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A new class of α-Farey maps and an application to normal numbers

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We define two types of the α-Farey maps F_α and $F_{\alpha,\flat}$ for $0 < \alpha < \frac{1}{2}$, which were previously defined only for $\frac{1}{2} \leq \alpha \leq 1$ by Natsui (2004). Then, for each $0 < \alpha < \frac{1}{2}$, we construct the natural extension maps on the plane and show that the natural extension of $F_{\alpha,\flat}$ is metrically isomorphic to the natural extension of the original Farey map. As an application, we show that the set of normal numbers associated with α-continued fractions does not vary by the choice of α, $0 < \alpha < 1$. This extends the result by Kraaikamp and Nakada (2000).

Keywords: α-continued fraction expansions; Farey map; natural extension; normal numbers

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1. Introduction

The main purpose of this paper is to extend the notion of the α-Farey map to $0 < \alpha < \frac{1}{2}$, and discuss its properties with applications. We start with a simple introduction of the theory of the regular continued fraction map.

Let x be a real number, then it is well-known that the *simple* or *regular continued fraction* (RCF) *expansion* of x yields a finite (if $x \in \mathbb{Q}$) or infinite (if $x \in \mathbb{R} \setminus \mathbb{Q}$) sequence of rational convergents (p_n/q_n) with extremely good approximation properties; see e.g. [10, 15, 16, 32–34]. The RCF-expansion of x can be obtained using the so-called *Gauss map* $G : [0, 1] \to [0, 1)$, defined as follows:

$$G(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

For $0 < x < 1$, the digits (or: partial quotients) $a_n = a_n(x)$ of the RCF-expansion of x are defined for $n \geq 1$ by $a_n(x) = \lfloor \frac{1}{G^{n-1}(x)} \rfloor$, where $\lfloor \frac{1}{0} \rfloor = \infty$ and $\frac{1}{\infty} = 0$. For $x \in \mathbb{R}$, we define $a_0 = a_0(x) = \lfloor x \rfloor$; if $x \notin (0, 1)$, we define for $n \geq 1$ the digit $a_n(x)$ by setting $a_n(x) := a_n(x - a_0)$.

It is well-known that for $x \in (0, 1)$ the simple continued fraction expansion of x easily follows from the above definitions of G and $a_n(x)$:

$$x = \frac{1}{|a_1(x)} + \frac{1}{|a_2(x)} + \cdots + \frac{1}{|a_n(x)} + \cdots .$$

We put

$$\frac{p_n(x)}{q_n(x)} = \frac{1}{|a_1(x)} + \frac{1}{|a_2(x)} + \cdots + \frac{1}{|a_n(x)},$$

where $p_n(x)$, $q_n(x) \in \mathbb{N}$ and where we assume that $(p_n(x), q_n(x)) = 1$. This rational number $p_n(x)/q_n(x)$ is called the nth principal convergent of x. It is also well known that for $n \geq 1$,

$$\begin{cases} p_n(x) = a_n(x)p_{n-1}(x) + p_{n-2}(x), \\ q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x), \end{cases}$$

with $p_{-1}(x) = 1$, $p_0(x) = 0$, $q_{-1}(x) = 0$, $q_0(x) = 1$. If $a_n(x) \geq 2$ for $n \geq 1$, then

$$\frac{\ell \cdot p_{n-1}(x) + p_{n-2}(x)}{\ell \cdot q_{n-1}(x) + q_{n-2}(x)}, \quad 1 \leq \ell < a_n(x) \tag{1.1}$$

is called the (n, ℓ)-mediant (or intermediate) convergent of x.

Setting

$$\Theta_n(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right| \quad \text{for } n \geq 0,$$

one can easily show that for all irrational x and all $n \geq 1$ one has that $0 < \Theta_n(x) < 1$; see e.g. [10, 16]. Several classical results on these *approximation coefficients* $\Theta_n(x)$ have been obtained for all $n \geq 1$ and all irrational x; just to mention a few:

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$$\min\{\Theta_{n-1}(x), \Theta_n(x)\} < \tfrac{1}{2}, \text{ (Vahlen, 1913)}$$

and

$$\min\{\Theta_{n-1}(x), \Theta_n(x), \Theta_{n+1}(x)\} < \tfrac{1}{\sqrt{5}}, \quad \text{(Borel, 1903)}.$$

Borel's result is a consequence of

$$\min\{\Theta_{n-1}(x), \Theta_n(x), \Theta_{n+1}(x)\} < \tfrac{1}{\sqrt{a_{n+1}^2+4}},$$

which was obtained independently by various authors; see also Chapter 4 in [10]. That the sequence $(p_n(x)/q_n(x))$ converges extremely fast to x follows from $0 < \Theta_n(x) < 1$ for all n, and the fact that the sequence $(q_n(x))$ grows exponentially fast. An old result by Legendre further underlines the Diophantine qualities of the RCF: let $x \in \mathbb{R}$, and let $p \in \mathbb{Z}$, $q \in \mathbb{N}$, such that $(p, q) = 1$, and suppose we moreover have that

$$\left| x - \frac{p}{q} \right| < \frac{1}{2} \frac{1}{q^2},$$

then p/q is a RCF-convergent of x. I.e., there exists an n such that $p = p_n(x)$ and $q = q_n(x)$. Here the constant $1/2$ is best possible. In 1904, Fatou stated (and this was published in 1918 by Grace; see [14]), that if $\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$, then p/q is either an RCF-convergent, or an extreme mediant; i.e., an (n, ℓ)-mediant convergent of x from (1.1) with $\ell = 1$ or $\ell = a_n - 1$. Further refinements of this result can be found in [3].

The (n, ℓ)-mediant convergents of x from (1.1) can be obtained by the so-called Farey-map F. The notion of the Farey map was introduced in 1989 by Ito in [17] and by Feigenbaum, Procaccia and Tel in [13] independently. In particular, the metric properties of F were discussed by Ito in [17]; see also [4, 5, 7, 9, 11].

To introduce F, we write G as the composition of two maps: an *inversion* $R : (0, 1] \to [1, \infty)$ and a translation $S : [1, \infty) \to (0, 1]$, defined as:

$$R(x) = \frac{1}{x} \quad \text{for } x \in (0, 1]$$

and

$$S(x) = x - k, \quad \text{if } x \in [k, k+1) \text{ for some } k \in \mathbb{N}.$$

If we furthermore define that $0 \mapsto 0$, we clearly have that $G(x) = (S \circ R)(x)$ for $x \in (0, 1]$. Note that the latter map S can be written as the k-fold composition of a map $S_1 : [1, \infty) \to [0, \infty)$, defined as $S_1(x) = x - 1$ for $x \geq 1$, so that $S(x) = (\underbrace{S_1 \circ \cdots \circ S_1}_{k})(x)$.

Next, we extend the inversion R to $[1, \infty)$: $R(x) = \frac{1}{x}$ for $x \in [1, \infty)$, so that we map $[1, \infty)$ bijectively on the bounded interval $(0, 1]$. With this extended definition of the map R, we define the map $F : (0, 1] \to [0, 1)$ as a "slow continued fraction map," given by

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$$F(x) = \begin{cases} (R \circ S_1 \circ R)(x) = \dfrac{x}{1-x}, & \text{if } x \in [0, 1/2), \\ G(x) = (S_1 \circ R)(x) = \dfrac{1-x}{x}, & \text{if } x \in [1/2, 1], \end{cases}$$

see also [11], where the relation between F and the so-called *Lehner continued fraction* is investigated.

For a given matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$, we define its associated linear fractional transformation as:

$$A(x) = \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}} \quad \text{for } x \in \mathbb{R}. \tag{1.2}$$

The map F "yields" the mediant convergents together with the principal (i.e., RCF) convergents in the following manner. For each $x \in (0, 1)$, $F(x)$ is either $\frac{x}{1-x}$ or $\frac{1-x}{x}$, which is a linear fractional transformation associated with matrices $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, respectively. We put

$$A_n = A_n(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \left(= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \right), & \text{if } 0 < F^{n-1}(x) < \frac{1}{2}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \left(= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \right), & \text{if } \frac{1}{2} < F^{n-1}(x) < 1, \end{cases}$$

thus yielding a sequence of matrices $(A_n : n \geq 1)$. Viewing this sequence as a sequence of linear fractional transformations, we obtain a sequence of rationals $(t_n : n \geq 1)$ with $t_n = (A_1 A_2 \cdots A_n)(-\infty)$ for each $n \geq 1$. It is not hard to see that this sequence is

$$\begin{aligned}
&\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{a_1 - 1}, \\
&\frac{0}{1} = \frac{p_0}{q_0}, \frac{1}{a_1 + 1} = \frac{1 \cdot p_1 + p_0}{1 \cdot q_1 + q_0}, \frac{2 \cdot p_1 + p_0}{2 \cdot q_1 + q_0}, \ldots, \frac{(a_2 - 1) \cdot p_1 + p_0}{(a_2 - 1) \cdot q_1 + q_0}, \\
&\frac{p_1}{q_1}, \frac{1 \cdot p_2 + p_1}{1 \cdot q_2 + q_1}, \frac{2 \cdot p_2 + p_1}{2 \cdot q_2 + q_1}, \ldots, \frac{(a_3 - 1) \cdot p_2 + p_1}{(a_3 - 1) \cdot q_2 + q_1}, \\
&\qquad\qquad \vdots \\
&\frac{p_{n-1}}{q_{n-1}}, \frac{1 \cdot p_n + p_{n-1}}{1 \cdot q_n + q_{n-1}}, \ldots, \frac{\ell \cdot p_n + p_{n-1}}{\ell \cdot q_n + q_{n-1}}, \ldots, \frac{(a_{n+1} - 1) \cdot p_n + p_{n-1}}{(a_{n+1} - 1) \cdot q_n + q_{n-1}}, \\
&\frac{p_n}{q_n}, \frac{1 \cdot p_{n+1} + p_n}{1 \cdot q_{n+1} + q_n}, \ldots\ldots\ldots,
\end{aligned}$$

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i.e., we have the sequence of the mediant convergents together with the principal convergents of x. We will find it again in §3 as a special case of Nakada's α-expansions from [26], with $\alpha = 1$.

Apart from the *regular continued fraction expansion* there is a bewildering amount of other continued fraction expansion: continued fraction expansions with *even* (or *odd*) partial quotients, the *optimal continued fraction expansion*, the Rosen fractions, and many more. In this paper we will look at a family of continued fraction algorithms, introduced by Nakada in 1981 in [26] with the natural extensions as planer maps. These continued fraction expansions are parameterized by a parameter $\alpha \in (0, 1]$, the case $\alpha = 1$ being the RCF. After their introduction, the natural extension of the Gauss map played an important role in solving a conjecture by Hendrik Lenstra, which was previously proposed by Wolfgang Doeblin (see [6], and also [10, 16] for more details on the proof and various corollaries of this *Doeblin-Lenstra* conjecture). The notion of the natural extension planer maps lead to various generalization, e.g. the so-called S-expansions, introduced by Kraaikamp in [21]. The papers mentioned here, and various other papers at the time dealt with the case $\frac{1}{2} \le \alpha \le 1$. At that time, there was no discussion on α-continued fractions for $0 < \alpha < \frac{1}{2}$ except for a 1999 paper by Moussa, Cassa and Marmi [25], dealing with $\sqrt{2} - 1 < \alpha < \frac{1}{2}$. Later on, after two papers published in 2008 by Luzzi and Marmi ([24]), and Nakada and Natsui ([29]), the interest to work on α-continued fractions was rekindled, but then for parameters $\alpha \in (0, \frac{1}{2})$; see e.g. [12, 23].

In 2004, Natsui introduced and studied the so-called α-Farey maps F_α in [30, 31] for parameters $\alpha \in [\frac{1}{2}, 1)$. These maps F_α relate to the α-expansion maps G_α from [26] as the Farey-map F relates to the Gauss-map G. In this paper, we investigate these α-Farey maps F_α for $0 < \alpha < \frac{1}{2}$.

Recall from [26] that the α-continued fraction map G_α, for $0 < \alpha \le 1$, is defined as follows (We refer to papers by A. Abrams, S. Katok and I. Ugarcovici [2] and by S. Katok and I. Ugarcovici [18–20], where a similar idea is applied to a different type (2-parameter family) of continued fraction maps). Let $\alpha \in (0, 1]$ fixed, then for $x \in \mathbb{I}_\alpha = [\alpha - 1, \alpha)$ we define the map G_α as

$$G_\alpha(x) = \begin{cases} -\frac{1}{x} - \lfloor -\frac{1}{x} + 1 - \alpha \rfloor, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ \frac{1}{x} - \lfloor \frac{1}{x} + 1 - \alpha \rfloor, & \text{if } x > 0. \end{cases} \tag{1.3}$$

For $x \in \mathbb{I}_\alpha$, we put $a_{\alpha,n}(x) = \left\lfloor \frac{1}{|G_\alpha^{n-1}(x)|} + 1 - \alpha \right\rfloor$ and $\varepsilon_{\alpha,n}(x) = \operatorname{sgn}(x)$. Then we have for $x \in \mathbb{I}_\alpha \setminus \{0\}$ that:

$$G_\alpha(x) = \frac{\varepsilon_{\alpha,n}(x)}{x} - a_{\alpha,n}(x).$$

From this one easily finds that:

$$x = \frac{\varepsilon_{\alpha,1}(x)|}{|a_{\alpha,1}(x)} + \frac{\varepsilon_{\alpha,2}(x)|}{|a_{\alpha,2}(x)} + \cdots + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n}(x)} + \cdots,$$

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which we call the α-continued fraction expansion of x. We define the nth principal convergent as

$$\frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)} = \frac{\varepsilon_{\alpha,1}(x)|}{|a_{\alpha,1}(x)} + \frac{\varepsilon_{\alpha,2}(x)|}{|a_{\alpha,2}(x)} + \cdots + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n}(x)} \qquad \text{for } n \geq 1,$$

where $p_{\alpha,n}(x) \in \mathbb{Z}$, $q_{\alpha,n}(x) \in \mathbb{N}$ and $(p_{\alpha,n}(x), q_{\alpha,n}(x)) = 1$. Moreover, whenever $a_{\alpha,n}(x) \geq 2$ we also define the mediant convergents as

$$\frac{\ell \cdot p_{\alpha,n-1}(x) + \varepsilon_{\alpha,n}(x) p_{\alpha,n-2}(x)}{\ell \cdot q_{\alpha,n-1}(x) + \varepsilon_{\alpha,n}(x) q_{\alpha,n-2}(x)} \quad \text{for } 1 \leq \ell < a_n(x).$$

To get these mediant convergents, we consider the Farey type map F_α, and as in the case $\alpha = 1$ we show how it is related with G_α; note that $G_1 = G$ and $F_1 = F$. As in the case $\alpha = 1$, we consider *inversions* and a *translation*. The inversions are now defined by

$$R_-(x) = -\frac{1}{x} \text{ for } x \in [\alpha - 1, 0), \text{and } R(x) = \frac{1}{x} \text{ for } x > 0,$$

while the translation is now defined by

$$S_1(x) = x - 1 \quad \text{for } x > \alpha.$$

Of course, we again define that $0 \mapsto 0$. From this process, we have the map F_α defined on $\left[\alpha - 1, \frac{1}{\alpha}\right]$ by

$$F_\alpha(x) = \begin{cases} (R \circ S_1 \circ R_-)(x) & = -\frac{x}{1+x}, \text{ if } x \in [\alpha - 1, 0), \\ (R \circ S_1 \circ R)(x) & = \frac{x}{1-x}, \text{ if } x \in [0, \frac{1}{1+\alpha}], \\ (S_1 \circ R)(x) & = \frac{1-x}{x}, \text{ if } x \in (\frac{1}{1+\alpha}, \frac{1}{\alpha}], \end{cases} \tag{1.4}$$

see Figure 1. We will see in §2 that the mediant convergents are induced from F_α. However, we should note that if $\varepsilon_{\alpha,n+1}(x) = -1$,

$$\frac{1 \cdot p_{\alpha,n}(x) - p_{\alpha,n-1}(x)}{1 \cdot q_{\alpha,n}(x) - q_{\alpha,n-1}(x)} = \frac{(a_{\alpha,n}(x) - 1) \cdot p_{\alpha,n-1}(x) + \varepsilon_{\alpha,n}(x) p_{\alpha,n-2}(x)}{(a_{\alpha,n}(x) - 1) \cdot q_{\alpha,n-1}(x) + \varepsilon_{\alpha,n}(x) q_{\alpha,n-2}(x)},$$

which means that we get the same rational number more than once as a mediant convergent. To avoid such repetitions, Natsui in 2004 introduced in [30] another type of a Farey like map $F_{\alpha,\flat}$ for $\frac{1}{2} \leq \alpha < 1$, which is an induced transformation of F_α, and was defined on $[\alpha - 1, 1]$ by

$$F_{\alpha,\flat}(x) = \begin{cases} -\frac{x}{1+x}, & \text{if } \alpha - 1 \leq x < 0, \\ \frac{x}{1-x}, & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1-2x}{x}, & \text{if } \frac{1}{2} \leq x \leq \frac{1}{1+\alpha}, \\ \frac{1-x}{x}, & \text{if } \frac{1}{1+\alpha} < x \leq 1. \end{cases} \tag{1.5}$$

