SEPARATING SUBVERSION FORCING AXIOMS

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Abstract. We study a family of variants of Jensen's *subcomplete forcing axiom*, SCFA, and *subproper forcing axiom*, SubPFA. Using these, we develop a general technique for proving nonimplications of SCFA, SubPFA and their relatives and give several applications. For instance, we show that SCFA does not imply $\mathsf{MA}^+(\sigma\text{-closed})$ and SubPFA does not imply Martin's Maximum.

§1. Introduction. In this article, we study variants of subcomplete and subproper forcing classes with an eye towards investigating and distinguishing their forcing principles. Subcomplete and subproper forcing are two classes of forcing notions introduced by Jensen in [16] in connection with the extended Namba problem (see [17, Section 6.4][1]. Both are iterable with revised countable support and generalize significantly σ-closed and proper forcing notions, respectively, while allowing, under some circumstances, new cofinal ω-sequences of ordinals to be added to uncountably cofinal cardinals. As such, each comes with a forcing axiom (consistent relative to a supercompact cardinal). The forcing axiom for subcomplete forcing, in particular, dubbed SCFA by Jensen in [14, 17] is especially interesting as it is consistent with $\diamondsuit$ while implying some of the strong, structural consequences of MM (see [17, Section 4]. Since their initial introduction subcomplete and subproper forcing have been tied to several applications and received further treatment (see, for instance, [7, 9, 11, 12]).

Unfortunately, there is a fly in the ointment of the birth of the theory, initially present in [16, Lemma 1, p. 18] in the form of a missing needed assumption of CH (see also [17, Chapter 3, p. 154]. A consequence of this error led to the (false) conclusion that the SCFA implied the failure of $\square_{\omega_1}$ when in fact a careful reading of the proof of that result shows that SCFA implies the failure of $\square_{2^{\aleph_0}}$ (hence the conclusion under CH), the gap was first observed by Cox. An initial starting point for us in this work was to determine if the gap was fixable and discovered that it was not. Indeed, SCFA is consistent with $\square_{\omega_1}$.

THEOREM 1.1 (See Theorem 3.1). *Assuming the consistency of a supercompact cardinal*, SCFA *does not imply the failure of* $\square_{\aleph_1}$ *when* CH *fails.*

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[1]See Definition 1.4 below for precise definitions.

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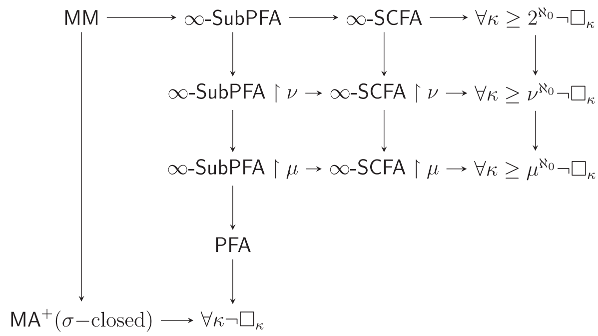

FIGURE 1. Subversion forcing axioms, $\Box$ principles and their relations. An arrow means direct implication.

This result led to a general method of separating various principles related to SCFA and this method is, in essence, the subject of the present work. See [10] for a very detailed and meticulous discussion of the error as well as its propagation in the literature and corrections.

Already in [12], the second author and Fuchs found (seemingly) more general classes, dubbed "∞-subcomplete" and "∞-subproper" each containing their non "∞" version, respectively, and proved a variety of iteration and preservation theorems. The main theorem in that work was that the forcing axiom for ∞-subcomplete forcing notions, ∞-SCFA, is compatible with a large variety of behavior on $\aleph_1$ when CH fails. For instance, $\aleph_1 = \mathfrak{d} < \mathfrak{c} = \aleph_2$ and the existence of Souslin trees are both consistent with ∞-SCFA $+\neg$CH. All of these results also hold for SCFA as well (with no ∞).

In this article, we combine the ∞-versions of these forcing classes with further parametrization "above μ" for cardinals μ, initially investigated, somewhat sparingly, by Jensen in [15, Chapter 3]. This leads to a large family of forcing axioms ∞-SubPFA $\restriction \mu$ and ∞-SCFA $\restriction \mu$, where ∞-SubPFA and ∞-SCFA coincide with ∞-SubPFA $\restriction 2^{\aleph_0}$ and ∞-SCFA $\restriction 2^{\aleph_0}$, respectively. The main outcome of this work is an investigation into how these axioms relate to one another and to other, more well known axioms such as MM and $\mathsf{MA}^+(\sigma$-closed$)$. Formal definitions will be given in the second part of this introduction and Section 2 but the definitions of these axioms alongside well known results provide almost immediately that the following diagram of implications holds with $2^{\aleph_0} \leq \nu < \mu$ cardinals.

The main result of this work is that essentially no arrows are missing from Figure 1. above.

MAIN THEOREM 1.1. *Let $2^{\aleph_0} \leq \nu \leq \lambda < \mu = \lambda^+$ be cardinals with $\nu^\omega < \mu$. Assuming the consistency of a supercompact cardinal, the implications given in Figure 1. are complete in the sense that if no composition of arrows exists from one axiom to another then there is a model of* ZFC *in which the implication fails*[2].

[2]Except for the trivial $\forall\kappa\neg\Box_\kappa \to \forall\kappa \geq 2^{\aleph_0}\neg\Box_\kappa$ which did not fit aesthetically into the picture.

As a corollary of this theorem and its proof, we obtain separations of several "subversion" forcing principles from other, more well-studied reflection principles and forcing axioms. As noted above, in particular, this corrects the aforementioned error in the literature by showing SCFA to be consistent with $\square_{\omega_1}$. Another sample application is the following.

COROLLARY 1.2. *Assuming the consistency of a supercompact cardinal*, SCFA *does not imply* $\mathsf{MA}^+(\sigma\text{– closed})$.

The rest of this article is organized as follows. In the next section of this introduction, we give relevant background and terminology. In the next section, we introduce the variants ∞-subcompleteness and ∞-subproperness above μ and discuss some of their properties. In Section 3, we study the forcing axioms associated with these classes and show, among other things, that they are distinct as well as the fact ∞-SCFA implies neither $\mathsf{MA}^+(\sigma\text{– closed})$ nor $\neg\square_\kappa$ for any $\kappa < 2^\omega$. In Section 4, we continue this investigation and show that ∞-SubPFA does not imply MM. Section 5 concludes with some final remarks and open problems.

1.1. Preliminaries. We conclude this introduction with the key definitions we will use throughout, beginning with that of subproperness and subcompleteness. These are these two classes of forcing notions defined by Jensen in [17] which have found several applications (see, e.g., [7, 11, 16, 21]). More discussion of these concepts can be found in [17] or [12]. Before beginning with the definition, we will need one preliminary definition. Below, we denote by ZFC^- the axioms of ZFC without the power set axiom[3].

DEFINITION 1.3. A transitive set N (usually a model of ZFC^-) is *full* if there is an ordinal γ so that $L_\gamma(N) \models \mathsf{ZFC}^-$ and N is regular in $L_\gamma(N)$ i.e., for all $x \in N$ and $f \in L_\gamma(N)$ if $f : x \to N$ then $\mathrm{ran}(f) \in N$.

DEFINITION 1.4. Let $\mathbb{P}$ be a forcing notion and let $\delta(\mathbb{P})$ be the least size of a dense subset of $\mathbb{P}$.

(1) We say that $\mathbb{P}$ is *subcomplete* if for all sufficiently large θ, $\tau > \theta$ so that $H_\theta \subseteq N := L_\tau[A] \models \mathsf{ZFC}^-$, $s \in N$, $\sigma : \bar{N} \prec N$ countable, transitive, and full with $\sigma(\bar{\mathbb{P}}, \bar{s}, \bar{\theta}) = \mathbb{P}, s, \theta$, if $\bar{G} \subseteq \bar{\mathbb{P}} \cap \bar{N}$ is generic then there is a $p \in \mathbb{P}$ so that if $p \in G$ is $\mathbb{P}$-generic over V then in $V[G]$ there is a $\sigma' : \bar{N} \prec N$ so that
  1. $\sigma'(\bar{\mathbb{P}}, \bar{s}, \bar{\theta}, \bar{\mu}) = \mathbb{P}, s, \theta, \mu$
  2. $\sigma' “\bar{G} \subseteq G$
  3. $\mathrm{Hull}^N(\delta(\mathbb{P}) \cup \mathrm{ran}(\sigma)) = \mathrm{Hull}^N(\delta(\mathbb{P}) \cup \mathrm{ran}(\sigma'))$.

(2) We say that $\mathbb{P}$ is *subproper* if for all sufficiently large θ, $\tau > \theta$ so that $H_\theta \subseteq N := L_\tau[A] \models \mathsf{ZFC}^-$, $s \in N$, $p \in N \cap \mathbb{P}$, $\sigma : \bar{N} \prec N$ countable, transitive and full with $\sigma(\bar{p}, \bar{\mathbb{P}}, \bar{s}, \bar{\theta}) = p, \mathbb{P}, s, \theta$, there is a $q \in \mathbb{P}$ so that $q \leq p$ and if $q \in G$ is $\mathbb{P}$-generic over V then in $V[G]$ there is a $\sigma' : \bar{N} \prec N$ so that
  1. $\sigma'(\bar{p}, \bar{\mathbb{P}}, \bar{s}, \bar{\theta}) = p, \mathbb{P}, s, \theta$
  2. $(\sigma')^{-1} “G$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$
  3. $\mathrm{Hull}^N(\delta(\mathbb{P}) \cup \mathrm{ran}(\sigma)) = \mathrm{Hull}^N(\delta(\mathbb{P}) \cup \mathrm{ran}(\sigma'))$.

