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RESEARCH ARTICLE

Random independent sets in triangle-free graphs

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Abstract

We establish several new results on the existence of probability distributions on the independent sets in triangle-free graphs where each vertex is present with a given probability. This setting was introduced and studied under the name of "fractional coloring with local demands" by Kelly and Postle and is closely related to the well-studied fractional chromatic number of graphs.

Our first main result strengthens Shearer's classic bound on independence number, proving that for every triangle-free graph G there exists a distribution over independent sets where each vertex v appears with probability $(1 - o(1))\frac{\ln d_G(v)}{d_G(v)}$, resolving a conjecture by Kelly and Postle. This in turn implies new upper bounds on the fractional chromatic number of triangle-free graphs with a prescribed number of vertices or edges, which resolves a conjecture by Cames van Batenburg et al. and addresses yet another one by the same authors.

Our second main result resolves Harris' conjecture on triangle-free d-degenerate graphs, showing that such graphs have fractional chromatic number at most $(4 + o(1))\frac{d}{\ln d}$. As previously observed by various authors, this in turn has several interesting consequences. A notable example is that every triangle-free graph with minimum degree d contains a bipartite induced subgraph of minimum degree $\Omega(\log d)$. This settles a conjecture by Esperet, Kang, and Thomassé.

The main technique employed to obtain our results is the analysis of carefully designed random processes on vertex-weighted triangle-free graphs that preserve weights in expectation. The analysis of these processes yields weighted generalizations of the aforementioned results that may be of independent interest.

Contents

1. Introduction

One of the most fundamental problems in all of combinatorics concerns bounding the famous Ramsey number $R(\ell, k)$, which may be defined as the smallest number n such that every graph on n vertices contains either a clique of size ℓ or an independent set of size k. The first highly challenging instance

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of this general problem is the determination of the Ramsey numbers $R(3,k)$, posed as a prize-money question by Erdős already back in 1961 [28] (see also this Erdős problem page entry). Currently, the best known asymptotic bounds are

$$\left(\frac{1}{4}-o(1)\right)\frac{k^2}{\ln k} \le R(3,k) \le (1+o(1))\frac{k^2}{\ln k}.$$

The lower bound was established by an analysis of the famous *triangle-free process* independently by Fiz Pontiveros, Griffiths, and Morris [31] and Bohman and Keevash [10] in 2013. Proving an upper bound on $R(3,k)$ is equivalent to establishing a lower bound on the *independence number* $\alpha(G)$ (i.e., the size of a largest independent set in G) for all triangle-free graphs on n vertices. In 1980, Ajtai, Komlós, and Szemerédi [2] famously proved that every triangle-free graph G on n vertices with average degree $\overline{d}$ satisfies $\alpha(G) \ge c\frac{\ln \overline{d}}{\overline{d}}$, where $c > 0$ is some small absolute constant, which is easily seen to imply that $R(3,k) \le O\left(\frac{k^2}{\log k}\right)$. The stronger upper bound on $R(3,k)$ stated above is due to a strengthening of the result of Ajtai, Komlós and Szemerédi, established in a landmark result by Shearer in 1983. Namely, Shearer [51] improved the constant factor c in this bound significantly by showing that $\alpha(G) \ge \frac{(1-\overline{d})+\overline{d}\ln \overline{d}}{(\overline{d}-1)^2}n = (1-o(1))\frac{\ln \overline{d}}{\overline{d}}n$. Further refining this result, Shearer [52] proved in 1991 that every triangle-free graph G satisfies $\alpha(G) \ge \sum_{v\in V(G)} g(d_G(v))$, where $d_G(v)$ denotes the degree of v in G and $g(d) = (1-o(1))\frac{\ln d}{d}$ is a recursively defined function. This bound is slightly better than Shearer's first bound in terms of the average degree for graphs with unbalanced degree sequences.

These two classic bounds on the independence number of triangle-free graphs due to Shearer have essentially remained the state of the art on the topic for four decades, and due to their ubiquity have found widespread application as a tool across many areas of extremal and probabilistic combinatorics. By a result of Bollobás [11], it is known that Shearer's bounds are tight up to a multiplicative factor of 2. Because of the relation to the Ramsey numbers $R(3,k)$ discussed above, any constant factor improvement of Shearer's longstanding bounds would be a major breakthrough in Ramsey theory. Due to this, a lot of research has been devoted to finding strengthenings and generalizations of Shearer's bounds: We refer to [23, 39] for recent surveys covering Shearer's bound and relations to the hard-core model in statistical mechanics as well as the theory of graph coloring and to [3, 19, 20, 22, 24, 26, 49] for some extensions and generalizations of Shearer's bounds.

The study of lower bounds on the independence number is closely connected to the theory of graph coloring. Recall that in a *proper graph coloring* vertices are assigned colors such that neighboring vertices have distinct colors, and the *chromatic number* $\chi(G)$ of a graph G is the smallest amount of colors required to properly color G. It is easily seen by the definition (by considering a largest "color class") that every graph G on n vertices has an independent set of size at least $\frac{n}{\chi(G)}$. An even stronger lower bound on the independence number is provided by the well-known *fractional chromatic number* $\chi_f(G)$ of the graph. The fractional chromatic number has many different equivalent definitions (see the standard textbook [50] on fractional coloring as a reference). Here, we shall find the following definition convenient: $\chi_f(G)$ is the minimum real number $r \ge 1$ for which there exists a probability distribution on the independent sets of G such that a random independent set I sampled from this distribution contains any given vertex $v \in V(G)$ with probability at least $\frac{1}{r}$. By considering the expected size of a random set drawn from such a distribution, one immediately verifies that $\alpha(G) \ge \frac{n}{\chi_f(G)}$ holds for every graph G. In general, the latter lower bound $\frac{n}{\chi_f(G)}$ on the independence number is stronger than the lower bound $\frac{n}{\chi(G)}$, as there are graphs (such as the Kneser graphs [7, 44]) for which $\chi_f(G)$ is much smaller than $\chi(G)$.

Given these lower bounds of the independence number in terms of the (fractional) chromatic number, it is natural to ask whether there are analogues or strengthenings of Shearer's bounds that provide corresponding *upper* bounds for the (fractional) chromatic number. A prime example of such a result is a recent breakthrough of Molloy [46], who proved that $\chi(G) \le (1+o(1))\frac{\Delta}{\ln \Delta}$ for every triangle-free graph

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G with maximum degree Δ, where the $o(1)$-term vanishes as $\Delta \to \infty$. This strengthened a longstanding previous bound of the form $O\left(\frac{\Delta}{\ln \Delta}\right)$ due to Johansson [38] and recovers Shearer's independence number bound in the case of regular graphs in a stronger form. As with Shearer's bound, it is known that Molloy's bound is optimal up to a factor of 2, and improving the constant 1 to any constant below 1 would be a major advance in the field. Several interesting strengthenings and generalizations of Johansson's and Molloy's results have been proved in the literature, see, for example, [15, 4, 5, 6, 8, 12, 13, 19, 35, 36, 49] for some selected examples.

Our results.

Our first main result concerns the following conjecture from 2018 posed by Kelly and Postle [40] that claims a local strengthening of Shearer's bounds that can also be seen as a degree-sequence generalization of Molloy's bound for fractional coloring[1].

Conjecture 1.1 (Local fractional Shearer/Molloy, cf. Conjecture 2.2 in [40]). *For every triangle-free graph there exists a probability distribution on its independent sets such that every vertex $v \in V(G)$ appears with probability at least $(1 - o(1))\frac{\ln d_G(v)}{d_G(v)}$ in a random independent set sampled from the distribution. Here, the $o(1)$ term represents any function that tends to 0 as the degree grows.*

To see that this conjecture indeed forms a strengthening of Shearer's bounds, note that the expected size of a random independent set drawn from a distribution as given by the conjecture is

$$\sum_{v \in V(G)} (1 - o(1))\frac{\ln d_G(v)}{d_G(v)},$$

which recovers Shearer's second (stronger) lower bound [52] on the independence number up to lower-order terms. But on top of that, and this explains the word "local" in the name of the conjecture, the distribution in Conjecture 1.1 guarantees that *every* vertex can be expected to be contained in the random independent set a good fraction of the time (and lower degree vertices are contained proportionally more frequently). This relates back to the previously discussed fractional chromatic number, and, for instance, directly implies that $\chi_f(G) \le (1 + o(1))\frac{\Delta(G)}{\ln \Delta(G)}$ for every triangle-free graph, which recovers the fractional version of Molloy's bound.

