9

Surface Layer Integrals and Conservation Laws

In this chapter, we introduce *surface layer integrals* as an adaptation of surface integrals to causal fermion systems and causal variational principles. The mathematical structure of a surface layer integral fits nicely with the analytic structures (namely, the EL equations and the linearized field equations as introduced in Chapters 7 and 8). This will become apparent in *conservation laws*, which generalize Noether's theorem and the symplectic form to the setting of causal variational principles. Moreover, we shall introduce a so-called *nonlinear surface layer integral*, which makes it possible to compare two measures ρ and $\tilde{\rho}$ at a given time. Finally, we will explain how *two-dimensional* surface integrals can be described by surface layer integrals.

9.1 The Concept of a Surface Layer Integral

In daily life, we experience space and objects therein. These objects are usually described by densities, and integrating these densities over space gives particle numbers, charges, the total energy, etc. In mathematical terms, the densities are typically described as the normal components of vector fields on a Cauchy surface, and conservation laws express that the values of these integrals do not depend on the choice of the Cauchy surface, that is,

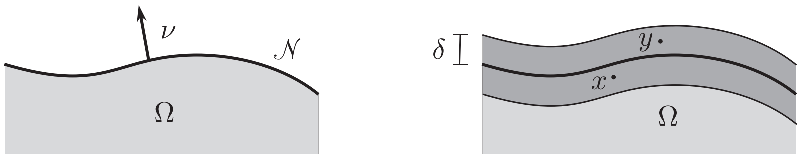
$$\int_{\mathscr{N}_1} J^k \nu_k \; \mathrm{d}\mu_{\mathscr{N}_1}(x) = \int_{\mathscr{N}_2} J^k \nu_k \; \mathrm{d}\mu_{\mathscr{N}_2}(x) \,, \tag{9.1}$$

where $\mathscr{N}_1$ and $\mathscr{N}_2$ are two Cauchy surfaces, ν is the future-directed normal and $\mathrm{d}\mu_{\mathscr{N}_{1/2}}$ is the induced volume measure.

In the setting of causal variational principles, surface integrals like (9.1) are undefined. Instead, one considers so-called *surface layer integrals*, which we now introduce. In general terms, a surface layer integral is a double integral of the form

$$\int_\Omega \bigg(\int_{M \setminus \Omega} (\cdots) \; \mathcal{L}(x,y) \; \mathrm{d}\rho(y) \bigg) \; \mathrm{d}\rho(x) \,, \tag{9.2}$$

where one variable is integrated over a subset $\Omega \subset M$, and the other variable is integrated over the complement of Ω. Here, $(\cdots)$, stands for a differential operator acting on the Lagrangian to be specified later. In order to explain the basic idea, we begin with the additional assumption that the Lagrangian is of *short range* in the following sense. We let $d \in C^0(M \times M, \mathbb{R}^+_0)$ be a suitably chosen distance function on M. Then, the assumption of short range can be quantified by demanding that $\mathcal{L}$ should vanish on distances larger than δ, that is,



$$\int_{\mathcal{N}} \cdots \, d\mu_{\mathcal{N}} \qquad\qquad \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \, \cdots \, \mathcal{L}(x,y)$$

Figure 9.1 A surface integral and a corresponding surface layer integral. From [61]. Reproduced with permission from Springer Nature.

$$d(x,y) > \delta \quad \Longrightarrow \quad \mathcal{L}(x,y) = 0\,. \tag{9.3}$$

Under this assumption, in the surface layer integral (9.2), only pairs (x, y) of distance at most δ contribute, with x lying in Ω and y lying in the complement $M \setminus \Omega$. As a consequence, the integral only involves points in a layer around the boundary of Ω of width δ, that is,

$$x, y \in B_\delta\big(\partial\Omega\big)\,. \tag{9.4}$$

Therefore, a double integral of the form (9.2) can be regarded as an approximation of a surface integral on the length scale δ, as shown in Figure 9.1. In the setting of causal variational principles, such surface layer integrals take the role of surface integrals.