[3] There is a subtlety here, see [13]. As usual, we mean by ZFC^- the theory of ZFC without the powerset axiom and the replacement scheme replaced by the collection scheme, see [13] for full details.

Note that the special case, where $\sigma = \sigma'$ is properness (for subproperness) and (up to forcing equivalence) σ-closedness (for subcomplete). To explicate this in the later case, we recall the definition of *completeness*, which is due to Shelah originally though we take Jensen's definition[4] from [17, p. 112].

DEFINITION 1.5. A forcing notion $\mathbb{P}$ is said to be *complete* if for all sufficiently large θ $\mathbb{P} \in H_\theta$ and all countable, transitive $\sigma : \bar{N} \prec H_\theta$ with $\sigma(\bar{\mathbb{P}}) = \mathbb{P}$, if $\bar{G}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$ then there is a $p \in \mathbb{P}$ forcing that $\sigma``\bar{G} \subseteq G$.

It's clear that σ-closed forcing notions are complete. What is less clear (though equally true) is that conversely if $\mathbb{P}$ is complete it is forcing equivalent to a σ-closed forcing notion, a result due to Jensen (see [17, Lemma 1.3, Chapter 3]. In this sense, therefore, subcompleteness is the "subversion" of σ-closedness.

It was pointed out in [12] that the "Hulls" condition 3) in both definitions is somewhat unnatural. It is never used in applications and appears solely for the purpose of proving the iteration theorem, [17, Theorem 3]. In [12] Fuchs and the second author showed that by iterating with Miyamoto's *nice iterations* this condition could be avoided. As such, it makes sense to define the following.

DEFINITION 1.6. Let $\mathbb{P}$ be a forcing notion.

(1) We say that $\mathbb{P}$ is ∞-*subcomplete* if for all sufficiently large θ, $\tau > \theta$ so that $H_\theta \subseteq N := L_\tau[A] \models \mathsf{ZFC}^-$, $s \in N$, $\sigma : \bar{N} \prec N$ countable, transitive, and full with $\sigma(\bar{\mathbb{P}}, \bar{s}, \bar{\theta}) = \mathbb{P}, s, \theta$, if $\bar{G} \subseteq \bar{\mathbb{P}} \cap \bar{N}$ is generic then there is a $p \in \mathbb{P}$ so that if $p \in G$ is $\mathbb{P}$-generic over V then in $V[G]$ there is a $\sigma' : \bar{N} \prec N$ so that
 1. $\sigma'(\bar{\mathbb{P}}, \bar{s}, \bar{\theta}, \bar{\mu}) = \mathbb{P}, s, \theta, \mu$;
 2. $\sigma'``\bar{G} \subseteq G$.

(2) We say that $\mathbb{P}$ is ∞-*subproper* if for all sufficiently large θ, $\tau > \theta$ so that $H_\theta \subseteq N := L_\tau[A] \models \mathsf{ZFC}^-$, $s \in N$, $p \in N \cap \mathbb{P}$, $\sigma : \bar{N} \prec N$ countable, transitive, and full with $\sigma(\bar{p}, \bar{\mathbb{P}}, \bar{s}, \bar{\theta}) = p, \mathbb{P}, s, \theta$, there is a $q \in \mathbb{P}$ so that $q \leq p$ and if $q \in G$ is $\mathbb{P}$-generic over V then in $V[G]$ there is a $\sigma' : \bar{N} \prec N$ so that
 1. $\sigma'(\bar{p}, \bar{\mathbb{P}}, \bar{s}, \bar{\theta}) = p, \mathbb{P}, s, \theta$;
 2. $(\sigma')^{-1}``G$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$.

To be clear, this is just the same as the definitions of the "non-∞" versions, simply with the additional "Hulls" condition removed. As mentioned, these classes come with an iteration theorem.

THEOREM 1.7 (Theorem 3.19 (for Subcomplete) and Theorem 3.20 (for Subproper) of [12]). *Let γ be an ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ be a nice iteration in the sense of Miyamoto so that for all $\alpha < \gamma$, we have $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is ∞-subproper (respectively, ∞-subcomplete). Then, $\mathbb{P}_\gamma$ is ∞-subproper (respectively, ∞-subcomplete).*

We note that the above theorem in the case of ∞-subproper forcing was originally proved first independently by Miyamoto in [20]. A consequence of this theorem (initially observed for the non ∞-versions by Jensen) is that, modulo a supercompact cardinal, these classes have a consistent forcing axiom.

[4]Note that Jensen defines completeness using Boolean algebras but the definition we give below can easily be seen to be equivalent.

DEFINITION 1.8. Let Γ be a class of forcing notions. The *forcing axiom for* Γ, denoted $\mathsf{FA}(\Gamma)$ is the statement that for all $\mathbb{P}$ in Γ and any ω_1-sequence of dense subsets of $\mathbb{P}$, say $\{D_i \mid i < \omega_1\}$ there is a filter $G \subseteq \mathbb{P}$ which intersects every D_i.

If Γ is the class of (∞-)subproper forcing notions we denote $\mathsf{FA}(\Gamma)$ by (∞-)SubPFA. Similarly, if Γ is the class of (∞-)subcomplete forcing notions we denote $\mathsf{FA}(\Gamma)$ by (∞-)SCFA.

It is not known whether up to forcing equivalence each class is simply equal to its "∞"-version or if their corresponding forcing axioms are equivalent. However, since the "∞" versions are more general (or appear to be) and avoid the unnecessary technicality of computing hulls, we will work with them in this article. Nearly, everything written here could be formulated for the "non-∞" versions equally well, though we leave the translation to the particularly persnickety reader.

If $\Gamma \subseteq \Delta$ then $\mathsf{FA}(\Delta)$ implies $\mathsf{FA}(\Gamma)$ so we get the following collection of implications, which are part of Figure 1.

PROPOSITION 1.9. $\mathsf{MM} \to \infty$-SubPFA $\to$ PFA *and* $\mathsf{MM} \to \infty$-SubPFA $\to$ ∞-SCFA

Here, MM, known as *Martin's Maximum* and introduced in [5], is the forcing axiom for forcing notions which preserve stationary subsets of ω_1 (all ∞-subproper forcing notions have this property) and PFA is the forcing axiom for proper forcing notions. It is known from the work of Jensen (see also [12] that none of the above implications can be reversed with the exception of whether SubPFA implies MM. In this article, we will show the consistency of SubPFA $+ \neg$MM, see Theorem 4.1 below.

On that note, we move to our last preliminary. Many of the theorems in this article involve showing that we can preserve some fragment of ∞-SCFA (or ∞-SubPFA) via a forcing killing another fragment of it. Towards this end, we will need an extremely useful theorem due to Cox. Below, recall that a class of forcing notions Γ is *closed under restrictions* (see [2, Definition 39]) if for all $\mathbb{P} \in \Gamma$, and all $p \in \mathbb{P}$ the lower cone $\mathbb{P} \upharpoonright p := \{q \in \mathbb{P} \mid q \leq p\} \in \Gamma$. One can check that both the classes of ∞-subcomplete and ∞-subproper forcing notions (as well as the restrictions "above μ" defined in Section 2) have this property.

THEOREM 1.10 (Cox, see [2, Theorem 20]). *Let Γ be a class of forcing notions closed under restrictions and assume $\mathsf{FA}(\Gamma)$ holds. Let $\mathbb{P}$ be a forcing notion. Suppose that for every $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a forcing notion in Γ there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\mathbb{R}}$ for a forcing notion so that the following hold*:

(1) $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ *is in* Γ,
(2) ***If*** $j : V \to N$ *is a generic elementary embedding*, $\theta \geq |\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}|^+$ *is regular in* V *and*
 a) H_θ^V *is in the wellfounded part of* N;
 b) $j``H_\theta^V \in N$ *has size* ω_1 *in* N;
 c) $\mathrm{crit}(j) = \omega_2^V$;
 d) *There exists a* $G * H * K$ *in* N *that is* $(H_\theta^V, \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}})$*-generic.*
 Then, in N *the set* $j``G \subseteq j(\mathbb{P})$ *that* $j``G$ *has a lower bound in* $j(\mathbb{P})$ *i.e., there is a* $p \in j(\mathbb{P}) \cap N$ *so that* $p \leq r$ *for each* $r \in j``G$,

Then, $\Vdash_{\mathbb{P}} \mathsf{FA}(\Gamma)$ *i.e.,* $\mathbb{P}$ *preserves the forcing axiom for* Γ.

See [2] for more on strengthenings and generalizations of this wide ranging theorem. In particular, a more general version stated in that article accounts for "+-versions" of forcing axioms by carrying stationary sets through the list of assumptions. Since we won't use this here, we omit it.

