. In this variational principle, we minimize a functional, the so-called *causal action*, under variations of the measure ρ. The minimality property will then impose strong conditions on the possible form of this measure. The mathematical structure of the causal action is similar to the action (5.18) given in our example of the lattice system. Its detailed form, however, is the result of many computations and longer considerations, as will be outlined in Section 5.8.

For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. However, in general, it is no longer a symmetric operator because $(xy)^* = yx$, and this is different from xy unless x and y commute. As a consequence, the eigenvalues of the operator xy are in general complex. We denote these eigenvalues counting algebraic multiplicities by $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$ (more specifically, denoting the rank of xy by $k \leq 2n$, we choose $\lambda_1^{xy}, \ldots, \lambda_k^{xy}$ as all the nonzero eigenvalues and set $\lambda_{k+1}^{xy}, \ldots, \lambda_{2n}^{xy} = 0$). We introduce the Lagrangian and the causal action by

$$\text{causal Lagrangian:} \qquad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \Big(\big|\lambda_i^{xy}\big| - \big|\lambda_j^{xy}\big| \Big)^2 \tag{5.35}$$

$$\text{causal action:} \qquad \mathcal{S}(\rho) = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(x)\, \mathrm{d}\rho(y)\,. \tag{5.36}$$

The *causal action principle* is to minimize $\mathcal{S}$ by varying the measure ρ under the following constraints:

$$\text{volume constraint:} \qquad \rho(\mathcal{F}) = \text{const} \tag{5.37}$$

$$\text{trace constraint:} \qquad \int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho(x) = \text{const} \tag{5.38}$$

$$\text{boundedness constraint:} \quad \mathcal{T}(\rho) := \iint_{\mathcal{F}\times\mathcal{F}} \bigg(\sum_{j=1}^{2n} \big|\lambda_j^{xy}\big| \bigg)^2 \mathrm{d}\rho(x)\, \mathrm{d}\rho(y) \leq C\,, \tag{5.39}$$

where C is a given parameter (and tr denotes the trace of a linear operator on $\mathcal{H}$ of finite rank). As already mentioned, we postpone the physical explanation of the detailed form of the Lagrangian to Section 5.8. The constraints can be understood mathematically as being needed in order to get a well-posed variational principle with nontrivial minimizers. This will be explained in Chapter 12 (see, in particular, Section 12.4; also Exercise 5.4 is related).

Before going on, for clarity, we point out that the mathematical structure of the causal action principle is quite different from other variational principles considered in physics and mathematics. There does not seem to be a direct way of deriving or even motivating the causal action principle from other known action principles or Lagrangians. The only way to get the connection to the known physical equation is by studying suitable limiting cases of the causal action principle and the corresponding Euler–Lagrange (EL) equations (it will be outlined in Chapters 21 and 22 how to get a connection to classical field theory and quantum field theory, respectively).

In order to make the causal action principle mathematically well defined, one needs to specify the class of measures in which to vary ρ. To this end, on $\mathcal{F}$ we consider the topology induced by the operator norm

$$\|A\| := \sup \big\{ \|Au\|_{\mathcal{H}} \text{ with } \|u\|_{\mathcal{H}} = 1 \big\}, \tag{5.40}$$

for basics, see the preliminaries in Sections 2.1 and 2.2. In this topology, the Lagrangian as well as the integrands in (5.38) and (5.39) are continuous. The σ-algebra generated by the open sets of $\mathcal{F}$ consists of the so-called Borel sets. A *regular Borel measure* is a measure on the Borel sets with the property that it is continuous under approximations by compact sets from inside and by open sets from outside (for basics, see the preliminaries in Section 2.3). The right prescription is to vary ρ within the class of regular Borel measures on $\mathcal{F}$.

One must distinguish two settings:

(a) The *finite-dimensional setting*: $\dim \mathcal{H} < \infty$ and $\rho(\mathcal{F}) < \infty$.
In this case, we will prove the existence of minimizing measures in Chapter 12. This will also clarify the significance of the constraints (see in particular the examples in Section 12.4).

(b) The *infinite-dimensional setting*: $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) = \infty$.
In this setting, it is an obvious complication that the volume constraint (5.37) is infinite. Likewise, the other constraints as well as the causal action may diverge. These divergences can be avoided by restricting attention to variations that change the measure only on a set of finite volume. By doing so, the differences between the action and the constraints are well defined and finite (this method will be introduced in Sections 6.3 and 12.8).

With this in mind, the remaining problem is to deal with infinite-dimensional Hilbert spaces. The question whether physics is to be described on the fundamental level by finite- or infinite-dimensional Hilbert spaces seems of a more philosophical nature, and we shall not enter this question here. One way of getting along with the finite-dimensional setting is to take the point of view that, on a fundamental physical level, the total volume is finite and the Hilbert space $\mathcal{H}$ is finite-dimensional, whereas the infinite-dimensional setting merely is a mathematical idealization needed in order to describe systems in infinite volume involving an infinite number of quantum particles. Even if this point of view is taken, the infinite-dimensional case is of independent mathematical interest and should also be the appropriate effective description in many physical situations. This case also seems to be mathematically sensible. However, the existence theory has not yet been developed. But at least, it is known that the EL equations corresponding to the causal action principle still have a mathematical meaning in the infinite-dimensional setting (for details, see [45]).

We now explain how the spacetime of a causal fermion system is endowed with a topological and causal structure. Recall that, given a minimizing measure ρ, *spacetime* $M \subset \mathcal{F}$ is defined as the support of ρ (see (5.24); this is illustrated in

Exercise 2.18). Thus, the spacetime points are symmetric linear operators on $\mathcal{H}$. On M we consider the topology induced by $\mathcal{F}$ (generated by the sup-norm (5.40) on $\mathrm{L}(\mathcal{H})$). Moreover, the measure $\rho|_M$ restricted to M can be regarded as a volume measure on spacetime. This turns spacetime into a *topological measure space*. Furthermore, one has the following notion of causality:

Definition 5.6.1 (Causal structure) *For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. We denote its nontrivial eigenvalues (counting algebraic multiplicities) by $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$. The points x and y are called* **spacelike** *separated if all the λ_j^{xy} have the same absolute value. They are said to be* **timelike** *separated if the λ_j^{xy} are all real and do not all have the same absolute value. In all other cases (i.e., if the λ_j^{xy} are not all real and do not all have the same absolute value), the points x and y are said to be* **lightlike** *separated.*

Restricting the causal structure of $\mathcal{F}$ to M, we get causal relations in spacetime.

Before going on, we point out that it is not obvious whether and in which sense this definition of causality agrees with the usual notion of causality in Minkowski space (or, more generally, in a Lorentzian spacetime). In order to get the connection, one can consider the causal fermion system constructed in Section 5.5 with the Hilbert space $\mathcal{H} \subset \mathcal{H}_m$ chosen as the subspace of all negative-energy solutions of the Dirac equation (thereby realizing the concept of the Dirac sea as explained in Section 1.5). Then the above "spectral definition" of causality goes over to the causal structure of Minkowski space in the limiting case $\varepsilon \searrow 0$. Since the detailed computations for getting this correspondence are a bit lengthy, we do not present them here but refer the interested reader instead to [45, Section 1.2].

The Lagrangian (5.35) is compatible with the above notion of causality in the following sense. Suppose that two points $x, y \in \mathcal{F}$ are spacelike separated. Then, the eigenvalues λ_i^{xy} all have the same absolute value. As a consequence, the Lagrangian (5.35) vanishes. Thus, pairs of points with spacelike separation do not enter the action. This can be seen in analogy to the usual notion of causality where points with spacelike separation cannot influence each other. This analogy is the reason for the notion "causal" in "causal fermion system" and "causal action principle."

A causal fermion system also distinguishes a *direction of time*. In order to see this, for $x \in \mathcal{F}$, we let π_x be the orthogonal projection in $\mathcal{H}$ on the subspace $x(\mathcal{H}) \subset \mathcal{H}$ and introduce the functional

$$\mathcal{C} \,:\, M \times M \to \mathbb{R}\,, \qquad \mathcal{C}(x,y) := i \,\mathrm{tr}\, \big(y\, x\, \pi_y\, \pi_x - x\, y\, \pi_x\, \pi_y \big)\,. \tag{5.41}$$

Obviously, this functional is anti-symmetric in its two arguments, making it possible to introduce the notions

$$\left\{ \begin{array}{ll} y \text{ lies in the } \textit{future} \text{ of } x & \quad \text{if } \mathcal{C}(x,y) > 0 \\ y \text{ lies in the } \textit{past} \text{ of } x & \quad \text{if } \mathcal{C}(x,y) < 0\,. \end{array} \right. \tag{5.42}$$

We remark that the detailed form of the functional (5.41) is not obvious; it must be justified by working out that it gives back the time direction of Minkowski space in a suitable limiting case (for details, see Exercise 5.8 and [45, §1.2.5]).

By distinguishing a direction of time, we get a structure similar to a causal set (see, e.g., [17]). However, in contrast to a causal set, our notion of "lies in the future of" is not necessarily transitive.

5.7 Basic Inherent Structures

It is the general concept that a causal fermion system describes spacetime as well as all structures therein (like the causal and metric structures, particles, fields, etc.). Thus all these structures must be constructed from the basic objects of the theory alone, using the information already encoded in the causal fermion system. We refer to these constructed structures as being *inherent* in the causal fermion system. We now introduce and explain the most important of these structures: the *spin spaces*, the *physical wave functions* and the *kernel of the fermionic projector*. Other inherent structures will be introduced later in this book (see Chapters 9–11); for a more complete account, we also refer to [45, Chapter 1].