We point out that the causal Lagrangian is *not* of short range in the sense (9.3). But it decays on a length scale that typically coincides with the Compton scale $1/m$ (where m denotes the rest mass of the Dirac particles). With this in mind, the above consideration and the qualitative picture of a surface layer integral in Figure 9.1 apply to the causal action principle as well.

9.2 A Noether-Like Theorem

In modern physics, the connection between symmetries and conservation laws is of central importance. For continuous symmetries, this connection is made mathematically precise by Noether's theorem (see [122] or the textbooks [95, Section 13.7] and [7, Chapter III]). As shown in [61], the connection between symmetries and conservation laws can be extended to the setting of causal variational principles. As we shall see, both the statement and the proof are quite different from the classical Noether theorem; this is why we refer to our result as a *Noether-like theorem*.

The first step is to formulate a symmetry condition for the Lagrangian of a causal variational principle. Similar to the procedure in Section 7.3, one could describe the symmetry by a group of diffeomorphisms. For our purposes, the correct setting would be to consider a one-parameter group of diffeomorphisms Φ_τ on $\mathcal{F}$, that is,

$$\Phi : \mathbb{R} \times \mathcal{F} \to \mathcal{F} \qquad \text{with} \qquad \Phi_\tau \Phi_{\tau'} = \Phi_{\tau+\tau'} \tag{9.5}$$

(we usually write the first argument as a subscript, that is, $\Phi_\tau(x) \equiv \Phi(\tau, x)$). The symmetry condition could be imposed by demanding that the Lagrangian be invariant under this one-parameter group in the sense that

$$\mathcal{L}(x,y) = \mathcal{L}\big(\Phi_\tau(x), \Phi_\tau(y)\big) \qquad \text{for all } \tau \in \mathbb{R} \text{ and } x, y \in \mathcal{F}\,. \tag{9.6}$$

It turns out that this condition is unnecessarily strong for two reasons. First, it suffices to consider families that are defined locally for $\tau \in (-\tau_{\max}, \tau_{\max})$. Second, the mapping Φ does not need to be defined on all of $\mathcal{F}$. Instead, it is more appropriate to impose the symmetry condition only on spacetime $M \subset \mathcal{F}$. This leads us to consider instead of (9.5) a mapping

$$\Phi : (-\tau_{\max}, \tau_{\max}) \times M \to \mathcal{F} \qquad \text{with} \qquad \Phi_0 = \mathrm{id}_M\,. \tag{9.7}$$

We refer to Φ_τ as a **variation** of M in $\mathcal{F}$. Next, we need to specify what we mean by "smoothness" of this variation. This is a subtle point because, as explained in the example of the causal variational principle on the sphere in Section 6.1, the support of a minimizing measure will, in general, be singular. Moreover, the function ℓ defined by (7.3), in general, will only be Lipschitz continuous. Our Noether-like theorems only require that this function be differentiable in the direction of the variations:

Definition 9.2.1 *A variation Φ_τ of the form* (9.7) *is* **continuously differentiable** *if the composition*

$$\ell \circ \Phi \;:\; (-\tau_{\max}, \tau_{\max}) \times M \to \mathbb{R}, \tag{9.8}$$

is continuous and if its partial derivative $\partial_\tau(\ell \circ \Phi)$ exists and is continuous.

The next question is how to adapt the symmetry condition (9.6) to the mapping Φ defined only on $(-\tau_{\max}, \tau_{\max}) \times M$. This is not obvious because setting $\tilde{x} = \Phi_\tau(x)$ and using the group property, the condition (9.6) can be written equivalently as

$$\mathcal{L}\big(\Phi_{-\tau}(\tilde{x}), y\big) = \mathcal{L}\big(\tilde{x}, \Phi_\tau(y)\big) \qquad \text{for all } \tau \in \mathbb{R} \text{ and } \tilde{x}, y \in \mathcal{F}\,. \tag{9.9}$$

But if we restrict attention to pairs $x, y \in M$, the equations in (9.6) and (9.9) are different. For the general mathematical formulation, it is preferable to weaken the condition (9.6) starting from the expression in (9.9).

