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SOME REMARKS ON MINIMAL ASYMPTOTIC BASES OF ORDER THREE

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Abstract

We study a question on minimal asymptotic bases asked by Nathanson ['Minimal bases and powers of 2', *Acta Arith.* **49** (1988), 525–532].

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1. Introduction

Let A be a subset of $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $h \geq 2$, let

$$hA = \{a_1 + \cdots + a_h : a_i \in A, i = 1, \ldots, h\}$$

and, for $c \in \mathbb{N}$, let

$$A - c = \{a - c : a \in A\}.$$

The set A is called an *asymptotic basis* of order h if hA contains all sufficiently large integers. Let P be a subset of an asymptotic basis A of order h. We say that P is *necessary* if $A \setminus P$ is not an asymptotic basis of order h and *unnecessary* if $A \setminus P$ is an asymptotic basis of order h. An asymptotic basis A of order h is *minimal* if $\{a\}$ is necessary for every $a \in A$. Let W be a nonempty subset of $\mathbb{N}$. Denote by $\mathcal{F}^*(W)$ the set of all finite, nonempty subsets of W and by $A(W)$ the set of all numbers of the form $\sum_{f \in F} 2^f$, where $F \in \mathcal{F}^*(W)$.

In 1988, Nathanson [8] gave a construction of minimal asymptotic bases of order h.

THEOREM 1.1 [8]. *Let $h \geq 2$. For $i = 0, 1, \ldots, h-1$, let $W_i = \{n \in \mathbb{N} : n \equiv i \pmod{h}\}$. Then $\cup_{i=0}^{h-1} A(W_i)$ is a minimal asymptotic basis of order h.*

Nathanson posed the following problem in [8]. (Jia and Nathanson restated this problem in [3].)

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PROBLEM 1.2 [8]. Characterise the partitions $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ with the property that $A = A(W_0) \cup \cdots \cup A(W_{h-1})$ is a minimal asymptotic basis of order h.

In 2011, Chen and Chen [1] resolved Problem 1.2 for $h = 2$ and partially for $h \geq 3$.

THEOREM 1.3. *Let $\mathbb{N} = W_1 \cup W_2$ be a partition with $0 \in W_1$ such that W_1 and W_2 are infinite. Then $A = A(W_1) \cup A(W_2)$ is a minimal asymptotic basis of order two if and only if either W_1 contains no consecutive integers or W_2 contains consecutive integers or both.*

THEOREM 1.4. *Let $h \geq 2$ and let t be the least integer with $t > \log h / \log 2$. Let $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ be a partition such that each set W_i is infinite and contains t consecutive integers for $i = 1, \ldots, h$. Then $A = A(W_0) \cup \cdots \cup A(W_{h-1})$ is a minimal asymptotic basis of order h.*

For other related problems on minimal asymptotic bases, see [2, 4–7, 9]. Up to now, there are few results on Problem 1.2. We focus on Problem 1.2 for $h = 3$.

Let $\mathbb{N} = W_0 \cup W_1 \cup W_2$ be a partition such that each set W_i $(i = 0, 1, 2)$ is infinite. There are four possible classes of problems to consider.

Class 1. Each W_i contains no consecutive integers.

Class 2. Each W_i contains consecutive integers.

Class 3. One of the W_i contains consecutive integers; the other two W_i contain no consecutive integers.

Class 4. One of the W_i contains no consecutive integers; the other two W_i contain consecutive integers.

Theorem 1.1 gives an example of a minimal asymptotic basis belonging to Class 1. Theorem 1.4 shows that, for $h = 3$, the answer to Problem 1.2 is affirmative for Class 2. We study Class 3 of Problem 1.2 for $h = 3$ and obtain the following two results.