The causal action principle depends crucially on the eigenvalues of the operator product xy with $x, y \in \mathcal{F}$. For computing these eigenvalues, it is convenient not to consider this operator product on the (possibly infinite-dimensional) Hilbert space $\mathcal{H}$, but instead to restrict attention to a finite-dimensional subspace of $\mathcal{H}$, chosen such that the operator product vanishes on the orthogonal complement of this subspace. This construction leads us to the spin spaces and to the kernel of the fermionic projector, which we now introduce. For every $x \in \mathcal{F}$ we define the *spin space* S_x as the image of the operator x, that is,

$$S_x := x(\mathcal{H}) \; ; \tag{5.43}$$

it is a subspace of $\mathcal{H}$ of dimension at most $2n$ (see Figure 5.7).

Moreover, we let

$$\pi_x \;:\; \mathcal{H} \to S_x \tag{5.44}$$

be the orthogonal projection in $\mathcal{H}$ on the subspace $S_x \subset \mathcal{H}$. For any $x, y \in M$ we define the *kernel of the fermionic projector* $P(x, y)$ by (see Figure 5.8).

$$P(x,y) = \pi_x \, y|_{S_y} \;:\; S_y \to S_x \tag{5.45}$$

where π_x is again the orthogonal projection on the subspace $x(\mathcal{H}) \subset \mathcal{H}$. Taking the trace of (5.45) in the case $x = y$, one finds that

$$\mathrm{tr}(x) = \mathrm{Tr}_{S_x}\big(P(x,x)\big) \,, \tag{5.46}$$

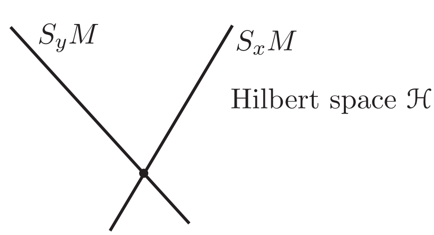

Figure 5.7 The spin spaces.

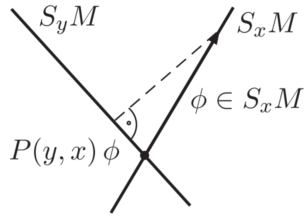

Figure 5.8 The kernel of the fermionic projector.

making it possible to express the integrand of the trace constraint (5.38) in terms of the kernel of the fermionic projector. In order to also express the eigenvalues of the operator xy in terms of the kernel of the fermionic projector, we introduce the *closed chain* A_{xy} as the product

$$A_{xy} = P(x,y)\,P(y,x) \;:\; S_x \to S_x \,. \tag{5.47}$$

The closed chain can be computed in more detail using the formula (5.45). In preparation, we note that, from the definition of π_x as the orthogonal projection to the image of x (5.44), it follows immediately that $\pi_x x = x$. Taking the adjoint of this relation, we conclude that

$$\pi_x\, x = x = x\, \pi_x \,. \tag{5.48}$$

Using these identities, we can compute the closed chain by

$$A_{xy} = (\pi_x y)(\pi_y x)|_{S_x} = \pi_x\, yx|_{S_x} \,. \tag{5.49}$$

Applying this equation iteratively and using again (5.48), we obtain for the p^{th} power of the closed chain

$$(A_{xy})^p = \pi_x\, (yx)^p|_{S_x} \,. \tag{5.50}$$

Taking the trace, one sees in particular that

$$\mathrm{Tr}_{S_x}(A_{xy}^p) = \mathrm{tr}\big((yx)^p\big) = \mathrm{tr}\big((xy)^p\big), \tag{5.51}$$

where the last identity simply is the invariance of the trace under cyclic permutations. Since all our operators have finite rank, for any $x, y \in \mathcal{F}$ there is a finite-dimensional subspace I of $\mathcal{H}$ such that xy maps I to itself and vanishes on the orthogonal complement of I. For example, one can choose I as the span of the image of xy and the orthogonal complement of the kernel of xy,

$$I = \mathrm{span}\big\{ (xy)(\mathcal{H}), \ker(xy)^\perp \big\} \,. \tag{5.52}$$

Then, the nontrivial eigenvalues of the operator product xy are the nonzero roots of the characteristic polynomial of the restriction $xy|_I : I \to I$. The coefficients of this characteristic polynomial (like the trace, the determinant, etc.) are symmetric polynomials in the eigenvalues and can therefore be expressed in terms of traces of powers of the operator $xy|_I : I \to I$ (for details, see Exercise 5.9). Using this result similarly for the characteristic polynomial of A_{xy} and using (5.51), we conclude that the eigenvalues of the closed chain coincide with the nontrivial eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ of the operator xy in Definition 5.6.1 (including multiplicities). In particular, one sees that kernel of the fermionic projector encodes the causal structure of M. The above argument also implies that the operator products xy and yx are isospectral. This shows that the causal structure is symmetric in x and y. The main advantage of working with the kernel of the fermionic projector is that the closed chain (5.47) is a linear operator on a vector space of dimension at most $2n$, making it possible to compute the $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ as the eigenvalues of a matrix (in finite dimensions).

Next, it is very convenient to choose inner products on the spin spaces in such a way that the kernel of the fermionic projector is symmetric in the sense that

$$P(x,y)^* = P(y,x) \,, \tag{5.53}$$

where the star denotes the adjoint with respect to yet to be specified inner products on the spin spaces. This identity indeed holds if on the spin space S_x (and similarly on S_y) one chooses the *spin inner product* $\prec .|.\succ_x$ defined by

$$\prec u|v \succ_x := -\langle u|xv\rangle_{\mathcal{H}} \qquad \text{(for all } u,v \in S_x) \,. \tag{5.54}$$

Due to the factor x on the right, this definition really makes the kernel of the fermionic projector symmetric, as is verified by the computation

$$\begin{aligned} \prec u \,|\, P(x,y)\, v \succ_x &= -\langle u \,|\, x\, P(x,y)\, v\rangle_{\mathcal{H}} = -\langle u \,|\, xy\, v\rangle_{\mathcal{H}} \\ &= -\langle \pi_y\, x\, u \,|\, y\, v\rangle_{\mathcal{H}} = \prec P(y,x)\, u \,|\, v \succ_y \,, \end{aligned} \tag{5.55}$$

where we again used (5.48) (and $u \in S_x$, $v \in S_y$). The spin space $(S_x, \prec .|.\succ_x)$ is an *indefinite* inner product of signature (p,q) with $p,q \leq n$ (for textbooks on indefinite inner product spaces, see [16, 94]). In this way, indefinite inner product spaces arise naturally when analyzing the mathematical structure of the causal action principle.

The kernel of the fermionic projector plays a central role in the analysis for several reasons:

- The Lagrangian can be expressed in terms of $P(x,y)$ (via the closed chain (5.47) and its eigenvalues).
- Being a mapping from one spin space to another, $P(x,y)$ gives relations between different spacetime points. In this way, it carries geometric information. This will be explained in Chapter 11 (see also [55] or the introductory survey paper [47]).
- The kernel of the fermionic projector also encodes all the wave functions of the system. In order to see the connection, for a vector $u \in \mathcal{H}$ one introduces the corresponding *physical wave function* ψ^u as (see Figure 5.9)

$$\psi^u \;:\; M \to \mathcal{H} \,, \qquad \psi^u(x) = \pi_x u \in S_x \,. \tag{5.56}$$

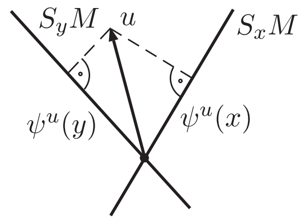

Figure 5.9 The physical wave function.

Then, choosing an orthonormal basis (e_i) of $\mathcal{H}$ and using the completeness relation as well as (5.54), one obtains for any $\phi \in S_y$

$$\begin{aligned} P(x,y)\,\phi = \pi_x y|_{S_y}\,\phi &= \sum_i \pi_x e_i\,\langle e_i | y\,\phi\rangle_{\mathcal{H}} \\ &= -\sum_i \psi^{e_i}(x)\,{\prec} \psi^{e_i}(y)\,|\,\phi {\succ}_y\,, \end{aligned} \tag{5.57}$$

showing that $P(x,y)$ is indeed composed of all the physical wave functions, that is, in bra/ket notation

$$P(x,y) = -\sum_i |\psi^{e_i}(x){\succ}\;{\prec}\psi^{e_i}(y)|\,. \tag{5.58}$$

We remark that knowing the kernel of the fermionic projector in spacetime makes it possible to reconstruct the causal fermion system (the detailed construction can be found in [80, Section 1.1.2]). We also note that the representation of the kernel of the fermionic projector (5.58) also opens the door to the detailed study of causal fermion systems in Minkowski space as carried out in [45]; see also Exercises 5.14–5.17.

Taking a slightly different perspective, one can say that all structures of the causal fermion system are encoded in the *family of physical wave functions* ψ^u with $u \in \mathcal{H}$ as defined in (5.56). In order to make this statement precise, it is most convenient to introduce the *wave evaluation operator* $\Psi(x)$ at the spacetime point $x \in M$ by

$$\Psi(x)\;:\;\mathcal{H} \to S_x\,, \qquad u \mapsto \psi^u(x) = \pi_x u\,. \tag{5.59}$$

Clearly, using (5.56), the wave evaluation operator can be written simply as

$$\Psi(x) = \pi_x\,. \tag{5.60}$$

The wave evaluation operator describes the family of all physical wave functions. Indeed, applying the wave evaluation operator to a vector u and varying the point x, we get back the corresponding physical wave function ψ^u. We next compute the adjoint of $\Psi(x)$,

$$\Psi(x)^* : S_x \to \mathcal{H}\,. \tag{5.61}$$

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Taking into account the corresponding inner products, we obtain for any $\phi \in S_x$ and $u \in \mathcal{H}$,

$$\langle \Psi(x)^* \phi | u \rangle_{\mathcal{H}} = \prec \phi | \Psi(x)\, u \succ_x \overset{(5.54)}{=} -\langle \phi | x\, \Psi(x)\, u \rangle_{\mathcal{H}} \,. \tag{5.62}$$

This shows that

$$\Psi(x)^* = -x|_{S_x} \,. \tag{5.63}$$

Combining (5.60) and (5.63) and comparing with (5.45), one sees that

$$x = -\Psi(x)^*\, \Psi(x) \qquad \text{and} \qquad P(x,y) = -\Psi(x)\, \Psi(y)^* \,. \tag{5.64}$$

In this way, all the spacetime point operators and the kernel of the fermionic projector can be constructed from the wave evaluation operator. Moreover, the conclusion after (5.51) that the eigenvalues of the closed chain coincide with the nontrivial eigenvalues of the operator product xy can be seen more directly from the computation

$$\begin{aligned} A_{xy} &= P(x,y)\, P(y,x) = \Psi(x)\Psi(y)^*\Psi(y)\Psi(x)^* \\ &= -\Psi(x)\big(y\, \Psi(x)^*\big) \simeq -\Psi(x)^*\Psi(x)\, y = xy \,, \end{aligned} \tag{5.65}$$

where by $\simeq$ we mean that the operators are isospectral (in the sense that they have the same nonzero eigenvalues with the same algebraic multiplicities). Here, we used that for any two matrices $A \in \mathbb{C}^{p \times q}$ and $B \in \mathbb{C}^{q \times p}$, the matrix product AB is isospectral to BA (for details, see Exercise 5.5).