Definition 9.2.2 *A variation Φ_τ of the form* (9.7) *is a* **symmetry of the Lagrangian** *if*

$$\mathcal{L}\big(x, \Phi_\tau(y)\big) = \mathcal{L}\big(\Phi_{-\tau}(x), y\big) \qquad \textit{for all } \tau \in (-\tau_{\max}, \tau_{\max}) \textit{ and } x, y \in M\,. \tag{9.10}$$

We now state and prove our Noether-like theorem.

Theorem 9.2.3 *Let ρ be a critical measure and Φ_τ a continuously differentiable symmetry of the Lagrangian. Then, for any compact subset $\Omega \subset M$,*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big) \Big)\Big|_{\tau=0} = 0\,. \tag{9.11}$$

Proof Integrating (9.10) over $\Omega \times \Omega$ gives

$$\begin{aligned}
0 &= \iint_{\Omega\times\Omega} \Big(\mathcal{L}\big(x, \Phi_\tau(y)\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big) \Big) \, \mathrm{d}\rho(x) \, \mathrm{d}\rho(y) \\
&= \iint_{\Omega\times\Omega} \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big)\Big) \, \mathrm{d}\rho(x) \, \mathrm{d}\rho(y) \\
&= \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y) \, \chi_\Omega(y) \, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big)\Big) \,, \qquad (9.12)
\end{aligned}$$

where, in the first step, we used the Lagrangian is symmetric in its two arguments and that the integration range is symmetric in x and y. We rewrite the characteristic function $\chi_\Omega(y)$ as $1-(1-\chi_\Omega(y))$, multiply out and use the definition of ℓ, (7.3). We thus obtain

$$\begin{aligned}
0 &= \int_\Omega \Big(\ell\big(\Phi_\tau(x)\big) - \ell\big(\Phi_{-\tau}(x)\big) \Big) \, \mathrm{d}\rho(x) \\
&\quad - \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y) \, \chi_{M\setminus\Omega}(y) \, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big)\Big) \,. \qquad (9.13)
\end{aligned}$$

We thus obtain the identity

$$\begin{aligned}
&\int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y) \, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big)\Big) \\
&\qquad = \int_\Omega \Big(\ell\big(\Phi_\tau(x)\big) - \ell\big(\Phi_{-\tau}(x)\big) \Big) \, \mathrm{d}\rho(x) \,. \qquad (9.14)
\end{aligned}$$

Using that $\ell(\Phi_\tau(x))$ is continuously differentiable (see Definition 9.2.1) and that Ω is compact, we conclude that the right-hand side of this equation is differentiable at $\tau = 0$. Moreover, we are allowed to interchange the τ-differentiation with integration. The EL equations (7.9) imply that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \ell\big(\Phi_\tau(x)\big)\Big|_{\tau=0} = 0 = \frac{\mathrm{d}}{\mathrm{d}\tau} \ell\big(\Phi_{-\tau}(x)\big)\Big|_{\tau=0} \,. \qquad (9.15)$$

Hence, the right-hand side of (9.14) is differentiable at $\tau = 0$, and the derivative vanishes. This gives the result. $\square$

This theorem requires a detailed explanation. We first clarify the connection to surface layer integrals. To this end, let us assume for technical simplicity that Φ_τ and the Lagrangian are differentiable in the sense that the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \Phi_\tau(x)\big|_{\tau=0} =: u(x) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}\tau} \mathcal{L}\big(\Phi_\tau(x), y\big)\big|_{\tau=0}, \qquad (9.16)$$

exist for all $x, y \in M$ and are continuous on M, respectively, $M \times M$. Then, one may exchange differentiation and integration in (9.11) and apply the chain rule to obtain

$$\int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y) \, D_{1,u} \mathcal{L}(x, y) = 0 \,, \qquad (9.17)$$