Adding to that, Conjecture 1.1 connects to several other notions of graph coloring discussed in detail by Kelly and Postle, see in particular [40, Proposition 1.4] which provides many different equivalent formulations of Conjecture 1.1. One of these involves the notion of *fractional coloring with local demands* introduced by Dvořák, Sereni and Volec [27]. Following Kelly and Postle [40], given a graph G and a so-called *demand function* $h : V(G) \to [0, 1]$ that assigns to each vertex its individual "demand," an *h-coloring* of a graph G is a mapping $c : V(G) \to 2^{[0,1]}$ that assigns to every vertex $v \in V(G)$ a measurable subset $c(v) \subseteq [0, 1]$ of measure at least $h(v)$, in such a way that adjacent vertices in G are assigned disjoint subsets. Since the function h does not have to be constant but can depend on local information concerning the vertex v in G, this setting extends the usual paradigm of graph coloring in a local manner. Kelly and Postle [40, Proposition 1.4] proved that Conjecture 1.1 is equivalent to the statement that every triangle-free graph has an h-coloring, where $h : V(G) \to [0, 1]$ is a function depending only on the vertex-degrees such that $h(v) = (1 - o(1))\frac{\ln d_G(v)}{d_G(v)}$. We refer to the extensive introduction of [40] for further applications of the conjecture.

In one of their main results, Kelly and Postle [40, Theorem 2.3] proved a relaxation of Conjecture 1.1, replacing the bound $(1 - o(1))\frac{\ln d_G(v)}{d_G(v)}$ with the asymptotically weaker $\left(\frac{1}{2e} - o(1)\right)\frac{\ln d_G(v)}{d_G(v) \ln \ln d_G(v)}$. As the first main result of this paper, we fully resolve Conjecture 1.1.

[1]For presentation purposes, we here chose to present Conjecture 2.2 by Kelly and Postle from [40] in a rephrased version, using Proposition 1.4 from their paper which yields several equivalent variants of their conjecture.

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Theorem 1.2. *For every triangle-free graph G there exists a probability distribution $\mathcal{D}$ on the independent sets of G such that*

$$\mathbb{P}_{I\sim D}[v \in I] \geq (1 - o(1))\frac{\ln(d_G(v))}{d_G(v)}$$

for every $v \in V(G)$. Here the $o(1)$ represents a function of $d_G(v)$ that tends to 0 as the degree grows.

A pleasing consequence of Theorem 1.2 is that it can also be used to fully resolve another conjecture about fractional coloring raised in 2018 by Cames van Batenburg, de Joannis de Verclos, Kang, and Pirot [16]: Motivated by the aforementioned problem of estimating the Ramsey number $R(3, k)$, in 1967 Erdős asked the fundamental question of determining the maximum chromatic number of triangle-free graphs on n vertices. An observation of Erdős and Hajnal [29] combined with Shearer's bound implies an upper bound $(2\sqrt{2}+o(1)))\sqrt{\frac{n}{\ln n}}$ for this problem. In recent work of Davies and Illingworth [21], this upper bound was improved by a $\sqrt{2}$-factor to the current state of the art $(2+o(1))\sqrt{\frac{n}{\ln n}}$. The current best lower bound for this quantity is $(1/\sqrt{2} - o(1))\sqrt{\frac{n}{\ln n}}$, coming from the aforementioned lower bounds on $R(3, k)$ [31, 10].

Cames van Batenburg et al. [16] studied the natural analogue of this question for fractional coloring and made the following conjecture.

Conjecture 1.3 (cf. Conjecture 4.3 in [16]). *As $n \to \infty$, every triangle-free graph on n vertices has fractional chromatic number at most $(\sqrt{2} + o(1))\sqrt{\frac{n}{\ln n}}$.*

In one of their main results [16, Theorem 1.4], Cames van Batenburg et al. proved the fractional version of the result of Davies and Illingworth, namely an upper bound $(2+o(1))\sqrt{\frac{n}{\ln n}}$ on the fractional chromatic number. Using a connection between Conjectures 1.1 and 1.3 proved by Kelly and Postle [40, Proposition 5.2], we are able to confirm Conjecture 1.3 too.

Theorem 1.4. *The maximum fractional chromatic number among all n-vertex triangle-free graphs is at most*

$$(\sqrt{2} + o(1))\sqrt{\frac{n}{\ln(n)}}.$$

We also prove a similar upper bound on the fractional chromatic number of triangle-free graphs in terms of the number of edges, as follows.

Theorem 1.5. *The maximum fractional chromatic number among triangle-free graphs with m edges is at most*

$$(18^{1/3} + o(1))\frac{m^{1/3}}{(\ln m)^{2/3}}.$$

Theorem 1.5 comes very close to confirming another conjecture of Cames van Batenburg et al. [16, Conjecture 4.4], stating that every triangle-free graph with m edges has fractional chromatic number at most $(16^{1/3} + o(1))m^{1/3}/(\ln m)^{2/3}$. In fact, after a personal communication with the authors of [16], it turned out that the constant $16^{1/3}$ seems to be due to a miscalculation on their end. In particular, it was claimed in [16] that the conjectured bound on the fractional chromatic number can be verified in the special case of d–regular triangle-free graphs using the upper bound $\chi_f(G) \leq \min((1 + o(1))d/\ln d, n/d)$. However, assuming $n = (1 + o(1))d^2/\ln d$ and thus $m = (1/2 + o(1))d^3/\ln d$, this upper bound simplifies to $(1 + o(1))d/\ln d = (1 + o(1))(2m)^{1/3}/(\ln m^{1/3})^{2/3} = (1 + o(1))(18m)^{1/3}/(\ln m)^{2/3}$, matching our bound in Theorem 1.5.

Let us now turn to our second main result. A broad generalization of the standard upper bound $\chi(G) \leq \Delta(G) + 1$ on the chromatic number in terms of the maximum degree is the upper bound $\chi(G) \leq d + 1$ which holds for every *d-degenerate* graph. Given Johansson's and Molloy's improved

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upper bounds for the chromatic number of the form $O(\Delta(G)/\ln\Delta(G))$ for triangle-free graphs, it is tempting to suspect that similarly an upper bound $O(d/\ln d)$ holds for the chromatic number of triangle-free d-degenerate graphs. However, this turns out to be too strong: A number of articles ranging back to at least the 1940s, see for instance [25, 55, 47], present constructions of d-degenerate triangle-free graphs with chromatic number $d + 1$. However, all these constructions turn out to have relatively small fractional chromatic number. And indeed, a well-known conjecture by Harris [33] posits that this is part of a general phenomenon.

Conjecture 1.6 (cf. Conjecture 6.2 in [33]). *Suppose that G is d-degenerate and triangle-free. Then* $\chi_f(G) = O(d/\log d)$.

This conjecture has gained quite some attention in recent years. It is known to imply various other conjectures and strengthenings of known results in the literature [30, 33, 37, 41, 43] including another well-known conjecture by Esperet, Kang and Thomassé [30, Conjecture 1.5] that any triangle-free graph with minimum degree d contains an induced bipartite subgraph of minimum degree $\Omega(\log d)$. Currently the best known lower bound on the conjecture by Esperet et al. is $\Omega(\log d/\log\log d)$ due to Kwan, Letzter, Sudakov and Tran [41], though Girão and Hunter (personal communication) recently announced upcoming work improving this to average degree $(1 - o(1))\ln d$. See also [24, 40]. Despite this attention, Harris' conjecture itself has remained wide open with no significant improvement of the trivial upper bound $d + 1$ having existed in the literature thus far.

As our second main result, we fully resolve Conjecture 1.6. More precisely, we prove the following.

Theorem 1.7. *Suppose G is a triangle-free and d-degenerate graph. Then* $\chi_f(G) \le (4 + o(1))\frac{d}{\ln d}$, *where the* $o(1)$ *term tends to* 0 *as d increases.*

As already mentioned, this result is known to have some nice consequences. For instance, a direct application of the theorem implies that any triangle-free graph with minimum degree d contains an induced bipartite subgraph of average degree at least $(\frac{1}{4} - o(1))\ln d$ (and thus one of minimum degree at least $(\frac{1}{8} - o(1))\ln d$). We refer to [30, Theorem 3.1] for further details on the calculations. This proves [30, Conjecture 1.5], improving on the previously best known bound [41] by a factor $\Theta(\log\log d)$. We note that the factor $\frac{1}{4}$ can likely be improved by a more careful analysis, but we do not attempt this here.