A typical application of Theorem 1.10 is when $\mathbb{P}$ adds some object witnessing some "nonreflective" behavior and $\mathbb{R}$ adds the nonreflective behavior to the full generic for $\mathbb{P}$ which allows $j``G$ to have a lower bound. For instance, a classic result of Beaudoin (see [1, Theorem 2.6]) states that PFA is consistent with a nonreflecting stationary subset of ω_2, i.e., a subset whose intersection with every point of uncountable cofinality below ω_2 is not stationary. In this case, the $\mathbb{P}$ would be the natural forcing to add such a nonreflecting set, and $\mathbb{R}$ would be the forcing to shoot a club through the compliment of the generic stationary set added by $\mathbb{P}$. The meat of Theorem 1.10 is then that the forcing $\mathbb{P}$ preserves PFA if $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper for any proper $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ (which it is). A variation of this argument is made as part of Theorem 4.1, see Section 4 for details.

§2. ∞-Subcompleteness and ∞-Subproperness above μ. Most theorems in this article filter through the notions of ∞-*subcompleteness (respectively,* ∞-*subproperness) above* μ for a cardinal μ. These are technical strengthenings of ∞-subcompleteness (respectively, ∞-subproperness). In this section, we define these strengthenings as well as make some elementary observations which will be used in rest of the article.

DEFINITION 2.1. Let μ be a cardinal and $\mathbb{P}$ a forcing notion.

(1) We say that $\mathbb{P}$ is ∞-*subcomplete above* μ if for all sufficiently large θ, $\tau > \theta$ so that $H_\theta \subseteq N := L_\tau[A] \models \mathsf{ZFC}^-$, $s \in N$, $\sigma : \bar{N} \prec N$ countable, transitive, and full with $\sigma(\bar{\mathbb{P}}, \bar{s}, \bar{\theta}, \bar{\mu}) = \mathbb{P}, s, \theta, \mu$, if $\bar{G} \subseteq \bar{\mathbb{P}} \cap \bar{N}$ is generic then there is a $p \in \mathbb{P}$ so that if $p \in G$ is $\mathbb{P}$-generic over V then in $V[G]$ there is a $\sigma' : \bar{N} \prec N$ so that
   1. $\sigma'(\bar{\mathbb{P}}, \bar{s}, \bar{\theta}, \bar{\mu}) = \mathbb{P}, s, \theta, \mu$;
   2. $\sigma'``\bar{G} \subseteq G$;
   3. $\sigma' \restriction \bar{\mu} = \sigma \restriction \bar{\mu}$.

(2) We say that $\mathbb{P}$ is ∞-*subproper above* μ if for all sufficiently large θ, $\tau > \theta$ so that $H_\theta \subseteq N := L_\tau[A] \models \mathsf{ZFC}^-$, $s \in N$, $p \in N \cap \mathbb{P}$, $\sigma : \bar{N} \prec N$ countable, transitive, and full with $\sigma(\bar{p}, \bar{\mathbb{P}}, \bar{s}, \bar{\theta}, \bar{\mu}) = p, \mathbb{P}, s, \theta, \mu$, there is a $q \in \mathbb{P}$ so that $q \leq p$ and if $q \in G$ is $\mathbb{P}$-generic over V then in $V[G]$ there is a $\sigma' : \bar{N} \prec N$ so that
   1. $\sigma'(\bar{p}, \bar{\mathbb{P}}, \bar{s}, \bar{\theta}, \bar{\mu}) = p, \mathbb{P}, s, \theta, \mu$;
   2. $(\sigma')^{-1}``G$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$;
   3. $\sigma' \restriction \bar{\mu} = \sigma \restriction \bar{\mu}$.

Concretely being ∞-subcomplete above μ simply means that $\mathbb{P}$ is ∞-subcomplete and, moreover, for any $\sigma : \bar{N} \prec N$ the corresponding σ' (in $V[G]$) witnessing the ∞-subcompleteness can be arranged to agree with σ "up to μ" i.e., on the ordinals below $\sigma^{-1}``\mu$ (and idem for ∞-subproperness). The "non-∞" versions of these classes were first introduced by Jensen in [16] and were investigated further by Fuchs in [8] who made several of the elementary observations we repeat below. The terminology "above μ" was used by Fuchs as well as in [16, Chapter 2] while in other places, e.g., [15] Jensen uses the terminology "μ-subcomplete". Following the first convention, we have moved the parameter μ to the end to avoid the awkwardness of "μ-∞-subcomplete/μ-∞-subproper". The following is immediate from the definitions.

Observation 2.2. *Let $\mu < \nu$ be cardinals. If $\mathbb{P}$ is ∞-subcomplete (respectively, ∞-subproper) above ν then it is ∞-subcomplete (respectively, subproper) above μ and it is ∞-subcomplete (respectively, ∞-subproper) (without any restriction).*

It is easy to see that being ∞-subcomplete (respectively, ∞-subproper) is equivalent to being ∞-subcomplete (respectively, ∞-subproper) above ω_1, however more is true, an observation due independently to the first author and Fuchs (see [8, Observation 4.2], note also [8, Observation 4.7] which is relevant here).

Proposition 2.3. *Let $\mathbb{P}$ be a forcing notion. $\mathbb{P}$ is ∞-subcomplete (respectively, ∞-subproper) if and only if $\mathbb{P}$ is ∞-subcomplete above $2^{\aleph_0}$ (respectively, ∞-subproper above $2^{\aleph_0}$).*

As noted above, this proposition (in the case of subcompleteness) is proved as [8, Observation 4.2] but we give a detailed proof in order to help the reader get accustomed to ∞-subversion forcing as well as to include the mild difference of subproperness. However, let us note that essentially the point is that, using the definable well order in $L_\tau[A]$, the reals of $\bar{N}$ code the cardinality of the continuum.

Proof. We prove the case of ∞-subproperness and leave the reader to check the case of ∞-subcompleteness since the latter, in its non "∞-version" can already be found in the literature. Let $\mathbb{P}$ be a forcing notion. It is immediate as noted above that if $\mathbb{P}$ is ∞-subproper above 2^ω then it is ∞-subproper so we need to check just the reverse direction. Thus, assume that $\mathbb{P}$ is ∞-subproper and let $\tau > \theta$ be cardinals so that $\sigma : \bar{N} \prec N := L_\tau[A]$ with $H_\theta \subseteq N$ be as in the definition of ∞-subproperness. Finally, let $p \in \mathbb{P}$ force that there is a $\sigma' : \bar{N} \prec N$ so that $\sigma'(\bar{\mathbb{P}}) = \mathbb{P}$ and $\sigma'^{-1}G := \bar{G}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$ for any generic $G \ni p$ (the existence of such a condition is the heart of the definition of ∞-subproperness of course). We need to show that p forces that $\sigma' \restriction 2^{\aleph_0} = \sigma \restriction 2^{\aleph_0}$, where, to be clear, $2^{\aleph_0}$ denotes the cardinal (as computed in $\bar{N}$) which bijects onto the continuum (as defined in $\bar{N}$). To avoid confusion, let us denote the cardinal $2^{\aleph_0} = \kappa$ (in V and hence N) and the preimage of κ in $\bar{N}$ under σ as $\bar{\kappa}$.

Fix a $G \ni p$ generic and work in $V[G]$ with σ' etc as described in the previous paragraph. First, note that by the absoluteness of ω we have that for all reals $x \in \bar{N}$ it must be the case that $\sigma(x) = \sigma'(x) = x$ (and being a real is absolute between $\bar{N}$ and $V/V[G]$). Moreover, since $N = L_\tau[A]$ there is a definable well order of the universe, and, in particular, there is a definable bijection of the reals onto κ, say $f : 2^\omega \to \kappa$. By elementarity in $\bar{N}$, there is a definable bijection $\bar{f} : 2^\omega \cap \bar{N} \to \bar{\kappa}$. But since f is definable we have $\sigma(\bar{f}) = \sigma'(\bar{f}) = f$ and hence for all $\alpha \in \bar{\kappa}$ we get $\sigma(\alpha) = \sigma(\bar{f}(\bar{f}^{-1}(\alpha))) = \sigma(\bar{f})(\sigma(\bar{f}^{-1}(\alpha))) = \sigma'(\bar{f}(\bar{f}^{-1}(\alpha))) = \sigma'(\alpha)$, as needed. Since the only assumption on G was that $p \in G$ we have, back in V that p forces this situation which completes the proof. $\dashv$

Jensen showed that Namba forcing is ∞-subcomplete above ω_1 assuming CH while it is not even ∞-subproper above ω_2 in ZFC, a consequence of the next observation, which essentially appears in [19, Theorem 2.12].

LEMMA 2.4. *Let μ be a cardinal.*

(1) *If $\mathbb{P}$ is ∞-subproper above μ then any new countable set of ordinals less than μ added by $\mathbb{P}$ is covered by an old countable set of ordinals (less than μ). In particular, if $\Vdash_{\mathbb{P}}$"$\mathrm{cf}(\mu) = \omega$" then $\mathrm{cf}(\mu) = \omega$ (in V).*

(2) *If $\mathbb{P}$ is ∞-subcomplete above μ then $\mathbb{P}$ adds no new countable sets of ordinals below μ.*

PROOF. The proofs of both are similar to the corresponding proofs that every new countable set of ordinals added by a proper forcing notion is contained in an old countable set of ordinals and σ-closed forcing notions do not add new countable sets of ordinals at all, respectively. The point is that to show the corresponding fact "below μ" one only needs ∞-subproperness (respectively, ∞-subcompleteness) above μ, see [19, Theorem 2.12] for details. ⊣

As mentioned before Lemma 2.4, an immediate consequence is the following.