THEOREM 1.5. *Let $W_0 = \{n \in \mathbb{N} \mid n \equiv 0, 1 \pmod 6\}$, $W_1 = \{n \in \mathbb{N} \mid n \equiv 2, 4 \pmod 6\}$ and $W_2 = \{n \in \mathbb{N} \mid n \equiv 3, 5 \pmod 6\}$. Then $A = A(W_0) \cup A(W_1) \cup A(W_2)$ is a minimal asymptotic basis of order three.*

REMARK 1.6. By a similar proof to that of Theorem 1.5, for any $i \in \{0, 1, 2, 3, 4, 5\}$, if $W_0 = \{n \in \mathbb{N} \mid n \equiv i, i + 1 \pmod 6\}$, $W_1 = \{n \in \mathbb{N} \mid n \equiv i + 2, i + 4 \pmod 6\}$ and $W_2 = \{n \in \mathbb{N} \mid n \equiv i + 3, i + 5 \pmod 6\}$, then $A = A(W_0) \cup A(W_1) \cup A(W_2)$ is a minimal asymptotic basis of order three.

THEOREM 1.7. *Let $\mathbb{N} = W_0 \cup W_1 \cup W_2$ be a partition such that each set W_i is infinite for $i \in \{0, 1, 2\}$. Suppose that W_0 contains consecutive integers, W_1 and W_2 contain no two consecutive integers and $W_1 - 1 \subseteq W_0$. Then $A = A(W_0) \cup A(W_1) \cup A(W_2)$ is not a minimal asymptotic basis of order three.*

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2. A lemma

For $W \subseteq \mathbb{N}$, set $W(x) = |\{n \in W \mid n \leq x\}|$.

LEMMA 2.1 [8, Lemma 1].

(a) *If W_1 and W_2 are disjoint subsets of $\mathbb{N}$, then $A(W_1) \cap A(W_2) = \emptyset$.*

(b) *If $W \subseteq \mathbb{N}$ and $W(x) = \theta x + O(1)$ for some $\theta \in (0, 1]$, then there exist positive constants c_1 and c_2 such that*

$$c_1 x^\theta < A(W)(x) < c_2 x^\theta$$

for all x sufficiently large.

(c) *Suppose that $\mathbb{N} = W_0 \cup W_1 \cup \cdots \cup W_{h-1}$, where $W_i \neq \emptyset$ for $i = 0, \ldots, h-1$. Then $A = A(W_0) \cup A(W_1) \cup \cdots \cup A(W_{h-1})$ is an asymptotic basis of order h. Indeed, $hA = \{n \in \mathbb{N} \mid n \geq h\}$ and $h(A \cup \{0\}) = \mathbb{N}$.*

3. Proof of Theorem 1.5

By Lemma 2.1, A is an asymptotic basis of order three. To prove that A is minimal, it is sufficient to prove that $\{x\}$ is necessary for every $x \in A$. Let $x \in A$. Then $x \in A(W_u)$ for some $u \in \{0, 1, 2\}$ and so x has a unique 2-adic representation of the form

$$x = \sum_{f \in F_u} 2^f,$$

where F_u is a finite, nonempty subset of W_u. Let f_u be the maximal element of the set F_u. Then there exists a unique $k \in \mathbb{N}$ such that

$$f_u = 6k + v_u \tag{3.1}$$

for some $v_u \in \{0, 1, 2, 3, 4, 5\}$. If $x \in A(W_0)$, then choose

$$m = x + \Big(\sum_{i=0}^{k}(2^{6i+2} + 2^{6i+4}) + 2^{6t+2}\Big) + \Big(\sum_{i=0}^{k}(2^{6i+3} + 2^{6i+5}) + 2^{6t+3}\Big). \tag{3.2}$$

If $x \in A(W_1)$, then choose

$$m = \Big(\sum_{i=0}^{k+1} 2^{6i} + \sum_{i=0}^{k} 2^{6i+1} + 2^{6t}\Big) + x + \Big(\sum_{i=0}^{k}(2^{6i+3} + 2^{6i+5}) + 2^{6t+3}\Big). \tag{3.3}$$

If $x \in A(W_2)$, then choose

$$m = \Big(\sum_{i=0}^{k+1}(2^{6i} + 2^{6i+1}) + 2^{6t}\Big) + \Big(\sum_{i=0}^{k+1} 2^{6i+2} + \sum_{i=0}^{k} 2^{6i+4} + 2^{6t+2}\Big) + x. \tag{3.4}$$

In all cases, t is any positive integer greater than $k + 1$.