In addition, Harris [33] observed that Theorem 1.7 can be extended to the setting where the triangle-free condition is relaxed to G being *locally sparse*, similar to the extension of the upper bound for the chromatic number of triangle-free graphs presented in [4]. More precisely, we say that a d-degenerate graph G has *local triangle bound* y if each vertex in G is the last vertex of at most y triangles, where *last* refers to the degeneracy ordering of the graph. Combining Theorem 1.7 with [33, Lemma 6.3], it follows that

$$\chi_f(G) = O\left(\frac{d}{\ln(d^2/y)}\right)$$

for any d-degenerate graph G with local triangle bound y. This in turn proves various relationships between the chromatic number and the triangle count of a graph. We refer to [33, Section 6] for more details. We remark that Harris formally stated his results in a slightly weaker form, namely with "local triangle bound" referring to the maximum number of triangles containing a vertex in the graph. However, looking into his arguments [33] it is not hard to check that they work just as well for the modified definition of local triangle bound given above.

In fact, Theorem 1.7 can be seen as a special case of the following generalization of Harris' conjecture.

Theorem 1.8. *Let G be a triangle-free graph with a vertex ordering* $v_1, v_2, \ldots, v_n$. *Suppose* $p : V(G) \to [0, 1]$ *satisfies*

$$p(v_i) \le \prod_{v_j \in N_L(v_i)} (1 - p(v_j))$$

for all vertices v_i, where $N_L(v_i)$ denotes the set of neighbors v_j of v_i with $j < i$. Then there exists a probability distribution $\mathcal{D}$ over the independent sets of G such that

$$\mathbb{P}_{I\sim\mathcal{D}}[v_i \in I] \geq \frac{p(v_i)}{4},$$

for all vertices v_i.

It is not too hard to see that this statement implies Theorem 1.7. In fact, Theorem 1.8 also implies, up to constant factors, the *local fractional Shearer bound* as can be seen by ordering the vertices of any triangle-free graph decreasingly by their degrees and setting $p(v_i) = \Theta\left(\frac{\ln d_G(v_i)}{d_G(v_i)}\right)$. In particular, this matches the bound in Theorem 1.2 up to a constant factor. Beyond this, Theorem 1.8 appears to be a very natural extension of Harris' conjecture which may be of independent interest.

High-level proof ideas.

The key concept to prove our two main results, Theorem 1.2 and 1.7, is to equip the triangle-free graphs under consideration with positive vertex-weights, and attempt to prove extensions of our claims in these generalized setups (see Theorems 2.1 and 4.1, respectively). In both of these settings, we will construct random independent sets I by iteratively/inductively applying the following type of operation: Pick some vertex v and include it in I with probability depending on its current weight. If v is added to I, set the weight of its neighbors to 0 (or, equivalently, remove its neighbors from G). Otherwise, update the vertex-weights in G such that weights are preserved in expectation. This is particularly useful, as it limits the influence of the event that $v \in I$ from any update outside the neighborhood of v, which in turn allows us to derive lower bounds on the probability that $v \in I$ in terms of the initial weights of v and its neighbors.

Organization.

The rest of the paper is structured as follows: In Section 2 we establish a key technical result, namely Theorem 2.1, that goes beyond Theorem 1.2 and generalizes it to a vertex-weighted setting. In the following Section 3 we then derive our first three results (Theorems 1.2, 1.4, and 1.5) from Theorem 2.1. In Section 4, we then proceed to present a random process on weighted graphs with a fixed linear vertex-ordering. Analyzing this process then yields our second key technical result, Theorem 4.1. Finally, our second main result Theorem 1.7 as well as Theorem 1.8 can be quickly deduced as special cases of this more general statement.

We conclude the paper in Section 5 with some discussion of open problems and future research directions.

2. Key technical result for Theorem 1.2

In this section, we present the proof of a key technical result, Theorem 2.1 below, which generalizes Theorem 1.2 to a vertex-weighted setting.

In the following, we denote by[2] $f : [0,\infty) \to \mathbb{R}_+$ the unique continuous extension of $x \to \frac{(1-x)+x\ln(x)}{(x-1)^2}$ from $[0,\infty) \setminus \{0,1\}$ to $[0,\infty)$. It is not hard to check that f exists and has the following properties:

- $f(0) = 1, f(1) = \frac{1}{2}$.
- f is convex.
- f is strictly monotonically decreasing.

[2]We remark that the function f is the same function that was used by Shearer in his first paper [51].

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- f is continuously differentiable on $(0, \infty)$ and satisfies the following differential equation:

$$x(x-1)f'(x) + (x+1)f(x) = 1$$

  for every $x > 0$.
- $|xf'(x)| < 1$ for every $x > 0$.
- $f(x) = (1 - o(1))\frac{\ln(x)}{x}$ as $x \to \infty$.

In the following, given a weight function $w : V(G) \to \mathbb{R}_+$ on the vertices of a graph G and a subset $X \subseteq V(G)$, $w(X) := \sum_{v \in X} w(v)$ denotes the total weight of X.

In this section our goal shall be to establish the following statement, which represents a main technical contribution of the paper at hand and from which Theorems 1.2, 1.4, and 1.5 can be deduced (this will be done in the following section). We believe that the more general result offered by Theorem 2.1 may be of independent interest.

Theorem 2.1. *For every triangle-free graph G and every strictly positive weight function $w : V(G) \to \mathbb{R}_+$ on the vertices there exists a probability distribution $\mathcal{D}$ on the independent sets of G such that*

$$\mathbb{P}_{I \sim \mathcal{D}}[v \in I] \geq f\left(\frac{w(N_G(v))}{w(v)}\right)$$

for every vertex $v \in V(G)$.

Proof. We prove the statement by induction on $|V(G)|$. In the base case $|V(G)| = 1$, there is a unique vertex v of G, so $w(N_G(v)) = w(\emptyset) = 0$ and hence our target probability of the appearance of v in a randomly drawn independent set is $f(0) = 1$. This is easily achieved by letting $\mathcal{D}$ be the probability distribution that always picks $\{v\}$, establishing the induction base.

For the induction step, let us assume that G is a triangle-free graph on at least two vertices and that we have already proven the claim of the theorem for all triangle-free graphs with strictly less vertices than G.

Let $K \subseteq [0, 1]$ be the set of all $\delta \in [0, 1]$ such that for every strictly positive weight function $w : V(G) \to \mathbb{R}_+$ there exists a probability distribution $\mathcal{D}$ on the independent sets of G such that $\mathbb{P}_{I \sim \mathcal{D}}[v \in I] \geq f\left(\frac{w(N_G(v))}{w(v)}\right) - \delta$ for every $v \in V(G)$. Since f takes values in $[0, 1]$, we trivially have $1 \in K$. Furthermore, we claim that the set K is closed (and thus compact). To see this, note that $K = \bigcap_{w:V(G) \to \mathbb{R}_+} K_w$, where K_w is the set of all $\delta \in [0, 1]$ for which there exists a probability distribution $\mathcal{D}$ on the independent sets of G satisfying $\mathbb{P}_{I \sim \mathcal{D}}[v \in I] \geq f\left(\frac{w(N_G(v))}{w(v)}\right) - \delta$ for every $v \in V(G)$. Since intersections of closed sets are closed, it suffices to show that K_w is closed for every fixed $w : V(G) \to \mathbb{R}_+$. Now, consider the following linear program ($\mathcal{I}(G)$ denotes the collection of all independent sets in G):

$$\begin{aligned} &\min \ y \\ \text{s.t.} \ & y + \sum_{I \in \mathcal{I}(G): v \in I} x_I \geq f\left(\frac{w(N_G(v))}{w(v)}\right) \ (\forall v \in V(G)), \\ & \sum_{I \in \mathcal{I}(G)} x_I = 1 \\ & x_I \geq 0 \ (\forall I \in \mathcal{I}(G)). \end{aligned}$$

It can easily be checked that this linear program is bounded and feasible, and hence has a unique optimum y^*. Further, since the constraints of the program encode a probability distribution $\mathcal{D}$ on independent sets with $\mathbb{P}_{I \sim \mathcal{D}}[v \in I] \geq f\left(\frac{w(N_G(v))}{w(v)}\right) - y$, we can see that $K_w = [y^*, 1]$ is indeed a closed set as desired.

This shows that K is indeed compact and hence has a unique minimum $\delta_0 \in K$. Our goal is to show that $0 \in K$ (equivalently, $\delta_0 = 0$), since this clearly establishes the induction hypothesis for G. So, toward a contradiction, let us assume $\delta_0 > 0$ in the following.