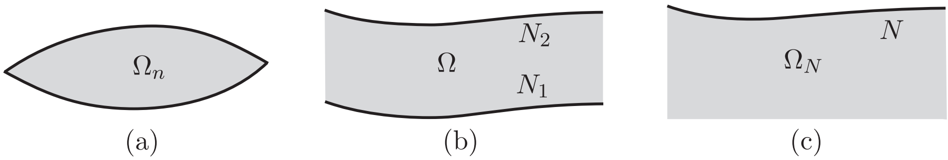


Figure 9.2 Choices of spacetime regions: Lens-shaped region (a), the region between two Cauchy surfaces (b) and the past of a Cauchy surface (c).

where $D_{1,u}$ is the partial derivative at x in the direction of the vector field $u(x)$. This expression is a surface layer integral as in (9.2). In general, the derivatives in (9.16) need *not* exist because we merely imposed the weaker differentiability assumption of Definition 9.2.1. In this case, the statement of the theorem implies that the derivative of the integral in (9.11) exists and vanishes.

We next explain the connection to conservation laws. Let us assume that M admits a sensible notion of "spatial infinity" and that the vector field $\partial_\tau \Phi \in \Gamma(M, T\mathcal{F})$ has suitable decay properties at spatial infinity. Then, one can choose a sequence $\Omega_n \subset M$ of compact sets that form an exhaustion of a set Ω that extends up to spatial infinity (see Figure 9.2 (a) and (b)). Considering the surface layer integrals for Ω_n and passing to the limit, one also concludes that the surface layer integral corresponding to Ω vanishes. Let us assume that the boundary $\partial\Omega$ has two components N_1 and N_2 (as in Figure 9.2 (b)). Then, Theorem 9.2.3 implies that the surface layer integrals over N_1 and N_2 coincide (where the surface layer integral over N is defined as the surface layer integral corresponding to a set Ω_N with $\partial\Omega_N = N$ as shown in Figure 9.2 (c)). In other words, the quantity

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\Omega_N} \mathrm{d}\rho(x) \int_{M \setminus \Omega_N} \mathrm{d}\rho(y) \, \Big(\mathcal{L}\big(\Phi_\tau(x), y\big) - \mathcal{L}\big(\Phi_{-\tau}(x), y\big) \Big) \Big|_{\tau=0}, \tag{9.18}$$

is well defined and independent of the choice of N. In this setting, the surfaces N can be interpreted as Cauchy surfaces, and the conservation law of Theorem 9.2.3 means that the surface layer integral is preserved under the time evolution.

As a concrete example, the unitary invariance of the causal action principle gives rise to a conservation law, which corresponds to current conservation. This example will be considered in detail in Section 9.4. We finally remark that the conservation laws for *energy-momentum* can also be obtained from Theorem 9.2.3, assuming that the causal fermion system has symmetries as described by generalized Killing symmetries. We refer the interested reader to [61, Section 6].

9.3 A Class of Conservation Laws in the Smooth Setting

In the previous section, we saw that surface layer integrals can be used to formulate a Noether-like theorem, which relates symmetries to conservation laws. In this section, we shall derive conservation laws even in the absence of symmetries. Instead, these conservation laws are closely tied to the structure of the linearized field equations as derived in Section 8.1. In order to focus on the essence of the construction, we again restrict attention to the smooth setting (6.10). The basic idea of the construction is explained in the following proposition:

Proposition 9.3.1 *Let ρ be a critical measure and $\Omega \subset M$ be compact. Then for any solution $\mathfrak{v} \in \mathfrak{J}^{\text{lin}}$ of the linearized field equations* (8.15)*,*

$$\gamma^\Omega_\rho(\mathfrak{v}) := \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) = \int_\Omega \nabla_{\mathfrak{v}}\, \mathfrak{s}\, d\rho\,. \tag{9.19}$$

Proof In view of the anti-symmetry of the integrand,

$$\int_\Omega d\rho(x) \int_\Omega d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) = 0\,. \tag{9.20}$$