By Lemma 2.1(c), for each $i \in \{0, 1, 2\}$, there are a $j_i \in \{0, 1, 2\}$ and an $m_i \in A(W_{j_i})$ so that

$$m = m_0 + m_1 + m_2. \tag{3.5}$$

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For $i = 0, 1, 2$, let $c_i^{(n)}$ be the least nonnegative residue of m_i modulo 2^n. Write $M = \{m_0, m_1, m_2\}$. We shall show that, for any $j \in \{0, 1, 2\}$,

$$M \not\subseteq \bigcup_{i\in\{0,1,2\}\setminus\{j\}} A(W_i).$$

Case 1: $x \in A(W_0)$. By (3.1), $f_0 = 6k$ or $f_0 = 6k + 1$.

Suppose that $M \subseteq A(W_1) \cup A(W_2)$. Then

$$\sum_{i=0}^{2} c_i^{(f_0+1)} \le 3 \cdot \sum_{i=0}^{k-1} (2^{6i+3} + 2^{6i+5}) = \sum_{i=0}^{k-1} 2^{6i+4} + \sum_{i=1}^{k} 2^{6i} + \sum_{i=0}^{k-1} (2^{6i+3} + 2^{6i+5}).$$

By (3.2),

$$m \equiv x + \sum_{i=0}^{k-1} (2^{6i+2} + 2^{6i+4}) + \sum_{i=0}^{k-1} (2^{6i+3} + 2^{6i+5}) \pmod{2^{f_0+1}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(f_0+1)} \pmod{2^{f_0+1}}$, which contradicts (3.5).

Suppose that $M \subseteq A(W_0) \cup A(W_2)$. Then

$$\sum_{i=0}^{2} c_i^{(6k+5)} \le 3 \cdot \Big(\sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5} \Big) = \sum_{i=0}^{k} 2^{6i+4} + \sum_{i=1}^{k} 2^{6i} + \sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5}.$$

By (3.2),

$$m \equiv x + \sum_{i=0}^{k} (2^{6i+2} + 2^{6i+4}) + \sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5} \pmod{2^{6k+5}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6k+5)} \pmod{2^{6k+5}}$, which contradicts (3.5).

Suppose that $M \subseteq A(W_0) \cup A(W_1)$. Then

$$\sum_{i=0}^{2} c_i^{(6(k+1))} \le 3 \cdot \sum_{i=0}^{k} (2^{6i+2} + 2^{6i+4}) = \sum_{i=0}^{k} (2^{6i+2} + 2^{6i+4}) + \sum_{i=0}^{k} (2^{6i+3} + 2^{6i+5}).$$

By (3.2),

$$m \equiv x + \sum_{i=0}^{k} (2^{6i+2} + 2^{6i+4}) + \sum_{i=0}^{k} (2^{6i+3} + 2^{6i+5}) \pmod{2^{6(k+1)}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6(k+1))} \pmod{2^{6(k+1)}}$, which contradicts (3.5).

Case 2: $x \in A(W_1)$. By (3.1), $f_1 = 6k + 2$ or $f_1 = 6k + 4$.