5.8 How Did the Causal Action Principle Come About?

Causal fermion systems and the causal action principle came to light as a result of many considerations and computations carried out over several years. We now give an outline of these developments, also explaining the specific form of the causal action principle.

The starting point for the considerations leading to causal fermion systems was the belief that in order to overcome the conceptual problems of quantum field theory, the structure of spacetime should be modified. Moreover, instead of starting from differential equations in a spacetime continuum, one should formulate the physical equations using the new structures of spacetime, which might be nonsmooth or discrete. A more concrete idea in this direction was that the spacetime structures should be encoded in the family of wave functions which is usually associated to the Dirac sea (for basics, see Section 1.5). Thus, instead of disregarding the sea states, one should take all these wave functions into account. The mutual interaction of all these wave functions should give rise to the structures of spacetime as we experience them.

The first attempts toward making this idea more precise go back to the early 1990s. The method was to consider families of Dirac solutions (the formalism of quantum fields was avoided in order to keep the setting as simple and nontechnical as possible). In order to describe such a family mathematically, the corresponding two-point kernel $P(x, y)$ was formed

$$P(x, y) := -\sum_{l=1}^{f} |\psi_l(x) \succ \prec \psi_l(y)| \tag{5.66}$$

where $\psi_1, \ldots, \psi_f$ are suitably normalized solutions of the Dirac equation; for preliminaries, see Section 1.3. The kernel $P(x, y)$ is also referred to as the *kernel of the fermionic projector*. In the Minkowski vacuum, this kernel is formed of all the states of the Dirac sea. Then the sum goes over to an integral over the lower mass shell

$$P^{\text{vac}}(x, y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, (\not{k} + m) \, \delta(k^2 - m^2) \, \Theta(-k_0) \, \mathrm{e}^{-\mathrm{i}k(x-y)}, \tag{5.67}$$

this integral is well defined as the Fourier transform of a tempered distribution; see the preliminaries in Section 2.4. Likewise, a system involving particles and antiparticles is described by "occupying additional states of positive energy" and by "creating holes in the Dirac sea," respectively. Thus, more technically, one sets

$$P(x, y) = P^{\text{vac}}(x, y) - \sum_a |\psi_a(x) \succ \prec \psi_a(y)| + \sum_b |\phi_b(x) \succ \prec \phi_b(y)| \,, \tag{5.68}$$

where ψ_a and ϕ_b are suitably normalized Dirac solutions of positive and negative energy, respectively. In case a bosonic interaction is present, the kernel of the fermionic projector should no longer satisfy the vacuum Dirac equation, but the Dirac equation in the presence of a, say, external potential $\mathcal{B}$. Clarifying the dependence on the bosonic potential with an additional tilde, we write the resulting Dirac equation as

$$(\mathrm{i}\not{\partial} + \mathcal{B} - m) \, \tilde{P}(x, y) = 0 \,. \tag{5.69}$$

Analyzing the distribution $\tilde{P}(x, y)$ in Minkowski space reveals the following facts:

(a) The kernel $\tilde{P}(x, y)$ contains all the information on the wave functions of the particles and antiparticles of the system. This statement can be understood from the representation (5.68) in which all these wave functions appear. Alternatively, the wave functions can be reconstructed from $\tilde{P}(x, y)$ as being the image of the corresponding integral operator on $C_0^\infty(\mathscr{M}, S\mathscr{M})$

$$\phi \mapsto \int_{\mathscr{M}} \tilde{P}(.\,, y) \, \phi(y) \; \mathrm{d}^4 x \,. \tag{5.70}$$

(b) The kernel $\tilde{P}(x, y)$ has singularities on the light cone. The detailed form of the singularities involves integrals of the potential $\mathcal{B}$ and its derivatives along the light cone. In particular, knowing the kernel $\tilde{P}(x, y)$ makes it possible to reconstruct the potential $\mathcal{B}$ at every spacetime point. These statements follow immediately by looking at the so-called light-cone expansion of $\tilde{P}(x, y)$ (see Chapter 19 in this book or [45, Section 2.2 and Appendix B]).

(c) The singularity structure of $\tilde{P}(x,y)$ encodes the causal structure of Minkowski space. This can be seen again from the light-cone expansion of $\tilde{P}(x,y)$ (see again Chapter 19 in this book or [45, Section 2.2 and Appendix B]).

These findings show that, at least for Dirac systems in the presence of classical bosonic potentials, the kernel $\tilde{P}(x,y)$ contains all the information on the physical system. This led to the concept to regard $\tilde{P}(x,y)$ as the basic physical object in spacetime. The more familiar structures and objects like Minkowski space with its causal structure, the Dirac equation, the classical field equations for the bosonic fields (like the Maxwell or Einstein equations), however, should no longer be considered as being fundamental. Consequently, the physical equations should be formulated directly in terms of the kernel of the fermionic projector.

Formalizing this idea in a clean way also made it necessary to disregard or to prescind from the usual spacetime structures. This led to the *principle of the fermionic projector* as formulated around 1990 (see the unpublished preprint [35] and the monograph [41]). We here present a slightly different but equivalent formulation which is somewhat closer to the setting of causal fermion systems. Let M be a discrete set (i.e., a point set without additional structures), the *discrete spacetime*. Moreover, for every $x \in M$ one chooses an indefinite inner product space $(S_x, \prec .|. \succ_x)$, referred to as the *spin space* at x (usually, one chooses the dimensions and signatures of all spin spaces to be the same, but this is not crucial for the construction). Next, we consider a collection of *wave functions* $(\psi_a)_a$, each being a mapping which to every discrete spacetime point $x \in M$ associates a vector $\psi_a(x) \in S_x$ of the corresponding spin space. Out of these wave functions, one can form the kernel of the fermionic projector

$$P(x,y) := -\sum_a |\psi_a(x) \succ \prec \psi_a(y)| \;:\; S_y \to S_x \,. \tag{5.71}$$

The principle of the fermionic projector asserts that the physical equations should be formulated purely in terms of the kernel of the fermionic projector in discrete spacetime.

The next question was how precisely these physical equations should look like. This was a difficult question which took many years to be answered. Apart from the structural requirements coming from the principle of the fermionic projector, the following considerations served as guiding principles[3]:

(i) In analogy to classical field theory, a *variational approach* should be used. One main advantage is the resulting connection between symmetries and conservation laws (corresponding to the classical Noether theorem), which seems of central importance in physical applications.

[3] Of course, it is also an important requirement that our variational principle should give agreement with quantum field theory. But this connection was not used for finding the causal action principle. It was worked out more recently; for more details, see Chapter 22.

(ii) Classical field theory should be obtained in a certain limiting case. More specifically, the EL equations coming from our variational principle should reproduce the *Maxwell* and *Einstein equations*.
(iii) Also, the *Dirac equation* should be recovered in a certain limiting case.

More mathematically, the strategy was to form composite expressions of the kernel of the fermionic projector. Choosing n points $x_1, \ldots, x_n \in M$, one can form the *closed chain*

$$A_{x_1,\ldots,x_n} := P(x_1, x_2)\, P(x_2, x_3) \cdots P(x_{n-1}, x_n)\, P(x_n, x_1) \;:\; S_{x_1} \to S_{x_1} \,. \tag{5.72}$$

Being an endomorphism of the spin space, one can compute the eigenvalues of the closed chain and form a Lagrangian $\mathcal{L}[A_{x_1,\ldots,x_n}]$ as a symmetric function of these eigenvalues. Summing over the spacetime points gives an ansatz for the

$$n\text{-point action} \qquad \mathcal{S} = \sum_{x_1,\ldots,x_n \in M} \mathcal{L}[A_{x_1,\ldots,x_n}] \,. \tag{5.73}$$

This general ansatz can be made more specific and concrete by considering *gauge phases*. This consideration was motivated by the fact that the kernel of the fermionic projector $\tilde{P}(x,y)$ formed of Dirac solutions involves gauge phases. More specifically, choosing the potential in the Dirac equation (5.69) as an electromagnetic potential, that is, $\mathcal{B} = \not{A}$, then the leading contribution to the kernel are gauge phases described by line integrals over the electromagnetic potential,

$$\tilde{P}(x,y) = \mathrm{e}^{-\mathrm{i} \int_x^y A_j \xi^j}\, P^{\mathrm{vac}}(x,y) + \cdots \,, \tag{5.74}$$

where

$$\int_x^y A_j \xi^j = \int_0^1 A_j(\alpha y + (1-\alpha) x)\, (y-x)^j \; \mathrm{d}\alpha, \tag{5.75}$$

(this can again be seen from the light-cone expansion; more specifically, see [45, §2.2.4]). Here $\cdots$ stands for many other contributions to $\tilde{P}(x,y)$ which involve derivatives of the potential (like the field tensor, the Maxwell current, etc.). All these additional contributions are small in the sense that they are less singular on the light cone. These findings will be made precise by the Hadamard and light-cone expansions of the kernel of the fermionic projector in Chapter 21 of this book. At this stage, we do not need to be specific. All we need is that gauge phases come into play, which involves integrals of the potential along the line segment joining the points x and y.