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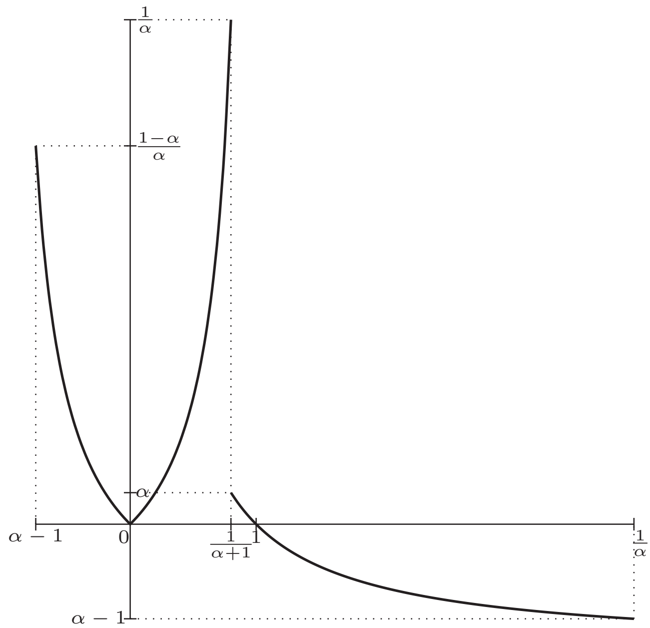

Figure 1. The map F_α for $\alpha = \frac{1}{4}$

The definition of $F_{\alpha,\flat}$ as given in (1.5) does not work for the case $0 < \alpha < \frac{1}{2}$, since the image of $[\alpha - 1, 0)$ under $F_{\alpha,\flat}$ is not contained in $[\alpha - 1, 1]$. Indeed, $F_{\alpha,\flat}(\alpha - 1) = \frac{1-\alpha}{\alpha} > 1$ for $\alpha < 1/2$. For this reason, we modify the above definition of $F_{\alpha,\flat}$ slightly; see (2.3) in §2. Both are induced transformations, but with a slightly different definition. In the sequel, we first show that F_α certainly induces the mediant convergents and is well-defined for $0 < \alpha < 1/2$. Then we introduce a simple variant of $F_{\alpha,\flat}$. We will show that dynamically these maps are isomorphic to the Farey map F in the following sense. In §3, we construct a planer map $\hat{F}_\alpha$ which is the natural extension of F_α and then construct in §4 the natural extension of $F_{\alpha,\flat}$ (denoted by $\hat{F}_{\alpha,\flat}$) as an induced map of $\hat{F}_\alpha$. Then we show that for $0 < \alpha < 1$, all $\hat{F}_{\alpha,\flat}$ are metrically isomorphic to $\hat{F}_1$. One of the points which we have to be careful is that the first coordinates of planer maps of the "mediant convergent maps $\hat{F}_{\alpha,\cdot}$, $0 < \alpha < 1$" are not the "mediant convergent maps $F_{\alpha,\cdot}$.", though the first coordinate of the natural extension maps $\hat{G}_\alpha$ are exactly the α-continued fraction maps G_α.

In §5, we apply the idea of the planer maps to show some results on α-continued fractions, which are already known for $\frac{1}{2} \leq \alpha \leq 1$ but not for $0 < \alpha < \frac{1}{2}$. We recall some results on normal numbers and on mixing properties of G_α; see [22] and [28], respectively. The first application of the α-Farey map is a relation among normal numbers with respect to α-continued fractions for different values of α. In [22], it was shown that the set of normal numbers with respect to G_α is the same for any $\frac{1}{2} \leq \alpha \leq 1$. It is natural to ask whether we can extend the result to $0 < \alpha \leq 1$. However, the proof used in [22] does not work. The main point is that the sequence of the principal convergents $\left(\frac{p_{\alpha,n}}{q_{\alpha,n}} : n \geq 1\right)$ is a subsequence

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of $\left(\frac{p_n}{q_n} : n \geq 1\right)$ for $\frac{1}{2} \leq \alpha \leq 1$. This also holds for $\sqrt{2}-1 \leq \alpha < \frac{1}{2}$ but not anymore for $0 < \alpha < \sqrt{2}-1$. Thus it is easy to follow the proof used in [22] for $\sqrt{2}-1 \leq \alpha < \frac{1}{2}$ but not possible for α below $\sqrt{2}-1$. At this point, we need the α-mediant convergents to discuss normality. The second application is the following. In [28], we show that the ϕ-mixing property fails for a.e. $\alpha \in [\frac{1}{2}, 1]$ using the above normal number result. Indeed, the result implies that for a.e. $\alpha \in [\frac{1}{2}, 1]$, $\{G_\alpha^n(\alpha-1) : n \geq 1\}$ is dense in $\mathbb{I}_\alpha$. Then for any $\varepsilon > 0$, we can find $n \geq 1$ such that the size of either the interval $[\alpha-1, G_\alpha^n(\alpha-1)]$ or $[G_\alpha^n(\alpha-1), \alpha)$ is less than ε. The property called "matching" plays an important role there. It was proved in [26], however it seems that nobody, not even the author of [26], noticed the importance of this property until [29] appeared (after [28]!). It is also easy to see the matching property for $\sqrt{2}-1 \leq \alpha < \frac{1}{2}$ holds, but not easy for α below $\sqrt{2}-1$. After [29] was published, in [8] the complete characterization of the set of α's which have the matching property was given together with the proof of a conjecture from [29]. Actually, the matching property holds for almost all $\alpha \in (0,1)$. Together with the result from §5.1, we show in §5.2 that G_α is not ϕ-mixing for almost every $\alpha \in (0,1)$. In §5 the construction of the natural extension $\hat{F}_{\alpha,\flat}$ of $F_{\alpha,\flat}$ as a planer map plays an important role.

In this paper, we change the notation in [30] and [31] to adjust for the names of Gauss and Farey:

[30, 31]		this note
T_α	$\to$	G_α
G_α	$\to$	F_α
F_α	$\to$	$F_{\alpha,\flat}$

2. Basic properties of the α-Farey map F_α, $0 < \alpha < 1$

First of all, note that there is a strong relation between the maps G_α from (1.3) and F_α from (1.4). For any $\alpha \in (0,1)$, we get G_α as a induced transformation of F_α. This induced transformation is defined as follows.

For each $\alpha \in (0,1)$ and $x \in \mathbb{I}_\alpha = [\alpha-1, \alpha)$, we put $j(0) = 0$ if $x = 0$, and $j(x) = j_\alpha(x) = k$ if

(i) $x \neq 0$,
(ii) $F_\alpha^\ell(x) \notin (\frac{1}{1+\alpha}, \frac{1}{\alpha}]$, $0 \leq \ell < k$,
(iii) $F_\alpha^k(x) \in (\frac{1}{1+\alpha}, \frac{1}{\alpha}]$.

Note that from definition (1.4) of F_α we then have that $F_\alpha^{k+1}(x) \in \mathbb{I}_\alpha$, which is the domain of G_α; see (1.3). In case $\frac{\sqrt{5}-1}{2} < \alpha < 1$ we further define $j(x) = 0$ whenever $x \in [\frac{1}{\alpha+1}, \alpha)$; see also Remark 1(*i*). As usual, we set that $F_\alpha^0(x) = x$. Now the induced transformation $F_{\alpha,J}$ is defined as:

$$F_{\alpha,J}(x) = F_\alpha^{j(x)+1}(x) \quad \text{for } x \in \mathbb{I}_\alpha.$$

The next proposition generalizes the result in [30], where α was restricted to the interval $[\frac{1}{2}, 1]$.

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PROPOSITION 1. *For any $0 < \alpha \le 1$, we have $G_\alpha(x) = F_{\alpha,J}(x)$ for any $x \in \mathbb{I}_\alpha$.*

Proof. Since $F_\alpha(0) = 0$, $F_{\alpha,J}(0) = 0 = G_\alpha(0)$ is trivial. Next we consider the case $x \in [\alpha-1, 0)$. If $-\frac{1}{x} \in [(n-1)+\alpha, n+\alpha)$, then $F_\alpha(x) = (R \circ S_1 \circ R_-)(x) \in (\frac{1}{(n-1)+\alpha}, \frac{1}{(n-2)+\alpha}]$. Thus we get $F_\alpha^{n-1}(x) \in (\frac{1}{1+\alpha}, \frac{1}{\alpha}]$ inductively, and $j(x) = n-1$ in this case. We also see that $F_\alpha^{n-1}(x) = -\frac{x}{(n-1)x+1}$ and then $F_{\alpha,J}(x) = F_\alpha^{j(x)+1}(x) = F_\alpha\left(-\frac{x}{(n-1)x+1}\right) = \left(-\frac{1}{x} - (n-1)\right) - 1 = -\frac{1}{x} - n = G_\alpha(x)$. For $x \in (0, \alpha)$, the same proof holds since $F_\alpha(x) = (R \circ S_1 \circ R)(x) \in (\frac{1}{(n-1)+\alpha}, \frac{1}{(n-2)+\alpha}]$ when $\frac{1}{x} \in [(n-1)+\alpha, n+\alpha)$. The rest of the proof is straightforward. □

REMARK 1. We gather some results on $j(x)$ for various values of α.

(i) $\frac{\sqrt{5}-1}{2} < \alpha < 1$.
In this case $\frac{1}{1+\alpha} < \alpha$ holds. It is for this reason we defined $j(x) = 0$ for $x \in [\frac{1}{\alpha+1}, \alpha)$. On the other hand, $j(x) \ge 2$ for $\alpha - 1 \le x < 0$.

(ii) $0 < \alpha \le \frac{\sqrt{5}-1}{2}$.
In this case we have $\frac{1}{1+\alpha} \ge \alpha$ and $1+\alpha < \frac{1}{1-\alpha}$, which show that $j(x) = 0$ *only* for $x = 0$, and $j(x) = 1$ for $x \in [\alpha-1, -\frac{1}{1+\alpha}) \cup [\frac{1}{2+\alpha}, \alpha]$.

(iii) $0 < \alpha < \sqrt{2} - 1$.
We see that $j(x) \ge 2$ for $x \in (0, \alpha)$. △

Setting $A^- = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $A^+ = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $A^R = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, in view of (1.2) and (1.4) we can write F_α as:

$$F_\alpha(x) = \begin{cases} A^- x, & \text{if } x \in [\alpha-1, 0), \\ A^+ x, & \text{if } x \in [0, \frac{1}{1+\alpha}], \\ A^R x, & \text{if } x \in (\frac{1}{1+\alpha}, \frac{1}{\alpha}]. \end{cases}$$

Define

$$A_n(x) = A(F_\alpha^{n-1}(x)) = \begin{cases} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = (A^-)^{-1}, & \text{if } F_\alpha^{n-1}(x) \in [\alpha-1, 0), \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } F_\alpha^{n-1}(x) = 0, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (A^+)^{-1}, & \text{if } F_\alpha^{n-1}(x) \in (0, \frac{1}{1+\alpha}], \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (A^R)^{-1}, & \text{if } F_\alpha^{n-1}(x) \in (\frac{1}{1+\alpha}, \frac{1}{\alpha}], \end{cases}$$

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for $x \in \mathbb{I}_\alpha$ and $n \geq 1$. We identify x with $(A_1(x), A_2(x), \ldots, A_n(x), \ldots)$. We will show that

$$\lim_{n\to\infty} (A_1(x)A_2(x)\cdot\cdots A_n(x))(-\infty) = x.$$

Put $\begin{pmatrix} s_n & u_n \\ t_n & v_n \end{pmatrix} = A_1(x)A_2(x)\cdots A_n(x)$ with $\begin{pmatrix} s_0 & u_0 \\ t_0 & v_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Suppose that

$$x = \frac{\varepsilon_{\alpha,1}(x)|}{|a_{\alpha,1}(x)} + \frac{\varepsilon_{\alpha,2}(x)|}{|a_{\alpha,2}(x)} + \cdots + \frac{\varepsilon_{\alpha,n}(x)|}{|a_{\alpha,n}(x)} + \cdots$$

is the α-continued fraction expansion of $x \in \mathbb{I}_\alpha$. Recall that $\varepsilon_{\alpha,n}(x) = \mathrm{sgn}(G_\alpha^{n-1}(x))$ and $a_{\alpha,n}(x) = \lfloor \frac{1}{|G_\alpha^{n-1}(x)|} + 1 - \alpha \rfloor$ for $G_\alpha^{n-1}(x) \neq 0$. Then, from Proposition 1, it is easy to see that $(A_1(x), A_2(x), \ldots, A_n(x), \ldots)$ is of the form

$$(A^\pm, \underbrace{A^+, \ldots, A^+}_{a_{\alpha,1}(x)-2}, A^R, A^\pm, \underbrace{A^+, \ldots, A^+}_{a_{\alpha,2}(x)-2}, A^R, \ldots), \tag{2.1}$$

unless $A_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ appears in (2.1) for some $m \geq 1$ (which happens when $x \in \mathbb{Q}$). Here $A^\pm = A^-$ or A^+ according to $\varepsilon_{\alpha,n}(x) = -1$ or $+1$, respectively. If $a_{\alpha,n}(x) = 1$, then we read $a_{\alpha,n} - 2 = 0$ and delete $A^\pm$ before A^+. More precisely,

$$\begin{aligned} A_{\sum_{j=1}^n a_{\alpha,j}(x)}(x) &= A^R, \\ A_{\sum_{j=1}^n a_{\alpha,j}(x)+1}(x) &= \begin{cases} A^-, & \text{if} \quad \varepsilon_{\alpha,n}(x) = -1, \\ A^+, & \text{if} \quad \varepsilon_{\alpha,n}(x) = +1 \quad \text{and} \quad a_{\alpha,n+1}(x) \geq 2, \end{cases} \\ A_{\sum_{j=1}^n a_{\alpha,j}(x)+\ell}(x) &= A^+ \qquad \text{if} \;\; 2 \leq \ell < a_{\alpha,n+1}, \end{aligned}$$

with $\sum_{j=1}^0 a_{\alpha,j}(x) = 0$. As usual, we have for the G_α-convergents of x that:

$$\begin{pmatrix} 0 & \varepsilon_{\alpha,1}(x) \\ 1 & a_{\alpha,1}(x) \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{\alpha,2}(x) \\ 1 & a_{\alpha,2}(x) \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon_{\alpha,n}(x) \\ 1 & a_{\alpha,n}(x) \end{pmatrix} = \begin{pmatrix} p_{\alpha,n-1}(x) & p_{\alpha,n}(x) \\ q_{\alpha,n-1}(x) & q_{\alpha,n}(x) \end{pmatrix}.$$

We have the following result.

LEMMA 1. *For $k, n \in \mathbb{N}$, let $\Pi_k(x) := A_1(x)A_2(x)\cdots A_k(x)$, and $S_n(x) = \sum_{j=1}^n a_{\alpha,j}(x)$. Then, if $k = S_n(x)$,*

$$\Pi_k(x) = \begin{pmatrix} p_{\alpha,n-1}(x) & p_{\alpha,n}(x) \\ q_{\alpha,n-1}(x) & q_{\alpha,n}(x) \end{pmatrix}.$$

Furthermore, if $\ell \geq 1$ and $k = S_n(x) + \ell < S_{n+1}(x)$, we have

$$\Pi_k(x) = \begin{pmatrix} \ell p_{\alpha,n}(x) + \varepsilon_{\alpha,n+1}(x)p_{\alpha,n-1}(x) & p_{\alpha,n-1}(x) \\ \ell q_{\alpha,n}(x) + \varepsilon_{\alpha,n+1}(x)q_{\alpha,n-1}(x) & q_{\alpha,n-1}(x) \end{pmatrix}.$$

Proof. The assertion of this lemma follows from an easy induction and is essentially due to the fact that G_α is an induced transformation of F_α. □

With this result in mind, and analogously to the regular case (which is $\alpha = 1$), we call

$$(A_1(x)A_2(x)\cdots A_k(x))(-\infty) = \frac{\ell p_{\alpha,n}(x) + \varepsilon_{\alpha,n+1}p_{\alpha,n-1}(x)}{\ell q_{\alpha,n}(x) + \varepsilon_{\alpha,n+1}q_{\alpha,n-1}(x)}$$

the $((n+1,\ell)$th) α-mediant convergent of x for $\ell > 0$; see also (1.1) for the case $\alpha = 1$.