Suppose that $M \subseteq A(W_1) \cup A(W_2)$. Then

$$\sum_{i=0}^{2} c_i^{(6k+2)} \le 3 \cdot \sum_{i=0}^{k-1} (2^{6i+3} + 2^{6i+5}) = \sum_{i=1}^{k} 2^{6i} + \sum_{i=0}^{k-1} 2^{6i+4} + \sum_{i=0}^{k-1} (2^{6i+3} + 2^{6i+5}).$$

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By (3.3),

$$m \equiv \sum_{i=0}^{k}(2^{6i} + 2^{6i+1}) + \sum_{f\in F_1, f<6k+2} 2^f + \sum_{i=0}^{k-1}(2^{6i+3} + 2^{6i+5}) \pmod{2^{6k+2}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6k+2)} \pmod{2^{6k+2}}$, which contradicts (3.5).

Suppose that $M \subseteq A(W_0) \cup A(W_2)$. If $f_1 = 6k+2$, then

$$\begin{aligned}\sum_{i=0}^{2} c_i^{(6k+4)} &\le 2\cdot\sum_{i=0}^{k}(2^{6i} + 2^{6i+1}) + \sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5} \\ &= \sum_{i=0}^{k}(2^{6i} + 2^{6i+1}) + \sum_{i=0}^{k}(2^{6i} + 2^{6i+1}) + \sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5}.\end{aligned}$$

By (3.3),

$$m \equiv \sum_{i=0}^{k}(2^{6i} + 2^{6i+1}) + x + \sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5} \pmod{2^{6k+4}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6k+4)} \pmod{2^{6k+4}}$, which contradicts (3.5). If $f_1 = 6k+4$, then

$$\begin{aligned}\sum_{i=0}^{2} c_i^{(6k+5)} &\le 3\cdot\Big(\sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5}\Big) \\ &= \sum_{i=1}^{k} 2^{6i} + \sum_{i=0}^{k-1} 2^{6i+4} + 2^{6k+4} + \sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5}.\end{aligned}$$

By (3.3),

$$m \equiv \sum_{i=0}^{k}(2^{6i} + 2^{6i+1}) + x + \sum_{i=0}^{k} 2^{6i+3} + \sum_{i=0}^{k-1} 2^{6i+5} \pmod{2^{6k+5}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6k+5)} \pmod{2^{6k+5}}$, which contradicts (3.5).

Suppose that $M \subseteq A(W_0) \cup A(W_1)$. Then

$$\sum_{i=0}^{2} c_i^{(6(k+1)+1)} \le \sum_{i=0}^{k+1} 2^{6i} + \sum_{i=0}^{k} 2^{6i+1} + 2\cdot\sum_{i=0}^{k}(2^{6i+2} + 2^{6i+4}).$$

By (3.3),

$$m \equiv \sum_{i=0}^{k+1} 2^{6i} + \sum_{i=0}^{k} 2^{6i+1} + x + \sum_{i=0}^{k}(2^{6i+3} + 2^{6i+5}) \pmod{2^{6(k+1)+1}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6(k+1)+1)} \pmod{2^{6(k+1)+1}}$, which contradicts (3.5).

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Case 3: $x \in A(W_2)$. By (3.1), $f_2 = 6k+3$ or $f_2 = 6k+5$.

Suppose that $M \subseteq A(W_1) \cup A(W_2)$. Then

$$\sum_{i=0}^{2} c_i^{(6k+3)} \le \sum_{i=0}^{k} 2^{6i+2} + \sum_{i=0}^{k-1} 2^{6i+4} + 2 \cdot \sum_{i=0}^{k-1} (2^{6i+3} + 2^{6i+5}).$$

By (3.4),

$$m \equiv \sum_{i=0}^{k} (2^{6i} + 2^{6i+1}) + \sum_{i=0}^{k} 2^{6i+2} + \sum_{i=0}^{k-1} 2^{6i+4} + \sum_{f \in F_2, f < 6k+3} 2^f \pmod{2^{6k+3}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6k+3)} \pmod{2^{6k+3}}$, which contradicts (3.5).