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Let us define $\delta := \delta_0 - \frac{\delta_0^2}{8}$. Then, since $\delta \in (0, \delta_0)$ and hence $\delta \notin K$, there exists a positive weight function $w : V(G) \longrightarrow \mathbb{R}_+$ such that there exists no probability distribution on the independent sets of G for which every vertex v is contained in an independent set drawn from the distribution with probability at least $f\left(\frac{w(N_G(v))}{w(v)}\right) - \delta$. Since the latter formula is scale-invariant, we may assume without loss of generality throughout the rest of the proof that $w(V(G)) = 1$.

Let us pick and fix some $\varepsilon \in (0, 1)$ (for now arbitrarily, later on we will assign a concrete value). Let $w' : V(G) \longrightarrow \mathbb{R}^+$ be a modified vertex-weighting of G, defined as $w'(v) := w(v) \cdot \exp(\varepsilon w(N_G(v)))$ for every $v \in V(G)$.

Since $\delta_0 \in K$, there must exist a probability distribution $\mathcal{D}$ on the independent sets of G such that

$$\mathbb{P}_{I\sim\mathcal{D}}[v \in I] \geq f\left(\frac{w'(N_G(v))}{w'(v)}\right) - \delta_0$$

for every $v \in V(G)$.

For a vertex $u \in V(G)$, let us denote by $\overline{N}_G(u) := \{u\} \cup N_G(u)$ the *closed neighborhood* of u and by $G_u := G - \overline{N}_G(u)$ the graph obtained from G by deleting this closed neighborhood. By the inductive assumption, for every $u \in V(G)$ there exists a probability distribution $\mathcal{D}_u$ on G_u such that $\mathbb{P}_{I\sim\mathcal{D}_u}[v \in I] \geq f\left(\frac{w'(N_{G_u}(v))}{w'(v)}\right)$ for every $v \in V(G_u)$.

Let us now define $\varepsilon := \frac{\delta_0}{4} \in (0, 1)$, and let us consider the following process to generate a random independent set I of G:

- With probability $1 - \varepsilon$ (we call this event A), draw I randomly from the distribution $\mathcal{D}$ and return I.
- With probability ε (we call this event $B := A^c$), do the following: First, sample randomly a vertex $u \in V(G)$ where u equals any given vertex x with probability exactly $w(x)$. Then, randomly draw an independent set I_u from the distribution $\mathcal{D}_u$ and return the independent set $I := \{u\} \cup I_u$.

In the following, let $\mathcal{D}'$ denote the probability distribution on independent sets of G that is induced by the random independent set I created according to the above process. By our choice of the weight function w, there must exist some vertex $v \in V(G)$ such that

$$\mathbb{P}_{I\sim\mathcal{D}'}[v \in I] < f\left(\frac{w(N_G(v))}{w(v)}\right) - \delta.$$

Our intermediate goal is to give a lower bound on $\mathbb{P}_{I\sim\mathcal{D}'}[v \in I]$.

To estimate this probability, we stick with the random process described above. We then have

$$\begin{aligned}\mathbb{P}_{I\sim\mathcal{D}'}[v \in I] &= (1-\varepsilon)\mathbb{P}_{I\sim\mathcal{D}'}[v \in I|A] + \varepsilon\mathbb{P}_{I\sim\mathcal{D}'}[v \in I|B] \\ &= (1-\varepsilon)\mathbb{P}_{I\sim\mathcal{D}}[v \in I] + \varepsilon \sum_{x\in V(G)} \mathbb{P}_{I\sim\mathcal{D}'}[v \in I|B \wedge \{u = x\}]w(x) \\ &= (1-\varepsilon)\mathbb{P}_{I\sim\mathcal{D}}[v \in I] + \varepsilon w(v) + \varepsilon \sum_{x\in V(G)\setminus\overline{N}_G(v)} \mathbb{P}_{I\sim\mathcal{D}_x}[v \in I]w(x).\end{aligned}$$

By our choice of the distributions $\mathcal{D}$ and $\mathcal{D}_x$, $x \in V(G)$, we have

$$\begin{aligned}&(1-\varepsilon)\mathbb{P}_{I\sim\mathcal{D}}[v \in I] + \varepsilon \sum_{x\in V(G)\setminus\overline{N}_G(v)} \mathbb{P}_{I\sim\mathcal{D}_x}[v \in I]w(x) \\ &\geq (1-\varepsilon)\left(f\left(\frac{w'(N_G(v))}{w'(v)}\right) - \delta_0\right) + \varepsilon \sum_{x\in V(G)\setminus\overline{N}_G(v)} f\left(\frac{w'(N_{G_x}(v))}{w'(v)}\right)w(x) \\ &= -(1-\varepsilon)\delta_0 + (1-\varepsilon)f\left(\frac{w'(N_G(v))}{w'(v)}\right) + \sum_{x\in V(G)\setminus\overline{N}_G(v)} \varepsilon w(x) f\left(\frac{w'(N_{G_x}(v))}{w'(v)}\right).\end{aligned}$$

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Since $(1-\varepsilon)+\sum_{x\in V(G)\setminus\overline{N}_G(v)}\varepsilon w(x)=(1-\varepsilon)+\varepsilon\Big(1-w(\overline{N}_G(v))\Big)=1-\varepsilon w(\overline{N}_G(v))$, the convexity of f implies that

$$(1-\varepsilon)f\left(\frac{w'(N_G(v))}{w'(v)}\right)+\sum_{x\in V(G)\setminus\overline{N}_G(v)}\varepsilon w(x)f\left(\frac{w'(N_{G_x}(v))}{w'(v)}\right)\geq$$
$$\Big(1-\varepsilon w(\overline{N}_G(v))\Big)f\left(\frac{(1-\varepsilon)w'(N_G(v))+\sum_{x\in V(G)\setminus\overline{N}_G(v)}\varepsilon w(x)w'(N_{G_x}(v))}{w'(v)\Big(1-\varepsilon w(\overline{N}_G(v))\Big)}\right).$$

The next claim gives a simple upper bound for the expression in the argument of f above.

Claim 2.2. We have that

$$\frac{(1-\varepsilon)w'(N_G(v))+\sum_{x\in V(G)\setminus\overline{N}_G(v)}\varepsilon w(x)w'(N_{G_x}(v))}{w'(v)\Big(1-\varepsilon w(\overline{N}_G(v))\Big)}\leq\frac{w(N_G(v))}{w(v)}\cdot e^{\varepsilon(w(v)-w(N_G(v)))}.$$

Proof. We have

$$\begin{aligned}
&(1-\varepsilon)w'(N_G(v))+\sum_{x\in V(G)\setminus\overline{N}_G(v)}\varepsilon w(x)w'(N_{G_x}(v))\\
&=\sum_{y\in N_G(v)}(1-\varepsilon)w'(y)+\sum_{x\in V(G)\setminus\overline{N}_G(v)}\varepsilon w(x)\sum_{y\in N_G(v)\setminus\overline{N}_G(x)}w'(y)\\
&=\sum_{y\in N_G(v)}\left((1-\varepsilon)+\sum_{x\in V(G)\setminus(\overline{N}_G(v)\cup\overline{N}_G(y))}\varepsilon w(x)\right)w'(y)\\
&=\sum_{y\in N_G(v)}\Big(1-\varepsilon+\varepsilon(1-w(\overline{N}_G(v)\cup\overline{N}_G(y)))\Big)w'(y)\\
&=\sum_{y\in N_G(v)}\Big(1-\varepsilon w(\overline{N}_G(v)\cup\overline{N}_G(y))\Big)w'(y).
\end{aligned}$$

Note that for every $y\in N_G(v)$, we have $\overline{N}_G(v)\cup\overline{N}_G(y)=N_G(v)\cup N_G(y)$. Furthermore, since G is triangle-free, the sets $N_G(v)$ and $N_G(y)$ are disjoint, and thus we have $w(\overline{N}_G(v)\cup\overline{N}_G(y))=w(N_G(v))+w(N_G(y))$. This implies

$$\begin{aligned}
&\frac{(1-\varepsilon)w'(N_G(v))+\sum_{x\in V(G)\setminus\overline{N}_G(v)}\varepsilon w(x)w'(N_{G_x}(v))}{w'(v)\Big(1-\varepsilon w(\overline{N}_G(v))\Big)}\\
&=\frac{1}{w'(v)}\sum_{y\in N_G(v)}\frac{1-\varepsilon w(N_G(v))-\varepsilon w(N_G(y))}{1-\varepsilon w(\overline{N}_G(v))}w'(y)\\
&=\frac{1}{w'(v)}\sum_{y\in N_G(v)}\left(1-\varepsilon\frac{w(N_G(y))-w(v)}{1-\varepsilon w(\overline{N}_G(v))}\right)w'(y)\\
&\leq\frac{1}{w'(v)}\sum_{y\in N_G(v)}(1-\varepsilon(w(N_G(y))-w(v)))w'(y)\\
&\leq\frac{1}{w'(v)}\sum_{y\in N_G(v)}\exp(-\varepsilon(w(N_G(y))-w(v)))\cdot w(y)\exp(\varepsilon w(N_G(y)))
\end{aligned}$$