Let us analyze the effect of the gauge phases on the closed chain (5.72). First of all, the closed chain is *gauge invariant*. Indeed, if one considers a pure gauge potential $A_j = \partial_j \Lambda$, then the gauge phases in (5.74) simplify to

$$\tilde{P}(x,y) = \mathrm{e}^{-\mathrm{i}\Lambda(y) + \mathrm{i}\Lambda(x)}\, P^{\mathrm{vac}}(x,y) \,, \tag{5.76}$$

and the phase factors of neighboring factors cancel in (5.72). This consideration also gives a relation between local gauge invariance and the fact that the adjacent factors in (5.72) must coincide. In the case $n = 1$, the kernel of the fermionic projector is evaluated only on the diagonal $P(x,x)$. This turns out to be too simple for formulating physical equations, as can be understood from the fact that no relations between spacetime points are taken into account. If $n \geq 3$, the gauge phases in (5.72) can be rewritten using Stokes' theorem as *flux integrals* of the electromagnetic field through the two-dimensional polygon with vertices $x_1, \ldots, x_n$. Analyzing the situation in some more detail, one finds that the resulting EL equations will be satisfied only if all fluxes vanish. This implies that the electromagnetic potential must be a pure gauge potential. In other words, the case $n \geq 3$ does not allow for an interaction via gauge potentials. This is the reason why this case was disregarded (for some more details on this argument, see [41, Remark 6.2.5]).

After these considerations, we are left with the

$$\text{two-point action} \qquad \mathcal{S} = \sum_{x,y \in M} \mathcal{L}[A_{xy}] \,, \tag{5.77}$$

where A_{xy} is the closed chain formed of two points,

$$A_{xy} := P(x,y)\, P(y,x) \,. \tag{5.78}$$

In this case, the polygon with vertices x and y degenerates to a straight line, implying that the flux through this polygon vanishes as desired. The starting point for a more quantitative analysis was to choose the Lagrangian formed by taking products and sums of traces of powers of the closed chain. A typical example is the Lagrangian

$$\mathcal{L}[A_{xy}] := \mathrm{Tr}_{S_x}\left(A_{xy}^2\right) - c\left(\mathrm{Tr}_{S_x}(A_{xy})\right)^2 \tag{5.79}$$

with a real parameter c. In such examples, the Lagrangian is a symmetric polynomial in the eigenvalues of the closed chain. The methods and results of this early analysis can be found in the unpublished preprints [35, 36].

Generally speaking, the study of such polynomial Lagrangians seemed a promising strategy toward formulating physically sensible equations. However, the more detailed analysis revealed the basic problem that *chiral gauge phases* come into play: As just explained after (5.76), the closed chain and therefore also the Lagrangian are gauge invariant for the electromagnetic potential. However, the situation changes if chiral gauge potentials are considered. Here, chiral gauge potentials are left- or right-handed potentials A_L and A_R which can be inserted into the Dirac equation by generalizing (1.29) to

$$\big(\mathrm{i}\partial\!\!\!/ + \chi_R A\!\!\!/_L + \chi_L A\!\!\!/_R - m\big)\psi = 0 \,, \tag{5.80}$$

where $\chi_{L/R}$ are the chiral projection operators (1.56) (for details, see, e.g., [45, §2.2.3]). In physics, the electroweak interaction involves left-handed gauge potentials. In this case, the left- and right-handed components of $P(x,y)$ involve phase transformations by the left- and right-handed gauge potentials, respectively. When forming the closed chain (5.78), the left- and right-handed components of $P(x,y)$ are multiplied together. As a consequence, the closed chain involves relative phases of the left- and right-handed gauge potentials, that is, phase factors of the form

$$\mathrm{e}^{\pm \mathrm{i} \int_x^y (A_L - A_R)_j \xi^j} \,, \tag{5.81}$$

where A_L and A_R are the left- and right-handed gauge potentials (here for simplicity again Abelian). As a consequence, also the eigenvalues of the closed chain are multiplied by these relative phases. The traces of powers of the closed chain as in (5.79) are still real-valued (this is because the phase factors always come as complex conjugate pairs), but they do not have fixed signs. Working out the EL equations, one sees that they also involve the relative gauge phases, making it difficult to allow for chiral gauge fields. In order to bypass these difficulties, from around 1999 on Lagrangians were considered which involved *absolute values* of the eigenvalues of the closed chain. This had two major advantages:

(a) The *chiral gauge phases drop out* of the Lagrangian.
(b) It became natural to formulate nonnegative Lagrangians. As a consequence, in the variational principle one *minimize the action* instead of merely looking for critical points.
(c) A connection to *causality* was obtained. In order to see how this comes about, we give a simple computation in the Minkowski vacuum. Suppose that the points x and y are either timelike or spacelike separated. Then, $P(x,y)$ is well defined and finite even without regularization and, due to Lorentz symmetry, it has the form

$$P(x,y) = \alpha\, \xi_j \gamma^j + \beta\, \mathbb{1} \tag{5.82}$$

with two complex-valued functions α and β (where again $\xi = y - x$, and γ^j are the Dirac matrices). Taking the adjoint with respect to the spin inner product, we see that

$$P(y,x) = \overline{\alpha}\, \xi_j \gamma^j + \overline{\beta}\, \mathbb{1}\,. \tag{5.83}$$

As a consequence,

$$A_{xy} = P(x,y)\, P(y,x) = a\, \xi_j \gamma^j + b\, \mathbb{1} \tag{5.84}$$

with two real parameters a and b given by

$$a = \alpha \overline{\beta} + \beta \overline{\alpha}\,, \qquad b = |\alpha|^2\, \xi^2 + |\beta|^2 \tag{5.85}$$

(here $\xi^2 = \xi^i \xi_i$ denotes the Minkowski inner product, which may be negative). Applying the formula $(A_{xy} - b\mathbb{1})^2 = a^2\, \xi^2\, \mathbb{1}$, the roots of the characteristic polynomial of A_{xy} are computed by

$$b \pm \sqrt{a^2\, \xi^2}\,. \tag{5.86}$$

Therefore, the eigenvalues of the closed chain are either real, or else they form a complex conjugate pair. Moreover, one gets a connection to causality: By explicit computation in Minkowski space one sees that a is nonzero (the details can be found in [55, proof of Lemma 4.3]). Therefore, if ξ is timelike (i.e., $\xi^2 > 0$) then the relations (5.85) and (5.86) show that the eigenvalues are distinct, both real and have the same sign. If ξ is spacelike, on the other hand, the eigenvalues are complex and have the same absolute value. In this way, one gets agreement with the spectral definition of causality in

Definition 5.6.1. Moreover, choosing a Lagrangian which depends only on differences of absolute values of the eigenvalues vanishes for spacelike separation, making it possible to build causality into the action principle.

The further analysis led to the class of Lagrangians

$$\mathcal{L} = \sum_{i,j} \Big(\big|\lambda_i^{xy}\big|^p - \big|\lambda_j^{xy}\big|^p \Big)^2 \tag{5.87}$$

with a parameter $p \in \mathbb{N}$, where the λ_i^{xy} are the eigenvalues of A_{xy} (again counted with algebraic multiplicities). The case $p = 1$ gives the causal Lagrangian (5.35) (albeit with the difference of working instead of the local correlation operators with the kernel of the fermionic projector; the connection will be explained below). The decision for $p = 1$ was taken based on the so-called *state stability analysis*, which revealed that the vacuum Dirac sea configuration (5.67) is a local minimizer of the causal action only if $p = 1$ (for details, see [41, Section 5.5]). Now that the form of the causal action was fixed, the monograph [41] was completed and published. The causal action principle is given in this book as an example of a variational principle in discrete spacetime (see [41, Section 3.5]). The boundedness constraint (5.39) already appears, and the causal Lagrangian (5.35) arises when combining the Lagrangian with the Lagrange multiplier term corresponding to the boundedness constraint. The volume constraint (5.37) is also implemented, however in discrete spacetime simply as the condition that the number of spacetime points be fixed (and ρ-integrals are replaced by sums over the spacetime points). The trace constraint, however, was not yet recognized as being necessary and important.

After the publication of the monograph [41], the causal action principle was analyzed in more detail and more systematically, starting from simple systems and proceeding to more realistic physical models, concluding with systems showing all the interactions of the standard model and gravity (see [45, Chapters 3–5]). This detailed study also led to the causal action principle in the form given in Section 5.6. The path from the monograph [41] to the present formulation in [45] is outlined in [41, Preface to second online edition]. We now mention a few points needed for the basic understanding.

One major conceptual change compared to the setting in indefinite inner product spaces was to recognize that an underlying Hilbert space structure is needed in order for the causal variational principle to be mathematically well defined. This became clear when working on the existence theory in discrete spacetime [42]. This Hilbert space structure is built in most conveniently by working instead of the kernel of the fermionic projector with the local correlation operators which relate the Hilbert space scalar product to the spin inner product by

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} = - \prec \psi(x) | \phi(x) \succ_x . \tag{5.88}$$

Using that the operator product $F(x)F(y)$ has the same nontrivial eigenvalues as the closed chain A_{xy} given by (5.78) (as we already observed in Section 5.7 after (5.47)), the causal action principle can also be formulated in terms of the local correlation operators $F(x)$ with $x \in M$. Moreover, it turned out that measure-theoretic methods can be used to generalize the setting such as to allow for the description of not only discrete, but also continuous spacetimes. In this formulation, the sums over the discrete spacetime points are replaced by integrals with respect to a measure μ on M. This setting was first introduced in [43] when working out the existence theory. In this formulation, the only a priori structure of spacetime is that of a measure space (M, μ). The local correlation operators give rise to a mapping

$$F : M \to \mathcal{F} , \quad x \mapsto F(x) , \tag{5.89}$$

where $\mathcal{F}$ is the subset of finite rank operators on $\mathcal{H}$ which are symmetric and (counting multiplicities) have at most n positive and at most n negative eigenvalues (where n is introduced via the signature (n, n) of the indefinite inner product in (5.88)). This analysis also revealed the significance of the trace constraint. As the final step, instead of working with the measure μ, the causal action can be expressed in terms of the push-forward measure $\rho = F_* \mu$, being a measure on $\mathcal{F}$ (defined by $\rho(\Omega) = \mu(F^{-1}(\Omega))$). Therefore, it seems natural to leave out the measure space (M, μ) and to work instead directly with the measure ρ on $\mathcal{F}$.