Suppose that $M \subseteq A(W_0) \cup A(W_2)$. Then

$$\begin{aligned}\sum_{i=0}^{2} c_i^{(6(k+1)+3)} &\le 2 \cdot \sum_{i=0}^{k+1} (2^{6i} + 2^{6i+1}) + \sum_{i=0}^{k+1} 2^{6i} + \sum_{i=0}^{k} 2^{6i+1} \\ &= \sum_{i=0}^{k+1} (2^{6i} + 2^{6i+1}) + \sum_{i=0}^{k+1} 2^{6i+2} + \sum_{i=0}^{k} 2^{6i+1}.\end{aligned}$$

By (3.4),

$$m \equiv \sum_{i=0}^{k+1} (2^{6i} + 2^{6i+1}) + \sum_{i=0}^{k+1} 2^{6i+2} + \sum_{i=0}^{k} 2^{6i+4} + x \pmod{2^{6(k+1)+3}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6(k+1)+3)} \pmod{2^{6(k+1)+3}}$, which contradicts (3.5).

Suppose that $M \subseteq A(W_0) \cup A(W_1)$. If $f_2 = 6k+3$, then

$$\begin{aligned}\sum_{i=0}^{2} c_i^{(6k+4)} &\le 3 \cdot \Big(\sum_{i=0}^{k} 2^{6i+2} + \sum_{i=0}^{k-1} 2^{6i+4} \Big) \\ &= \sum_{i=0}^{k-1} (2^{6i+3} + 2^{6i+5}) + \sum_{i=0}^{k} 2^{6i+2} + \sum_{i=0}^{k-1} 2^{6i+4} + 2^{6k+3}.\end{aligned}$$

By (3.4),

$$m \equiv \sum_{i=0}^{k} (2^{6i} + 2^{6i+1}) + \sum_{i=0}^{k} 2^{6i+2} + \sum_{i=0}^{k-1} 2^{6i+4} + x \pmod{2^{6k+4}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6k+4)} \pmod{2^{6k+4}}$, which contradicts (3.5). If $f_2 = 6k+5$, then

$$\sum_{i=0}^{2} c_i^{(6(k+1)+1)} \le \sum_{i=0}^{k+1} 2^{6i} + \sum_{i=0}^{k} 2^{6i+1} + 2 \cdot \sum_{i=0}^{k} (2^{6i+2} + 2^{6i+4}).$$

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By (3.4),

$$m \equiv \sum_{i=0}^{k+1} 2^{6i} + \sum_{i=0}^{k} 2^{6i+1} + \sum_{i=0}^{k} (2^{6i+2} + 2^{6i+4}) + x \pmod{2^{6(k+1)+1}}.$$

Thus, $m \not\equiv \sum_{i=0}^{2} c_i^{(6(k+1)+1)} \pmod{2^{6(k+1)+1}}$, which contradicts (3.5).

In all, we have proved that $M \not\subseteq \bigcup_{i\in\{0,1,2\}\setminus\{j\}} A(W_i)$ for any $j \in \{0, 1, 2\}$, that is, $m_i = x$ for some $i \in \{0, 1, 2\}$. Moreover, the 2-adic representation of m is unique and thus $m \notin 3(A \setminus \{x\})$.

This completes the proof of Theorem 1.5.

4. Proof of Theorem 1.7

By Lemma 2.1, A is an asymptotic basis of order three. Choose $w \in W_0$ such that $w - 1 \in W_0$ and set $a = 2^w$. We will show that $A \setminus \{a\}$ is an asymptotic basis of order three, so A is not a minimal asymptotic basis of order three. For every sufficiently large integer n, we have $n = a_1 + a_2 + a_3$, where $a_1, a_2, a_3 \in A$. If $a_i \neq a$ for all $i \in \{1, 2, 3\}$, then $n \in 3(A \setminus \{a\})$. So, it suffices to show that if $a_1, a_2 \in A$ and $n = a + a_1 + a_2$, then $n \in 3(A \setminus \{a\})$ for all but at most finitely many integers a_1, a_2. By symmetry, we need to discuss the following six cases.