LEMMA 2.5. *Namba forcing is not ∞-subproper above ω_2. In particular, Namba forcing is not ∞-subproper if* CH *fails.*

We do not know whether this lifts to the forcing axiom level. In other words, the following is open though seems unlikely given Lemma 2.5.

QUESTION 1. *Does* SCFA *imply the forcing axiom for Namba forcing when* CH *fails?*

Finally, we end this section with some observations about the associated forcing axioms for the classes we have been discussing.

DEFINITION 2.6. Let μ be a cardinal. Denote by ∞-SubPFA $\restriction \mu$ the forcing axiom for forcing notions $\mathbb{P}$ which are ∞-subproper above μ and ∞-SCFA $\restriction \mu$ the same for $\mathbb{P}$ which are ∞-subcomplete above μ.

The following is immediate by Observation 2.2.

PROPOSITION 2.7. *Let $\mu < \nu$ be cardinals. We have that ∞-SCFA implies ∞-SCFA $\restriction \mu$ implies ∞-SCFA $\restriction \nu$. Similarly, for the variants of ∞-SubPFA.*

In the next section, we will show that (in many cases) the reverse implications do not hold. Before doing this, let us note the following which was essentially known but requires piecing together from several places in the literature (and sifting through errors given by the initial mistake detailed above).

THEOREM 2.8 (Essentially Jensen, [14]). *Let $2^{\aleph_0} \leq \nu \leq \kappa < \mu = \kappa^+$ be cardinals with $\nu^\omega < \mu$. The forcing axiom ∞-SCFA $\restriction \nu$ implies the failure of $\square_\kappa$ and even that there is no nonreflecting stationary subset of $\kappa^+ \cap \mathrm{cof}(\omega)$.*

We remark that the definitions of $\square_\kappa$ and "nonreflecting stationary set" are given in Sections 3 and 4, respectively, where we use them.

PROOF. This is essentially known though it needs to be pieced together from a few sources—particularly taking into account the error discussed before, again (see [10]). First, in [14], Jensen uses the forcing notion (at κ) from [17, Lemma 6.3 of Section 3.3] to obtain the failure of $\square_\kappa$ from SCFA. Indeed, it's easy to see that this forcing notion implies the nonexistence of reflecting stationary sets and much more. See [6] for a detailed discussion of the effect of SCFA on square principles. As noted before, there is a missing assumption in the subcompleteness of the relevant forcing—namely, that $\kappa > 2^{\aleph_0}$. Second, [9, Lemma 3.5], which contains no errors as written, implies that the forcing notion needed is indeed ∞-subcomplete above ν under the cardinal arithmetic assumptions mentioned in the theorem statement. See the proof of [9, Lemma 3.5] and the discussion therein for more details. ⊣

§3. Separating the ∞-SCFA $\restriction \mu$ Principles. In this section, we show that under certain cardinal arithmetic assumptions ∞-SCFA $\restriction \nu$ does not imply ∞-SCFA $\restriction \mu$ for $\mu < \nu$. Before proving this general theorem, we introduce our technique with the simple example of separating ∞-SCFA $\restriction \omega_1$ from ∞-SCFA $\restriction \omega_2$. This involves showing that adding a $\square_{\omega_1}$-sequence to a model of ∞-SCFA preserves ∞-SCFA $\restriction \omega_2$. By contrast, note that Theorem 2.8 proves that SCFA + CH implies the failure of $\square_{\omega_1}$. Let us remark one more time that, as stated in the introduction the fact that SCFA can coexist with a $\square_{\omega_1}$-sequence closes the door on the aforementioned error by showing that the argument cannot be resurrected when CH fails.

This case is treated as a warm-up and we extract from it a more general lemma for preservation of axioms of the form ∞-SCFA $\restriction \mu^+$ from which the other separation results are then derived.

3.1. The case of ∞-SCFA $\restriction \omega_2$: Adding a $\square_{\omega_1}$ sequence. Recall that for an uncountable cardinal λ a $\square_\lambda$-sequence is a sequence $\langle C_\alpha \mid \alpha \in \lambda^+ \cap \mathrm{Lim}\rangle$ so that for all α the following hold:

(1) C_α is club in α;
(2) $\mathrm{ot}(\alpha) \leq \lambda$;
(3) For each $\beta \in \lim(C_\alpha)$ we have that $C_\alpha \cap \beta = C_\beta$.

We recall the poset $\mathbb{P}_0$ from [3, Example 6.6] for adding a square sequence. Conditions $p \in \mathbb{P}_0$ are functions so that the domain of p is $\beta + 1 \cap \mathrm{Lim}$ for some $\beta \in \lambda^+ \cap \mathrm{Lim}$ and

(1) For all $\alpha \in \mathrm{dom}(p)$ we have that $p(\alpha)$ is club in α with order type $\leq \lambda$; and
(2) If $\alpha \in \mathrm{dom}(p)$ then for each $\beta \in \lim(p(\alpha))$ we have $p(\alpha) \cap \beta = p(\beta)$.

The order is end extension. We remark that a moment's reflection confirms that this poset is σ-closed. Moreover, it is $<\lambda^+$-strategically closed (see [3]). In particular, it preserves cardinals up to λ^+.

THEOREM 3.1. *Assume ∞-SCFA $\restriction \omega_2$ and let $\mathbb{P}_0$ be the forcing notion defined above for adding a $\square_{\omega_1}$-sequence. Then, $\Vdash_{\mathbb{P}_0} \infty$-SCFA $\restriction \omega_2$. In particular, if the existence of a supercompact cardinal is consistent with* ZFC *then ∞-SCFA $\restriction \omega_2 + \square_{\omega_1}$ is consistent as well.*

Before proving this theorem, we need to define one more poset. Recall that if $G \subseteq \mathbb{P}_0$ is generic and $\vec{C}_G = \langle C_\alpha \mid \alpha \in \lambda^+ \cap \mathrm{Lim}\rangle$ is the generic $\square_\lambda$-sequence added by G then for any cardinal $\gamma < \lambda$ we can *thread the square sequence* via the following poset, $\mathbb{T}_{G,\gamma}$. Conditions are closed, bounded subsets $c \subseteq \lambda^+$ so that c has order type $< \gamma$, and for all limit points $\beta \in c$ we have that $\beta \cap c = C_\beta$. See [4, Section 6] and [18, p. 7] for more on this threading poset. The point is the following.

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Fact 3.2 ([4, Lemma 6.9]). *Let $\gamma < \lambda$ be cardinals, $\mathbb{P}_0$ the forcing notion described above for adding a $\square_\lambda$-sequence and $\dot{\mathbb{T}}_{\dot{G},\gamma}$ be the $\mathbb{P}_0$-name for the forcing to thread the generic square sequence with conditions of size $< \gamma$. Then, $\mathbb{P}_0 * \dot{\mathbb{T}}_{\dot{G},\gamma}$ has a dense $< \gamma$-closed subset.*

We can now prove Theorem 3.1.

Proof. We let $\mathbb{P}_0$ be the forcing described above for adding a $\square_{\omega_1}$-sequence (so $\lambda = \omega_1$). Let $\gamma = \aleph_1$ so in $V^{\mathbb{P}_0}$ the threading poset $\dot{\mathbb{T}} := \dot{\mathbb{T}}_{\dot{G},\aleph_1}$ consists of countable closed subsets of ω_2. We want to apply Theorem 1.10 to $\mathbb{P}_0$. Note that if $\dot{\mathbb{Q}}$ is a $\mathbb{P}_0$-name for an ∞-subcomplete above ω_2 forcing notion, then $\dot{\mathbb{T}} = \dot{\mathbb{T}}_{\dot{G},\aleph_1}$ is absolute between $V^{\mathbb{P}_0}$ and $V^{\mathbb{P}_0 * \dot{\mathbb{Q}}}$ by Lemma 2.4 (2).

Claim 3.3. *It is enough to show that for any $\mathbb{P}_0$-name $\dot{\mathbb{Q}}$ for a forcing notion which is ∞-subcomplete above ω_2, the three step $\mathbb{P}_0 * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ is ∞-subcomplete above ω_2.*

Proof of Claim. This is because $\mathbb{T}$ adds a lower bound to $j``G$ as described in the statement of Theorem 1.10. In more detail, let $\dot{\mathbb{Q}}$ be a $\mathbb{P}_0$-name for a forcing notion which is ∞-subcomplete above ω_2, we want to show that for $\dot{\mathbb{R}} = \dot{\mathbb{T}}$ the hypotheses of Theorem 1.10 are satisfied assuming that $\mathbb{P}_0 * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ is ∞-subcomplete above ω_2. Since this is exactly the first clause we only need to concern ourselves with the second one. Recall that, relativized to this situation, this says that **if** $j : V \to N$ is a generic elementary embedding, $\theta \geq |\mathbb{P}_0 * \dot{\mathbb{Q}} * \dot{\mathbb{T}}|^+$ is regular in V and

a) H_θ^V is in the wellfounded part of N;
b) $j``H_\theta^V \in N$ has size ω_1 in N;
c) $\mathrm{crit}(j) = \omega_2;^V$
d) There exists a $G * H * K$ in N that is $(H_\theta^V, \mathbb{P}_0 * \dot{\mathbb{Q}} * \dot{\mathbb{T}})$-generic.

Then, N believes that $j``G$ has a lower bound in $j(\mathbb{P}_0)$.