These considerations led to the general definition of causal fermion systems in Section 5.4, where the physical system is described by a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ and the measure ρ on $\mathcal{F}$. The causal action principle takes the form as stated in Section 5.6.

5.9 Underlying Physical Concepts

We now briefly explain a few physical concepts behind causal fermion systems and the causal action principle. The aim is to convey the reader the correct physical picture in a nontechnical way. Doing so already here makes it necessary to anticipate some ideas on a qualitative level which will be introduced more systematically and thoroughly later in this book.

It is a general feature of causal fermion systems that the usual distinction between the structure of spacetime itself (being modelled by Minkowski space or a Lorentzian manifold) and structures in spacetime (like wave functions and matter fields) ceases to exist. Instead, all these structures are described as a whole by *a single object*: the measure ρ on $\mathcal{F}$. Spacetime and all structures therein are different manifestations of this one object. The dynamics of spacetime and of all objects in spacetime are described in a unified and holistic manner by the causal action principle. Clearly, in order to get a connection to the conventional description of physics, one still needs to construct the familiar physical objects from a causal fermion system. Also, one needs to rewrite the dynamics as described by the causal action principle in terms of these familiar physical objects. This study is a main objective of this book. As already exemplified in Section 5.7 by the spin spaces and physical wave functions, the strategy is to identify suitable inherent structures in a causal fermion system, which then may be given suitable names. This must be done carefully in such a way that these names convey the correct physical picture. Ultimately, the inherent structures serve the purpose of getting a better understanding of the causal action principle. As we shall see, this will be achieved by reformulating the EL equations of the causal action principle in terms of the inherent structures. When this is done, the physical names of the inherent structures will also be justified by showing that they agree with the familiar physical objects in specific limiting cases and generalize these objects in a sensible way.

In view of this unified description of all physical structures by a single mathematical object, it is difficult to describe the essence of causal fermion systems using the familiar notions from physics. One simple way of understanding the causal action principle is to focus on the structure of the *physical wave functions* and the kernel of the fermionic projector. Clearly, the resulting picture is a bit oversimplified, because it only captures part of the structures encoded in a causal fermion system. Nevertheless, it conveys a good and the correct intuition of what the causal action principle is about. We saw in Section 5.7 that a causal fermion system gives rise to the family of physical wave functions $(\psi^u)_{u \in \mathcal{H}}$ (see (5.56)). The kernel of the fermionic projector (5.58) is built up of all the physical wave functions and thus describes the whole family. It gives rise to the closed chain (5.47), which in turn determines the causal action and the constraints. In this way, the causal action principle becomes a variational principle for the family of physical wave functions. Thus the interaction described by the causal action principle can be understood as a direct mutual interaction of all the physical wave functions. In simple terms, the causal action principle aims at bringing the family of wave functions into an "optimal" configuration. For such optimal configurations, the family of wave functions gives rise to the spacetime structures as we know them: the causal and metric structure, the bosonic fields, and all that.

The last step can be understood more concretely starting from Dirac's hole theory and the picture of the *Dirac sea* (for basics, see again Section 1.5). In our approach, the Dirac sea is taken literally. Thus all the states of the Dirac sea correspond to physical wave functions. All the information contained in these wave functions induces spacetime with the familiar structures. As a simple example, the bosonic potentials $\mathcal{B}$ are determined via the Dirac equation (5.69) from the family of wave functions as described by $\tilde{P}(x,y)$. Clearly, in order to make this picture precise, one needs to verify that, in a certain limiting case, the kernel of the fermionic projector corresponding to a minimizer of the causal action principle indeed satisfies a Dirac equation of the form (5.69) and thus gives rise to a potential $\mathcal{B}$. This will be one of the objectives of the later chapters in this book.

We now discuss which *physical principles* enter the approach, and how they were incorporated. Causal fermion systems evolved from an attempt to combine several physical principles in a coherent mathematical setting. As a result, these principles appear in a specific way:

- The **principle of causality**: A causal fermion system gives rise to a causal structure (see Definition 5.6.1). The causal action principle is compatible with this notion of causality in the sense that the pairs of points with spacelike separation do not enter the EL equations. In simple terms, points with spacelike separation do not interact.

- The **local gauge principle**: Already in the above discussion of how the causal action principle came about, we mentioned that the Lagrangian is gauge invariant in the sense that gauge phases drop out of the Lagrangian (see the explanation after (5.76) in Section 5.8). When starting from a general causal fermion system, **local gauge invariance** becomes apparent when representing the physical wave functions in bases of the spin spaces. More precisely, choosing a pseudo-orthonormal basis $(\mathfrak{e}_\alpha(x))_{\alpha=1,\ldots,\dim S_x}$ of S_x, a physical wave function can be represented as

$$\psi(x) = \sum_{\alpha=1}^{\dim S_x} \psi^\alpha(x)\, \mathfrak{e}_\alpha(x) \tag{5.90}$$

with component functions $\psi^1, \ldots, \psi^{\dim S_x}$. The freedom in choosing the basis $(\mathfrak{e}_\alpha)$ is described by the group of unitary transformations with respect to the indefinite spin inner product. This gives rise to the transformations

$$\begin{aligned} \mathfrak{e}_\alpha(x) &\to \sum_\beta U^{-1}(x)^\beta_\alpha\; \mathfrak{e}_\beta(x)\,, \\ \psi^\alpha(x) &\to \sum_\beta U(x)^\alpha_\beta\, \psi^\beta(x) \end{aligned} \tag{5.91}$$

with $U \in \mathrm{U}(p,q)$. As the basis $(\mathfrak{e}_\alpha)$ can be chosen independently at each space-time point, one obtains *local gauge transformations* of the wave functions, where the gauge group is determined to be the isometry group of the spin inner product. The causal action is *gauge invariant* in the sense that it does not depend on the choice of spinor bases.

- The **Pauli exclusion principle** is incorporated in a causal fermion system, as can be seen in various ways. One formulation of the Pauli exclusion principle states that every fermionic one-particle state can be occupied by at most one particle. In this formulation, the Pauli exclusion principle is respected because every wave function can either be represented in the form ψ^u (the state is occupied) with $u \in \mathcal{H}$ or it cannot be represented as a physical wave function (the state is not occupied). Via these two conditions, the fermionic projector encodes for every state the occupation numbers 1 and 0, respectively, but it is impossible to describe higher occupation numbers.

  More technically, one may obtain the connection to the fermionic Fock space formalism by choosing an orthonormal basis $u_1, \ldots, u_f$ of $\mathcal{H}$ and forming the f-particle Hartree–Fock state

$$\Psi := \psi^{u_1} \wedge \cdots \wedge \psi^{u_f}\,. \tag{5.92}$$

  Clearly, the choice of the orthonormal basis is unique only up to the unitary transformations

$$u_i \to \tilde{u}_i = \sum_{j=1}^f U_{ij}\, u_j \quad \text{with} \quad U \in \mathrm{U}(f)\,. \tag{5.93}$$

Due to the anti-symmetrization, this transformation changes the corresponding Hartree–Fock state only by an irrelevant phase factor,

$$\psi^{\tilde{u}_1} \wedge \cdots \wedge \psi^{\tilde{u}_f} = \det U \; \psi^{u_1} \wedge \cdots \wedge \psi^{u_f} \; . \tag{5.94}$$

Thus the configuration of the physical wave functions can be described by a fermionic multiparticle wave function. The Pauli exclusion principle becomes apparent in the total anti-symmetrization of this wave function.

Clearly, the above Hartree–Fock state does not account for quantum entanglement. Indeed, the description of entanglement requires more general Fock space constructions (this will be described in more detail in Chapter 22).

- The **equivalence principle**: Starting from a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$, spacetime M is given as the support of the measure ρ. Thus spacetime is a topological space (with the topology on M induced by the operator norm on $\mathrm{L}(\mathcal{H})$). In situations when spacetime has a smooth manifold structure, one can describe spacetime by choosing coordinates. However, there is no distinguished coordinate systems, giving rise to the freedom of performing general coordinate transformations. The causal action as well as all the constraints are invariant under such transformations. In this sense, the equivalence principle is implemented in the setting of causal fermion systems.

However, other physical principles are missing. For example, the principle of locality is not included. Indeed, the causal action principle is *nonlocal*, and locality is recovered only in the continuum limit. Moreover, our concept of causality is quite different from *causation* (in the sense that the past determines the future) or *microlocality* (stating that the observables of spacelike separated regions must commute).

5.10 A Summary of the Basic Concepts and Objects

In this section we summarize all important concepts of the preceding sections. You may use this as a reference list for frequently used concepts and objects.

Basic concept	Summary and Comments
Causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$	A separable Hilbert space $\mathcal{H}$, a natural number $n \in \mathbb{N}$, the set $\mathcal{F}$ of symmetric linear operators on $\mathcal{H}$ with at most n positive and n negative eigenvalues as well as a measure ρ defined on a σ-algebra on $\mathcal{F}$ forms a **causal fermion system**.

Remarks:

- The structure of a causal fermion system provides a general framework for describing generalized spacetimes. Concrete physical systems correspond to specific choices of $\mathcal{H}$, n and the measure ρ.
- $\mathcal{H}$ should be considered as the Hilbert space spanned by all one-particle wave functions realized in our system (the *physical wave functions*).
- We are mainly interested in the case $n = 2$ (at most two positive and two negative eigenvalues). This case allows for the description of Dirac spinors in four-dimensional spacetimes.