Case 1: $a_1, a_2 \in A(W_0)$. Write

$$a_1 = \sum_{i\in I} 2^i, \quad a_2 = \sum_{j\in J} 2^j,$$

where I, J are finite, nonempty subsets of W_0. If $I \cap J = \emptyset$, then

$$n = 2^{w-1} + 2^{w-1} + (a_1 + a_2) \in 3(A \setminus \{a\}).$$

Now suppose that $I \cap J \neq \emptyset$.

Subcase 1.1: $I, J \not\subseteq \{w, w - 1\}$. If $|I| \geq 2$, then

$$n = (2^w + 2^{i_0}) + \sum_{i\in I\setminus\{i_0\}} 2^i + \sum_{j\in J} 2^j$$

for some $i_0 \in I$ and $i_0 = w$ if $w \in I$, so $n \in 3(A \setminus \{a\})$. If $I = \{i\}$, then $i \neq w, w - 1$ and

$$n = 2^{w-1} + (2^{w-1} + 2^i) + \sum_{j\in J} 2^j \in 3(A \setminus \{a\}).$$

Subcase 1.2: $I, J \subseteq \{w, w - 1\}$. There are only finitely many integers n in this case.

Subcase 1.3: $I \subseteq \{w, w - 1\}$ *and* $J \not\subseteq \{w, w - 1\}$.

Subcase 1.3.1: $I = \{w\}$. Then

$$n = 2^{w-1} + (2^{w-1} + 2^w) + \sum_{j\in J} 2^j \in 3(A \setminus \{a\}).$$

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Subcase 1.3.2: $I = \{w-1\}$. Since $I \cap J \neq \emptyset$, we have $|J| \geq 2$. If $|J| = 2$, then $J = \{w-1, j\}$ for some $j \neq w$ and

$$n = (2^w + 2^{w-1}) + 2^{w-1} + 2^j \in 3(A \setminus \{a\}).$$

If $|J| \geq 3$, choose a $j_0 \in J$ such that $j_0 \neq w$. Then

$$n = (2^w + 2^{w-1}) + 2^{j_0} + \sum_{j \in J \setminus \{j_0\}} 2^j \in 3(A \setminus \{a\}).$$

Subcase 1.3.3: $I = \{w, w-1\}$. Then

$$n = 2^{w+1} + 2^{w-1} + \sum_{j \in J} 2^j \in 3(A \setminus \{a\}).$$

Case 2: $a_1 \in A(W_0), a_2 \in A(W_1)$. Write

$$a_1 = \sum_{s \in S} 2^s, \quad a_2 = \sum_{t \in T} 2^t,$$

where S, T are finite, nonempty subsets of W_0 and W_1, respectively.

Subcase 2.1: $|S| \geq 3$. Then

$$n = (2^w + 2^{s_0}) + \sum_{s \in S \setminus \{s_0\}} 2^s + \sum_{t \in T} 2^t$$

for some $s_0 \in S$ and hence $n \in 3(A \setminus \{a\})$.

Subcase 2.2: $|S| = 2$. Write $S = \{s_1, s_2\}$. If $w, w-1 \notin S$, then

$$n = (2^w + 2^{s_1}) + 2^{s_2} + \sum_{t \in T} 2^t \in 3(A \setminus \{a\}).$$

If $w \in S$ and $w-1 \notin S$, then

$$n = 2^{w-1} + (2^{w-1} + 2^{s_1} + 2^{s_2}) + \sum_{t \in T} 2^t \in 3(A \setminus \{a\}).$$