So fix some θ and $j : V \to N$ as described in a) to d). Note that $j``G = G$ by c) and the fact that G is coded as a subset of ω_2^V. Thus, it suffices to find a lower bound of G in $j(\mathbb{P}_0)$. The point is now though that since $G * H * K \in N$ we can, in particular, form $\bigcup K \in N$ which is a club subset of $\omega_2^V = \sup_{p \in G} \mathrm{dom}(p)$ and coheres with all of the elements of G, and hence $(\bigcup G) \cup \langle \omega_2^V, \bigcup K \rangle$ is as needed. ⊣

Let us now show that $\mathbb{P}_0 * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ is ∞-subcomplete above ω_2. Let $\tau > \theta$ be sufficiently large cardinals and $\sigma : \bar{N} \prec N = L_\tau[A] \supseteq H_\theta$ be as in the definition of ∞-subcompleteness above ω_2. Let $\sigma(\bar{\mathbb{P}}_0, \dot{\bar{\mathbb{Q}}}, \dot{\bar{\mathbb{T}}}) = \mathbb{P}_0, \dot{\mathbb{Q}}, \dot{\mathbb{T}}$. Let $\bar{G} * \bar{H} * \bar{K}$ be $\bar{\mathbb{P}}_0 * \dot{\bar{\mathbb{Q}}} * \dot{\bar{\mathbb{T}}}$-generic over $\bar{N}$. There are few things to note. First, let us point out that $\bar{G}$ and $\bar{K}$ are (coded as) subsets of $\bar{\omega}_2$, the second, uncountable cardinal from the point of view of $\bar{N}$ (so $\sigma(\bar{\omega}_2) = \omega_2$). Next note that $\mathbb{P}_0 * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ is isomorphic to $\mathbb{P}_0 * \dot{\mathbb{T}} * \dot{\mathbb{Q}}$ since both $\dot{\mathbb{Q}}$ and $\dot{\mathbb{T}}$ are in $V^{\mathbb{P}_0}$, and the same for the "bar" versions in $\bar{N}$ (i.e., we have a product not an iteration for the second and third iterands). Now, note that since $\mathbb{P}_0 * \dot{\mathbb{T}}$ has a σ-closed dense subset, $\sigma``\bar{G} * \bar{K}$ has a lower bound (in N), say (p, t) (t is in the ground model and the σ-closed dense subset is simply the collection of conditions whose second coordinate is a check name decided by p). By σ-closedness (which again is implied completeness) (p, t), forces that there is a unique lift of $\sigma : \bar{N} \prec N$ to some $\sigma_0 : \bar{N}[\bar{G}] \prec N[G]$ with $\sigma_0(\bar{G}) = G$ for any $\mathbb{P}_0$-generic $G \ni p$ (technically we need to work in the extension by $\mathbb{P}_0 * \dot{\mathbb{T}}$, but we only want to specify the embedding of the $\bar{\mathbb{P}}_0$ extension). Fix such a G (from which σ_0 is defined) and work in $V[G]$. Note that $\sigma_0``\bar{K} = \sigma``\bar{K}$ has $t \in N$ as a lower bound. Now, in $V[G]$ (NOT $V[G][K]$), we have that $\mathbb{Q} := \dot{\mathbb{Q}}^G$ is ∞-subcomplete above ω_2 as $\mathbb{P}$ forced this to be so by assumption. Therefore, in $V[G]$, we can apply the definition of ∞-subcompleteness to $\sigma_0 : \bar{N}[\bar{G}] \prec N[G]$ to obtain a condition $\dot{q}^G := q \in \mathbb{Q}$ so that if $H \ni q$ is $\mathbb{Q}$-generic over $V[G]$ then in $V[G][H]$ there is a $\sigma_1 : \bar{N}[\bar{G}] \prec N[G]$ so that $\sigma_1(\bar{G}, \bar{\mathbb{P}}_0, \dot{\bar{\mathbb{Q}}}^{\bar{G}}, \dot{\bar{\mathbb{T}}}^{\bar{G}}) = G, \mathbb{P}_0, \mathbb{Q}, \mathbb{T}$, where $\mathbb{T} \in V[G]$ is $\dot{\mathbb{T}}^G$, $\sigma_1``\bar{H} \subseteq H$, and $\sigma_1 \restriction \bar{\omega}_2 = \sigma \restriction \bar{\omega}_2$. Note also that by condensation we have that $\bar{N} = L_{\bar{\tau}}[\bar{A}]$ and hence we can ensure that $\sigma_1 \restriction \bar{N} : \bar{N} \prec N$. Let us denote by σ_2 this restriction $\sigma_1 \restriction \bar{N}$. As this is an element of $V[G][H]$ there is, in V a $\mathbb{P}_0 * \dot{\mathbb{Q}}$-name for this embedding, which we will call $\dot{\sigma_2}$.

Now, by the first observation above, we know that since $\bar{G}$ and $\bar{K}$ are coded as subsets of $\bar{\omega}_2$ so it must be the case that in fact $\sigma_2 \restriction \bar{G} = \sigma \restriction \bar{G}$ and idem for $\bar{K}$. In particular, (p, t) is still a lower bound of $\sigma_1``\bar{G} * \bar{K}$. But putting all of these observations together now ensures that the triple $(p, \dot{q}, t) \in \mathbb{P}_0 * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ forces that $\sigma_2 := \sigma_1 \restriction \bar{N}$ is as needed to witness that the three step is ∞-subcomplete above ω_2 as needed. ⊣

Note the following corollary of Theorem 3.1.

Corollary 3.4. *The forcing axiom* ∞-SCFA *does not imply* $\mathsf{MA}^+(\sigma\text{– closed})$ *assuming the consistency of a supercompact cardinal. In particular,* ∞-SCFA *does not imply* SCFA^+.

Proof. Begin with a model of ∞-SCFA $+ 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ (for instance a model of MM). Force with $\mathbb{P}_0$ to preserve these axioms and add a $\square_{\omega_1}$-sequence. Then, ∞-SCFA $\restriction \omega_2$ and $\square_{\omega_1}$ hold in the extension by Theorem 3.1. But, since $\mathbb{P}_0$ does not collapse cardinals (by $2^{\aleph_1} = \aleph_2$) or add reals, the continuum is still $\aleph_2$ hence ∞-SCFA holds yet $\mathsf{MA}^+(\sigma\text{-closed})$ fails since this axiom implies that $\square_\kappa$ fails for all κ (see [5]. ⊣

3.2. The general case. The proof of Theorem 3.1 can be generalized in many ways. Observe that very little about $\mathbb{P}_0$ and $\dot{\mathbb{T}}$ were used. In fact, essentially the same proof as above really shows the following general metatheorem.

Theorem 3.5. *Let μ be an uncountable cardinal. Let $\mathbb{P}$ be a poset whose conditions as well as any generic G can be coded by subsets of μ^+ and let $\dot{\mathbb{R}}$ be a $\mathbb{P}$-name for a poset which is forced to be so that all of its conditions and any generic K are coded by subsets of μ^+. Assume moreover, that $\mathbb{P} * \dot{\mathbb{R}}$ has a σ-closed dense subset and $\Vdash_{\mathbb{P}} \dot{\mathbb{R}} \subseteq V$ i.e., all of the elements of $\dot{R}$ are in the ground model*[5]*. Then, for every $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a ∞-subcomplete above μ^+ poset the poset $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is ∞-subcomplete above μ^+.*

Consequently, if $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ satisfies (2) of Theorem 1.10 and ∞-SCFA $\restriction \mu^+$ *holds then $\mathbb{P}$ preserves* ∞-SCFA $\restriction \mu^+$.

Proof. This is really just an abstraction of what we have already seen. Let $\tau > \theta$ be sufficiently large cardinals and $\sigma : \bar{N} \prec N = L_\tau[A] \supseteq H_\theta$ be as in the definition of ∞-subcompleteness above μ^+. Let $\sigma(\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \dot{\bar{\mathbb{R}}}, \bar{\mu}) = \mathbb{P}, \dot{\mathbb{Q}}, \dot{\mathbb{R}}, \mu$. Let $\bar{G} * \bar{H} * \bar{K}$ be

[5]For instance, in the case of Theorem 3.1, this follows from the strategic closure.