Remark 2. For $0 < \alpha \le \frac{\sqrt{5}-1}{2}$, $a_{\alpha,n}(x) \ge 2$ for any $x \in \mathbb{I}_\alpha$ and $n \ge 1$.

From Lemma 1, the convergence of the mediant convergents follows:

Proposition 2. *We have*

$$\lim_{k\to\infty} (\Pi_k(x))(-\infty) = x.$$

Proof. If x is rational, then the assertion follows easily. So we estimate $|x - (\Pi_k(x))(-\infty)|$ for an irrational x. We note that $(q_{\alpha,n}(x) : n \ge 1)$ is strictly increasing for any $x \in \mathbb{I}_\alpha \setminus \mathbb{Q}$, which follows from the fact that $a_{\alpha,n}(x) \ge 2$ if $\varepsilon_{\alpha,n}(x) = -1$ (for any α, $0 < \alpha \le 1$). This implies $\lim_{n\to\infty} q_{\alpha,n}(x) = \infty$ if x is irrational. As for the RCF, see e.g. [10] or (1.1.14) in [16], x can be written as:

$$\frac{(\ell p_{\alpha,n}(x) \pm p_{\alpha,n-1}(x))F_\alpha^k(x) + p_{\alpha,n-1}(x)}{(\ell q_{\alpha,n}(x) \pm q_{\alpha,n-1}(x))F_\alpha^k(x) + q_{\alpha,n-1}(x)} \text{or} \quad \frac{p_{\alpha,n-1}(x)G_\alpha^n(x) + p_{\alpha,n}(x)}{q_{\alpha,n-1}(x)G_\alpha^n(x) + q_{\alpha,n}(x)}. \tag{2.2}$$

The estimate of the latter is easy since $(\Pi_k(x))(-\infty) = \frac{p_{\alpha,n-1}(x)}{q_{\alpha,n-1}(x)}$, $|G_\alpha^n(x)| < \max(\alpha, 1-\alpha) < 1$, and $|p_{\alpha,n-1}(x)q_{\alpha,n}(x) - p_{\alpha,n}(x)q_{\alpha,n-1}(x)| = 1$. Anyway, it is the convergence estimate of the α-continued fraction expansion of x. In the former case, we see that $|x - (\Pi_k(x))(-\infty)|$ is equal to:

$$\left| \frac{(\ell p_{\alpha,n}(x) \pm p_{\alpha,n-1}(x))F_\alpha^k(x) + p_{\alpha,n-1}(x)}{(\ell q_{\alpha,n}(x) \pm q_{\alpha,n-1}(x))F_\alpha^k(x) + q_{\alpha,n-1}(x)} - \frac{\ell p_{\alpha,n}(x) \pm p_{\alpha,n-1}(x)}{\ell q_{\alpha,n}(x) \pm q_{\alpha,n-1}(x)} \right|,$$

with $k = \sum_{j=1}^n a_{\alpha,j}(x) + \ell$. This can be estimated as

$$\begin{aligned} &\left| \frac{\ell}{((\ell q_{\alpha,n}(x) \pm q_{\alpha,n-1}(x))F_\alpha^k(x) + q_{\alpha,n-1}(x))\,(\ell q_{\alpha,n}(x) \pm q_{\alpha,n-1}(x))} \right| \\ &= \left| \frac{1}{\left(q_{\alpha,n}(x) \pm \frac{q_{\alpha,n-1}(x)}{\ell}\right)((\ell q_{\alpha,n}(x) \pm q_{\alpha,n-1}(x))F_\alpha^k(x) + q_{\alpha,n-1}(x))} \right| \\ &< \frac{1}{q_{\alpha,n-1}(x)} \quad \to \quad 0 \end{aligned}$$

Here we used the fact $F_\alpha^k(x) \ge 0$ for k not of the form $\sum_{j=1}^n a_{\alpha,j}(x)$. □

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As mentioned in the introduction, the $(n+1, a_{\alpha,n+1}(x)-1)$th convergent is the same as the $(n+2,1)$th convergent if $\varepsilon_{\alpha,n+2}(x) = -1$, i.e.,

$$\frac{(a_{\alpha,n+1}(x)-1)p_{\alpha,n}(x) + \varepsilon_{\alpha,n+1}(x)p_{\alpha,n-1}(x)}{(a_{\alpha,n+1}(x)-1)q_{\alpha,n}(x) + \varepsilon_{\alpha,n+1}(x)q_{\alpha,n-1}(x)} = \frac{p_{\alpha,n+1}(x) - p_{\alpha,n}(x)}{q_{\alpha,n+1}(x) - q_{\alpha,n}(x)}.$$

In this sense, the map F_α makes a duplication if $\varepsilon_{\alpha,n}(x) = -1$. This duplication is k-fold if $(\varepsilon_{\alpha,n+\ell}(x), a_{\alpha,n+\ell}(x)) = (-1, 2)$ for $1 \le \ell \le k$, i.e.,

$$\begin{aligned} \frac{p_{\alpha,n}(x) - p_{\alpha,n-1}(x)}{q_{\alpha,n}(x) - q_{\alpha,n-1}(x)} &= \frac{p_{\alpha,n+1}(x) - p_{\alpha,n}(x)}{q_{\alpha,n+1}(x) - q_{\alpha,n}(x)} = \cdots \\ &= \frac{p_{\alpha,n+k}(x) - p_{\alpha,n+k-1}(x)}{q_{\alpha,n+k}(x) - q_{\alpha,n+k-1}(x)}. \end{aligned}$$

We avoid this duplication using a suitable induced transformation. First, let us recall the definition of $F_{\alpha,\flat}$ for $\frac{1}{2} \le \alpha < 1$; see (1.5). One can see that this map skips the $(n, a_{n+1}(x)-1)$th mediant convergent of $x \in \mathbb{I}_\alpha$ with $\varepsilon_{n+1}(x) = -1$. An important observation in [30] is that for $\frac{1}{2} \le \alpha < 1$ we have that $-\frac{x}{1+x} < 1$ for any $\alpha - 1 \le x < 0$. This does not apply anymore when $0 < \alpha < \frac{1}{2}$, as the definition of $F_{\alpha,\flat}$ should be on the interval $[\alpha-1, 1]$. To achieve this we “speed up” F_α and modify the definition of $F_{\alpha,\flat}$ as follows:

$$F_{\alpha,\flat}(x) = F_\alpha^{K(x)}(x),$$

with $K(x) = \min\{k \ge 1 : F_\alpha^k(x) \in [\alpha-1, 1]$.

For $\alpha - 1 \le x < -\frac{1}{2}$, $F_\alpha(x) = -\frac{x}{1+x} > 1$ (see also (1.4)), so $F_\alpha^2(x) \in \mathbb{I}_\alpha$. Thus $K(x) = 2$ and we have that $F_\alpha^{K(x)}(x) = \frac{1+2x}{x}$ in this case. For $x \in [-\frac{1}{2}, \frac{1}{2})$, one can easily see that $F_\alpha(x) \in [0,1]$ and the same holds also for $x \in [\frac{1}{1+\alpha}, 1]$. For $x \in [\frac{1}{2}, \frac{1}{1+\alpha})$, we find that $K(x) = 2$ and $F_{\alpha,\flat}(x) = \frac{1-2x}{x}$. Consequently, our new definition of $F_{\alpha,\flat}$ is the following:

$$F_{\alpha,\flat}(x) = \begin{cases} F_\alpha^2(x) = -\frac{1+2x}{x}, & \text{if } \alpha - 1 \le x < -\frac{1}{2}, \\ F_\alpha(x) = -\frac{x}{1+x}, & \text{if } -\frac{1}{2} \le x < 0, \\ F_\alpha(x) = \frac{x}{1-x}, & \text{if } 0 \le x < \frac{1}{2}, \\ F_\alpha^2(x) = \frac{1-2x}{x}, & \text{if } \frac{1}{2} \le x < \frac{1}{1+\alpha}, \\ F_\alpha(x) = \frac{1-x}{x}, & \text{if } \frac{1}{1+\alpha} \le x \le 1. \end{cases} \tag{2.3}$$

Clearly from (2.3) we have that for every $x \in [\alpha-1, 1]$ the sequence $(F_{\alpha,\flat}^k(x))_{k\ge0}$, which is the orbit of x under $F_{\alpha,\flat}$, is a subsequence of the sequence $(F_\alpha^n(x))_{n\ge0}$ (the orbit of x under F_α). But then for every $x \in [\alpha-1, 1]$ fixed there exists a (unique) monotonically increasing function $\hat{k} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, such that $F_{\alpha,\flat}^k(x) = F_\alpha^{\hat{k}(k)}(x)$. Setting $\hat{k} = \hat{k}(k)$ for $k = 0, 1, \ldots,$ we put

$$\Pi_{\flat,k}(x) = \Pi_{\hat{k}}(x).$$

From this definition, it is easy to derive the following.

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PROPOSITION 3. *For any $k \geq 1$, whenever $\varepsilon_{\alpha,n+1}(x) = -1$, the $(n+1, a_{n+1}-1)$th mediant convergent does not appear for any $n \geq 1$ in $(\Pi_{\flat,k}(x)(-\infty) : k \geq 1)$ and all other mediant convergents and all principal convergents of* x *appear in it.*

Another possibility is by skipping $p_{n+1}(x) - p_n(x)$ and $q_{n+1}(x) - q_n(x)$ instead of $(a_{n+1}(x)-1))p_n(x)+\varepsilon_{n+1}(x)p_{n-1}(x)$ and $(a_{n+1}(x)-1))q_n(x)+\varepsilon_{n+1}(x)q_{n-1}(x)$ if $\varepsilon_{n+1}(x) = -1$. This can be done by the jump transformation $F_{\alpha,\sharp}$, defined as follows:

$$F_{\alpha,\sharp}(x) = \begin{cases} F_\alpha^2(x), & \text{if } x < 0, \\ F_\alpha(x), & \text{if } x \geq 0. \end{cases} \tag{2.4}$$

Note that the map $F_{\alpha,\sharp}$ from (2.4) is explicitly given by

$$F_{\alpha,\sharp}(x) = \begin{cases} -\frac{x}{1+2x}, & \text{if } \alpha - 1 \leq x < 0, \\ \frac{x}{1-x}, & \text{if } 0 \leq x < \frac{1}{1+\alpha}, \\ \frac{1-x}{x}, & \text{if } \frac{1}{1+\alpha} \leq x \leq \frac{1}{\alpha}. \end{cases} \tag{2.5}$$

This is well-defined for any $0 < \alpha < 1$. Indeed, the map $F_{\alpha,\sharp}$ from (2.5) skips $F_\alpha^{k+1}(x)$ if $F_\alpha^k(x) < 0$, which implies that there exists an $n \geq 1$ such that $G_\alpha^n(x) = F_\alpha^k(x)$ and $\varepsilon_n(x) = -1$. Thus we see that $\frac{p_n(x)-p_{n-1}(x)}{q_n(x)-q_{n-1}(x)}$ has been skipped in the sequence of the mediant convergents. In this note, we do not further consider this map $F_{\alpha,\sharp}$ since the discussion is almost the same as that of $F_{\alpha,\flat}$.

Now we consider $\frac{s_k(x)}{t_k(x)} = \Pi_k(x)(-\infty)$ with $s_k(x), t_k(x) \in \mathbb{Z}$, coprime, $t_k > 0$. From this and (2.2) we derive that

$$t_k^2(x) \left| x - \frac{s_k(x)}{t_k(x)} \right| = \left| F_\alpha^k(x) - \Pi_k^{-1}(-\infty) \right|^{-1}$$

for $x \in [\alpha - 1, \frac{1}{\alpha}]$ and $n \geq 1$. Note that $F_\alpha^k(x)$ can be interpreted as *the future of* x *at time* k, while $\Pi_k^{-1}(-\infty)$ is like *the past of* x *at time* k; see also Chapter 4 in [10]. For this reason, it is interesting to find the closure of the set

$$\left\{ \left(F_\alpha^k(x), \Pi_k^{-1}(x)(-\infty) \right) : \ x \in [\alpha - 1, \frac{1}{\alpha}], \ k > 0 \right\}.$$

This leads us to consider the following maps:

$$\hat{F}_\alpha(x,y) = \begin{cases} \left(-\frac{x}{1+x}, -\frac{y}{1+y} \right), & \text{if } \alpha - 1 \leq x < 0, \\ \left(\frac{x}{1-x}, \frac{y}{1-y} \right), & \text{if } 0 \leq x < \frac{1}{1+\alpha}, \\ \left(\frac{1-x}{x}, \frac{1-y}{y} \right), & \text{if } \frac{1}{1+\alpha} \leq x < \frac{1}{\alpha}, \end{cases} \tag{2.6}$$

and

$$\hat{F}_{\alpha,\flat}(x,y) = \begin{cases} \left(-\frac{1+2x}{x}, -\frac{1+2y}{y}\right), & \text{if } \alpha - 1 \le x < -\frac{1}{2}, \\ \left(-\frac{x}{1+x}, -\frac{y}{1+y}\right), & \text{if } -\frac{1}{2} \le x < 0, \\ \left(\frac{x}{1-x}, \frac{y}{1-y}\right), & \text{if } 0 \le x < \frac{1}{2}, \\ \left(\frac{1-2x}{x}, \frac{1-2y}{y}\right), & \text{if } \frac{1}{2} \le x \le \frac{1}{1+\alpha}, \\ \left(\frac{1-x}{x}, \frac{1-y}{y}\right), & \text{if } \frac{1}{1+\alpha} < x \le 1, \end{cases} \tag{2.7}$$

where (x, y) is in a 'reasonable domain' of the definition of each map, respectively. The question is to find this 'reasonable domain' for each case. This will be done in Theorem 1 for $\hat{F}_\alpha$ and in Theorem 2 for $\hat{F}_{\alpha,\flat}$. For example, for $\hat{F}_\alpha$ the domain will be the closure of

$$\left\{ \left(F_\alpha^k(x), (A_1(x) \cdots A_k(x))^{-1}(-\infty)\right) : \ x \in \left[\alpha - 1, \frac{1}{\alpha}\right], \ k > 0 \right\}$$

so that $\hat{F}_\alpha$ is bijective except for a set of Lebesgue measure *0*. From this point of view, $\hat{F}_\alpha$ is the planar representation of the natural extension of F_α in the sense of Ergodic theory. Another point of view is that the characterization of quadratic surds by the periodicity of the map. Indeed, it is easy to see that $x \in (0, 1)$ is strictly periodic by the iteration of F if and only if it is a quadratic surd and its algebraic conjugate is negative. We can characterize the set of quadratic surds in a similar way with $\hat{G}^*_\alpha$, see the next section, and also $\hat{F}_\alpha$. We can apply the above to the construction of the natural extension of $F_{\alpha,\flat}$. Indeed, it is obtained as an induced transformation of $\hat{F}_\alpha$. In the next section, we give a direct construction of the natural extension of $\hat{F}_\alpha$ as a tower of the natural extension $\hat{G}^*_\alpha$ of G_α.

3. The natural extension of F_α for $0 < \alpha < \frac{1}{2}$

As the case $\frac{1}{2} \le \alpha \le 1$ was discussed in [30, 31], in the rest of this paper we will focus on the case $0 < \alpha < \frac{1}{2}$. We give some figures in the case of $\alpha = \sqrt{2} - 1$ for better understanding of the construction. We selected this value of α as an example as this is historically the first "more difficult" case; for $\alpha \in (\sqrt{2} - 1, 1]$ the natural extensions are simply connected regions which are the union of finitely many overlapping rectangles, while for $\alpha = \sqrt{2} - 1$ the natural extension consists of two disjoint rectangles; see [12, 24, 25]. In [12] it is shown that for $\alpha \in \left(\frac{\sqrt{10}-3}{2}, \sqrt{2} - 1\right)$ there is a countably infinite number of disjoint connected regions. For $0 < \alpha < \sqrt{2} - 1$ it is not so easy to describe Ω_α explicitly; see the discussion at the end of § 2, and also [12, 23–25].