If $w-1 \in S$ and $w \notin S$, then $S = \{w-1, s\}$ and

$$n = (2^w + 2^{w-1}) + 2^s + \sum_{t \in T} 2^t \in 3(A \setminus \{a\}).$$

If $w, w-1 \in S$, then

$$n = 2^{w+1} + 2^{w-1} + \sum_{t \in T} 2^t \in 3(A \setminus \{a\}).$$

Subcase 2.3: $|S| = 1$. Let $S = \{s\}$. If $s \neq w-1$, then

$$n = 2^{w-1} + (2^{w-1} + 2^s) + \sum_{t \in T} 2^t \in 3(A \setminus \{a\}).$$

If $s = w - 1$, then

$$n = \begin{cases} (2^w + 2^{w-1}) + 2^{t_0} + \sum_{t \in T \setminus \{t_0\}} 2^t, & \text{if } |T| \geq 2, \\ (2^w + 2^{w-1}) + 2^{t_0 - 1} + 2^{t_0 - 1}, & \text{if } T = \{t_0\} \end{cases}$$

and hence $n \in 3(A \setminus \{a\})$ except for finitely many integers n.

Case 3: $a_1 \in A(W_0), a_2 \in A(W_2)$. Proceeding as in Case 2, $n \in 3(A \setminus \{a\})$ except for finitely many integers n.

Case 4: $a_1 \in A(W_1), a_2 \in A(W_2)$. Write

$$a_1 = \sum_{k \in K} 2^k,$$

where K is a finite, nonempty subset of W_1. Since $w, w-1 \in W_0$, it follows that $w - 1 \notin K - 1$. Since $K - 1 \subseteq W_0$,

$$n = \Big(2^{w-1} + \sum_{k \in K} 2^{k-1}\Big) + \Big(2^{w-1} + \sum_{k \in K} 2^{k-1}\Big) + a_2 \in 3(A \setminus \{a\}).$$

Case 5: $a_1, a_2 \in A(W_1)$. Proceeding as in Case 4, $n \in 3(A \setminus \{a\})$ except for finitely many integers n.

Case 6: $a_1, a_2 \in A(W_2)$. Write

$$a_1 = \sum_{u \in U} 2^u, \quad a_2 = \sum_{v \in V} 2^v,$$

where U, V are finite, nonempty subsets of W_2. If $U \cap V = \emptyset$, then

$$n = 2^{w-1} + 2^{w-1} + (a_1 + a_2) \in 3(A \setminus \{a\}).$$

Now suppose that $U \cap V \neq \emptyset$. We claim that $x + 1 \in W_0$ for all $x \in U \cap V$. Otherwise, since $W_1 - 1 \subseteq W_0$, if there exists an $x \in U \cap V$ such that $x + 1 \in W_1$, then $x = (x+1) - 1 \in W_0$, which is a contradiction. Let $U_0 = (U \cup V) \setminus (U \cap V)$.

Subcase 6.1: $U_0 \neq \emptyset$. If $|U \cap V| \geq 2$, then $w \notin (U \cap V) + 1$ and

$$n = (2^w + 2^{u_0+1}) + \sum_{u \in (U \cap V) \setminus \{u_0\}} 2^{u+1} + \sum_{u \in U_0} 2^u$$

for some $u_0 \in U \cap V$; hence, $n \in 3(A \setminus \{a\})$. If $|U \cap V| = 1$, let $U \cap V = \{u\}$. If $u + 1 \neq w - 1$, then

$$n = 2^{w-1} + (2^{w-1} + 2^{u+1}) + \sum_{u \in U_0} 2^u \in 3(A \setminus \{a\}).$$

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If $u + 1 = w - 1$, then

$$n = \begin{cases} (2^w + 2^{w-1}) + 2^{u_0} + \sum_{u \in U_0 \setminus \{u_0\}} 2^u, & \text{if } |U_0| \geq 2, \\ (2^w + 2^{w-1}) + 2^{u_0 - 1} + 2^{u_0 - 1}, & \text{if } U_0 = \{u_0\} \end{cases}$$

and hence $n \in 3(A \setminus \{a\})$ except for finitely many integers n.

Subcase 6.2: $U_0 = \emptyset$. Then

$$n = 2^{w-1} + 2^{w-1} + \sum_{u \in U \cap V} 2^{u+1} \in 3(A \setminus \{a\}).$$

This completes the proof of Theorem 1.7.

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