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$$= \frac{1}{w'(v)} \exp(\varepsilon w(v)) w(N_G(v))$$
$$= \frac{w(N_G(v))}{w(v)} \cdot \exp(\varepsilon(w(v) - w(N_G(v)))),$$

where we used the definition of w' in the last and third to last line. This concludes the proof of the claim. □

Using Claim 2.2 and the previously established inequalities (using that f is monotonically decreasing), it follows that $\mathbb{P}_{I\sim\mathcal{D}'}[v \in I]$ is lower-bounded by

$$\varepsilon w(v) - (1-\varepsilon)\delta_0 + \Big(1 - \varepsilon w(\overline{N}_G(v))\Big) f\left(\frac{w(N_G(v))}{w(v)} \cdot \exp(\varepsilon(w(v) - w(N_G(v))))\right).$$

Let us now go about estimating the above expression. By Taylor expansion, it is not hard to verify that the inequality $\exp(z) \le 1 + z + z^2$ holds for every $z \in [-1, 1]$. Let us set $x := \frac{w(N_G(v))}{w(v)}$, $z := \varepsilon(w(v) - w(N_G(v)))$ and $y := x \cdot \exp(z)$. Note that since f is convex and differentiable, we have the inequality $f(y) \ge f(x) + f'(x)(y - x)$. Since $w(v), w(N_G(v)) \le w(V(G)) = 1$, we obtain $|z| \le \varepsilon < 1$ and thus $y \le x(1 + z + z^2)$. Since f is monotonically decreasing, we have $f'(x) < 0$. Putting these facts together, it follows that

$$f\left(\frac{w(N_G(v))}{w(v)} \cdot \exp(\varepsilon(w(v) - w(N_G(v))))\right) = f(y)$$
$$\ge f(x) + f'(x)(y - x)$$
$$\ge f(x) + f'(x) \cdot x \cdot (z + z^2)$$
$$\ge f(x) + xzf'(x) - \varepsilon^2,$$

where we used that $|x \cdot f'(x)| \le 1$ for every $x > 0$ and that $|z| \le \varepsilon$ in the last line. Plugging this estimate into the above lower bound for $\mathbb{P}_{I\sim\mathcal{D}'}[v \in I]$ and using that by our choice of v, we have $\mathbb{P}_{I\sim\mathcal{D}'}[v \in I] < f\left(\frac{w(N_G(v))}{w(v)}\right) - \delta$, we find:

$$f(x) - \delta > \mathbb{P}_{I\sim\mathcal{D}'}[v \in I]$$
$$\ge -(1-\varepsilon)\delta_0 + \varepsilon w(v) + (1 - \underbrace{\varepsilon(w(v) + w(N_G(v))))}_{\le \varepsilon} \cdot (f(x) + \underbrace{xzf'(x) - \varepsilon^2}_{\le \varepsilon})$$
$$> -(1-\varepsilon)\delta_0 + \varepsilon w(v) + f(x) + xzf'(x) - 2\varepsilon^2 - \varepsilon(w(v) + w(N_G(v)))f(x).$$

Rearranging yields

$$\delta_0 - \delta > \varepsilon\delta_0 - 2\varepsilon^2 + \varepsilon w(v) + xzf'(x) - \varepsilon(w(v) + w(N_G(v)))f(x).$$

Using that $x = \frac{w(N_G(v))}{w(v)}$ and $z = \varepsilon w(v)(1 - x)$, we can simplify as follows.

$$\varepsilon w(v) + xzf'(x) - \varepsilon(w(v) + w(N_G(v)))f(x)$$
$$= \varepsilon w(v)(1 + x(1-x)f'(x) - (1+x)f(x)) = 0,$$

where we used the differential equation satisfied by f in the last step. Hence, we have proven the inequality $\delta_0 - \delta > \varepsilon\delta_0 - 2\varepsilon^2$. Recalling our definitions $\delta := \delta_0 - \frac{\delta_0^2}{8}$ and $\varepsilon := \frac{\delta_0}{4}$ we can now see that the above inequality implies $\frac{\delta_0^2}{8} > \frac{\delta_0^2}{8}$, which is absurd. This is the desired contradiction, which shows that

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our initial assumption, namely that $\delta_0 > 0$, was wrong. Hence, we have shown that $\delta_0 = 0$, establishing the inductive claim for G. This concludes the proof of the theorem by induction. □

3. Proofs of Theorems 1.2, 1.4, and 1.5

In this section we use Theorem 2.1 established in the previous section to deduce Theorems 1.2, 1.4, and 1.5. Let us start with Theorem 1.2, which is a simple corollary of Theorem 2.1 by using the all-1 weight assignment.

Proof of Theorem 1.2. Let G be any given triangle-free graph, and let $w : V(G) \to \mathbb{R}_+$ be defined as $w(v) := 1$ for every $v \in V(G)$. Then $\frac{w(N_G(v))}{w(v)} = d_G(v)$ for every vertex $v \in V(G)$, and hence by Theorem 2.1 there exists a probability distribution $\mathcal{D}$ on independent sets of G such that

$$\mathbb{P}_{I\sim\mathcal{D}}[v \in I] \geq f(d_G(v))$$

for every $v \in V(G)$. Since $f(x) = (1 - o(1))\frac{\ln(x)}{x}$, this establishes Theorem 1.2. □

Next, let us deduce Theorem 1.4. This, in fact, can be derived from Theorem 1.2 using the following relationship between Conjectures 1.1 and 1.3 proved by Kelly and Postle [40, Proposition 5.2]:

Proposition 3.1. *For every $\varepsilon, c > 0$, the following holds for sufficiently large n. Let G be a triangle-free graph on n vertices with demand function h such that $h(v) \geq c\frac{\ln d_G(v)}{d_G(v)}$ for every $v \in V(G)$. If G has an h-coloring, then*

$$\chi_f(G) \leq (\sqrt{2/c} + \varepsilon)\sqrt{\frac{n}{\ln n}}.$$

With this statement at hand, we can now easily deduce Theorem 1.4.

Proof of Theorem 1.4. The statement of Theorem 1.4 is equivalent to showing that for every fixed $\delta > 0$ and n sufficiently large in terms of δ, every triangle-free graph G on n vertices satisfies $\chi_f(G) \leq (\sqrt{2}+\delta)\sqrt{\frac{n}{\ln n}}$. Let $\varepsilon > 0$ and $0 < c < 1$ (only depending on δ) be chosen such that $\sqrt{2/c}+\varepsilon < \sqrt{2}+\delta$. By Proposition 3.1 there exists $n_0 = n_0(\varepsilon, c) \in \mathbb{N}$ such that every triangle-free graph G with $n \geq n_0$ vertices that admits an h-coloring for some demand function h satisfying $h(v) \geq c\frac{\ln d_G(v)}{d_G(v)}$ for all $v \in V(G)$ has fractional chromatic number at most $(\sqrt{2/c}+\varepsilon)\sqrt{\frac{n}{\ln n}} \leq (\sqrt{2}+\delta)\sqrt{\frac{n}{\ln n}}$. By [40, Proposition 1.4 (c)] the latter statement is equivalent to the following: Every triangle-free graph on $n \geq n_0$ vertices that admits a probability distribution on its independent sets such that each vertex v is included with probability at least $c\frac{\ln d_G(v)}{d_G(v)}$ in a randomly drawn independent set has fractional chromatic number at most $(\sqrt{2}+\delta)\sqrt{\frac{n}{\ln n}}$.

Since $c < 1$, Theorem 1.2 implies that there exists a constant $D = D(c)$ such that every triangle-free graph of minimum degree at least D admits a probability distribution on its independent sets where each vertex v is included in a randomly drawn independent set with probability at least $c\frac{\ln d_G(v)}{d_G(v)}$. Putting this together with the statement above, we immediately obtain that every triangle-free graph on $n \geq n_0$ vertices with minimum degree at least D has fractional chromatic number at most $(\sqrt{2}+\delta)\sqrt{\frac{n}{\ln n}}$.