$\bar{\mathbb{P}} * \dot{\bar{\mathbb{Q}}} * \dot{\bar{\mathbb{R}}}$-generic over $\bar{N}$. As in Theorem 3.1, note that first of all $\bar{G}$ and $\bar{K}$ are (coded as) subsets of $\bar{\mu}^+$ (note $\bar{\mu}^+$, the successor of $\bar{\mu}$ as computed in $\bar{N}$ is the same as $\bar{\mu^+}$, the preimage of μ^+ under σ by elementarity). Next, note that $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is isomorphic to $\mathbb{P} * \dot{\mathbb{R}} * \dot{\mathbb{Q}}$ since both $\dot{\mathbb{Q}}$ and $\dot{\mathbb{R}}$ are in $V^{\mathbb{P}}$, and the same for the "bar" versions in $\bar{N}$ (i.e., we have a product not an iteration for the second and third iterands), just as before. Now, note that since $\mathbb{P} * \dot{\mathbb{R}}$ has a σ-closed dense subset, there is a condition $(p, t) \in \mathbb{P} * \dot{\mathbb{R}}$ forcing $\sigma``\bar{G} * \bar{K}$ to be contained in the generic and moreover, this condition is a lower bound on $\sigma``\bar{G} * \bar{K}$ by elementarity: since $\bar{N}$ thinks $\bar{\mathbb{P}} * \dot{\bar{\mathbb{R}}}$ has a σ-closed dense subset densely many of the conditions in $\bar{G} * \bar{K}$ are in this set and hence their images are in the real σ-closed dense subset of $\mathbb{P} * \dot{\mathbb{R}}$ which in turn implies that we can find the lower bound. Note this condition (p, t) is in N and by the assumption that $\dot{\mathbb{R}}$ is forced to be contained in the ground model, we can assume that $t \in V$ (and in fact in N). It follows that (p, t) forces that there is a unique lift of $\sigma : \bar{N} \prec N$ to some $\sigma_0 : \bar{N}[\bar{G}] \prec N[G]$ with $\sigma_0(\bar{G}) = G$ for any $\mathbb{P}$-generic $G \ni p$ (technically we need to work in the extension by $\mathbb{P} * \dot{\mathbb{R}}$, but we only want to specify the embedding of the $\bar{\mathbb{P}}$ extension). Fix such a G (from which σ_0 is defined) and work in $V[G]$. Note that $\sigma_0``\bar{K} = \sigma``\bar{K}$ and, as already stated, $t \in N$ is a lower bound. Since $\mathbb{Q} := \dot{\mathbb{Q}}^G$ was forced to be ∞-subcomplete above μ^+, working in $V[G]$, we can apply the definition of ∞-subcompleteness to $\sigma_0 : \bar{N}[\bar{G}] \prec N[G]$ to obtain a condition $\dot{q}^G := q \in \mathbb{Q}$ so that if $H \ni q$ is $\mathbb{Q}$-generic over $V[G]$ then in $V[G][H]$ there is a $\sigma_1 : \bar{N}[\bar{G}] \prec N[G]$ so that $\sigma_1(\bar{G}, \bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}^{\bar{G}}, \dot{\bar{\mathbb{R}}}^{\bar{G}}) = G, \mathbb{P}, \mathbb{Q}, \mathbb{R}$, where $\mathbb{R} \in V[G]$ is $\dot{\mathbb{R}}^G$, $\sigma_1``\bar{H} \subseteq H$, and $\sigma_1 \restriction \bar{\mu}^+ = \sigma \restriction \bar{\mu}^+$. Note also that by condensation, we have that $\bar{N} = L_{\bar{\tau}}[\bar{A}]$ and hence we can ensure that $\sigma_1 \restriction \bar{N} : \bar{N} \prec N$. Let us denote by σ_2 the embedding $\sigma_1 \restriction \bar{N}$ and let $\dot{\sigma_2}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for σ_2 in V.

Now, $\bar{G}$ and $\bar{K}$ are coded as subsets of $\bar{\mu}^+$ by assumption. Therefore, it must be the case that in fact $\sigma_1 \restriction \bar{G} = \sigma \restriction \bar{G}$ and idem for $\bar{K}$ - note the subtlety here $\bar{K}$ is not in $\bar{N}[\bar{G}]$ but is a subset of it. In particular, (p, t) is still a lower bound in $\sigma_1``\bar{G} * \bar{K}$. But putting all of these observations together now ensures that the triple $(p, \dot{q}, t) \in \mathbb{P} * \dot{\mathbb{Q}} * \mathbb{T}$ forces that $\dot{\sigma_2}$ is as needed to witness that the three step is ∞-subcomplete above μ^+ as needed. ⊣

Before moving to our main application, let us give another one at the level of ω_2.

THEOREM 3.6. *Assume* ∞-SCFA $\restriction \omega_2$. *The forcing* $\mathbb{S}_{\omega_2}$ *to add an* ω_2*-Souslin tree preserves* ∞-SCFA $\restriction \omega_2$.

PROOF (SKETCH). Let $\mathbb{S}_{\omega_2}$ be the standard forcing to add an ω_2-Souslin tree: conditions are binary trees $p \subseteq 2^{<\omega_2}$ of size $< \aleph_2$ ordered by end extension. This adds an ω_2-Souslin tree and is σ-closed. Let $\dot{T}_{\dot{G}}$ be the canonical name for the tree added i.e., if $G \subseteq \mathbb{S}_{\omega_2}$ is generic over V then $(\dot{T}_{\dot{G}})^G = \bigcup G$. Let $\dot{\mathbb{Q}}$ be a $\mathbb{S}_{\omega_2}$-name for a forcing notion which is ∞-subcomplete above ω_2. As before, it is enough to show that $\mathbb{S}_{\omega_2} * \dot{\mathbb{Q}} * \dot{T}_{\dot{G}}$ is ∞-subcomplete above ω_2, where $\dot{T}_{\dot{G}}$ is the name for the tree as a forcing notion, by essentially the same proof as in the case of Theorem 3.1. However, that this three step is ∞-subcomplete above ω_2 now follows almost immediately from Theorem 3.5. ⊣

We have the following corollary similar to Corollary 3.4 above by invoking a model of ∞-SCFA $+ 2^{\aleph_0} = \aleph_2$.

Corollary 3.7. *Assuming the consistency of a supercompact cardinal we have the consistency of* $\mathsf{SCFA} + \neg\mathsf{CH} + \neg\mathrm{TP}(\omega_2)$.

Here, $\mathrm{TP}(\omega_2)$ is the tree property at ω_2 i.e., no ω_2-Aronszajn trees. This result contrasts with [22, Corollary 4.1] which shows that under Rado's Conjecture, another forcing axiom-like statement compatible with CH, $\mathrm{TP}(\omega_2)$ is equivalent to $\neg\mathsf{CH}$.

The proof of Theorem 3.1, using Theorem 3.5 can be easily generalized to establish that for any cardinal μ adding a $\square_\mu$ sequence via $\mathbb{P}_0$ preserves ∞-$\mathsf{SCFA} \restriction \mu^+$.

Theorem 3.8. *Let μ be an uncountable cardinal and assume ∞-$\mathsf{SCFA} \restriction \mu^+$ holds. If $\mathbb{P}_0$ is the forcing from the previous subsection to add a $\square_\mu$-sequence then $\mathbb{P}_0$ preserves* ∞-$\mathsf{SCFA} \restriction \mu^+$.

Proof. In $V^{\mathbb{P}_0}$ let $\dot{\mathbb{T}} := \dot{\mathbb{T}}_{\dot{G},\aleph_1}$. We only give the proof of the claim obtained from Claim 3.3 by replacing ω_2 with μ. The other part of the proof—that the requisite three step forcing is ∞-subcomplete above μ^+ is an immediate consequence of Theorem 3.5.

Suppose $j : V \to N$, θ and $G * H * K$ are as in the proof of Claim 3.3. Let $\beta := (\mu^+)^V = \sup_{p\in G} \mathrm{dom}(p)$. Then, $\bigcup K \in N$ is a club subset of β and coheres with all of the elements of G. Note that all initial segments of $\bigcup K$ are countable sets in V. So $K^* := j``\bigcup K$ is club in $\beta^* := \sup(j``\beta)$ and coheres with all of the elements of $G^* := j``G$. Hence, $(\bigcup G^*) \cup \langle \beta^*, K^*\rangle$ is a lower bound of $j``G$ in $j(\mathbb{P}_0)$. ⊣

Putting all of these results together we get the following.

Theorem 3.9. *Let $2^{\aleph_0} \leq \nu \leq \kappa < \mu = \kappa^+$ be cardinals with $\nu^\omega < \mu$. Modulo the existence of a supercompact cardinal ∞-$\mathsf{SCFA} \restriction \mu + \neg\infty$-$\mathsf{SCFA} \restriction \nu$ is consistent.*

Proof. By Theorem 3.8, we know that ∞-$\mathsf{SCFA} \restriction \mu$ is consistent with $\square_\kappa$ hence it suffices to see that ∞-$\mathsf{SCFA} \restriction \nu$ implies the failure of $\square_\kappa$, but this is exactly the content of Theorem 2.8 above. ⊣

§4. Separating MM from SubPFA.

In this section, we prove the following result.

Theorem 4.1. *Assume there is a supercompact cardinal. Then, there is a forcing extension in which ∞-SubPFA holds but MM fails. In particular, modulo the large cardinal assumption, ∞-SubPFA does not imply MM.*

The idea behind this theorem is a combination of the proof technique from [1, Theorem 2.6] and the proof of Theorem 3.1. Starting from a model of MM, we will force to add a nonreflecting stationary set to $2^{\aleph_0}$ ($= \aleph_2$ since MM holds). This kills MM by the results of [5] but will preserve ∞-SubPFA by an argument similar to that of [1, Theorem 2.6]. In that paper Beaudoin proves that in fact PFA is consistent with a nonreflecting stationary subset of any regular cardinal κ. The interesting difference in the subproper case is that ∞-SubPFA (in fact SCFA) implies that there are no nonreflecting stationary subsets of any cardinal greater than the size of the continuum, see Theorem 2.8 above. In short, PFA is consistent with a nonreflecting stationary subset of every regular cardinal κ while ∞-SubPFA is only consistent with a nonreflecting stationary subset of ω_2. We begin by recalling the relevant definitions.