Spacetime M	By definition, we describe spacetime by the support of the measure $M := \operatorname{supp}(\rho)$.

Remarks:

- All points $x \in M$ are linear operators on $\mathcal{H}$. This fact implies that our spacetime is endowed with more structures and contains additional information.
- In order to describe systems in Minkowski space, we identify spacetime points $x \in M$ with corresponding points in Minkowski space $\mathscr{M}$ via a mapping $F^\varepsilon : \mathscr{M} \to M$ (for more details, see (5.33)).

The measure ρ	The measure ρ in Definition 5.4.1 is the most important object of the theory. It describes spacetime as well as all objects therein.

Remarks:

- A lot of structure is encoded in the measure ρ. In particular, it describes the behavior of spacetime on microscopic scales (Planck scale).
- In the example of causal fermion systems describing Minkowski space, the measure is obtained as the push-forward of the Minkowski volume measure $d\mu = d^4x$ under the local correlation map F^ε, that is, we set $\rho = F^\varepsilon_* \mu$.

The causal action	We define a Lagrangian $\mathcal{L}(x,y)$ for two spacetime points x and y using the eigenvalues $(\lambda_i^{xy})_{i=1,\ldots,2n}$ of the product xy, which is an operator of rank at most $2n$. The Lagrangian is given by $\mathcal{L}(x,y) := \frac{1}{4n}\sum_{i,j=1}^{2n}(\lvert\lambda_i^{xy}\rvert - \lvert\lambda_j^{xy}\rvert)^2$. Finally, the causal action is defined by taking the double integral $\mathcal{S}(\rho) := \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\mathrm{d}\rho(x)\mathrm{d}\rho(y)$.

Remarks:

- It may happen that the rank of the operator xy is smaller than $2n$. In this case, some of the eigenvalues $\lambda_1^{xy}, \ldots \lambda_{2n}^{xy}$ are zero.
- The action depends nonlinearly on the measure ρ. Since ρ describes spacetime and all objects therein, the action also depends on spacetime and on all these object.

The causal action principle	The causal action principle states that measures describing physical systems must be minimizers of the causal action under variations of ρ, respecting the constraints (5.37), (5.38) and (5.39).

Remarks:

- The EL equations corresponding to the causal action principle are the physical equations of the theory.
- By varying the measure ρ, we also vary spacetime as well as all structures therein.

The physical wave functions	Every vector $u \in \mathcal{H}$ can represented in spacetime by the physical wave function ψ^u defined by $\psi^u(x) = \pi_x u \in S_x$, where π_x denotes the orthogonal projection in $\mathcal{H}$ onto the subspace $x(\mathcal{H}) \subset \mathcal{H}$.
The kernel of the fermionic projector	For any spacetime point operator $x \in M$, we define the spin space S_x as its image $S_x := x(\mathcal{H})$. This gives rise to a mapping between spin spaces at different spacetime points $x, y \in M$ by $P(x,y) := \pi_x y\vert_{S_x} : S_y \to S_x$, The mapping $P(x,y)$ is the kernel of the fermionic projector. It can be expressed in terms of all physical wave functions by $P(x,y) = -\sum_i \lvert\psi^{e_i}(x)\succ\prec\psi^{e_i}(y)\rvert$, where the (e_i) form an orthonormal basis of $\mathcal{H}$.

Remarks:

- The kernel of the fermionic projector gives relations between spacetime points. In particular, it encodes the causal structure and the geometry of spacetime.
- In order to compute the Lagrangian, it is useful to form the *closed chain* $A(x,y) := P(x,y)P(y,x)$.

5.11 Exercises

Exercise 5.1 This exercise is devoted to the study of the variational principle (5.18) of the motivating example.

(a) Assume that the operators $F_1, \ldots, F_L$ are a minimizer of the action (5.18) under variations of $F_i \in \mathcal{F}$ with $\mathcal{F}$ according to (5.16). Given $i \in \{1, \ldots f\}$, represent F_i as

$$F_i = |\psi_i\rangle\langle\psi_i| \qquad \text{with } \psi_i \in \mathcal{H}\,. \tag{5.95}$$

Vary the vector ψ_i to derive the following EL equations,

$$\sum_{i,j=1}^{f} \operatorname{Tr}(F_j F_i)\, F_j\, \psi_i = 0\,. \tag{5.96}$$

(b) Deduce that all the matrices F_i must vanish. *Hint:* It is useful to first show that

$$\sum_{i,j=1}^{f} \big|\operatorname{Tr}(F_j F_i)\big|^2 = 0\,. \tag{5.97}$$

(c) In order to get nontrivial solutions, one can, for example, impose the constraint

$$\sum_{i=1}^{f} \operatorname{Tr}\left(F_i^2\right) = 1\,. \tag{5.98}$$

Derive the corresponding EL equations.

(d) The constraint (5.97) also makes it possible to prove existence of minimizer with a compactness argument. Work out this existence proof in detail.

Exercise 5.2 *(A causal fermion system on ℓ_2)* Let $\mathcal{H} = \ell_2$ be the Hilbert space of square-summable complex-valued sequences, equipped with the scalar product

$$\langle u|v\rangle = \sum_{i=1}^{\infty} \bar{u}_i\, v_i, \quad u = (u_i)_{i\in\mathbb{N}},\ v = (v_i)_{i\in\mathbb{N}}. \tag{5.99}$$

For any $k \in \mathbb{N}$, let $x_k \in \mathrm{L}(\mathcal{H})$ be the operator defined by

$$(x_k u)_k := u_{k+1}, \quad (x_k u)_{k+1} := u_k, \quad (x_k u)_i = 0 \text{ for } i \notin \{k, k+1\}. \tag{5.100}$$

In other words,

$$x_k u = \big(\underbrace{0, \ldots, 0}_{k\,-\,1\text{ entries}}, u_{k+1}, u_k, 0, \ldots\big) \tag{5.101}$$

Finally, let μ the counting measure on $\mathbb{N}$ (i.e., $\mu(X) = |X|$ equals the cardinality of $X \subset \mathbb{N}$.)

(a) Show that every operator x_k has rank two, is symmetric, and has one positive and one negative eigenvalue. Make yourself familiar with the concept that every operator is a point in $\mathcal{F}$ for spin dimension $n = 1$.

(b) Let $F : \mathbb{N} \to \mathcal{F}$ be the mapping which to every k associates the corresponding operator x_k. Show that the push-forward measure $\rho = F_*\,\mu$ defined by $\rho(\Omega) := \mu(F^{-1}(\Omega))$ defines a measure on $\mathcal{F}$. Show that this measure can also be characterized by

$$\rho(\Omega) = |\{k \in \mathbb{N} \mid x_k \in \Omega\}|. \tag{5.102}$$

(c) Show that $(\mathcal{H}, \mathcal{F}, \rho)$ is a causal fermion system of spin dimension one.
(d) Show that the support of ρ consists precisely of all the operators x_k. What is spacetime M? What is the causal structure on M? What is the resulting causal action?

Exercise 5.3 *(Boundedness of operators of finite rank)* Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be a Hilbert space and $(V, \|.\|)$ a normed space of finite dimension n. Moreover, let $A : \mathcal{H} \to V$ be a linear mapping.

(a) Show that the kernel of A is a closed subspace of $\mathcal{H}$. Show that its orthogonal complement $(\ker A)^\perp$ has dimension at most n.
(b) Derive a block matrix representation of A on $\mathcal{H} = (\ker A) \oplus (\ker A)^\perp$.
(c) Deduce that A is bounded, that is, that there is a constant $c > 0$ with $\|Au\| \leq c\,\|u\|_{\mathcal{H}}$ for all $u \in \mathcal{H}$.

Exercise 5.4 *(On the trace constraint)* This exercise shows that the trace constraint ensures that the action is nonzero. Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system of spin dimension n.

(a) Assume that $\operatorname{tr}(x) \neq 0$. Show that $\mathcal{L}(x, x) > 0$. (For a quantitative statement of this fact in the setting of discrete spacetimes, see [42, Proposition 4.3].)
(b) Assume that $\int_{\mathcal{F}} \operatorname{tr}(x)\, d\rho \neq 0$. Show that $\mathcal{S}(\rho) > 0$.

Exercise 5.5 *(On the spectrum of the closed chain)* This exercise is devoted to analyzing general properties of the spectrum of the closed chain.

(a) We let x and y be symmetric operators of finite rank on a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. Show that there is a finite-dimensional subspace $I \subset \mathcal{H}$ on which both x and y are invariant. By choosing an orthonormal basis of I and restricting the operators to I, we may represent both x and y by Hermitian matrices. Therefore, the remainder of this exercise is formulated for simplicity in terms of Hermitian matrices.
(b) Show that for any matrix Z, the characteristic polynomials of Z and of its adjoint Z^* (being the transposed complex conjugate matrix) are related by complex conjugation, that is, $\det(Z^* - \overline{\lambda}\,\mathbb{1}) = \overline{\det(Z - \lambda\,\mathbb{1})}$.
(c) Let X and Y be symmetric matrices. Show that the characteristic polynomials of the matrices XY and YX coincide.
(d) Combine (b) and (c) to conclude that the characteristic polynomial of XY has real coefficients, that is, $\det(XY - \overline{\lambda}\,\mathbb{1}) = \overline{\det(XY - \lambda\,\mathbb{1})}$. Infer that the spectrum of the matrix product XY is symmetric about the real axis, that is,

$$\det(XY - \lambda\,\mathbb{1}) = 0 \implies \det(XY - \overline{\lambda}\,\mathbb{1}) = 0\,. \tag{5.103}$$

(e) For the closed chain (5.47), the mathematical setting is somewhat different, because A_{xy} is a symmetric operator on the indefinite inner product space $(S_x, \prec.|.\succ_x)$. On the other hand, we saw after (5.47) that A_{xy} is isospectral to xy. Indeed, the symmetry result (5.103) can be used to prove a corresponding statement for A_{xy},

$$\det(A_{xy} - \lambda\, \mathbb{1}) = 0 \implies \det(A_{xy} - \overline{\lambda}\, \mathbb{1}) = 0\,. \tag{5.104}$$

This result is well known in the theory of indefinite inner product spaces (see, e.g., the textbooks [16, 94] or [42, Section 3]). In order to derive it from (5.103), one can proceed as follows: First, represent the indefinite inner product in the form $\prec.|.\succ = \langle .|S\,x\rangle$, where $\langle .|.\rangle$ is a scalar product and S is an invertible operator which is symmetric (with respect to this scalar product). Next, show that the operator $B := A_{xy}S$ is symmetric (again with respect to this scalar product). Finally, write the closed chain as $A_{xy} = BS^{-1}$ and apply (5.103).