Let n_1 be an integer chosen large enough such that $(\sqrt{2}+\delta)\sqrt{\frac{n_1}{\ln n_1}} > \max\{D+1, n_0\}$. We now claim that *every* triangle-free graph on $n \geq n_1$ vertices has fractional chromatic number at most $(\sqrt{2}+\delta)\sqrt{\frac{n}{\ln n}}$, which is the statement that we wanted to prove initially. Let G be any given triangle-free graph on $n \geq n_1$ vertices. Let G' be the subgraph of G obtained by repeatedly removing vertices of degree less than D from G, until no such vertices are left (G' is the so-called *D-core* of G). Then G' is a triangle-free graph that is either empty or has minimum degree at least D. Hence, we either have $|V(G')| < n_0$ and thus $\chi_f(G') < n_0$, or $|V(G')| \geq n_0$ and thus $\chi_f(G') \leq (\sqrt{2}+\delta)\sqrt{\frac{|V(G')|}{\ln|V(G')|}} \leq (\sqrt{2}+\delta)\sqrt{\frac{n}{\ln n}}$.

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Pause to verify that $\chi_f(G) \le \max\{\chi_f(G-v), d_G(v)+1\}$ holds for every graph G and every vertex $v \in V(G)$. Repeated application of this fact combined with the definition of G' now implies that

$$\chi_f(G) \le \max\{\chi_f(G'), D+1\} \le \max\left\{n_0, (\sqrt{2}+\delta)\sqrt{\frac{n}{\ln n}}, D+1\right\} = (\sqrt{2}+\delta)\sqrt{\frac{n}{\ln n}},$$

as desired. Here, we used our choice of n_1 and that $n \ge n_1$ in the last step. This concludes the proof. □

Finally, let us prove the upper bound on the fractional chromatic number of triangle-free graphs with a given number of edges stated in Theorem 1.5. Interestingly, it can be deduced by applying Theorem 2.1 with two different vertex-weight functions following a similar proof idea to Proposition 3.1.

Proof of Theorem 1.5. Let G be any given triangle-free graph with m edges. To prove the upper bound on the fractional chromatic number, without loss of generality it suffices to consider the case when G has no isolated vertices. By definition of the fractional chromatic number, we have to show that there exists a probability distribution on the independent sets of G for which a randomly drawn independent set contains any given vertex of G with probability at least $(1-o(1))(\ln m)^{2/3}/(18m)^{1/3}$. To construct such a distribution, we consider the following process to generate a random independent set I in G. With probability $1/3$ pick I as in Theorem 2.1 with the weight function defined as $w_1(v) := 1$ for every $v \in V(G)$, with probability $1/3$ we pick I as in Theorem 2.1 using the weight function $w_2(v) := d_G(v)$ for every $v \in V(G)$, and with probability $1/3$ we pick a random vertex u in G with $\mathbb{P}[u = v] = d_G(v)/2m$ for every $v \in V(G)$ and let I be its neighborhood (which is clearly an independent set in G).

It follows that

$$\mathbb{P}[v \in I] \ge \frac{1}{3} f(d_G(v)) + \frac{1}{3} f(S_G(v)/d_G(v)) + \frac{1}{3}\frac{S_G(v)}{2m},$$

for every $v \in V(G)$, where $S_G(v)$ denotes the sum of degrees over all neighbors of v in G. It suffices to show that the right-hand side is at least $(1-o(1))(\ln m)^{2/3}/(18m)^{1/3}$ for all vertices v.

Observe that if either $d_G(v) < m^{1/3}$ or $S_G(v)/d_G(v) < m^{1/3}$, then the desired inequality is already satisfied with room to spare from the first and second terms, respectively. Otherwise, if $d_G(v) \ge m^{1/3}$ and $S_G(v)/d(v) \ge m^{1/3}$, we have

$$\begin{aligned}
&\frac{1}{3} f(d_G(v)) + \frac{1}{3} f(S_G(v)/d_G(v)) + \frac{1}{3}\frac{S_G(v)}{2m} \\
&\quad = \frac{1}{3}\frac{(1-o(1))\ln d_G(v)}{d_G(v)} + \frac{1}{3}\frac{(1-o(1))\ln(S_G(v)/d_G(v))}{S_G(v)/d_G(v)} + \frac{1}{3}\frac{S_G(v)}{2m} \\
&\quad \ge \frac{1}{3}\frac{(1-o(1))\ln(m^{1/3})}{d(v)} + \frac{1}{3}\frac{(1-o(1))\ln(m^{1/3})}{S_G(v)/d_G(v)} + \frac{1}{3}\frac{S_G(v)}{2m} \\
&\quad \ge \left(\frac{(1-o(1))\ln(m^{1/3})}{d_G(v)} \cdot \frac{(1-o(1))\ln(m^{1/3})}{S_G(v)/d_G(v)} \cdot \frac{S_G(v)}{2m}\right)^{1/3},
\end{aligned}$$

where the last line follows by the AM–GM inequality. Simplifying yields a lower bound of $(1-o(1))\left(\frac{\ln(m)^2}{18m}\right)^{1/3}$, as desired. □

4. Proofs of Theorems 1.7 and 1.8

In this section, we present a new stochastic process for generating an independent set I in a graph G, and prove an accompanying key technical result, Theorem 4.1, which lower bounds the probability of any vertex being contained in I, under the assumption that G is triangle-free. Theorems 1.7 and 1.8 are both direct consequences of this statement.

Let us assume that G is a triangle-free graph with vertices $v_1, \ldots, v_n$. We denote by $N_L(v_i)$ the set of neighbors v_j of v_i where $j < i$. Similarly, $N_R(v_i)$ denotes the set of neighbors v_j of v_i where $j > i$. Let $w_0 : V(G) \to \mathbb{R}_+$ be any assignment of positive weights to the vertices of G.

Consider the following process: Initially assign vertices the weights $w(v_i) = w_0(v_i)$ for all $1 \le i \le n$. Then for each step i in $1, 2, \ldots, n$ do the following:

- With probability $1 - e^{-w(v_i)}$, put $w(v_j) = 0$ for all $v_j \in N_R(v_i)$.
- With probability $e^{-w(v_i)}$, multiply the weight of all $v_j \in N_R(v_i)$ by $e^{w(v_i)}$.

Let I be the set of vertices v_i for which the first option occurred. It is easy to see that I is an independent set. If $v_i \in I$, then at step i all vertices $v_j \in N_R(v_i)$ get assigned the weight 0 for the rest of the process, which means they enter the independent set with probability $1 - e^{-0} = 0$.

As will be proven next, the random independent set I generated in this fashion has the following property.

Theorem 4.1. *Let $w_0 : V(G) \to (0, 1)$ be any weight function satisfying*

$$\frac{1}{2}\ln(w_0(v_k)) + \sum_{v_i \in N_L(v_k)} w_0(v_i) \le 0$$

for every $1 \le k \le n$. Then we have

$$\mathbb{P}[v_k \in I] \ge \frac{w_0(v_k)}{2}$$

for every $1 \le k \le n$.

Consider a fixed vertex v_k. In order to prove Theorem 4.1, we will work toward establishing a lower bound on the probability that $v_k \in I$. We do this by considering a modified process, defined as follows. Initially assign the vertices weights $\tilde{w}(v_i) = w_0(v_i)$ for all $1 \le i \le n$. Then for each step i in $1, 2, \ldots, k-1$, do the following.

- If $v_i \notin N_L(v_k)$, do the same update rule as for w.
- If $v_i \in N_L(v_k)$, multiply the weight of all vertices $v_j \in N_R(v_i)$ by $e^{\tilde{w}(v_i)}$.

In other words, $\tilde{w}$ has the same update rule as w for any step i where $v_i \notin N_L(v_k)$. For any step i where $v_i \in N_L(v_k)$, the process follows the update rule of the second bullet point of w with probability 1.

Let us denote by $w_i(v_j)$ and $\tilde{w}_i(v_j)$ the weight of v_j after step i in the respective processes, let $\tilde{w}_0(v_j) := w_0(v_j)$, and let

$$X := \sum_{v_i \in N_L(v_k)} \tilde{w}_{k-1}(v_i).$$

By construction of $\tilde{w}$, we have the following relation to w.

Claim 4.2. For any function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ we have

$$\mathbb{E}_w\left[f(w_{k-1}(v_k))\right] = \mathbb{E}_{\tilde{w}}\left[f(\tilde{w}_{k-1}(v_k))e^{-X}\right].$$

Proof. We can encode each possible sequence of weight functions $(w_0, w_1, \ldots, w_{k-1})$ of the process w as a sequence $a \in \{1, 2\}^{k-1}$ where a_i denotes whether, in step i, randomness chooses the first or the second bullet point. In other words, $a_i = 1$ if and only if $v_i \in I$.