DEFINITION 4.2. Let κ be a cardinal of uncountable cofinality and $S \subseteq \kappa$. For a limit ordinal $\alpha < \kappa$ of uncountable cofinality, we say that S *reflects to* α if $S \cap \alpha$ is stationary in α. We say that S is *nonreflecting* if it does not reflect to any $\alpha < \kappa$ of uncountable cofinality.

FACT 4.3 (See [5, Theorem 9]). MM *implies that for every regular* $\kappa > \aleph_1$ *every stationary subset of* $\kappa \cap \mathrm{Cof}(\omega)$ *reflects.*

Compare this with the following, which was also noted in the proof of Theorem 2.8 above.

FACT 4.4 (See [17, Lemma 6, Section 4]). SCFA *implies that for every regular* $\kappa > 2^{\aleph_0}$ *every stationary subset of* $\kappa \cap \mathrm{Cof}(\omega)$ *reflects.*

REMARK 1. Again, in [17] it is claimed that SCFA implies that the above holds for all $\kappa > \aleph_1$, regardless of the size of the continuum. However, this too is incorrect without CH because of the error.

There is a natural forcing notion to add a nonreflecting stationary subset $S \subseteq \kappa \cap \mathrm{Cof}(\omega)$ for a fixed regular cardinal κ. The definition and basic properties are given in [3, Example 6.5]. We record the basics here for reference.

DEFINITION 4.5. Fix a regular cardinal $\kappa > \aleph_1$. The forcing notion $\mathbb{NR}_\kappa$ is defined as follows. Conditions are functions p with domain the set of countably cofinal ordinals below some ordinal $\alpha < \kappa$ mapping into 2 with the property that if $\beta \leq \sup(\mathrm{dom}(p))$ has uncountable cofinality then there is a set $c \subseteq \beta$ club in β which is disjoint from $p^{-1}(1) = \{\alpha \in \mathrm{dom}(p) \mid p(\alpha) = 1\}$. The extension relation is simply $q \leq_{\mathbb{NR}_\kappa} p$ if and only if $q \supseteq p$.

Proofs of the following can be found in [3].

PROPOSITION 4.6. *For any regular* $\kappa > \aleph_1$ *the forcing* $\mathbb{NR}_\kappa$ *has the following properties.*

(1) $\mathbb{NR}_\kappa$ *is* σ*-closed.*
(2) $\mathbb{NR}_\kappa$ *is* κ*-strategically closed and in particular preserves cardinals.*
(3) *If* $G \subseteq \mathbb{NR}_\kappa$ *is generic then* $S_G := \bigcup_{p \in G} p^{-1}(1)$ *is a nonreflecting stationary subset of* κ.

We neglect to give the definition of strategic closure since we will not need it beyond the fact stated above, see [4] or [3] for a definition.

Let κ be as above, $G \subseteq \mathbb{NR}_\kappa$ be generic over V and let $S_G := \bigcup_{p \in G} p^{-1}(1)$ be the generic nonreflecting stationary set. We want to define a forcing to kill S_G (this will be the "$\dot{\mathbb{R}}$" in our application of Theorem 1.10). Specifically, we will define a forcing notion $\mathbb{Q}_{S_G}$ so that forcing with $\mathbb{Q}_{S_G}$ will add a club to $\kappa \setminus S_G$ and hence kill the stationarity of S_G. Note that since S_G is nonreflecting its complement must also be stationary and indeed has to be *fat*, i.e., contain continuous sequences of arbitrary length $\alpha < \kappa$ cofinally high.

DEFINITION 4.7. Borrowing the notation from the previous paragraph define the forcing notion $\mathbb{Q}_{S_G}$ as the set of closed, bounded subsets of $\kappa \setminus S_G$ ordered by end extension.

Clearly, the above forcing generically adds a club to the complement of S_G thus killing its stationarity (see [3, Definition 6.10]). It is also ω-distributive.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Assume ∞-SubPFA holds (the consistency of this is the only application of the supercompact). Note that the continuum is $\aleph_2$ and will remain so in any cardinal preserving forcing extension which adds no reals. Let $\mathbb{P} = \mathbb{NR}_{\aleph_2}$, $G \subseteq \mathbb{P}$ be generic over V and work in $V[G]$. Obviously, in this model, we have "there is a nonreflecting stationary subset of $\aleph_2$" and thus MM fails by Fact 4.3. We need to show that ∞-SubPFA holds.

We will apply Theorem 1.10 much as in the proof of Theorem 3.1. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for an ∞-subproper forcing notion and let $\dot{\mathbb{R}}$ name $\mathbb{Q}_{S_{\dot{G}}}$ in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ (NOT just in $V^{\mathbb{P}}$ - this is different than the proof of Theorem 3.1 and crucial). By exactly the same argument as in the proof of Theorem 3.1, it suffices to show that $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is ∞-subproper (in V). This is because (2) from Theorem 1.10 follows from the fact that, borrowing the notation from the statement of that theorem applied to our situation $\dot{\mathbb{R}}$ shoots a club through the complement of S_G hence $j``S_G = S_G$ is nonstationary in its supremum and so has a lower bound in N.

So we show that $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is ∞-subproper. This is very similar to the proof of Theorem 3.1 or even Theorem 3.5 more generally but enough details are different to warrant repeating everything for completeness. Let $\tau > \theta$ be sufficiently large cardinals and $\sigma : \bar{N} \prec N = L_\tau[A] \supseteq H_\theta$ be as in the definition of ∞-subproperness. Let $\sigma(\bar{\mathbb{P}}, \dot{\bar{\mathbb{Q}}}, \dot{\bar{\mathbb{R}}}, \bar{\omega_2}) = \mathbb{P}, \dot{\mathbb{Q}}, \dot{\mathbb{R}}, \omega_2$. Let $(p_0, \dot{q}_0, \dot{r}_0)$ be a condition in $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ with $\sigma(\bar{p}_0, \dot{\bar{q}}_0, \dot{\bar{r}}_0) = (p_0, \dot{q}_0, \dot{r}_0)$. Applying the σ-closure of $\mathbb{P}$ we can find a $\bar{\mathbb{P}}$-generic $\bar{G}$ over $\bar{N}$ and a condition $p \leq p_0$ so that p is a lower bound on $\sigma``\bar{G}$ and, letting $\alpha = \sup(\sigma``\bar{\omega}_2)$, we have $p(\alpha) = 0$ (i.e., p forces α to not be in the generic stationary set). Let us assume $p \in G$ and note that this condition forces $\sigma``\bar{G} \subseteq G$ and hence $\sigma$ lifts uniquely to a $\tilde{\sigma} : \bar{N}[\bar{G}] \prec N[G]$ that $\tilde{\sigma}(\bar{G}) = G$ and $\alpha := \sup(\sigma``\bar{\omega}_2) \notin S_G$. Let $\bar{\mathbb{Q}} = \dot{\bar{\mathbb{Q}}}^{\bar{G}}$ as computed in $\bar{N}[\bar{G}]$ and let $\bar{q}_0 = \dot{\bar{q}}_0^{\bar{G}} \in \bar{N}[\bar{G}]$. Applying the fact that $\dot{\mathbb{Q}}$ is forced to be ∞-subproper let $q \leq q_0 = \tilde{\sigma}(\bar{q}_0)$ be a condition forcing that if $H \subseteq \mathbb{Q}$ is V-generic with $q \in H$ then there is a $\sigma' \in V[G][H]$ so that $\sigma' : \bar{N}[\bar{G}] \prec N[G]$ as in the definition of ∞-subproperness (with respect to $\tilde{\sigma}$). Note that as in the proof of Theorem 3.1, $\sigma' \restriction \bar{N} : \bar{N} \prec N$ and $\sigma' \restriction \bar{\omega}_2 = \sigma \restriction \bar{\omega}_2$. Let $\tilde{\sigma}' : \bar{N}[\bar{G}][\bar{H}] \to N[G][H]$ be the lift of σ', where $\bar{H} = (\sigma')^{-1}``H$.

Claim 4.8. *In $V[G][H]$, the set S_G does not contain a club.*

Proof of Claim. Since $\aleph_2$ is the continuum in $V[G]$ note that $\omega_2^{V[G]}$ remains uncountably cofinal in $V[G][H]$ (though of course it can be collapsed to ω_1). Suppose towards a contradiction that S_G contains a club and note that since we chose θ and τ to be sufficiently large with respect to the forcing (and, therefore, in particular, we can assume H_θ contains the powerset of ω_2) we have $N[G][H] \models$ "$\exists C$ which is club and $C \subseteq S_G$". By elementarity, there is a $\bar{C} \in \bar{N}[\bar{G}][\bar{H}]$ so that

$$\bar{N}[\bar{G}][\bar{H}] \models \bar{C} \subseteq \bar{S}_G \text{ is club},$$

where $\bar{H} := \sigma'^{-1}H$ is $\bar{\mathbb{Q}}$-generic over $\bar{N}[\bar{G}]$ by the definition of ∞-subcompleteness and the choice of q. But now note that if $C = \tilde{\sigma}'(\bar{C})$ then $C \cap \alpha$ is cofinal in α by elementarity so $\alpha \in C$ but $\alpha \notin S_G$ which is a contradiction. $\dashv$