We start with the domain Ω_α from [23], given by the closure

$$\Omega_\alpha = \overline{\left\{ \hat{G}^n_\alpha(x, -\infty) \,\middle|\, x \in [\alpha - 1, \alpha) \right\}},$$

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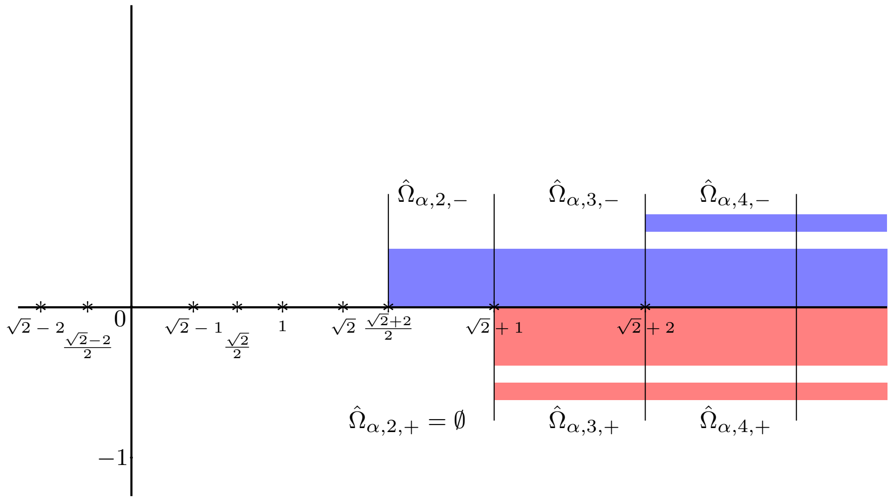

Figure 2. $\hat{\Omega}_{\alpha,k,\pm}$ for $\alpha = \sqrt{2}-1$

and the natural extension map $\hat{G}_\alpha : \Omega_\alpha \to \Omega_\alpha$, defined by

$$(x, y) \mapsto \begin{cases} \left(-\frac{1}{x} - b, \frac{1}{-y+b}\right), & \text{if } x < 0, \\ \left(\frac{1}{x} - b, \frac{1}{y+b}\right), & \text{if } x > 0, \end{cases}$$

for $(x, y) \in \Omega_\alpha$.

Next we change y to $-\frac{1}{y}$, i.e., we consider

$$\Omega_\alpha^* = \left\{(x, y) : \left(x, -\frac{1}{y}\right) \in \Omega_\alpha\right\},$$

see Figure 3, and the map $\hat{G}_\alpha^* : \Omega_\alpha^* \to \Omega_\alpha^*$, defined by:

$$(x, y) \mapsto \begin{cases} \left(-\frac{1}{x} - b, -\frac{1}{y} - b\right), & \text{if } x < 0, \\ \left(\frac{1}{x} - b, \frac{1}{y} - b\right), & \text{if } x > 0, \end{cases}$$

where $b = \lfloor \left|\frac{1}{x}\right| + \alpha - 1 \rfloor$; this gives another version of the natural extension, with which we work with for the rest of the paper. Recall from [23] that $\hat{G}_\alpha : \Omega_\alpha \to \Omega_\alpha$ (and therefore $\hat{G}_\alpha^*$) is bijective except for a set of Lebesgue measure *0*. The reason to move from Ω_α to Ω_α^* is that the first and second coordinate maps of G_α are similar. This allows for a more unified treatment. Also note that $\Omega_\alpha \subset [\alpha - 1, \alpha] \times [0, 1]$.

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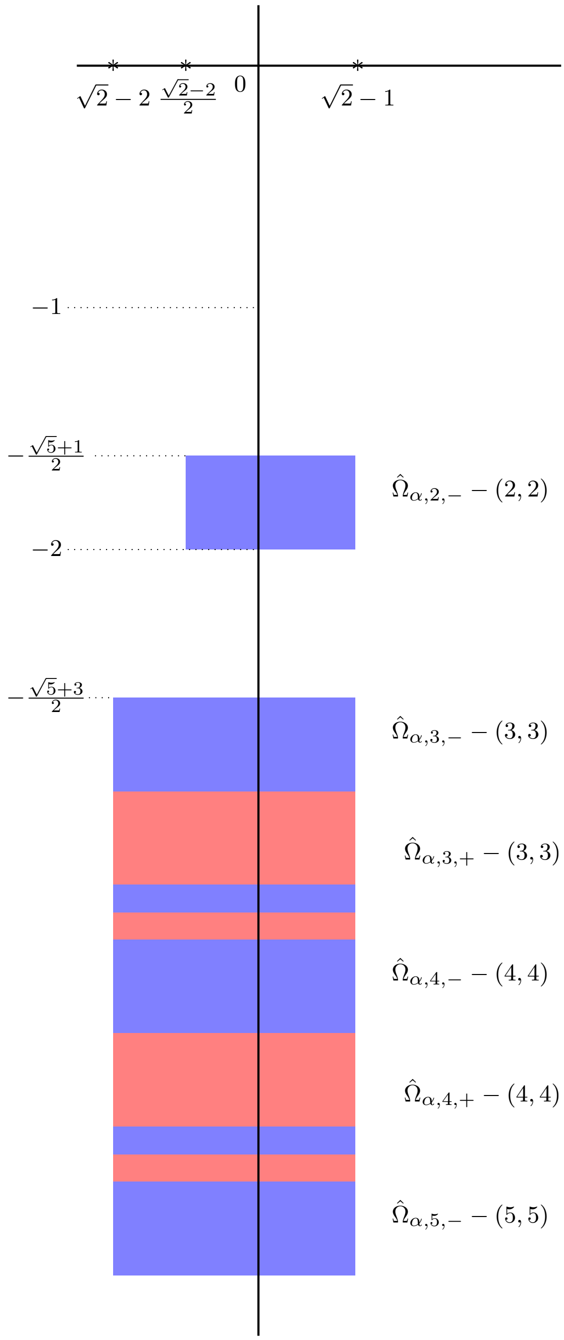

Figure 3. Ω_α^* for $\alpha = \sqrt{2}-1$

Although formally $\alpha \notin \mathbb{I}_\alpha$, we define k_0 as the first digit of α in the G_α-expansion of α, i.e., $\frac{1}{k_0+\alpha} \leq \alpha < \frac{1}{k_0-1+\alpha}$. Furthermore, we define cylinders by

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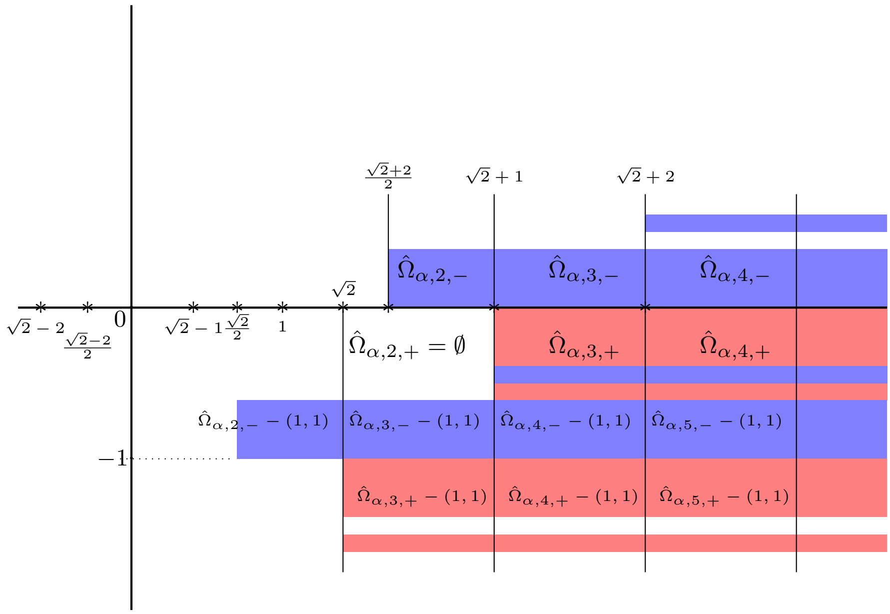

Figure 4. $\hat{\Omega}_{\alpha,k,\pm}-(1,1)$ for $\alpha=\sqrt{2}-1$

$$\begin{cases} \Omega^*_{\alpha,k_0,+} &= \{(x,y)\in\Omega^*_\alpha : \frac{1}{k_0+\alpha}<x<\alpha\}, \\ \Omega^*_{\alpha,k,+} &= \{(x,y)\in\Omega^*_\alpha : \frac{1}{k+\alpha}<x\le\frac{1}{(k-1)+\alpha}\},\ \text{if } k>k_0, \\ \Omega^*_{\alpha,2,-} &= \{(x,y)\in\Omega^*_\alpha : \alpha-1<x\le-\frac{1}{2+\alpha}\}, \\ \Omega^*_{\alpha,k,-} &= \{(x,y)\in\Omega^*_\alpha : -\frac{1}{(k-1)+\alpha}<x\le-\frac{1}{k+\alpha}\},\ \text{if } k\ge 3. \end{cases} \tag{3.1}$$

Then we put

$$\begin{cases} \hat{\Omega}_{\alpha,k,-} &= \left\{(x,y) : \left(-\frac{1}{x},-\frac{1}{y}\right)\in\Omega^*_{\alpha,k,-}\right\}, \\ \hat{\Omega}_{\alpha,k,+} &= \left\{(x,y) : \left(\frac{1}{x},\frac{1}{y}\right)\in\Omega^*_{\alpha,k,+}\right\}. \end{cases}$$

Note that if $(x,y)\in\hat{\Omega}_{\alpha,k,-}$ we have that $x>0$ and $y\ge 0$; see Figure 2. For convenience we put $\Omega^*_{\alpha,k,+}=\emptyset$ for $2\le k<k_0$. It is easy to see that

$$\Omega^*_\alpha = \left(\bigcup_{k=2}^{\infty}(\hat{\Omega}_{\alpha,k,-}-(k,k))\right)\cup\left(\bigcup_{k=k_0}^{\infty}(\hat{\Omega}_{\alpha,k,+}-(k,k))\right) \quad \text{(disj. a.e.)},$$

where (disj. a.e.) means "disjoint except for a set of measure 0". This disjointness follows from the next lemma.

Lemma 2. *For every $k\in\mathbb{N}$, $k\ge 2$, we have:*

$$\left(\hat{\Omega}_{\alpha,k+1,-}-(k+1,k+1)\right)\cap\left(\hat{\Omega}_{\alpha,k,+}-(k,k)\right)=\emptyset \quad \textit{disj. a.e.},$$

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or equivalently,

$$\left(\hat{\Omega}_{\alpha,k+1,-} - (1,1)\right) \cap \left(\hat{\Omega}_{\alpha,k,+} - (0,0)\right) = \emptyset \quad \textit{disj. a.e.};$$

see Figure 4.

Proof. We see

$$\left(\hat{\Omega}_{\alpha,k,\pm} - (k,k)\right) = \hat{G}^*_\alpha \left(\Omega^*_{\alpha,k,\pm}\right).$$

Then the assertion follows from the a.e.-bijectivity of $\hat{G}^*_\alpha$. □

For $j \geq 1$, we define

$$\Upsilon_{\alpha,j} = \bigcup_{k=j+1}^{\infty} \left(\hat{\Omega}_{\alpha,k,-} - (k-j, k-j)\right) \cup \bigcup_{k=j+1}^{\infty} \left(\hat{\Omega}_{\alpha,k,+} - (k-j, k-j)\right)$$

for $j \geq 2$, see Figure 5, and

$$\Upsilon_\alpha = \bigcup_{j=1}^{\infty} \Upsilon_{\alpha,j}.$$

From Lemma 2, this is "disj. a.e." We also see

$$\Upsilon_{\alpha,j} \cap \hat{\Omega}_{\alpha,j,+} = \emptyset \qquad \text{(disj. a.e.)},$$

which implies

$$\Omega^*_\alpha \cap (\Upsilon_\alpha)^{-1} = \emptyset \qquad \text{(disj. a.e.)},$$

where $(\Upsilon_\alpha)^{-1} = \left\{(x,y) : \left(\frac{1}{x}, \frac{1}{y}\right) \in \Upsilon_\alpha\right\}$. Note that $(\Upsilon_\alpha)^{-1} \subset \{(x,y) : x > 0\}$.

Now we will define the 'reasonable domain' V_α for the natural extension map $\hat{F}_\alpha$ from (2.6). We put $V_\alpha = \Omega^*_\alpha \cup (\Upsilon_\alpha)^{-1}$; see Figure 6. From the construction of V_α, it is not hard to see the following result.

THEOREM 1 *The dynamical system $(V_\alpha, \hat{F}_\alpha)$ together with the measure μ_α with density $\frac{dxdy}{(x-y)^2}$ is a representation of the natural extension of $([\alpha-1, \frac{1}{\alpha}), F_\alpha)$ with measure ν_α, which is the projection of μ_α on the first coordinate.*

Proof. We show the following below. Then the assertion of the theorem is proved in exactly the same way as in [31] in the case of $\frac{1}{2} \leq \alpha \leq 1$.

(i) The map $\hat{F}_\alpha$ defined on V_α is surjective.
(ii) The map $\hat{F}_\alpha$ is bijective except for a set of measure *0*.
(iii) The measure $\frac{dxdy}{(x-y)^2}$ is the absolutely continuous ergodic invariant measure.
(iv) The Borel σ-algebra $\mathcal{B}(V_\alpha)$ on V_α satisfies:

$$\mathcal{B}(V_\alpha) = \sigma\left(\bigvee_{n=0}^{\infty} \hat{F}^n_\alpha \pi_1^{-1} \mathcal{B}([\alpha-1,\alpha))\right),$$

where $\mathcal{B}([\alpha-1,\alpha))$ is the Borel σ-algebra on $[\alpha-1,\alpha)$ and $\pi_1 : V_\alpha \to [\alpha-1,\alpha)$ is the projection on the first coordinate.

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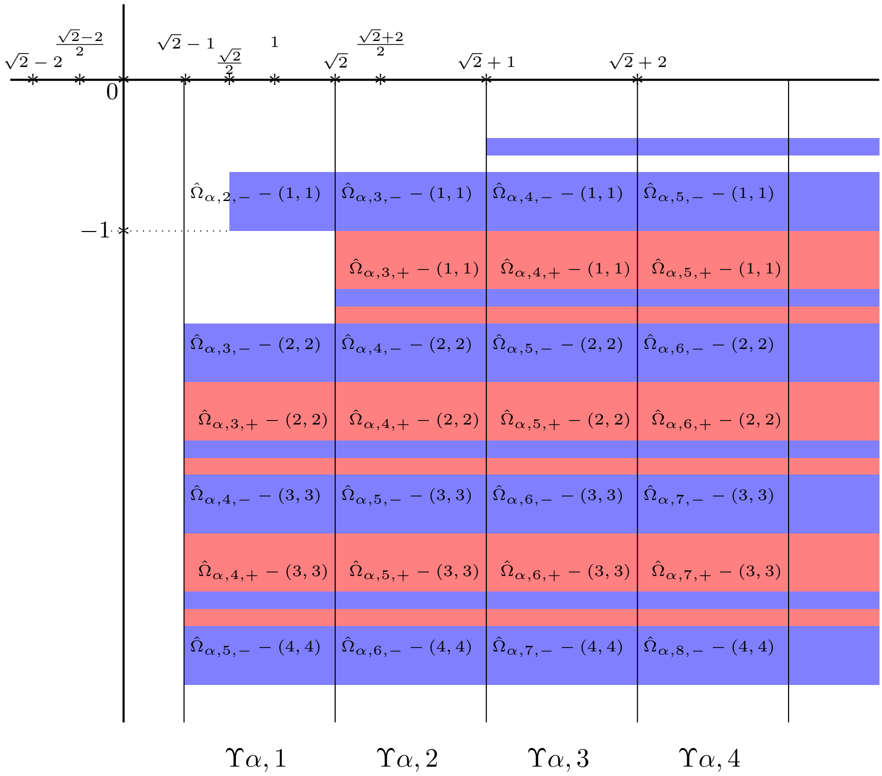

Figure 5. $\hat{\Omega}_{\alpha,k,\pm}-(\ell,\ell)$ and $\Upsilon_{\alpha,k}$ for $\alpha=\sqrt{2}-1$

For a.e. $(x,y)\in\Omega_\alpha^*$, $x\neq 0$, there exists a unique element $(x_0,y_0)\in\Omega_\alpha^*$ and a positive integer k such that

$$(x,y)=\hat{G}_\alpha^*(x_0,y_0)=\begin{cases}\left(-\frac{1}{x_0}-k,-\frac{1}{y_0}-k\right), & \text{if } x_0<0,\\ \left(\frac{1}{x_0}-k,\frac{1}{y_0}-k\right), & \text{if } x_0>0,\end{cases}$$

since $(\Omega_\alpha^*,\hat{G}_\alpha^*)$ is a natural extension of $(\mathbb{I}_\alpha,G_\alpha)$; see [23].