Exercise 5.6 *(Regular spacetime points)* Let $x \in \mathcal{F}$ have $p(x) \leq n$ negative and $q(x) \leq n$ positive eigenvalues. The pair $\text{sign}(x) := (p(x), q(x))$ is referred to as the *signature* of x. The operator x is said to be *regular* if $\text{sign}(x) = (n, n)$. The goal of this exercise is to show that the set $\mathcal{F}^{\text{reg}}$ of regular points is open in $\mathcal{F}$. Let us define the *positive* and *negative components* of x as the operators

$$x_\pm := \frac{x \pm |x|}{2}, \quad |x| := \sqrt{x^2}. \tag{5.105}$$

From the functional calculus, it follows that $x\,|x| = |x|\,x$. Prove the following statements.

(a) Let $\{e_i\,, i = 1, \ldots, m\}$ be an orthogonal set. Show that any vector set $\{h_i\,, i = 1, \ldots, m\}$ which fulfills the following condition is linearly independent,

$$\|e_i - h_i\| < \frac{\inf\{\|e_i\|\,,\, i = 1, \ldots, m\}}{m} \quad \text{for all } i = 1, \ldots, m. \tag{5.106}$$

(b) For every $x \in \mathcal{F}$,

$$x(\text{im}\, x_\pm) \subset \text{im}\, x_\pm \quad \text{and} \quad x_+\, x_- = 0. \tag{5.107}$$

Moreover, $x|_{\text{im}\, x_-}$ and $x|_{\text{im}\, x_+}$ are negative and positive definite, respectively.

(c) Let $x_0 \in \mathcal{F}$. Then, there is an orthonormal set $\{e_i \mid i = 1 \ldots \dim S_{x_0}\}$ of eigenvectors of x_0 such that

$$\begin{aligned} &\langle e_i|\, x_0\, e_i\rangle < 0 \qquad \text{for all } i \leq p(x_0), \\ &\langle e_i|\, x_0\, e_i\rangle > 0 \qquad \text{for all } p(x_0) < i \leq p(x_0) + q(x_0). \end{aligned} \tag{5.108}$$

(d) The following functions are continuous,

$$f_i : B_r(x_0) \ni x \mapsto f_i(x) := \begin{cases} x_-\, e_i & i \leq p(x_0) \\ x_+\, e_i & p(x_0) < i \leq p(x_0) + q(x_0)\,. \end{cases} \tag{5.109}$$

Hint: You can use the general inequality $\||A| - |B|\| \leq \|A^2 - B^2\|$

(e) There is a $r > 0$ such that $p(x) \geq p(x_0)$ and $q(x) \geq q(x_0)$ for every $x \in B_r(x_0)$. *Hint: Use the statements above.*
(f) Conclude that $\mathcal{F}^{\mathrm{reg}}$ is an open subset of $\mathcal{F}$.

Exercise 5.7 *(A causal causal fermion system on ℓ_2 – part 2)* We return to the example of Exercise 5.2. This time we equip it with a *Krein structure.*

(a) For any $k \in \mathbb{N}$, construct the spin space S_{x_k} and its spin scalar product.
(b) Given a vector $u \in \mathcal{H}$, what is the corresponding wave function ψ^u? What is the Krein inner product $< .,. >$?
(c) What is the topology on the Krein space $\mathcal{K}$? Does the wave evaluation operator $\Psi : u \mapsto \psi^u$ give rise to a well-defined and continuous mapping $\Psi : \mathcal{H} \to \mathcal{K}$? If yes, is it an embedding? Is it surjective?
(d) Repeat part (c) of this exercise for the causal fermion system obtained if the operators x_k are multiplied by k, that is,

$$x_k u := (0, \ldots, 0, k\, u_{k+1}, k\, u_k\, 0, \ldots). \tag{5.110}$$

Exercise 5.8 *(Time direction)* The ability to distinguish between past and future can be described in mathematical terms by the existence of an anti-symmetric functional $\mathcal{T} : M \times M \to \mathbb{R}$. One then says that

$$\begin{cases} y \text{ lies in the } \textit{future} \text{ of } x & \text{if } \mathcal{T}(x,y) > 0 \\ y \text{ lies in the } \textit{past} \text{ of } x & \text{if } \mathcal{T}(x,y) < 0 \,. \end{cases} \tag{5.111}$$

Can you think of simple nontrivial examples of such a functional which involve only products and linear combinations of the spacetime operators and the orthogonal projections on the corresponding spin spaces? *Hint:* One possible functional is

$$\mathcal{T}(x,y) := \operatorname{tr}\big(y\,\pi_x - x\,\pi_y\big), \tag{5.112}$$

this is considered further in [45, Exercise 1.22].

Exercise 5.9 This exercise is devoted to clarifying the connection between the characteristic polynomial and traces of powers of a matrix. We let A be an $N \times N$-matrix (not necessarily Hermitian) and denote the zeros of its characteristic polynomials counting multiplicities by $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, that is,

$$\det(\lambda\,\mathbb{1} - A) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_N)\,. \tag{5.113}$$

Moreover, we denote the coefficients of the characteristic polynomial by a_k, that is,

$$\det(\lambda\mathbb{1} - A) = \lambda^N + a_1 \lambda^{N-1} + \cdots + a_N\,. \tag{5.114}$$

(a) Show that the coefficients are symmetric polynomials in the eigenvalues of the form

$$a_n = c_n \sum_{\substack{B \subset \{1,\ldots,N\} \\ \text{with } \#B = n}} \prod_{k \in B} \lambda_k\,, \tag{5.115}$$

where the sum goes over all subsets of $\{1, \dots N\}$ with n elements, and c_n are combinatorial prefactors. Compute the c_n.

(b) Show that each coefficient a_n can be written in the form

$$a_n = d_n \operatorname{Tr}(A^n) + d_{1,n-1} \operatorname{Tr}(A) \operatorname{Tr}(A^{n-1}) \tag{5.116}$$

$$+ d_{1,1,n-2} \operatorname{Tr}(A) \operatorname{Tr}(A) \operatorname{Tr}(A^{n-2}) + \cdots \tag{5.117}$$

$$= \sum_{k=1}^{n} \sum_{\substack{1 \le j_1 \le \cdots \le j_k \\ \text{with } j_1 + \cdots + j_k = n}} d_{j_1,\dots,j_k} \operatorname{Tr}\left(A^{j_1}\right) \cdots \operatorname{Tr}\left(A^{j_k}\right) \tag{5.118}$$

with suitable combinatorial factors $d_n, d_{1,n-1}, \dots$. *Hint:* This formula can be derived in various ways. One method is to proceed inductively in n. Alternatively, one can use a dimensional argument.

Exercise 5.10 *(Embedding of $S_x\mathscr{M}$ into $S_{F(x)}$)* The goal of this exercise is to explain how the fibers of the spinor bundle $S\mathscr{M}$ are related to the spin spaces S_x of the corresponding causal fermion system. In order to keep the setting as simple as possible, we let $(\mathscr{M}, g)$ be Minkowski space and $\mathcal{H}$ a finite-dimensional subspace of the Dirac solution space $\mathcal{H}_m$, consisting of smooth wave functions of spatially compact support, that is,

$$\mathcal{H} \subset C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m \qquad \text{finite-dimensional}\,. \tag{5.119}$$

We again let $F(x)$ be the local correlation operators, that is,

$$\langle \psi | F(x) \phi \rangle = - \prec \psi(x) | \phi(x) \succ \qquad \text{for all } \psi, \phi \in \mathcal{H} \tag{5.120}$$

(since $\mathcal{H}$ consists of smooth functions, we may leave out the regularization operators). Defining the measure again by $d\rho = F_*(d^4x)$, we again obtain a causal fermion system of spin dimension $n = 2$. We next introduce the *evaluation map* e_x by

$$e_x \,:\, \mathcal{H} \to S_x\mathscr{M}\,, \qquad e_x(\psi) = \psi(x)\,. \tag{5.121}$$

Restricting the evaluation mapping to the spin space $S_{F(x)}$ at the spacetime point $F(x)$ (defined as in (5.43) as the image of the operator $F(x)$), we obtain a mapping

$$e_x|_{S_{F(x)}} \,:\, S_{F(x)} \to S_x\mathscr{M}\,. \tag{5.122}$$

(a) Show that $e_x|_{S_{F(x)}}$ is an isometric embedding.

(b) Show that for all $u \in \mathcal{H}$ and $x \in \mathscr{M}$,

$$e_x|_{S_{F(x)}}\big(\psi^u(F(x))\big) = u(x)\,. \tag{5.123}$$

Exercise 5.11 *(Identification of $S\mathscr{M}$ with SM)* In the setting of the previous exercise, we now make two additional assumptions:

(i) The mapping $F : \mathscr{M} \to \mathcal{F}$ is injective and its image is closed in $\mathcal{F}$.

(ii) The resulting causal fermion system is *regular* in the sense that for all $x \in \mathscr{M}$, the operator $F(x)$ has rank $2n$.