Adding this equation to the left-hand side of (9.19), we obtain

$$\begin{aligned}
\gamma^\Omega_\rho(\mathfrak{v}) &= \int_\Omega d\rho(x) \int_M d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) \\
&= 2\int_\Omega d\rho(x) \int_M d\rho(y)\, \big(\nabla_{1,\mathfrak{v}}\big)\mathcal{L}(x,y) \\
&\quad - \int_\Omega d\rho(x) \int_M d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) \\
&= \int_\Omega d\rho(x) \Big(2\,\nabla_{\mathfrak{v}} \Big(\ell(x) + \mathfrak{s}\Big) - \Big(\big(\Delta\mathfrak{v}\big)(x) + \nabla_{\mathfrak{v}}\, \mathfrak{s}\Big)\Big)\,,
\end{aligned} \tag{9.21}$$

where in the last line we used the definitions of ℓ and Δ (see (7.3) and (8.15)). Applying the restricted EL equations (7.13) and the linearized field equations (8.15) gives the result. □

Viewing γ^Ω_ρ as a linear functional on the linearized solutions, we also refer to $\gamma^\Omega_\rho(\mathfrak{v})$ as the *conserved one-form*. We remark that the identity (9.19) has a similar structure as the conservation law in the Noether-like theorem (9.11). In order to make the connection precise, one describes the symmetry Φ_τ infinitesimally by a jet $\mathfrak{v}$ with a vanishing scalar component,

$$\mathfrak{v}(x) := \frac{d}{d\tau}\big(0, \Phi_\tau(x)\big)\Big|_{\tau=0}\,. \tag{9.22}$$

Using the symmetry property (9.10), one verifies similarly to the proof of Lemma 8.2.1 that this jet satisfies the linearized field equations (8.15). Therefore, Proposition 9.3.1 applies, and the right-hand side vanishes because $\mathfrak{v}$ has no scalar component. We thus recover the identity obtained by carrying out the τ-derivative in (9.11).

We conclude that Proposition 9.19 is a generalization of Theorem 9.2.3. Instead of imposing symmetries, the identity (9.19) is a consequence of the linearized field equations. Again choosing Ω as the region between two Cauchy surfaces (see Figure 9.2), one obtains a relation between the surface layer integrals around N_1 and N_2. If the scalar component of $\mathfrak{v}$ vanishes, we obtain a conservation law. Otherwise, the right-hand side of (9.19) tells us how the surface layer integral changes in time.

We now generalize Proposition 9.3.1. The basic idea is to integrate anti-symmetric expressions in x and y that involve higher derivatives of the Lagrangian. We again restrict attention to the smooth setting (for the general proof, see [63]). Let $\tilde{\rho}_{s,t}$ with $s,t \in (-\delta,\delta)$ be a two-parameter family of measures that are solutions of the restricted EL equations. We assume that these measures are of the form

$$\tilde{\rho}_{s,t} = (F_{s,t})_* \big(f_{s,t}\, \rho \big) , \tag{9.23}$$

where $f_{s,t}$ and $F_{s,t}$ are smooth,

$$f \in C^\infty\big((-\delta,\delta)^2 \times \mathcal{F}, \mathbb{R}^+ \big) \qquad \text{and} \qquad F \in C^\infty\big((-\delta,\delta)^2 \times \mathcal{F}, \mathcal{F} \big) , \tag{9.24}$$

and are trivial in the case $s=t=0$ (6.22). Moreover, we need the following technical assumption:

(ta) For all $x \in M$, $p,q \geq 0$ and $r \in \{0,1\}$, the following partial derivatives exist and may be interchanged with integration,

$$\begin{aligned} &\int_M \partial_{s'}^r \partial_s^p \partial_t^q \mathcal{L}\big(F_{s+s',t}(x), F_{s,t}(y) \big) \Big|_{s'=s=t=0} \,\mathrm{d}\rho(y) \\ &\qquad = \partial_{s'}^r \partial_s^p \partial_t^q \int_M \mathcal{L}\big(F_{s+s',t}(x), F_{s,t}(y) \big) \,\mathrm{d}\rho(y) \Big|_{s'=s=t=0} . \end{aligned} \tag{9.25}$$

We now state a general identity between a surface layer integral and a volume integral, which was first obtained in [63]. It generalizes the result of Proposition 9.3.1 and gives rise to additional conservation laws for surface layer integrals, which will be analyzed subsequently (in Section 9.5). The proof of the following theorem also works out the mathematical essence of our conservation laws.