If $x_0<0$, then we see

$$\left(-\frac{1}{x_0}-(k-1),-\frac{1}{y_0}-(k-1)\right)\in\Upsilon_{\alpha,1}.$$

This implies $\alpha\leq-\frac{1}{x_0}-(k-1)<\alpha+1$. We put

$$(x_1,y_1)=\left(\left(-\frac{1}{x_0}-(k-1)\right)^{-1},\left(-\frac{1}{y_0}-(k-1)\right)^{-1}\right),$$

and have $\frac{1}{1+\alpha}<x_1\leq\frac{1}{\alpha}$. From (2.6), we have that $(x,y)=\hat{F}_\alpha(x_1,y_1)$.

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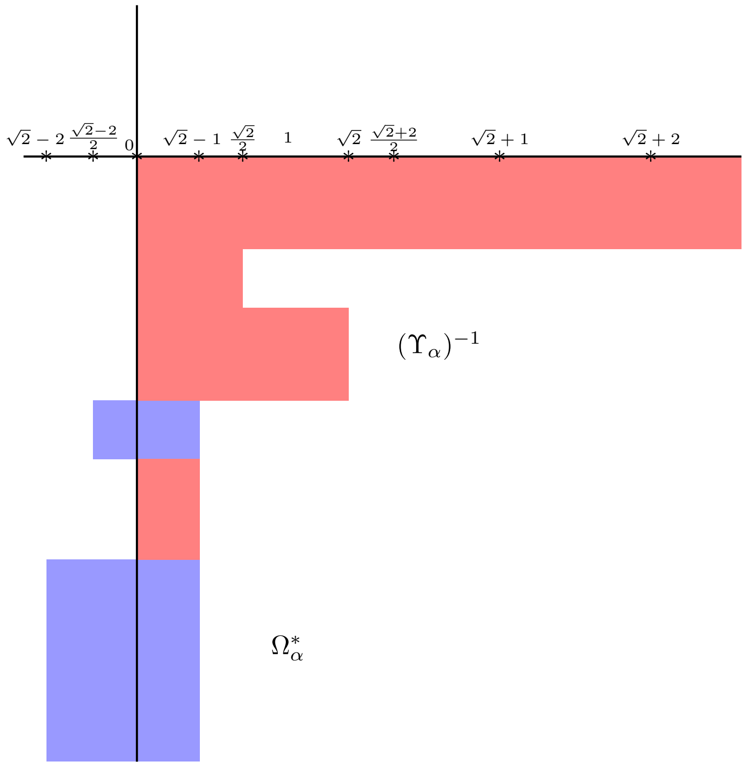

Figure 6. $V_\alpha = \Omega^*_\alpha \cup (\Upsilon_\alpha)^{-1}$ for $\alpha = \sqrt{2}-1$

If $x_0 > 0$, then

$$\left(\frac{1}{x_0} - (k-1), \frac{1}{y_0} - (k-1)\right) \in \Upsilon_{\alpha,1}$$

and $(x, y) = \hat{F}_\alpha(x_1, y_1)$ with

$$(x_1, y_1) = \left(\left(\frac{1}{x_0} - (k-1)\right)^{-1}, \left(\frac{1}{y_0} - (k-1)\right)^{-1}\right).$$

We note that in both cases we have

$$(x_1, y_1) \in \Upsilon^{-1}_{\alpha,1} \subset (\Upsilon_\alpha)^{-1}. \tag{3.2}$$

Next we consider the case $(x, y) \in \Upsilon^{-1}_\alpha$. This means $\left(\frac{1}{x}, \frac{1}{y}\right) \in \Upsilon_{\alpha,j}$ for some $j \geq 1$. We consider two cases.

Case (a): $\left(\frac{1}{x}, \frac{1}{y}\right) \in \hat{\Omega}_{\alpha,j+1,\pm} - (1, 1)$.

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In this case, there exists $(x_0, y_0) \in \hat{\Omega}_{\alpha,j+1,\pm}$ such that $\left(\frac{1}{x}, \frac{1}{y}\right) = (x_0 - 1, y_0 - 1)$. Thus $(x_1, y_1) = \left(\pm\frac{1}{x_0}, \pm\frac{1}{y_0}\right) \in \Omega^*_{\alpha,j+1}$, which imlies

$$\frac{1}{j+1+\alpha} < \pm x_1 \leq \frac{1}{j+\alpha} \leq \frac{1}{1+\alpha}.$$

Hence we find that $(x, y) = \hat{F}_\alpha(x_1, y_1)$. Here we see

$$(x_1, y_1) \in \Omega^*_\alpha. \tag{3.3}$$

Case (b): $\left(\frac{1}{x}, \frac{1}{y}\right) \in \hat{\Omega}_{\alpha,k,\pm} - (k-j, k-j)$ for $k > j+1$.

In this case, we have

$$(x_0, y_0) := \left(\frac{1}{x} + 1, \frac{1}{y} + 1\right) \in \Omega_{\alpha,k\pm} - (k-j-1, k-j-1) \tag{3.4}$$

and then

$$(x_1, y_1) := \left(\tfrac{1}{x_0}, \tfrac{1}{y_0}\right) \in \Upsilon^{-1}_\alpha. \tag{3.5}$$

This shows

$$(x, y) = \hat{F}_\alpha(x_1, y_1).$$

Consequently, we have the first statement. The second statement follows from (3.2), (3.3), (3.4) and (3.5). The third statement is also easy to obtain. Indeed, it is well-known that the measure given here is the absolutely continuous invariant measure for the direct product of the same linear fractional transformation. Because $\hat{G}^*_\alpha$ is an induced transformation of $\hat{F}_\alpha$ to Ω^*_α, the ergodicity of $\hat{F}_\alpha$ follows from that of $\hat{G}^*_\alpha$. The ergodicity of the latter is equivalent to that of G_α and it was proved by Luzzi and Marmi in [24]. For the last statement, note that (2.1) allows one to identify a point x with a one-sided sequence of matrices with entries $A^\pm$, A^R. As a result, F_α, $\hat{F}_\alpha$ can be seen as a one-sided resp. two sided shifts, from which the fourth statement follows. □

4. The natural extension of $F_{\alpha,\flat}$ for $0 < \alpha < \frac{1}{2}$

In the case of $\alpha = 1$, F_1 is the original Farey map F. We recall that

$$\hat{F}(x, y) = \begin{cases} \left(\frac{x}{1-x}, \frac{y}{1-y}\right), & \text{if } 0 \leq x < \frac{1}{2}, \\ \left(\frac{1-x}{x}, \frac{1-y}{y}\right), & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

defined on $V_1 = \{(x, y) : 0 \leq x \leq 1,\ -\infty \leq y \leq 0\}$ is the natural extension of F with the invariant measure $\hat{\mu}_1$ defined by $d\hat{\mu}_1 = \frac{dx\,dy}{(x-y)^2}$. In particular, $\hat{F}$ is bijective on V_1 except for a set of Lebesgue measure *0*. It is easy to see that F_1 and $F_{1,\flat}$ are the same. In the case of $\frac{1}{2} \leq \alpha < 1$, the complete description was given in [31]. Here we consider the case $0 < \alpha < \frac{1}{2}$ as a continuation of the previous section.

We put $V_{\alpha,\flat} = \{(x, y) \in V_\alpha,\ x \leq 1\}$ and consider the induced transformation $\hat{F}_{\alpha,\flat}$ of $\hat{F}_\alpha$ to $V_{\alpha,\flat}$; see Figure 7.

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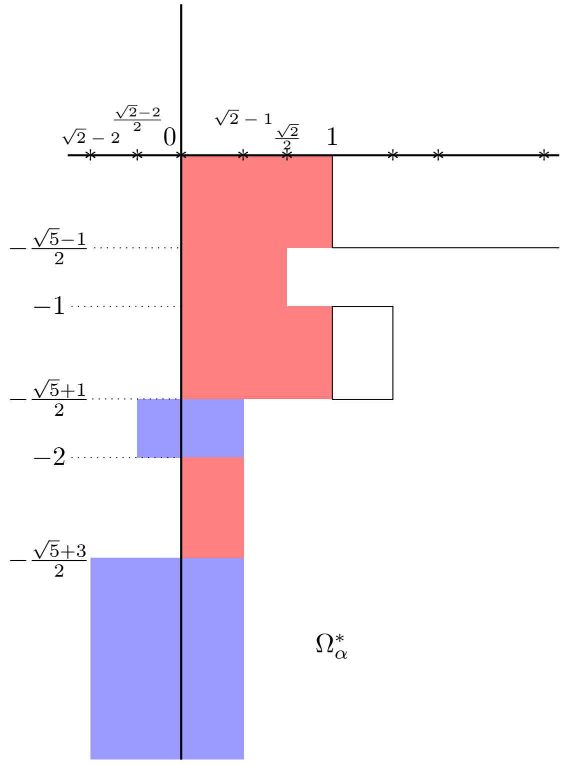

Figure 7. $V_{\alpha,\flat} = V_\alpha \cap \{(x, y) : x \le 1\}$ for $\alpha = \sqrt{2} - 1$

Recall the definition of the map $\hat{F}_{\alpha,\flat}$, as given in (2.7).

THEOREM 2 *The dynamical system $(V_{\alpha,\flat}, \hat{F}_{\alpha,\flat}, \mu_{\alpha,\flat})$ is a representation of the natural extension of $([\alpha - 1, 1], F_{\alpha,\flat}, \nu_{\alpha,\flat})$. Here the invariant measure $\mu_{\alpha,\flat}$ has density $\frac{dxdy}{(x-y)^2}$ on $V_{\alpha,\flat}$, and $\nu_{\alpha,\flat}$ is the projection of $\mu_{\alpha,\flat}$ on the first coordinate.*

Proof. Recall that $F_{\alpha,\flat}(x) = F_\alpha^{K(x)}(x)$ with $K(x) = \min\{K \ge 1 : F_\alpha^K(x) \in [\alpha - 1, 1]\}$, and for $(x, y) \in V_{\alpha,\flat}$, one has $x \in [\alpha - 1, 1]$. Since the first coordinate of $\hat{F}_\alpha(x, y)$ is $F_\alpha(x)$, we find $\hat{F}_{\alpha,\flat}(x, y) = \hat{F}_\alpha^{K(x)}(x, y)$. Here, we note that the first coordinate is $F_\alpha(x)$. Because of the general fact that an induced transformation of a bijective map is bijective, we see that $\hat{F}_{\alpha,\flat}$ is bijective. The rest of the proof follows from a standard argument. □

Put

$$\begin{cases} D_1 = (V_{\alpha,-} + (1,1)) & (\subset V_1) \\ D_2 = V_{\alpha,+} \setminus D_1 & (\subset V_1) \end{cases}$$

with $V_{\alpha,-} = \{(x, y) \in V_{\alpha,\flat} : x < 0\}$ and $V_{\alpha,+} = \{(x, y) \in V_{\alpha,\flat} : x \ge 0\}$. We write $D = D_1 \cup D_2$. By the definition we see that $D \subset V_1$. We define $\psi : D \to V_\alpha$ by

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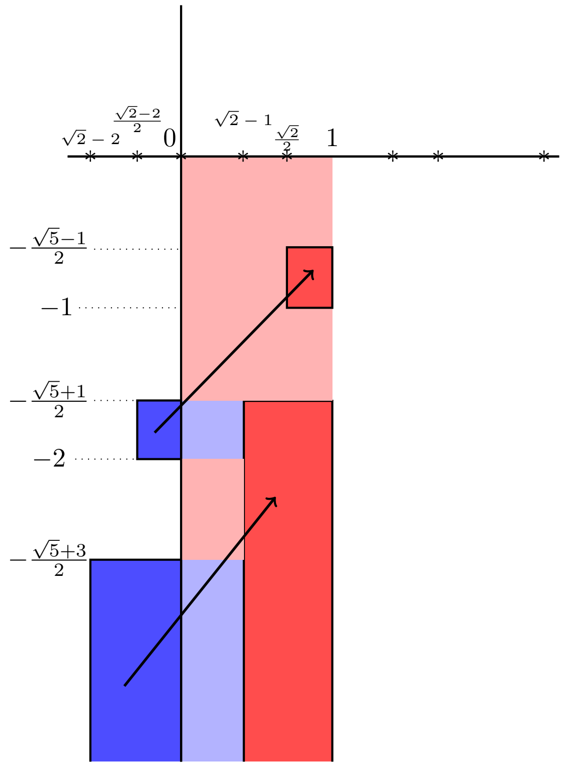

Figure 8. ψ^{-1} for $\alpha = \sqrt{2}-1$

$$\psi(x,y) = \begin{cases} (x-1, y-1), & \text{if } (x,y) \in D_1, \\ (x,y), & \text{if } (x,y) \in D_2, \end{cases} \tag{4.1}$$

see Figure 8. We note that

(i) D has positive Lebesgue measure since $V_{\alpha,-}$ has positive Lebesgue measure; see [23],
(ii) $(\psi)^{-1}(\mu_\alpha) = \mu_1|_D$,
(iii) ψ is injective.

Theorem 3 *We have $D = V_1$ and for a.e. $(x,y) \in V_1$,*

$$\left((\psi)^{-1} \circ \hat{F}_{\alpha,\flat} \circ \psi\right)(x,y) = \hat{F}(x,y). \tag{4.2}$$

In other words, for any $0 < \alpha < \frac{1}{2}$, $\left(V_{\alpha,\flat}, \hat{\mu}_{\alpha,\flat}, \hat{F}_{\alpha,\flat}\right)$ is metrically isomorphic to $\left(V_1, \hat{\mu}_1, \hat{F}\right)$ by the isomorphism $\psi : V_1 \to V_{\alpha,\flat}$.

Proof. We choose $(x,y) \in D$ such that both $\{\hat{F}^n(x,y) : n \geq 0\}$ and $\{\hat{F}^n_{\alpha,\flat}(\psi(x,y)) : n \geq 0\}$ are dense in V_1 and V_α, respectively. This is possible due to the fact that

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$\hat{F}$ and $\hat{F}_{\alpha,\flat}$ are ergodic with respect to μ_1 and μ_α, respectively. We see that the following hold:

(i) $(x_0, y_0) = \psi(x, y) \in V_{\alpha,-}$, $\alpha \leq x < \frac{1}{2}$
In this case, $F_{\alpha,\flat}(x_0) = \frac{2x-1}{1-x} < 0$ since $x_0 < -\frac{1}{2}$. Then we see $(\psi^{-1} \circ \hat{F}_{\alpha,\flat} \circ \psi)(x, y) = (\frac{x}{1-x}, \frac{y}{1-y}) = \hat{F}(x, y)$.

(ii) $(x_0, y_0) = \psi(x, y) \in V_{\alpha,-}$, $\frac{1}{2} \leq x < 1$
For $F_{\alpha,\flat}(x_0) = \frac{1-x}{x} > 0$, we see $(\psi^{-1} \circ \hat{F}_{\alpha,\flat} \circ \psi)(x, y) = (\frac{1-x}{x}, \frac{1-y}{y}) = \hat{F}(x, y)$.

(iii) $(x, y) \notin \psi^{-1}(V_{\alpha,-})$, $0 \leq x \leq \frac{1}{2}$
In this case, $\hat{F}_{\alpha,\flat}(\psi(x, y)) = \hat{F}_{\alpha,\flat}(x, y) = \left(\frac{x}{1-x}, \frac{y}{1-y}\right)$ and $\frac{x}{1-x} \geq 0$. Thus we have $(\psi^{-1} \circ \hat{F}_{\alpha,\flat} \circ \psi)(x, y) = \hat{F}(x, y)$.

(iv) $(x, y) \notin \psi^{-1}(V_{\alpha,-})$, $\frac{1}{2} \leq x \leq \frac{1}{1+\alpha}$
We see $\psi(x, y) = (x, y)$ again and $\hat{F}_{\alpha,\flat}(\psi(x, y)) = \hat{F}_{\alpha,\flat}(x, y) = \left(\frac{1-2x}{x}, \frac{1-2y}{y}\right)$. However, $\frac{1-2x}{x} < 0$. So we have

$$\left(\psi^{-1} \circ \hat{F}_{\alpha,\flat} \circ \psi\right)(x, y) = \left(\frac{1-2x}{x} + 1, \frac{1-2y}{y} + 1\right) = \left(\frac{1-x}{x}, \frac{1-y}{y}\right) = \hat{F}(x, y).$$

(v) $(x, y) \notin \psi^{-1}(V_{\alpha,-})$, $\frac{1}{1+\alpha} < x \leq 1$
In this case, we see

$$\hat{F}_{\alpha,\flat}(\psi(x, y)) = \hat{F}_{\alpha,\flat}(x, y) = \left(\frac{1-x}{x}, \frac{1-y}{y}\right)$$

and get $\left(\psi^{-1} \circ \hat{F}_{\alpha,\flat} \circ \psi\right)(x, y) = \hat{F}_{\alpha,\flat}(x, y)$.