Note that if $a_i = 1$ for any index i where $v_i \in N_L(v_k)$ then this sequence will result in $w_{k-1}(v_k) = 0$. Thus, such a sequence does not contribute to the value of $\mathbb{E}_w\left[f(w_{k-1}(v_k))\right]$. Similarly, if $a_i = a_j = 1$ for any two neighboring vertices v_i and v_j, then the probability of the corresponding sequence is 0, which means it also does not contribute to $\mathbb{E}_w\left[f(w_{k-1}(v_k))\right]$.

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Let $A \subseteq \{1,2\}^{k-1}$ denote the set of sequences that do not match either of the aforementioned conditions. Then any $a \in A$ can be interpreted as a possible sequence of weight functions $(w_0, \ldots, w_{k-1})$ and $(\tilde{w}_0, \ldots, \tilde{w}_{k-1})$ produced by either process w or $\tilde{w}$. Note that, by definition of w and $\tilde{w}$, the same sequence of choices a will produce the same sequence of weight functions in either process. Let us denote this common sequence by w^a, and let us denote by $\mathbb{P}_w[a]$ and $\mathbb{P}_{\tilde{w}}[a]$ the probabilities that the sequence of choices of the respective processes equals a.

By comparing the transition probabilities of w and $\tilde{w}$, we immediately get

$$\frac{\mathbb{P}_w[a]}{\mathbb{P}_{\tilde{w}}[a]} = \exp\left(-\sum_{v_i \in N_L(v_k)} w^a_{i-1}(v_i)\right) = \exp\left(-\sum_{v_i \in N_L(v_k)} w^a_{k-1}(v_i)\right)$$

for all $a \in A$, where the last equality follows by observing that no vertex v_i has its weight updated after step $i-1$. Thus

$$\begin{aligned}\mathbb{E}_w[f(w_{k-1}(v_k)] &= \sum_{a \in A} f(w^a_{k-1}(v_k))\mathbb{P}_w[a] \\ &= \sum_{a \in A} f(w^a_{k-1}(v_k)) \exp\left(-\sum_{v_i \in N_L(v_k)} w^a_{k-1}(v_i)\right)\mathbb{P}_{\tilde{w}}[a] \\ &= \mathbb{E}_{\tilde{w}}[f(\tilde{w}_{k-1}(v_k))e^{-X}].\end{aligned}$$

□

Claim 4.3. Suppose $v_i \in N_L(v_k)$. Then $\tilde{w}_t(v_i)$ is a martingale in t for $t = 0, \ldots, k-1$.

Proof. By definition of $\tilde{w}$, the only steps j where the value of $\tilde{w}(v_i)$ is updated are those where $v_j \in N_L(v_i)$. Note that $v_j \notin N_L(v_k)$ as otherwise v_i, v_j, v_k would form a triangle. Thus $\tilde{w}(v_i)$ is updated according to

$$\tilde{w}_j(v_i) = \begin{cases} 0 & \text{with probability } 1 - e^{-\tilde{w}_{j-1}(v_j)} \\ \tilde{w}_{j-1}(v_i)e^{\tilde{w}_{j-1}(v_j)} & \text{with probability } e^{-\tilde{w}_{j-1}(v_j)}. \end{cases}$$

It is easy to see that this is preserved in expectation. □

Claim 4.4.

$$\mathbb{E}_{\tilde{w}}[X] = \sum_{v_i \in N_L(v_k)} w_0(v_i).$$

Proof. By Claim 4.3, $\mathbb{E}_{\tilde{w}}[X] = \sum_{v_i \in N_L(v_k)} \mathbb{E}_{\tilde{w}}[\tilde{w}_{k-1}(v_i)] = \sum_{v_i \in N_L(v_k)} \mathbb{E}_{\tilde{w}}[\tilde{w}_0(v_i)]$. □

Claim 4.5.

$$\tilde{w}_{k-1}(v_k) = w_0(v_k)e^X.$$

Proof. By definition of $\tilde{w}$, $\tilde{w}(v_k)$ increases by a factor $e^{\tilde{w}_{i-1}(v_i)} = e^{\tilde{w}_{k-1}(v_i)}$ for each step i where $v_i \in N_L(v_k)$. For any other step, $\tilde{w}(v_k)$ is unchanged. □

Claim 4.6.

$$\mathbb{P}_w[v_k \in I] = \mathbb{E}_{\tilde{w}}\left[\left(1 - e^{-w_0(v_k)e^X}\right)e^{-X}\right].$$

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Proof. By the definition of w and I we have $\mathbb{P}_w[v_k \in I] = \mathbb{E}_w[1 - e^{-w_{k-1}(v_k)}]$. Let $f(x) = 1 - e^{-x}$. By Claim 4.2, noting that $f(0) = 0$, we get

$$\mathbb{E}_w[1 - e^{-w_{k-1}(v_k)}] = \mathbb{E}_w[f(w_{k-1}(v_k))] = \mathbb{E}_{\tilde{w}}[f(\tilde{w}_{k-1}(v_k))e^{-X}].$$

By Claim 4.5, $\tilde{w}_{k-1}(v_k) = w_0(v_k)e^X$. Combining these gives the desired equality. □

Proof of Theorem 4.1. By Claim 4.4, we know that X is a non-negative random variable satisfying

$$\mathbb{E}_{\tilde{w}}[X] = \sum_{v_i \in N_L(v_k)} w_0(v_i) \leq \frac{1}{2}\ln\left(\frac{1}{w_0(v_k)}\right).$$

Moreover, by Claim 4.6, we have that

$$\mathbb{P}_w[v_k \in I] = \mathbb{E}_{\tilde{w}}\left[\left(1 - e^{-w_0(v_k)e^X}\right)e^{-X}\right].$$

In order to estimate this expectation given the aforementioned conditions on X, we need the following somewhat technical inequalities.

Claim 4.7. For any $0 < t < 1.79328$, the following two inequalities hold.

1. $e^t < 1 + t + t^2$
2. $(1 - e^{-t})\left(1 - \frac{\ln(1/t)}{2\ln(1.79328/t)}\right) \geq \frac{t}{2}$

Proof. It is not hard to verify both inequalities by computer assistance, or by a direct proof if one replaces 1.79328 by a less ambitious constant. For the sake of clarity of the presentation, we omit explicit proofs. □

Claim 4.8. For any real numbers $x > 0$ and $0 < w < 1.79328$ we have

$$\left(1 - e^{-we^x}\right)e^{-x} \geq (1 - e^{-w})\left(1 - \frac{x}{\ln(1.79328/w)}\right).$$

Proof. Observe that $(1 - e^{-we^x})e^{-x}$ is non-negative. Moreover, it is easy to check that its second derivative in x equals

$$e^{-x-we^x}\left(e^{we^x} - 1 - we^x - w^2e^{2x}\right),$$

which, by Claim 4.7 (1), is negative whenever $we^x < 1.79328$, that is, $x < \ln(1.79328/w)$. Hence the inequality in the lemma holds for $0 \leq x \leq \ln(1.79328/w)$ as the inequality clearly holds at both endpoints, and the function is concave on the interval between these points. But for larger x, the inequality also holds as the right-hand side then turns negative. □

Given these inequalities, the theorem follows by straightforward calculations. By Claim 4.8 and since $\mathbb{E}_{\tilde{w}}[X] \leq \frac{1}{2}\ln\left(\frac{1}{w_0(v_k)}\right)$, we have:

$$\begin{aligned}\mathbb{P}_w[v_k \in I] &\geq \mathbb{E}_{\tilde{w}}\left[\left(1 - e^{-w_0(v_k)}\right)\left(1 - \frac{X}{\ln(1.79328/w_0(v_k))}\right)\right]\\ &\geq \left(1 - e^{-w_0(v_k)}\right)\left(1 - \frac{\ln(1/w_0(v_k))}{2\ln(1.79328/w_0(v_k))}\right),\end{aligned}$$

which by Claim 4.7 (2) is at least $\frac{w_0(v_k)}{2}$, as desired. □

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Proof of Theorem 1.7. Let G be a triangle-free d-degenerate graph with degeneracy order $v_1, \ldots, v_n$ such that $|N_L(v_i)| \le d$ for all vertices v_i. Assume $d \ge 2$. We apply Theorem 4.1 with $w_0 \equiv \frac{\ln d - \ln\ln d}{2d}$. One immediately checks that $0 < w_0(v_k) < 1$ and

$$\frac{1}{2}\ln w_0(v_k) + \sum_{v_i \in N_L(v_k)} w_0(v_i) \le \frac{1}{2}\ln\left(\frac{1}{2}(\ln d - \ln\ln d)\right) - \frac{1}{2}\ln\ln d \le 0,$$

which implies that

$$\mathbb{P}[v_k \in I] \ge \frac{1}{2} w_0(v_k) = \left(\frac{1}{4} - o(1)\right)\frac{\ln d}{d}. \qquad \square$$