Given the claim, we know that $\omega_2^V \setminus S_G$ is a stationary set in $V[G][H]$ and hence $\mathbb{R} := \dot{\mathbb{R}}^{G*H}$ is the forcing to shoot a club through a stationary set. Let $\bar{\mathbb{R}} \in \bar{N}[\bar{G}][\bar{H}]$ be $\dot{\bar{\mathbb{R}}}^{\bar{G}*\bar{H}}$. Note that for each $\beta \in \bar{N} \cap \bar{\omega}_2$ it is dense (in $\bar{N}[\bar{G}][\bar{H}]$) that there is a condition $\bar{r} \in \bar{\mathbb{R}}$ with $\beta \in \mathrm{dom}(\bar{r})$. It follows that if $\bar{K}$ is generic for $\bar{\mathbb{R}}$ over $\bar{N}[\bar{G}][\bar{H}]$ with $\bar{K} \ni \bar{r}_0 := \dot{\bar{r}}^{\bar{G}*\bar{H}}$ then $\tilde{\sigma'}``\bar{K}$ unions to a club in $\alpha \setminus S_G$. Since $\alpha \notin S_G$ we have that $r := \bigcup \tilde{\sigma'}``\bar{K} \cup \{\alpha\}$ is a condition in $\mathbb{R}$ which is a lower bound on $\tilde{\sigma'}``\bar{K}$ and hence $r \leq \dot{r}_0^{G*H}$. Finally, let $K \ni r$ be $\mathbb{R}$-generic over $V[G][H]$. It is now easy to check that the condition $(p, \dot{q}, \dot{r})$ and $\sigma' \restriction \bar{N}$ collectively witness the ∞-subproperness of $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ so we are done. ⊣

We note that by the same proof adding a nonreflecting stationary set of $\mu \cap \mathrm{Cof}(\omega)$ for larger cardinals μ, we can preserve ∞-SubPFA $\restriction \mu$. The following, therefore, holds.

THEOREM 4.9. *Let $2^{\aleph_0} \leq \mu \leq \lambda < \nu = \lambda^+$ be cardinals with $\mu^\omega < \nu$. Modulo the existence of a supercompact cardinal* ∞-SubPFA $\restriction \nu + \neg\infty$-SubPFA $\restriction \mu$ *is consistent.*

The proof of this Theorem finishes the proof of all nonimplications involved in Main Theorem 1.1.

§5. Conclusion and Open Questions. We view this article, alongside its predecessor [12] as showing, amongst other things, that the continuum forms an interesting dividing line for subversion forcing: below the continuum the "sub" plays no role as witnessed by the fact that the same nonimplications can hold as those that hold for the nonsub versions. Above, it adds considerable strength to the associated forcing axioms. However, as of now we only know how to produce models of SCFA in which the continuum is either $\aleph_1$ or $\aleph_2$. The most pressing question in this area is, therefore, whether consistently SCFA can co-exist with a larger continuum.

QUESTION 2. *Is* SCFA *consistent with the continuum $\aleph_3$ or greater*?

We note here that the most obvious attempt to address this question i.e., starting with a model of SCFA and adding $\aleph_3$-many reals with e.g., ccc forcing, does not work, an observation due to the first author.

LEMMA 5.1. *Suppose $\mathbb{P}$ is a proper forcing notion adding a real. Then,* SCFA *fails in $V^{\mathbb{P}}$.*

All that is needed about "properness" here is that being proper implies that stationary subsets of $\kappa \cap \mathrm{Cof}(\omega)$ are preserved. The proof of this is standard and generalizes the proof of Lemma 2.4 above (swapping subproper for proper and removing the bound by the continuum).

PROOF. Assume $\mathbb{P}$ is proper. Let G be a $\mathbb{P}$-generic filter over V. For a contradiction, assume SCFA holds in $V[G]$.

Take a regular cardinal $\nu > 2^\omega$ in $V[G]$. In V, take stationary partitions $\langle A_k : k < \omega \rangle$ of $\nu \cap \mathrm{Cof}(\omega)$ and $\langle D_i : i < \omega \rangle$ of ω_1. In $V[G]$, take a subset r of ω which is not in V. Let $\{k(i)\}_{i<\omega}$ be the increasing enumeration of r.

By [17, Lemma 7.1 of Section 4][6] in $V[G]$, there is an increasing continuous function $f : \omega_1 \to \nu$ such that $f[D_i] \subseteq A_{k(i)}$ for all $i < \omega$. Let $\alpha := \sup(\mathrm{range}(f))$. Then, in $V[G]$, we have that $r = \{k \in \omega : A_k \cap \alpha \text{ is stationary in } \alpha\}$.

But the set $\{k \in \omega : A_k \cap \alpha \text{ is stationary in } \alpha\}$ is absolute between V and $V[G]$ since $\mathbb{P}$ is proper and hence preserves stationary subsets of $\mathrm{Cof}(\omega)$ points. But then r is in V, which is a contradiction. ⊣

This shows that either SCFA implies the continuum is at most $\aleph_2$ - though given the results of this paper this seems difficult to prove by methods currently available—or else new techniques for obtaining $2^{\aleph_0} \geq \aleph_3$ are needed, which is well known to be in general an open and difficult area on the frontiers of set theory.

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REFERENCES

[1] R. E. Beaudoin, *The proper forcing axiom and stationary set reflection*. ***Pacific Journal of Mathematics***, vol. 149 (1991), no. 1, pp. 13–24.

[2] S. Cox, *Forcing axioms, approachability and stationary set reflection*. ***Journal of Symbolic Logic***, vol. 86 (2021), no. 2, pp. 499–530.

[3] J. Cummings, *Iterated forcing and elementary embeddings*, ***Handbook of Set Theory*** (M. Foreman and A. Kanamori, editors), Springer, Dordrect, 2010, pp. 775–883.

[4] J. Cummings, M. Foreman, and M. Magidor, *Squares, scales and stationary reflection*. ***Journal of Mathematical Logic***, vol. 1 (2001), no. 1, pp. 35–98.

[5] M. Foreman, M. Magidor, and S. Shelah, *Martin's maximum, saturated ideals, and nonregular ultrafilters I*. ***Annals of Mathematics***, vol. 127 (1988), no. 1, pp. 1–47.

[6] G. Fuchs, *Hierarchies of forcing axioms, the continuum hypothesis and square principles*. ***Journal of Symbolic Logic***, vol. 83 (2018), no. 1, pp. 256–282.

[7] ———, *Diagonal reflection on squares*. ***Archive for Mathematical Logic***, vol. 58 (2019), nos. 1–2, pp. 1–26.

[8] ———, *Aronszajn tree preservation and bounded forcing axioms*. ***Journal of Symbolic Logic***, vol. 86 (2021), no. 1, pp. 293–315.

[9] ———, *Canonical fragments of strong reflection principles*. ***Journal of Mathematical Logic***, vol. 21 (2021), no. 3, 2150023.

[10] ———, *Errata: On the role of the continuum hypothesis in forcing principles for subcomplete forcing*. ***Archive for Mathematical Logic***, vol. 63 (2024), pp. 509–521.

[11] G. Fuchs and K. Minden, *Subcomplete forcing, trees and generic absoluteness*. ***Journal of Symbolic Logic***, vol. 83 (2018), no. 3, pp. 920–938.

[12] G. Fuchs and C. B. Switzer, *Iteration theorems for subversions of forcing classes*. ***Journal of Symbolic Logic***, vol. 90 (2025), no. 1, pp. 1–51.

[13] V. Gitman, J. D. Hamkins, and T. A. Johnstone, *What is the theory* ZFC *without power set?* ***Mathematical Logic Quarterly***, vol. 62 (2016), nos. 4–5, pp. 391–406.

[14] B. R. Jensen, *Forcing axioms compatible with CH*. Handwritten Notes. available at http://www-irm.mathematik.hu-berlin.de/raesch/org/jensen.htm.

[15] ———, *Iteration theorems for subcomplete and related forcings*. Handwritten Notes. available at http://www-irm.mathematik.hu-berlin.de/raesch/org/jensen.html.

[6] Though note, as we have cautioned throughout this text, that this lemma requires the additional assumption of $\nu > 2^{\omega}$, which we are assuming. See [10, Lemma 3.29] for an error free statement.

[16] ———, *Subproper and subcomplete forcing*. Handwritten Notes. available at http://www-irm.mathematik.hu-berlin.de/raesch/org/jensen.html.

[17] R. B. JENSEN, *Subcomplete forcing and L-forcing*, ***E-Recursion, Forcing and C^*-Algebras*** (C. Chong, Q. Feng, T. A. Slaman, and W. H. Woodin, editors), volume 27 of Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, World Scientific, Singapore, 2014, pp. 83–182.

[18] C. LAMBIE-HANSON and M. MAGIDOR, *On the strengths and weaknessess of weak squares*, ***Appalachian Set Theory: 2006-2012*** (J. Cummings and E. Schimmerling, editors), Cambridge University Press, London, 2012, pp. 301–330.

[19] K. MINDEN, *The subcompletensss of diagonal Prikry forcing*. ***Archive for Mathematical Logic***, vol. 59 (2020), pp. 81–102.

[20] T. MIYAMOTO, *A class of preorders iterated under a type of RCS*. ***RIMS Kokyuroku***, vol. 1754 (2011), pp. 81–90.

[21] C. B. SWITZER, ***Alternative Cichoń diagrams and forcing axioms compatible with CH***. Ph.D. thesis, The Graduate Center, The City University of New York, 2020.

[22] V. TORRES-PÉREZ and L. WU, *Strong Chang's conjecture and the tree property at ω_2*. ***Topology and its Applications***, vol. 196 (2015), pp. 999–1004.

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