As a consequence, we find that $\left(\psi^{-1} \circ \hat{F}_{\alpha,\flat} \circ \psi\right)(x, y) = \hat{F}(x, y)$ and $\hat{F}(x, y) \in D$ for any $(x, y) \in D$. Since we have chosen $\{\hat{F}^n(x, y) : n \geq 0\}$ and $\{\hat{F}^n_{\alpha,\flat}(\psi(x, y)) : n \geq 0\}$ are dense in V_1 and V_α, respectively, we find that $D = V_1$ and $\psi(D) = V_\alpha$. Note that from (4.2) we see that $(x, y) \in \psi^{-1}(V_\alpha) = V_1 \cap \psi^{-1}(V_\alpha)$, then $\hat{F}(x, y) \in V_1 \cap \psi^{-1}(V_\alpha)$. Choose $(x, y) \in V_1 \cap \psi^{-1}(V_\alpha)$ such that the orbit $\left(\hat{F}^k(x, y)\right)$ is dense in V_1. Let $(x_0, y_0) \in V_1$, then $(x_0, y_0) = \lim_{k\to\infty} \hat{F}^{n_k}(x, y)$ for some subsequence (n_k). From the above, $\hat{F}^{n_k}(x, y) \in \psi^{-1}(V_\alpha)$. Since $\psi^{-1}(V_\alpha)$ is closed, taking limits we see that $(x_0, y_0) \in \psi^{-1}(V_\alpha)$. We conclude that $\psi^{-1}(V_\alpha) = V_1$. Finally we see that the choice of (x, y) implies that (4.2) holds for a.e. (x, y). This concludes the assertion of this theorem. □

5. Some applications

As stated in the Introduction, in §5.1 we extend the result from [22]. That is, we show that the set of normal numbers with respect to G_α is the same with that of

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$G_{\alpha'}$ for any α and α' in $(0, 1]$; see Theorem 4. To prove this result, we need the natural extensions of the α-Farey maps. In §5.2 we extend the result of [28], by proving that for a.e. α in $(0, 1)$, G_α is not ϕ-mixing; see §5.2 for the definition. To do so, we use the result of §5.1 together with a result from [8]. What we need are statements like "$G_\alpha^n(\alpha - 1)$ *is dense*" and "*there exist* n_0 *and* m_0 *such that* $G_\alpha^{n_0}(\alpha - 1) = G_\alpha^{m_0}(\alpha)$." The former follows from §5.1 and the latter from [8] for a.e. α.

5.1. Normal numbers

Given any finite sequence of non-zero integers $b_1, b_2, \ldots, b_n$, we define the cylinder set $\langle b_1, \ldots, b_n\rangle_\alpha$ as

$$\langle b_1, \ldots, b_n\rangle_\alpha = \{x \in \mathbb{I}_\alpha : c_{\alpha,1}(x) = b_1, \ldots, c_{\alpha,n}(x) = b_n\}, \tag{5.1}$$

where $c_{\alpha,j}(x) = \varepsilon_{\alpha,j}(x)a_{\alpha,j}(x)$ for $j = 1, 2, \ldots, n$. An irrational number $x \in \mathbb{I}_\alpha$ is normal with respect to G_α if for any cylinder set $\langle b_1, \ldots, b_n\rangle_\alpha$,

$$\lim_{N\to\infty} \frac{\#\{0 \le m \le N-1 : G_\alpha^m(x) \in \langle b_1, \ldots, b_n\rangle_\alpha\}}{N} = \mu_\alpha(\langle b_1, \ldots, b_n\rangle_\alpha)$$

holds, where μ_α is the absolutely continuous invariant probability measure for G_α. An irrational number $x \in (0, 1)$ is said to be α-normal if either $x \in [0, \alpha)$ and x is normal with respect to G_α, or $x \in [\alpha, 1)$ and $x - 1$ is normal with respect to G_α. In the sequel, we consider $\alpha = \lim_{\epsilon\downarrow 0}(\alpha - \epsilon)$ as an element of $\mathbb{I}_\alpha$.

Now we extend this notion to the *2*-dimensional case. We set $\epsilon_\alpha(x, y)$ to be equal to $\varepsilon_\alpha(x)$. Since $\hat{G}_\alpha^*$ is bijective (a.e.), we can define $\epsilon_{\alpha,n}$ and $a_{\alpha,n}$ for $n \le 0$ by $\epsilon_{\alpha,n}(x, y) = \epsilon_\alpha(\hat{G}_\alpha^{*\,n-1}(x, y))$ and $a_{\alpha,n}(x, y) = a_\alpha(\hat{G}_\alpha^{*\,n-1}(x, y))$.

We can define also $c_{\alpha,j}(x, y) = \epsilon_{\alpha,j}(x, y)a_{\alpha,j}(x, y)$. With these definitions, we extends the notion of a (k, ℓ)-cylinder set for $-\infty < k < \ell < \infty$ by:

$$\langle b_k, b_{k+1}, \ldots, b_\ell\rangle_{\alpha,(k,\ell)} = \{(x, y) \in \Omega_\alpha^* : c_{\alpha,k}(x) = b_k, \ldots, c_{\alpha,\ell}(x) = b_\ell\}.$$

Then we can define normality of an element $(x, y) \in \Omega_\alpha^*$: (x, y) is said to be normal with respect to $\hat{G}_\alpha^*$ if for any sequence of integers $(b_k, b_{k+1}, \ldots, b_\ell)$, $-\infty < k < \ell < \infty$

$$\lim_{N\to\infty} \frac{1}{N}\sharp\{1 \le n \le N : c_{k+n-1}(x, y) = b_k, \ldots, c_{\ell+n-1}(x, y) = b_\ell\}$$
$$= \hat{\mu}_\alpha(\langle b_k, b_{k+1}, \ldots, b_\ell\rangle_{\alpha,(k,\ell)}),$$

where $\hat{\mu}_\alpha$ is the absolutely continuous invariant probability measure for $\hat{G}_\alpha^*$, satisfying $d\hat{\mu}_\alpha = C_\alpha \frac{dxdy}{(x-y)^2}$ with the normalizing constant C_α. According to this definition, it is easy to see that (x, y) is normal with respect to $\hat{G}_\alpha^*$ if and only if x is normal with respect to G_α (independent of the choice of y). For example, one may choose $y = -\infty$. We will show the following result.

THEOREM 4 *The set of α-normal numbers is the same with that of 1-normal numbers with respect to $G = G_1$.*

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The proof for $\sqrt{2}-1 \le \alpha < \frac{1}{2}$ is basically the same as the case $\frac{1}{2} \le \alpha \le 1$. In what follows, we give the proof of this theorem mainly keeping in mind the case $0 < \alpha < \sqrt{2}-1$. In particular, for $0 < \alpha < \frac{3-\sqrt{5}}{2}$. (By [23] and [27], the size of Ω_α^* with respect to the measure $\frac{dxdy}{(x-y)^2}$ is equal to that of $\hat{G}^*_{\frac{1}{2}}$ for $\frac{3-\sqrt{5}}{2} \le \alpha < \frac{1}{2}$ and is larger than it for $0 < \alpha < \frac{3-\sqrt{5}}{2}$.)

We define an induced map $\hat{F}_{\alpha,\flat,2}$ of $\hat{F}_{\alpha,\flat}$. To do so, first we define an induced map $\hat{F}_{\alpha,\flat,1}$. We put

$$V_{\alpha,\flat,1} = \Omega_\alpha^* \cup \left\{ \left(-\frac{x}{1+x}, -\frac{y}{1+y} \right) : (x,y) \in \Omega_\alpha^*, -\tfrac{1}{2} \le x < 0 \right\}. \qquad (5.2)$$

We note that the second part of the right side is

$$\left\{ (x,y) : x \le 1,\ \left(\tfrac{1}{x}, \tfrac{1}{y}\right) \in \cup_{k=2}^{\infty} \hat{\Omega}_{\alpha,k,-} - (1,1) \right\}.$$

Hence we see $V_{\alpha,\flat,1} = V_{\alpha,\flat} \cap \{(x,y) : y \le -1\}$. Then $\hat{F}_{\alpha,\flat,1}$ is the induced map of $\hat{F}_{\alpha,\flat}$ to $\hat{V}_{\alpha,\flat,1}$. We will write it explicitly. Recall the definition of $\hat{\Omega}_{\alpha,k,\pm}$, (3.1).

(i) If $\alpha - 1 \le x < -\frac{1}{2}$, then $\hat{F}_{\alpha,\flat}(x,y) \in \Omega_\alpha^* \subset V_{\alpha,\flat,1}$, see (5.2), and $\hat{F}_{\alpha,\flat,1}(x,y) = \hat{F}_{\alpha,\flat}(x,y) = \hat{G}^*_\alpha(x,y)$.

(ii) If $-\frac{1}{2} \le x < 0$, then $\hat{F}_\alpha(x,y) = \left(-\frac{x}{1+x}, -\frac{y}{1+y}\right)$ which implies $0 < -\frac{x}{1+x} \le 1$ and $-\frac{y}{1+y} \le -1$. Thus we have $\hat{F}_{\alpha,\flat,1}(x,y) = \hat{F}_{\alpha,\flat}(x,y)$.

(iii) If $(x,y) \in \Omega^*_{\alpha,k,+}$ for some $k \ge k_0$, then $\hat{F}_\alpha(x,y) = \left(\frac{x}{1-x}, \frac{y}{1-y}\right) \in \hat{\Omega}_{\alpha,k,+}$. This shows $\frac{x}{1-x} < 1$ but $\frac{y}{1-y} > 1$. Hence $\hat{F}_\alpha(x,y) \notin V_{\alpha,\flat,1}$. The same hold for $\hat{F}^\ell_\alpha(x,y)$ for $2 \le \ell \le k-1$ and then $\hat{F}^k_\alpha(x,y) (= \hat{G}^*_\alpha(x,y)) \in \Omega^*_\alpha \subset V_{\alpha,\flat,1}$.

(iv) If $0 \le x \le 1$ and $(\frac{1}{x}, \frac{1}{y}) \in \hat{\Omega}_{\alpha,k,-} - (1,1)$, then $(\frac{1}{x}, \frac{1}{y}) - (\ell,\ell) \in \hat{\Omega}_{\alpha,k,-} - (\ell-1,\ell-1) =: (u_\ell, v_\ell)$ for $2 \le \ell \le k-1$. This implies $0 > -\frac{1}{v_\ell} > -1$ and so $\left(\frac{1}{u_\ell}, \frac{1}{v_\ell}\right) \in V_{\alpha,\flat,1}$. Moreover, $u_{k-1}^{-1} \in \mathbb{I}_\alpha$, where $(\frac{1}{x}, \frac{1}{y}) - (k-1,k-1) = (u_k, v_k)$. In other words, there exists $(u',v') \in \Omega^*_\alpha$ such that $\left(\frac{1}{u'}, \frac{1}{v'}\right) - (k,k) = \left(\frac{1}{x}, \frac{1}{y}\right) - (k-1,k-1)$. Hence we have

$$\hat{F}_{\alpha,\flat,1}(x,y) = \left(\tfrac{1}{x} - (k-1), \tfrac{1}{y} - (k-1)\right) \left(= \hat{G}^*_\alpha(u',v')\right).$$

Consequently, we see that $\hat{F}_{\alpha,\flat,1}(x,y)$ satisfies:

$$\begin{cases} \hat{F}_{\alpha,\flat}(x,y) = G^*_\alpha(x,y), & \text{if } \alpha - 1 \le x < -\frac{1}{2} \\ \hat{F}_{\alpha,\flat}(x,y), & \text{if } -\frac{1}{2} \le x < 0 \\ \hat{G}^*_\alpha(x,y), & \text{if } 0 \le x \text{ and } (x,y) \in \Omega^*_\alpha \\ \left(\frac{1}{x} - (k-1), \frac{1}{y} - (k-1)\right), & \text{if } 0 \le x \le 1, \text{and} \\ & (\frac{1}{x}, \frac{1}{y}) \in \hat{\Omega}_{\alpha,k,-} - (1,1). \end{cases} \qquad (5.3)$$

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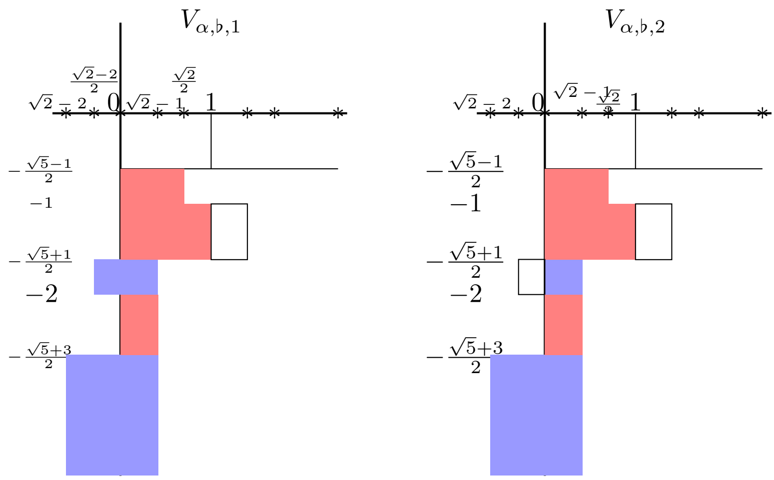

Figure 9. $V_{\alpha,\flat,1}$ and $V_{\alpha,\flat,2}$ for $\alpha = \sqrt{2}-1$

Next we consider $V_{\alpha,\flat,2} \subset V_{\alpha,\flat,1}$, which is defined as follows: $V_{\alpha,\flat,2} = V_{\alpha,\flat,-} \cup V_{\alpha,\flat,+}$, with

$$\begin{cases} V_{\alpha,\flat,-} & = & V_{\alpha,\flat,1} \cap \{(x,y) : x < 0, y \leq -2\} \\ & = & \Omega_\alpha^* \cap \{(x,y) : x < 0, y \leq -2\}, \\ \\ V_{\alpha,\flat,+} & = & V_{\alpha,\flat,1} \cap \{(x,y) : x \geq 0\}. \end{cases}$$

We let $\hat{F}_{\alpha,\flat,2}$ be the induced transformation on $V_{\alpha,\flat,2}$. We see that $V_{1,\flat,2} = V_{1,\flat,+} = W$, where W is defined as:

$$W = [0,1] \times [-\infty, -1] \tag{5.4}$$

and $V_{\alpha,\flat,-} = \emptyset$. Recall the map ψ as defined in (4.1), and notice that when ψ^{-1} is restricted to $V_{\alpha,\flat,2}$, one finds:

$$\psi^{-1}(x,y) = \begin{cases} (x+1, y+1), & \text{if} \quad (x,y) \in V_{\alpha,\flat,-}, \\ (x,y), & \text{if} \quad (x,y) \in V_{\alpha,\flat,+}. \end{cases}$$

Then from Theorem 3, we have $\psi^{-1}(V_{\alpha,\flat,2}) = W$, with W from (5.4).

THEOREM 5 *The induced map $\hat{F}_{\alpha,\flat,2}$ of $\hat{F}_{\alpha,\flat,1}$ to $V_{\alpha,\flat,2}$ is metrically isomorphic to the natural extension of the Gauss map G_1, where the absolutely continuous invariant probability measure for $\hat{F}_{\alpha,\flat,2}$ is given by $C_{\alpha,\flat,2}\frac{dxdy}{(x-y)^2}$, with the normalizing constant $C_{\alpha,\flat,2}$, see Figures 7, 9, and Figure 8.*

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Proof. It is easy to see that the induced map of $\hat{F}_1$ on W is the natural extension of the Gauss map G. Indeed we see, with W from (5.4), that $\hat{G}^* = \hat{F}_1|_W$:

$$(x,y) \mapsto \left(\tfrac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \tfrac{1}{y} - \left\lfloor \frac{1}{x} \right\rfloor\right)$$

is bijective on W on (a.e.). Since $\psi^{-1}V_{\alpha,\flat,2} = W$, the conjugacy $\psi^{-1} \circ \hat{F}_{\alpha,\flat,2} \circ \psi = \hat{G}_1^*$ follows from Theorem 3 and basic fact on induced transformations from Ergodic theory. □

The next step is the definition of normal numbers associated both with $\hat{F}_{\alpha,\flat,1}$ and $\hat{F}_{\alpha,\flat,2}$.