Proof of Theorem 1.8. We apply Theorem 4.1 with $w_0(v_i) := \frac{1}{2}p(v_i)$. Observe that

$$p(v_k) \le \prod_{v_i \in N_L(v_k)} (1 - p(v_i)) \le \exp\left(-\sum_{v_i \in N_L(v_k)} p(v_i)\right),$$

which implies that $\ln p(v_k) + \sum_{v_i \in N_L(v_k)} p(v_i) \le 0$ and hence $\frac{1}{2}\ln w_0(v_k) + \sum_{v_i \in N_L(v_k)} w(v_i) \le -\frac{1}{2}\ln 2 < 0$. Moreover, clearly $0 < w_0(v_i) < \frac{1}{2}$ for all v_i. Hence

$$\mathbb{P}[v_k \in I] \ge \frac{1}{2} w_0(v_k) = \frac{1}{4} p(v_k),$$

as desired. $\square$

5. Conclusion

In this final section, we would like to briefly mention some open problems and directions for future research.

First, it would be interesting to see to what extent our method used in the proof of Theorem 1.2 can be adapted to the more general setting of graphs with small clique number. Ajtai, Erdős, Komlós, and Szemerédi [1] proved a lower bound of $\Omega_r\left(\frac{\ln \overline{d}}{\overline{d}\ln\ln \overline{d}} n\right)$ for the independence number of n-vertex K_r-free graphs with average degree $\overline{d}$ (see also the later constant-factor improvement [53] due to Shearer). Johansson [38] and Molloy [46] established analogous upper bounds for the chromatic number of K_r-free graphs with maximum degree Δ of the form $O_r\left(\frac{\Delta \ln\ln\Delta}{\ln\Delta}\right)$. In both of these results, it remains a major open problem whether the additional $\ln\ln$-factors are necessary or can be omitted. Related to these questions, Kelly and Postle [40, Conjecture 2.4] posed the following conjecture (rephrased).

Conjecture 5.1. *For every $r \in \mathbb{N}$ there exists a constant $c = c(r) > 0$ such that every K_r-free graph G admits a probability distribution on its independent sets such that every vertex $v \in V(G)$ is contained in a random independent set drawn from the distribution with probability at least $c\frac{\ln d_G(v)}{d_G(v)}$.*

While this remains wide open, Kelly and Postle [40, Theorem 2.5] proved a weaker version, replacing $c\frac{\ln d_G(v)}{d_G(v)}$ with $c\frac{\ln d_G(v)}{d_G(v)(\ln\ln d_G(v))^2}$. As a first step, it would be interesting to see whether one could remove one of the two $\ln\ln$-factors in this result of Kelly and Postle, which would yield a fractional/local demand version of the aforementioned bounds of Ajtai, Erdős, Komlós and Szemerédi as well as of Molloy. It would also be very interesting to prove generalizations of Theorem 1.7 for K_r-free graphs for any $r \ge 4$.

Looking at our proof of Theorem 2.1, it seems likely that by driving the "step size" ε to zero, one can arrive at some explicit stochastic differential equation for the obtained distribution on random independent sets. It may be interesting to write down such an equation explicitly and see whether it has connections to other known distributions on independent sets.

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Other open problems closely related to our results in this paper can be phrased in the context of so-called *list packings*, see in particular the conjectures and open problems in the papers [15, 14] by Cambie et al. One of the open problems from these works related to Theorem 1.2 is whether for every triangle-free graph G and every assignment $L(\cdot)$ of color-lists to the vertices of G such that $|L(v)| \geq (C + o(1))\frac{d_G(v)}{\ln d_G(v)}$ for every $v \in V(G)$, there exists a probability distribution on the *proper* L-colorings of G such that every color in $L(v)$ is chosen with equal probability for every $v \in V(G)$.

Finally, given the resolution of Harris' conjecture, a natural remaining question is to determine the optimal leading constant C for the problem. In particular, by combining Theorem 1.7 with [11], we know that $\frac{1}{2} \leq C \leq 4$. It would appear that the most reasonable answer is $C = 1$. We state this as a conjecture.

Conjecture 5.2. *The following holds for any sufficiently large d.*

1. $\chi_f(G) \leq (1 + o(1))\frac{d}{\ln d}$ *for all d-degenerate triangle-free graphs G.*
2. *There exists a d-degenerate triangle-free graph G such that* $\chi_f(G) \geq (1 - o(1))\frac{d}{\ln d}$.

As some first evidence toward Conjecture 5.2, (1), we observe (as a further consequence of Theorem 2.1) that it holds when the degeneracy of the graph is replaced by the *spectral radius* $\rho(G)$, that is, the spectral radius of the adjacency matrix.

Theorem 5.3. *Every triangle-free graph G satisfies*

$$\chi_f(G) \leq (1 + o(1))\frac{\rho(G)}{\ln \rho(G)}.$$

Proof. Let G be any given triangle-free graph. We will show that $\chi_f(G) \leq \frac{1}{f(\rho(G))}$, where f is the function defined in Section 2. Since $f(x) = (1-o(1))\frac{\ln(x)}{x}$, this will verify the claim of Theorem 1.7. Pause to note that $\chi_f(G) = \max\{\chi_f(G_1), \ldots, \chi_f(G_c)\}$ and similarly $\rho(G) = \max\{\rho(G_1), \ldots, \rho(G_c)\}$ holds for every graph G with connected components $G_1, \ldots, G_c$. Hence, since f is monotonically decreasing, it suffices to show the inequality $\chi_f(G) \leq \frac{1}{f(\rho(G))}$ for all *connected* triangle-free graphs. So let G be such a graph, and let $A \in \mathbb{R}^{V(G)\times V(G)}$ be its adjacency matrix. By definition, A has non-negative entries, and hence we may apply the Perron–Frobenius theorem to find that $\rho(A) = \rho(G)$ is an eigenvalue of A and that there exists a corresponding eigenvector $\mathbf{u} \in \mathbb{R}^{V(G)}$ with non-negative entries. So we have $A\mathbf{u} = \rho(G)\mathbf{u}$, which reformulated means that

$$\sum_{x\in N_G(v)} \mathbf{u}_x = \rho(G)\mathbf{u}_v$$

for every $v \in V(G)$. This equality in particular implies that if at least one neighbor of a vertex v has a positive entry in $\mathbf{u}$, then so does v. Hence, since G is a connected graph, it follows that $\mathbf{u}_v > 0$ for every $v \in V(G)$. Now interpret the entries of the vector $\mathbf{u}$ as a strictly positive weight assignment to the vertices of G. Then, by Theorem 2.1, there exists a probability distribution $\mathcal{D}$ on the independent sets of G such that for every $v \in V(G)$, we have

$$\mathbb{P}_{I\sim D}[v \in I] \geq f\left(\frac{\sum_{x\in N_G(v)} \mathbf{u}_x}{\mathbf{u}_v}\right) = f(\rho(G)).$$

By definition of the fractional chromatic number, this implies that $\chi_f(G) \leq \frac{1}{f(\rho(G))}$, as desired. This concludes the proof. □

It is well-known that the spectral radius $\rho(G)$ is always sandwiched between the degeneracy of the graph and the maximum degree, and can be significantly smaller than the latter. Thus, Theorem 5.3 provides a first step toward Conjecture 5.2, (1). Moreover, it lines up nicely with a rich area of research that is concerned with spectral bounds on the (fractional) chromatic number, see, for example, Chapter 6 of the textbook on spectral graph theory [17] by Chung and [9, 18, 32, 34, 42, 45, 48] for some small

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selection of articles on the topic. For example, Theorem 5.3 relates to Wilf's classic spectral bound [54] on the chromatic number, which states that every connected graph G satisfies $\chi(G) \leq \rho(G) + 1$ with equality if and only if G is an odd cycle or a complete graph. In fact, we conjecture that the restriction to the fractional chromatic number in Theorem 5.3 is not necessary and that Wilf's bound can be improved for all triangle-free graphs as follows.

Conjecture 5.4. *Every triangle-free graph satisfies*

$$\chi(G) \leq (1 + o(1)) \frac{\rho(G)}{\ln \rho(G)}.$$

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