Theorem 9.3.2 *Let f and F be as in* (9.24) *and* (6.22) *which satisfy the above assumption* (ta)*. Moreover, assume that the measures $\tilde{\rho}_{s,t}$ given by* (9.23) *satisfy the restricted EL equations for all s and t. Then for every compact $\Omega \subset M$ and every $k \in \mathbb{N}$,*

$$\begin{aligned} I_{k+1}^\Omega := &\int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y) \\ &\quad \times \big(\partial_{1,s} - \partial_{2,s} \big) \big(\partial_{1,t} + \partial_{2,t} \big)^k f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y) \big)\, f_{s,t}(y) \Big|_{s=t=0} \\ = &\;\mathfrak{s} \int_\Omega \partial_s \partial_t^k f_{s,t}(x) \Big|_{s=t=0} \,\mathrm{d}\rho(x) . \end{aligned} \tag{9.26}$$

Proof Introducing the short notation

$$L\big(x_{s,t}, y_{s,t} \big) = f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y) \big)\, f_{s,t}(y) , \tag{9.27}$$

the restricted EL equations (7.13) read

$$\nabla_{\mathfrak{u}} \bigg(\int_M L\big(x_{s,t}, y_{s,t} \big) \,\mathrm{d}\rho(y) - \mathfrak{s}\, f_{s,t}(x) \bigg) = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J} . \tag{9.28}$$

In particular, for any $k \geq 0$ and any vector $v = v^s \partial_s + v^t \partial_s$, we obtain

$$\int_M \partial_{1,s} \big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y)\Big|_{s=t=0} = \mathfrak{s}\, \partial_s \partial_v^k f_{s,t}(x)\big|_{s=t=0} \tag{9.29}$$

$$\int_M \big(\partial_{1,v} + \partial_{2,v}\big)^{k+1} L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y)\Big|_{s=t=0} = \mathfrak{s}\, \partial_v^{k+1} f_{s,t}(x)\big|_{s=t=0}, \tag{9.30}$$

(the derivatives exist and can be exchanged with the integration according to the above assumption (ta)). Differentiating the last equation with respect to v^s and dividing by $k+1$, we obtain

$$\int_M \big(\partial_{1,s} + \partial_{2,s}\big)\big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y) = \mathfrak{s}\, \partial_s \partial_v^k f_{s,t}(x)\,. \tag{9.31}$$

Subtracting twice the identity (9.29), we obtain for any $k \geq 0$ the equation

$$\int_M \big(\partial_{1,s} - \partial_{2,s}\big)\big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big)\; \mathrm{d}\rho(y) = \mathfrak{s}\, \partial_s \partial_v^k f_{s,t}(x)\,. \tag{9.32}$$

Integrating the last equation over Ω gives

$$\begin{aligned} &\int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\; \big(\partial_{1,s} - \partial_{2,s}\big)\big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big) \\ &\qquad = \mathfrak{s} \int_\Omega \partial_s \partial_v^k f_{s,t}(x)\; \mathrm{d}\rho(x)\,. \end{aligned} \tag{9.33}$$

On the other hand, since the integrand is anti-symmetric in its arguments x and y, we also know that

$$\int_\Omega \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\; \big(\partial_{1,s} - \partial_{2,s}\big)\big(\partial_{1,v} + \partial_{2,v}\big)^k L\big(x_{s,t}, y_{s,t}\big) = 0\,. \tag{9.34}$$

Subtracting this equation from (9.33) and evaluating at $s = t = 0$ gives the result. □

Specializing the statement of this theorem to the case $k = 0$ and setting

$$\mathfrak{v} = \frac{\mathrm{d}}{\mathrm{d}s}\big(f_{s,t}, F_{s,t}\big)\Big|_{s=t=0}\,, \tag{9.35}$$

we recover the statement of Proposition 9.3.1. The case $k = 1$ will be studied in more detail in Section 9.5.