We define a digit function $\delta(x,y)$ and get a sequence $(\delta_n(x,y) : n \geq 1)$ in the following way:

$$\delta(x,y) = \begin{cases} \delta_{-,k} & \text{if } (x,y) \in \Omega^*_{\alpha,k,-},\ k \geq 2, \\ \delta_{+,k} & \text{if } (x,y) \in \Omega^*_{\alpha,k,+},\ k \geq k_0, \\ \delta_{0,2} & \text{if } (x,y) \in \left\{(x,y) : \left(\frac{1}{x}, \frac{1}{y}\right) \in \hat{\Omega}_{\alpha,2,-} - (1,1), x \leq 1\right\}, \\ \delta_{0,k} & \text{if } (x,y) \in \left\{(x,y) : \left(\frac{1}{x}, \frac{1}{y}\right) \in \hat{\Omega}_{\alpha,k,-} - (1,1)\right\},\ k > 2, \end{cases} \tag{5.5}$$

and $\delta_n(x,y) = \delta(\hat{F}^{n-1}_{\alpha,\flat,1}(x,y))$, $n \geq 1$, for $(x,y) \in V_{\alpha,\flat,1}$. It is easy to see that the set of sequences $(\delta_n(x,y))$ separates points of $V_{\alpha,\flat,1}$.

Let $(e_n : 1 \leq n \leq \ell)$ be a block of $\delta_{j,k}$'s. Then we define a cylinder set of length $\ell \geq 1$ by

$$\langle e_1, e_2, \ldots, e_\ell \rangle_{\alpha,\flat,1} = \{(x,y) \in V_{\alpha,\flat,1} : \delta_n(x,y) = e_n, 1 \leq n \leq \ell\}.$$

We denote by $\mu_{\alpha,\flat,1}$ the absolutely continuous invariant probability measure for $\hat{F}_{\alpha,\flat,1}$. An element $(x,y) \in V_{\alpha,\flat,1}$ is said to be α-1-Farey normal if

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sharp\{n : 1 \leq n \leq N,\ \hat{F}^{n-1}_{\alpha,\flat,1}(x,y) \in \langle e_1, e_2, \ldots, e_\ell \rangle_{\alpha,\flat,1}\} \\ = \mu_{\alpha,\flat,1}(\langle e_1, e_2, \ldots, e_\ell \rangle_{\alpha,\flat,1}) \end{aligned} \tag{5.6}$$

holds for every cylinder set of length $\ell \geq 1$. We can define the notion of the α-2-Farey normality in a similar way, compare (5.6) and (5.7). We may use the same notation $\delta(x,y)$ restricted on $V_{\alpha,\flat,2}$; c.f. (5.5). However, we use $\eta(x,y)$ to describe the difference between $\hat{F}_{\alpha,\flat,1}$ and $\hat{F}_{\alpha,\flat,2}$: from a digit function $\eta(x,y)$ we get a sequence $(\eta_n(x,y) : n \geq 1)$ as follows.

$$\eta(x,y) = \begin{cases} \delta_{-,k} & \text{if} \quad (x,y) \in \Omega^*_{\alpha,k,-},\ k \geq 2,\ y \leq -2, \\ \delta_{+,k} & \text{if} \quad (x,y) \in \Omega^*_{\alpha,k+},\ k \geq k_0, \\ \delta_{0,2} & \text{if} \quad (x,y) \in \left\{\left(\frac{1}{x}, \frac{1}{y}\right) \in \Omega^*_{\alpha,2,-} - (1,1), x \leq 1\right\}, \\ \delta_{0,k} & \text{if} \quad (x,y) \in \left\{\left(\frac{1}{x}, \frac{1}{y}\right) \in \Omega^*_{\alpha,k,-} - (1,1)\right\},\ k > 2, \end{cases}$$

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and $\eta_n(x,y) = \eta(\hat{F}^{n-1}_{\alpha,\flat,2}(x,y))$, $n \geq 1$, for $(x,y) \in V_{\alpha,\flat,2}$. It is easy to see that the set of sequences $(\eta_n(x,y))$ separates points of $V_{\alpha,\flat,2}$.

Let $(e_n : 1 \leq n \leq \ell)$ be a block $\delta_{j,k}$'s, then we define a cylinder set of length $\ell \geq 1$ by

$$\langle e_1, e_2, \ldots, e_\ell \rangle_{\alpha,\flat,2} = \{(x,y) \in V_{\alpha,\flat,2} : \eta_n(x,y) = e_n, 1 \leq n \leq \ell\}.$$

We denote by $\mu_{\alpha,\flat,2}$ the absolutely continuous invariant probability measure for $\hat{F}_{\alpha,\flat,2}$. An element $(x,y) \in V_{\alpha,\flat,2}$ is said to be α-2-Farey normal if

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sharp\{n : 1 \leq n \leq N,\ \hat{F}^{n-1}_{\alpha,\flat,2}(x,y) \in \langle e_1, e_2, \ldots, e_\ell \rangle_{\alpha,\flat,2}\} \\ = \mu_{\alpha,\flat,2}(\langle e_1, e_2, \ldots, e_\ell \rangle_{\alpha,\flat,2}) \end{aligned} \tag{5.7}$$

holds for every cylinder set of length $\ell \geq 1$. Here we have to be careful with the measures $\mu_{\alpha,\flat,1}$ and $\mu_{\alpha,\flat,2}$, which take different values only by the normalizing constants for any measurable set $A \subset V_{\alpha,\flat,2}$.

The proof of Theorem 4 is done in steps. We first prove that under the induced transformation $\hat{F}_{\alpha,\flat,1}$, α-1-Farey normality is equivalent to α-normality. After that we proceed to the induced system $\hat{F}_{\alpha,\flat,2}$, that is isomorphic to the Gauss map $\hat{G}_1$, and show that α-2-Farey normality is equivalent to normality w.r.t. $\hat{G}_1$. On the other hand, one can prove that for points in the domain of the $\hat{F}_{\alpha,\flat,2}$ map, a point is α-1-Farey normal if and only if it is α-2-Farey normal. From the above equivalences, one then concludes that α-normality is equivalent to *1*-normality.

Define $r_1 := 1$ and set for $j \geq 2$, $r_j = r_j(x,y) := n$ whenever $\hat{F}^{n-1}_{\alpha,\flat,1}(x,y) \in \Omega^*_\alpha$ and $\hat{F}^{m}_{\alpha,\flat,1}(x,y) \notin \Omega^*_\alpha$, $r_{j-1} \leq m < n$.

Lemma 3. *Suppose that*

$$(x,y) \in \left\{(x,y) : x \leq 1,\ \left(\tfrac{1}{x}, \tfrac{1}{y}\right) \in \bigcup_{k=2}^{\infty} \hat{\Omega}_{\alpha,k,-} - (1,1)\right\}.$$

Then (x,y) *is* α*-1-Farey normal if and only if* $\hat{F}_{\alpha,\flat,1}(x,y) \in \Omega^*_\alpha$ *is* α*-1-Farey normal.*

Proof. From the 4th line of the right side of (5.3), we have $\hat{F}_{\alpha,\flat,1}(x,y) \in \Omega^*_\alpha$. Then the equivalence of the normality is easy to follow. □

From this lemma, it is enough to restrict the α-1-Farey normality only for $(x,y) \in \Omega^*_\alpha$.

Lemma 4. *An element* $(x_0, y_0) \in \Omega^*_\alpha$ *is* α*-1-Farey normal if and only if* x_0 *is* α*-normal.*

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Proof. Suppose that $r_i = r_i(x_0, y_0)$, and decompose $\mathbb{N}$ as $\mathbb{N}_1 \cup \mathbb{N}_2$ with

$$\mathbb{N}_1 = \{r_i \,:\, i \geq 1\}(= \{n \in \mathbb{N} \,:\, \delta_n(x_0, y_0) = \delta_{\pm,k} \text{ for some } k\}),$$

and

$$\mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1 = \{n \in \mathbb{N} : \delta_n(x_0, y_0) = \delta_{0,k} \text{ for some } k\},$$

which corresponds to

$$\hat{F}^{n-1}_{\alpha,\flat,1}(x_0, y_0) \in \Omega^*_\alpha \text{ if } n \in \mathbb{N}_1,$$

and

$$\hat{F}^{n-1}_{\alpha,\flat,1}(x_0, y_0) \in \left\{(x, y) : \left(\tfrac{1}{x}, \tfrac{1}{y}\right) \in \cup_{k=2}^{\infty} \hat{\Omega}_{\alpha,k,-} - (1, 1)\right\} \text{ if } n \in \mathbb{N}_2.$$

Remark 5.1. We easily find that the following properties hold:

(i) $\hat{F}_{\alpha,\flat,1}(\langle\delta_{-,2}\rangle_{\alpha,\flat,1} \cap \{(x, y) : -\frac{1}{2} \leq x < 0\}) = \langle\delta_{0,2}\rangle_{\alpha,\flat,1}$,
(ii) $\hat{F}_{\alpha,\flat,1}(\langle\delta_{-,2}\rangle_{\alpha,\flat,1} \cap \{(x, y) : \alpha - 1 \leq x < -\frac{1}{2}\}) = \langle\delta_{-,k}\rangle_{\alpha,\flat,1}$ for some $k \geq 2$,
(iii) $\hat{F}_{\alpha,\flat,1}(\langle\delta_{-,k}\rangle_{\alpha,\flat,1}) = \langle\delta_{0,k}\rangle_{\alpha,\flat,1}$ for $k \geq 3$,
(iv) $\hat{F}_{\alpha,\flat,1}(\langle\delta_{+,k}\rangle_{\alpha,\flat,1}) \subset \Omega^*_\alpha$.

Furthermore, if $n \in \mathbb{N}_1$ and either $\delta_n(x_0, y_0) = \delta_{-,k}$, $k \geq 3$, or $\delta_n(x_0, y_0) = \delta_{-,2}$ and the first coordinate of $\hat{F}^{n-1}_{\alpha,\flat,1}(x_0, y_0)$ is in $\left(-\frac{1}{2}, 0\right)$, then $n + 1 \in \mathbb{N}_2$ and $n + 2 \in \mathbb{N}_1$. Otherwise $n + 1 \in \mathbb{N}_1$.

We note that $r_{k+1} - r_k = 1$ or *2* for any $k \geq 1$. Moreover, if $\delta_{n+1}(x_0, y_0) = \delta_{0,k}$ then $\delta_n(x_0, y_0) = \delta_{-,k}$.

The properties from Remark 5.1 show that we can reproduce $(\delta_n(x_0, y_0) : j \geq 1)$ if $(c_j : j \geq 1)$ is given as a sequence of integers, or equivalently, $(\delta_{r_j}(x_0, y_0) : j \geq 1)$; c.f. (5.1). Indeed, for $(x, y) \in \Omega^*_\alpha$ we see the following:

$$\begin{array}{lcl} \delta_n(x,y) = \delta_{-,k} & \iff & \delta_{n+1}(x,y) = \delta_{0,k} \text{ if } k \geq 3 \\ \delta_n(x,y) = \delta_{-,2} & \Rightarrow & \begin{cases} \delta_{n+1}(x,y) = \delta_{0,2}, \\ \text{and } \delta_{n+2}(x,y) = \delta_{+,k},\ k \geq k_0 \\ \qquad \text{or} \\ \delta_{n+1}(x,y) = \delta_{-,k},\ k \geq 2 \end{cases} \end{array} \tag{5.8}$$

and for $\ell \geq k_0$,

$$\delta_n(x,y) = \delta_{+,\ell} \ \Rightarrow \begin{cases} \delta_{n+1}(x,y) = \delta_{+,k} \ k \geq k_0 \\ \qquad \text{or} \\ \delta_{n+1}(x,y) = \delta_{-,k} \ k \geq 2. \end{cases} \tag{5.9}$$

So for any $(x, y) \in \Omega^*_\alpha$, from (5.8) and (5.9), we have

$$\begin{array}{lcl} \delta_n(x,y) = \delta_{0,k} & \Rightarrow & \delta_{n-1}(x,y) = \delta(\hat{F}^{-1}_{\alpha,\flat,1}(x,y)) = \delta_{-,k} \text{ if } k \geq 3, \\ \delta_n(x,y) = \delta_{0,2} & \Rightarrow & \delta_{n-1}(x,y) = \delta_{-,2} \text{ and } \delta_{n+1}(x,y) = \delta_{+,\ell},\ \ell \geq k_0. \end{array}$$

Let $(b_i : 1 \leq i \leq n)$ (see (5.1)) be a sequence of non-zero integers such that $\langle b_1, b_2, \ldots, b_n\rangle_\alpha \neq \emptyset$. From the above discussion, if $(x_0, y_0) \in \langle b_1, b_2, \ldots, b_n\rangle_\alpha$, we

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can construct a sequence $(\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_m)$ satisfying $\hat{\delta}_{r_i} = b_i$, where r_i is the ith occurrence of the form $\delta_{\pm,k}$ with $r_1 = 1$ (since we start in $\hat{\Omega}^*_\alpha$) and $r_n = m$ or $m-1$, and such that $(x_0, y_0) \in \langle \hat{\delta}_1, \ldots, \hat{\delta}_m \rangle_{\alpha,\flat,1}$. The latter happens when $\hat{\delta}_m = \delta_{0,2}$. Similarly, we can construct $(b_1, b_2, \ldots, b_n)$ from $(\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_m)$, where $b_n = \hat{\delta}_m$ or $b_n = \hat{\delta}_{m-1}$.

To show the statement of the lemma, first we assume that (x_0, y_0) is α-*1*-Farey normal. It is easy to see that:

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{n : 1 \le n \le N, n \in \mathbb{N}_2\} = \lim_{K\to\infty} \mu_{\alpha,\flat,1}\left(\bigcup_{k=2}^{K} \langle \delta_{0,k} \rangle_{\alpha,\flat,1}\right)$$
$$= \mu_{\alpha,\flat,1}\left(\bigcup_{k=2}^{\infty} \langle \delta_{0,k} \rangle_{\alpha,\flat,1}\right). \tag{5.10}$$

The equality (5.10) also shows that

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{n : 1 \le n \le N, n \in \mathbb{N}_1\} = \mu_{\alpha,\flat,1}(\hat{\Omega}^*_\alpha), \tag{5.11}$$

By the same argument, we can show that for any sequence $(\hat{\delta}_1, \ldots, \hat{\delta}_m)$,

$$\mu_{\alpha,\flat,1}(\langle \hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_m \rangle_{\alpha,\flat,1}) = \mu_{\alpha,\flat,1}(\langle b_1, b_2, \ldots, b_n \rangle_\alpha)$$

if $\hat{\delta}_m$ is of the form $\delta_{\pm,k}$, otherwise

$$\mu_{\alpha,\flat,1}(\langle \hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_m \rangle_{\alpha,\flat,1}) = \sum_{\ell_0}^{\infty} \mu_{\alpha,\flat,1}(\langle b_1, b_2, \ldots, b_n, \ell \rangle_\alpha).$$

We have

$$\frac{1}{N} \sharp\{j : 1 \le k \le N, \delta_j = b_1, \delta_{j+1} = b_2, \ldots, \delta_{j+n-1} = b_n\}$$
$$= \frac{\hat{N}}{N} \frac{1}{\hat{N}} \sharp\{j : 1 \le j \le \hat{N}, c_j = b_1, c_{j+1} = b_2, \ldots, c_{j+n-1} = b_n\} \tag{5.12}$$

where $\hat{N} = \max\{j : r_j \le N\}$ and $c_j = \varepsilon_j(x_0 \cdot a_j(x_0)$.

With this notation, the left side converges to $\mu_{\alpha,\flat,1}(\langle \hat{\delta}_1, \ldots, \hat{\delta}_m \rangle_{\alpha,\flat,1})$ and the first term of the right side goes to $\hat{\mu}_{\alpha,\flat,1}(\Omega^*_\alpha)$ (see (5.11)). Thus the second term of the right side goes to

$$\frac{\mu_{\alpha,\flat,1}(\langle \hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_m \rangle_{\alpha,\flat,1})}{\mu_{\alpha,\flat,1}(\Omega^*_\alpha)}.$$

as $N \to \infty$ and its numerator is

$$\mu_{\alpha,\flat,1}(\{(x,y) : c_{r_1}(c,y) = b_1, c_{r_2}(x,y) = b_2, \ldots, c_{r_n}(x,y) = b_n\}.$$

The definition of μ_α and $\mu_{\alpha,\flat,1}$ implies that $\frac{1}{\mu_{\alpha,\flat,1}(\hat{\Omega}^*_\alpha)}$ changes $\mu_{\alpha,\flat,1}$ to μ_α. Consequently, we get the limit of the second term as $\mu_\alpha(\langle b_1, b_2, \ldots, b_n \rangle_\alpha)$. This

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shows that(x_0, y_0) is normal with respect to $\hat{G}^*_\alpha$, which implies the α-normality of x_0.