Using the results of the previous exercise, explain how the following objects can be identified:

(a) x with $F(x)$
(b) $\mathscr{M}$ with M
(c) The spinor space $S_x\mathscr{M}$ with the corresponding spin space $S_{F(x)}$
(d) $u \in \mathcal{H}$ with its corresponding physical wave function ψ^u

Exercise 5.12 *(The space $C^0(M, SM)$)* A *wave function* ψ is defined a mapping from M to H such that $\psi(x) \in S_x$ for all $x \in M$. It is most convenient to define *continuity* of a wave function by the requirement that for all $x \in M$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\big\| \sqrt{|y|}\, \psi(y) - \sqrt{|x|}\, \psi(x) \big\|_{\mathcal{H}} < \varepsilon \qquad \text{for all } y \in M \text{ with } \|y - x\| \leq \delta\,. \tag{5.124}$$

Show that, using this definition, every physical wave function is continuous. Thus, denoting the space of continuous wave functions by $C^0(M, SM)$, we obtain an embedding

$$\mathcal{H} \hookrightarrow C^0(M, SM)\,. \tag{5.125}$$

Hint: You may use the inequality

$$\Big\| \sqrt{|y|} - \sqrt{|x|} \Big\| \leq \|y - x\|^{\frac{1}{4}}\, \|y + x\|^{\frac{1}{4}}\,. \tag{5.126}$$

Exercise 5.13 *(A causal fermion system in $\mathbb{R}^3$)* We choose $\mathcal{H} = \mathbb{C}^2$ with the canonical scalar product. Moreover, we choose let $\mathscr{M} = S^2 \subset \mathbb{R}^3$ and $d\mu$ the Lebesgue measure on $\mathscr{M}$. Consider the mapping

$$F : \mathscr{M} \to \mathrm{L}(\mathcal{H})\,, \qquad F(p) = 2 \sum_{\alpha=1}^{3} p^\alpha \sigma^\alpha + \mathbb{1}\,, \tag{5.127}$$

where σ^α are the three Pauli matrices (1.27).

(a) Show that for every $p \in S^2$,

$$\operatorname{tr}\big(F(p)\big) = 2\,, \qquad \operatorname{tr}\big(F(p)^2\big) = 10\,. \tag{5.128}$$

Conclude that the eigenvalues of $F(p)$ are equal to 1 ± 2.

(b) We introduce the measure ρ as the push-forward measure $\rho = F_*\mu$ (i.e., $\rho(\Omega) := \mu(F^{-1}(\Omega))$). Show that $(\mathcal{H}, \mathcal{F}, \rho)$ is a causal fermion system of spin dimension one.

(c) Show that the support of ρ coincides with the image of F. Show that it is homeomorphic to S^2.

This example is also referred to as the *Dirac sphere*; this and other similar examples can be found in [43, Examples 2.8 and 2.9] or [56, Example 2.2].

Exercise 5.14 *(The regularized fermionic projector in Minkowski space)* The goal of this exercise is to compute the kernel of the fermionic projector in the Minkowski vacuum for the simplest regularization, the $i\varepsilon$-regularization (5.30).

(a) Use the identifications of Exercises 5.10 and 5.11 to show that (5.58) holds in the example of Dirac wave functions in Minkowski space (as constructed in Section 5.5) but now with Dirac wave functions and the spin inner product thereon.

(b) More specifically, we now choose $\mathcal{H} = \mathcal{H}_m^-$ as the subspace of all negative-frequency solutions of the Dirac equation. Moreover, we choose the $i\varepsilon$-regularization (5.30). For clarity, we denote the corresponding kernel of the fermionic projector by $P^{2\varepsilon}(x,y)$. Show that

$$P^\varepsilon(x,y) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4k}{(2\pi)^4}\,(\not{k}+m)\,\delta(k^2-m^2)\,\Theta(-k^0)\,\mathrm{e}^{-\mathrm{i}k(x-y)}\,\mathrm{e}^{\varepsilon k^0}\,. \tag{5.129}$$

Hint: Work in a suitable orthonormal basis of the Hilbert space. Without regularization, the computation can be found in [45, Lemma 1.2.8].

(c) Show that $P^\varepsilon(x,y)$ can be written as $\not{v}^\varepsilon + \beta^\varepsilon$ with $v_j^\varepsilon, \beta^\varepsilon$ smooth functions of $\xi = y - x$.

(d) Compute $P^\varepsilon(x,x)$. Is this matrix invertible? How does it scale in ε? Why does this result show that the resulting causal fermion system is regular? *Hint:* The details can also be found in [45, Section 2.5]. For an alternative way of proving regularity, see Exercise 5.17.

(e) For ξ spacelike or timelike, that is, away from the lightcone, the limit $\varepsilon \searrow 0$ of (5.129) is well defined. More precisely, it can be shown that $v_j^\varepsilon \to \alpha\,\xi_j$ and $\beta^\varepsilon \to \beta$ pointwise, for α, β smooth complex functions. Find smooth real functions a, b such that

$$\lim_{\varepsilon\searrow 0} A_{xy}^\varepsilon = a\not{\xi} + b. \tag{5.130}$$

Exercise 5.15 *(Correspondence of the causal structure in Minkowski space I)*
Let $x, y \in \mathscr{M}$ be timelike separated vectors and assume that $\xi := y - x$ is normalized to $\xi^2 = 1$. As explained in Exercise 5.14, the limit $\varepsilon \searrow 0$ of the closed chain A_{xy}^ε takes the form $A := a\,\not{\xi} + b$. Consider the matrices

$$F_\pm := \frac{1}{2}\,(\mathbb{1} \pm \not{\xi}) \in \mathrm{L}(\mathbb{C}^4)\,. \tag{5.131}$$

Prove the following statements.

(a) The matrices $F_\pm$ have rank two and map to eigenspaces of A. What are the corresponding eigenvalues? Conclude that the points x and y are timelike separated in the sense of Definition 5.6.1.

(b) The matrices $F_\pm$ are idempotent and symmetric with respect to the spin inner product $\prec.|.\succ$.

(c) The image of the matrices $F_\pm$ is positive or negative definite (again with respect to the spin inner product).

(d) The image of F_+ is orthogonal to that of F_- (again with respect to the spin inner product).

(e) The eigenvalues of A are strictly positive. *Hint:* Use how the functions a and b came up in (5.130).

The result of (a)–(d) can be summarized by saying that the $F_\pm$ are the spectral projection operators of A. We remark that the findings also mean that the x and y are even *properly timelike separated* as introduced in [45, Definition 1.1.6].

Exercise 5.16 *(Correspondence of the causal structure in Minkowski space II)* We now let $x, y \in \mathscr{M}$ be spacelike separated vectors and assume that $\xi := y - x$ is normalized to $\xi^2 = -1$. Consider again the matrix $A := a \slashed{\xi} + b$ of Exercise 5.15 and set

$$F_\pm := \frac{1}{2}\,(\mathbb{1} \pm \mathrm{i}\,\slashed{\xi}) \in \mathrm{L}(\mathbb{C}^4)\,. \tag{5.132}$$

(a) The matrices $F_\pm$ have rank two and map to eigenspaces of A. What are the corresponding eigenvalues? Conclude that the points x and y are spacelike separated in the sense of Definition 5.6.1.
(b) The matrices $F_\pm$ are idempotent and $F_+^* = F_-$.
(c) The image of the matrices $F_\pm$ is null (in other words, it is a lightlike subspace of the spinor space).

These findings illustrate the more general statement that symmetric operators on an indefinite inner product space may have complex eigenvalues, in which case they form complex conjugate pairs.

Exercise 5.17 *(Spin spaces for the regularized Dirac sea vacuum)* We consider the causal fermion system constructed in Section 5.5, where we choose $\mathcal{H} = \mathcal{H}_m^-$ as the space of all negative-energy solutions of the Dirac equation. Moreover, we choose the $i\varepsilon$-regularization (5.30). For clarity, we denote the corresponding kernel of the fermionic projector by $P^\varepsilon(x,y)$. This causal fermion system is also referred to as the *regularized Dirac sea vacuum.*

(a) Let Σ_0 denote the Cauchy surface at time $t = 0$. Show that, for any $x \in \mathscr{M}$ and $\chi \in \mathbb{C}^4$,

$$(\mathrm{i}\slashed{\partial} - m)P^\varepsilon(\,\cdot\,, x)\chi = 0 \quad \text{and} \quad P^\varepsilon(\,\cdot\,, x)\chi\big|_{\Sigma_0} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)\,. \tag{5.133}$$

Conclude that $P^\varepsilon(\,\cdot\,, x)\chi \in \mathcal{H}_m^- \cap C^\infty(\mathbb{R}^4, \mathbb{C}^4)$.
(b) Convince yourself that

$$\mathfrak{R}_\varepsilon(P^\varepsilon(\,\cdot\,, x)\chi) = P^{2\varepsilon}(\,\cdot\,, x)\chi\,. \tag{5.134}$$

(c) Let $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_4\}$ denote the canonical basis of $\mathbb{C}^4$. Using Exercise 5.14 (b), show that the wave functions $P^\varepsilon(\,\cdot\,, x)\mathfrak{e}_\mu$ for $\mu = 1, 2, 3, 4$ are linearly independent.
(d) Let $S_x := F^\varepsilon(x)(\mathcal{H}_m^-)$ endowed with $\prec u, v \succ_x := -\langle u | F^\varepsilon(x) v\rangle$ be the *spin space* at $x \in \mathscr{M}$. Show that the following mapping is an isometry of indefinite inner products (i.e., injective and product preserving),

$$\Phi_x : S_x \ni u \mapsto \mathfrak{R}_\varepsilon u(x) \in \mathbb{C}^4. \tag{5.135}$$

Conclude that the causal fermion system is regular at $x \in \mathscr{M}$, that is, $\dim S_x = 4$, if and only if there exist vectors $u_\mu \in \mathcal{H}_m^-$, for $\mu = 1, 2, 3, 4$, such that the $\mathfrak{R}_\varepsilon u_\mu(x) \in \mathbb{C}^4$ are linearly independent.

(e) Conclude that the causal fermion system is regular at every spacetime point.