Next we suppose that (x_0, y_0) is not α-*1*-Farey normal. We want to show that (x_0, y_0) is also not α-normal. We check the equality (5.11) again. If $\frac{\hat{N}}{N}$ does not conveges to $\mu_{\alpha,\flat,1}(\Omega^*_\alpha)$, then it is easy to see that x_0 is not α-normal. On the other hand if $\lim_{N\to\infty} \frac{\hat{N}}{N} = \mu_{\alpha,\flat,1}(\Omega^*_\alpha)$, then, by the same argument, we see that the second term of the right side of (5.12) does not converge to $\mu_\alpha(\langle b_1, b_2, \ldots, b_n\rangle_\alpha)$, which shows that x_0 is not α-normal. Indeed, from Remark 5.1 we can construct a sequence $(\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_m)$ such that $\hat{\delta}_{r_j} = \delta_{\mathrm{sgn}(b_j), |b_j|}$ and $m = r_J = \hat{N}$, i.e., $\hat{\delta}_{r_j} = \delta_{\mathrm{sgn}, |b_j|}$, $1 \le j \le J$ and for other ℓ ($\ell \ne r_j$, $1 \le j \le J$) are of the form $\delta_{0,k}$, and it is determined uniquely by $\hat{\delta}_{\ell-1}$ since $\ell - 1$ is r_j for some $1 \le j \le J$. Indeed, $\hat{\delta}_\ell = \delta_{0,k}$ implies $\hat{\delta}_{\ell-1} = \delta_{-,k}$. Note that $\hat{\delta}_{\ell-1} = \delta_{-,2}$ does not mean $\hat{\delta}_\ell = \delta_{0,2}$ since $\ell - 1 = r_j$, $\ell = r_{j+1}$ can happen. But $\hat{\delta}_{\ell-1} = \delta_{-,k}$, $k \ge 3$, implies $r_j + 1 \ne r_{j+1}$. Then we can show the estimate in the above. □

The same idea shows that the α-1-Farey normality is equivalent to the α-2-Farey normality. Here we note that $\hat{F}_{\alpha,\flat,2}$ is an induced map of $\hat{F}_{\alpha,\flat,1}$ and for $(x, y) \in V_{\alpha,\flat,2}$, $\eta_n(x, y)\eta_{n+1}(x, y) \ne \delta_{-,2}\delta_{0,2}$. So in the η-code of a point (x, y) the digit $\delta_{0,2}$ serves as a marker for the missing preceding digit $\delta_{-,2}$ in the corresponding δ-code of (x, y).

LEMMA 5. *An element $(x, y) \in \Omega^*_\alpha$ is α-2-Farey normal if and only if it is α-1-Farey normal.*

Proof. Sketch of the proof. From the sequence $\delta_n(x, y)$, we can construct $\eta_m(x, y) \in V_{\alpha,\flat,2}$ by deleting the digit $\delta_{-,2}$ that is followed by a digit $\delta_{0,2}$. More precisely,

$$\begin{aligned}&\delta_{n-1}(x, y), \delta_n(x, y) = \delta_{-,2},\ \delta_{n+1}((x, y) = \delta_{0,-2}\\ \Rightarrow\ &\eta_m = \delta_{n-1}(x, y), \eta_{m+1}(x, y) = \delta_{n+1},\end{aligned}$$

where m is the cardinality of n such that $\delta_n\delta_{n+1} = \delta_{-,2}\delta_{0,2}$. On the other hand, given the sequence $(\eta_m(x, y) : -\infty < m < \infty)$, we can construct the sequence $(\delta_n(x, y) : -\infty < n < \infty)$ by inserting $\delta_{-,2}$ before every occurrence of every $\delta_{0,2}$. Following the proof of Lemma 4, we get the result. □

LEMMA 6. *An element $(x_0, y_0) \in \Omega^*_\alpha$ is α-2-normal if and only if $\psi^{-1}(x_0, y_0) \in W$ is normal with respect to $\hat{G}^*$.*

Proof. Following the above proofs, cylinder sets associated with $\hat{F}_{\alpha,\flat,2}$ are approximated by each other (using ψ and ψ^{-1}); see Theorem 5. Suppose that (x_0, y_0) is α-2-Farey normal. Every cylinder set associated with $\hat{G}^*(= \hat{G}^*_1)$ is a rectangle. To be more precise, a cylinder set $\langle a_1, a_2, \ldots, a_n\rangle_1$ is of the form $\left[\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) \times [-\infty, -1]$, or $\left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right] \times [-\infty, -1]$. We divide it into three parts such that $\eta_1(x, y) = \delta_{\sharp,k}$, $\sharp = +, 0$, and $-$. Then ψ^{-1}-image of each part is a countable union of $\hat{F}_{\alpha,\flat,2}$ cylinder sets, just as discussed in the above. Hence we can prove that (x_0, y_0) (or $(x_0 + 1, y_0 + 1)$ is normal with respect to $\hat{G}^*$ in the same way.

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Now suppose that (x_0, y_0) is **not** α-2-normal and $\psi^{-1}(x_0, y_0)$ is normal with respect to $\hat{G}^*$. Then there exist $\epsilon > 0$ and a cylinder set with respect to $\hat{F}_{\alpha,\flat,2}$ such that either

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \#\{n : 0 \le m \le N-1,\ \hat{F}^m_{\alpha,\flat,2}(x_0, y_0) \in \langle e_1, e_2, \ldots, e_\ell\rangle_{\alpha,\flat,2}\} \\ > \mu_{\alpha,\flat,2}(\langle e_1, e_2, \ldots, e_\ell\rangle_{\alpha,\flat,2}) + \epsilon,\end{aligned} \tag{5.13}$$

or

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \#\{n : 0 \le m \le N-1,\ \hat{F}^m_{\alpha,\flat,2}(x_0, y_0) \in \langle e_1, e_2, \ldots, e_\ell\rangle_{\alpha,\flat,2}\} \\ < \mu_{\alpha,\flat,2}(\langle e_1, e_2, \ldots, e_\ell\rangle_{\alpha,\flat,2}) - \epsilon,\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \#\{0 \le m \le N-1 : \hat{G^*}^m(\psi^{-1}(x_0, y_0)) \in \langle b_1, \ldots, b_n\rangle_{1,(1,n)}\} \\ = \hat{\mu}(\langle b_1, \ldots, b_n\rangle_{1,(1,n)})\end{aligned} \tag{5.15}$$

for any sequence of positive integers $(b_1, b_2, \ldots, b_n)$, where $\hat{\mu}$ is the measure defined by $\frac{1}{\log 2}\frac{dxdy}{(x-y)^2}$. Note that $\langle \cdots \rangle_{1,(1,n)}$ means a cylinder set with respect to $\hat{G}^* = \hat{G}^*_\alpha$ with $\alpha = 1$.

We start by assuming (5.13) and (5.15) hold, and show it will lead to a contradiction. Since the set of cylinder sets associated with $\hat{G}^*$ generates the Borel σ-algebra, there exist a finite number of pairwise disjoint cylinder sets

$$\langle b_{j,1}, b_{j,2}, \ldots, b_{j,k_j}\rangle_{1,(1,k_j)}, \qquad 1 \le j \le M < \infty \text{ and } 1 \le k_j < \infty,$$

such that

$$\psi^{-1}(\langle e_1, e_2, \ldots, e_\ell\rangle_{\alpha,\flat,2}) \subset \bigcup_{j=1}^{M} \langle b_{j,1}, b_{j,2}, \ldots, b_{j,k_j}\rangle_{1,(1,k_j)}$$

and

$$\hat{\mu}\left(\bigcup_{j=1}^{M} \langle b_{j,1}, b_{j,2}, \ldots, b_{j,k_j}\rangle_1\right) < \hat{\mu}(\psi^{-1}(\langle e_1, e_2, \ldots, e_\ell\rangle_{\alpha,\flat,2})) + \tfrac{1}{2}\epsilon.$$

Since ψ is an isomorphism (see Theorem 5), we have

$$\hat{F}_{\alpha,\flat,2}(x_0, y_0) = \psi \hat{G}^* \psi^{-1}(x_0, y_0)$$

and

$$\mu_{\alpha,\flat,2}(\langle e_1, e_2, \ldots, e_\ell\rangle_{\alpha,\flat,2}) = \hat{\mu}(\psi^{-1}(\langle e_1, e_2, \ldots, e_\ell\rangle_{\alpha,\flat,2})).$$

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Thus,

$$\lim_{N\to\infty}\frac{1}{N}\#\Big\{0\le m\le N-1:\ \hat{F}^m_{\alpha,\flat,2}(x_0,y_0)\in\langle e_1,e_2,\dots,e_\ell\rangle_{\alpha,\flat,2}\Big\}$$
$$\le \lim_{N\to\infty}\frac{1}{N}\#\Big\{0\le m\le N-1:\ \hat{G^*}^m(\psi^{-1}(x_0,y_0))\in\bigcup_{j=1}^{M}\langle b_{j,1},\dots,b_{j,k_j}\rangle_1\Big\}$$
$$=\hat{\mu}\left(\bigcup_{j=1}^{M}\langle b_{j,1},b_{j,2},\dots,b_{j,k_j}\rangle_1\right)$$
$$<\hat{\mu}(\psi^{-1}(\langle e_1,e_2,\dots,e_\ell\rangle_{\alpha,\flat,2}))+\tfrac{1}{2}\epsilon$$
$$=\mu_{\alpha,\flat,2}(\langle e_1,e_2,\dots,e_\ell\rangle_{\alpha,\flat,2})+\tfrac{1}{2}\epsilon.$$

Combining this with (5.13) yields $\epsilon<\frac{1}{2}\epsilon$, which is a contradiction.

On the other hand, if (5.14) and (5.15) hold, a proof similar to the one above but now approximating the cylinder $\psi^{-1}\left(\langle e_1,e_2,\dots,e_\ell\rangle_{\alpha,\flat,2}\right)$ from "inside" by a union of cylinders leads to the same contradiction. □

Proof. Proof of Theorem 4. This is a direct consequence of Lemmas 3, 4, 5 and 6. □

5.2. Non-ϕ-mixing property

We start with some definitions of mixing properties. Let $(\Omega,\mathfrak{B},P)$ be a probability space. For sub σ-algebras $\mathcal{A}$ and $\mathcal{B}\subset\mathfrak{B}$, we put

$$\phi(\mathcal{A},\mathcal{B})=\sup\left\{\left|\frac{P(A\cap B)}{P(A)}-P(B)\right|:A\in\mathcal{A},\ B\in\mathcal{B},\ P(A)>0\right\}.$$

Suppose that $(X_n: n\ge 1)$ is a stationary sequence of random variables. We denote by $\mathcal{F}^n_m$ the sub-σ algebra of $\mathfrak{B}$ generated by $X_m,X_{m+1},X_{m+2},\dots,X_n$. We define $\phi(n)=\sup_{m\ge1}\phi(\mathcal{F}^m_1,\mathcal{F}^\infty_{m+n})$ and $\phi^*(n)=\sup_{m\ge1}\phi(\mathcal{F}^\infty_{m+n},\mathcal{F}^m_1)$.

The process $(X_n: n\ge1)$ is said to be ϕ-mixing if $\lim_{n\to\infty}\phi(n)=0$ and reverse ϕ-mixing if $\lim_{n\to\infty}\phi^*(n)=0$, respectively.

G_α is said to be ϕ-mixing (or reverse ϕ-mixing) if $(a_{\alpha,n},\epsilon_{\alpha,n})$ is ϕ-mixing (or reverse ϕ-mixing), respectively. In [28], it is shown that G_α is not ϕ-mixing for a.e. α, $\frac{1}{2}\le\alpha\le1$. On the other hand, G_α is reverse ϕ-mixing for every α, $0<\alpha\le1$, which follows from [1].

In [28], it is shown that G_α is weak Bernoulli for any $\frac{1}{2}\le\alpha\le1$ but is not ϕ-mixing. It is not hard to show that G_α is weak Bernoulli for any $0<\alpha<\frac{1}{2}$ following the proof given in [28]. On the other hand, it follows that G_α is reverse ϕ-mixing for any $0<\alpha\le1$; see [1]. In the proof of the next Theorem we outline how one can extend the proofs of [28] to the case $0<\alpha<\frac{1}{2}$.

THEOREM 6 *For almost every α, $0<\alpha<1$, G_α is not ϕ-mixing.*

REMARK 5.2. One has ϕ-mixing whenever the orbit of $\alpha-1$ and the left-orbit of α are ultimately periodic. This is the case when α is rational or quadratic irrational.

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Proof. Sketch of the proof. The proof of the non-ϕ-property in [28] is based on the following two properties:

(i) For almost every α, $\frac{1}{2} \leq \alpha \leq 1$, $(G_\alpha^n(\alpha) : n \geq 0)$ is dense in $\mathbb{I}_\alpha$.
(ii) For every α, $\frac{1}{2} < \alpha < \frac{\sqrt{5}-1}{2}$, $G_\alpha^2(\alpha) = G_\alpha^2(\alpha - 1)$ and for every α, $\frac{\sqrt{5}-1}{2} \leq \alpha < 1$, $G_\alpha^2(\alpha) = G_\alpha(\alpha - 1)$, respectively.

The first statement follows from the fact that the set of normal numbers w.r.t. α is independent of α ([22]). Because of Theorem 4 above, we can extend (i) to almost every $0 < \alpha \leq 1$.

The second statement is generalized in [8]: for almost every α, there exists n, m such that $G_\alpha^n(\alpha) = G_\alpha^m(\alpha - 1)$. From this, we can show that thin cylinders exist for almost every α and for any $\delta > 0$. To be more precise, for any $\delta > 0$, a cylinder set $\mathcal{C} = \langle c_{\alpha,1}, c_{\alpha,2}, \ldots, c_{\alpha,\ell} \rangle_\alpha$ is said to be a δ-thin-cylinder if

a) $G_\alpha^\ell(\mathcal{C})$ is an interval,
b) $G_\alpha^\ell : \mathcal{C} \to G_\alpha^\ell(\mathcal{C})$ is bijective,

and

c) $|G_\alpha^\ell(\mathcal{C})| < \delta$.

Once we have a sequence of δ_n-thin cylinders with $\delta_n \searrow 0$, the proof is completely the same as one given in [28] if we choose α so that the matching property holds and $\alpha - 1$ is α-normal. For in this case, there exist n_0, m_0 such that $G_\alpha(\alpha - 1)^{n_0} = G_\alpha^{m_0}(\alpha)$ (matching property). Moreover, there exists $n_\delta > \max(n_0, m_0)$ such that $\min(|\alpha - G_\alpha^{n_\delta}(\alpha - 1)|, |(\alpha - 1) - G_\alpha^{n_\delta}(\alpha - 1)| < \delta)$, which follows from the normality of $\alpha - 1$.

We suppose that $|(\alpha - 1) - G_\alpha(\alpha - 1)| < \delta$. Because of the matching property, see [8], either $\langle c_1(\alpha - 1), c_2(\alpha - 1), \ldots, c_{n_\delta}(\alpha - 1) \rangle_\alpha$ or $\langle c_1(\alpha), c_2(\alpha), \ldots, c_{n_\delta}(\alpha) \rangle_\alpha$ is an δ-thin-cylinder set. This is because of the following: If α is normal, then it means α is not rational nor quadratic. The iteration G_α^n associated with α and G_α^m associated with $\alpha - 1$ are linear fractional transformations. Hence $\alpha \mapsto G_\alpha^n(\alpha)$ and $(\alpha \mapsto G_\alpha^m \circ \text{“} - 1\text{”})(\alpha)$ define the same linear fractional transformation, otherwise α is a fixed point of a linear fractional transformation which means α is either rational or quadratic. We denote by L_r, L_ℓ and S the linear fractional transformations which induce $G_\alpha^n(\alpha)$, $G_\alpha^m(\alpha - 1)$ and $x \mapsto x - 1$, respectively. Then $L_r(\langle c_1(\alpha), \ldots, c_{n_\delta}(\alpha) \rangle_\alpha) = (L_\ell \circ S)(\langle c_1(\alpha), \ldots, c_{n_\delta}(\alpha) \rangle_\alpha)$. This shows that $G_\alpha^n(\langle c_1(\alpha), \ldots, c_{n_\delta}(\alpha) \rangle_\alpha)$ and $G_\alpha^m(\langle c_1(\alpha - 1), \ldots, c_{n_\delta}(\alpha - 1) \rangle_\alpha)$ have one common end point $G_\alpha^n(\alpha)$ and no common inner point. In the case of $|\alpha - G_\alpha(\alpha - 1)| < \delta$, the same holds exactly. In this way, we can choose a sequence of δ_n-thin cylinders and Theorem 6 follows in exactly the same way as in [28]. □

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