We conclude this section by discussing the conservation law of Proposition 9.3.1 for *inner solutions* as considered in Section 8.3 (commutator jets will be considered afterward in Section 9.4). To this end, we need to assume again that spacetime has a smooth manifold structure. We first define an integration measure on the boundary of Ω.

Definition 9.3.3 *Let $\mathfrak{v} = (\operatorname{div} v, v) \in \mathfrak{J}_\rho^{\mathrm{in}}$ be an inner solution and $\Omega \subset M$ closed with smooth boundary $\partial\Omega$. On the boundary, we define the measure $d\mu(\mathfrak{v}, x)$ as the contraction of the volume form on M with v, that is, in local charts*

$$\mathrm{d}\mu(\mathfrak{v}, x) = h\, \epsilon_{ijkl}\, v^i\; \mathrm{d}x^j\, \mathrm{d}x^k\, \mathrm{d}x^l\,, \tag{9.36}$$

where ϵ_{ijkl} is the totally anti-symmetric symbol (normalized by $\epsilon_{0123} = 1$).

We now let $\mathfrak{v} = (\operatorname{div} v, v)$ be an inner solution. Then, the integral on the right-hand side of (9.19) reduces the integral over the divergence of the vector field v,

$$\int_\Omega \nabla_{\mathfrak{v}}\, \mathfrak{s}\; d\rho = \mathfrak{s} \int_\Omega \operatorname{div} v\; d\rho\,. \tag{9.37}$$

On the left-hand side of (9.19), on the other hand, as in Lemma 8.3.3 we can integrate by parts. But now boundary terms remain,

$$\begin{aligned}\gamma^\Omega_\rho(\mathfrak{v}) &= \int_{\partial\Omega} d\mu(\mathfrak{v},x) \int_{M\setminus\Omega} d\rho(y)\, \mathcal{L}(x,y) + \int_\Omega d\rho(x) \int_{\partial\Omega} d\mu(\mathfrak{v},y)\, \mathcal{L}(x,y) \\ &= \int_{\partial\Omega} d\mu(\mathfrak{v},x) \int_M d\rho(y)\, \mathcal{L}(x,y) = \mathfrak{s} \int_{\partial\Omega} d\mu(\mathfrak{v},x)\,, \end{aligned} \tag{9.38}$$

where in the last line we used the symmetry of $\mathcal{L}$ and employed the EL equations. In this way, the surface layer integral in (9.19) reduces to a usual surface integral over the hypersurface $\partial\Omega$. Moreover, combining (9.19) with (9.38) and (9.37), we get back the Gauss divergence theorem

$$\mathfrak{s} \int_{\partial\Omega} d\mu(\mathfrak{v},x) = \mathfrak{s} \int_\Omega \operatorname{div} v\; d\rho\,. \tag{9.39}$$

This illustrates that Proposition 9.3.1 is a generalization of the Gauss divergence theorem where the vector field is replaced with a general solution of the linearized field equations. The formulation with surface layer integrals has the further advantage that the result can be generalized in a straightforward way to non-smooth (e.g., discrete) spacetimes.

9.4 The Commutator Inner Product for Causal Fermion Systems

As a concrete example of a conservation law, we now consider *current conservation*. To this end, we consider the setting of causal fermion systems. As in Section 8.2, we again let $\mathcal{A}$ be a symmetric operator of finite rank on $\mathcal{H}$ and $\mathcal{U}_\tau$ be the corresponding one-parameter family of unitary transformations (8.22). Infinitesimally, this one-parameter family is described by the *commutator jet* $\mathfrak{v}$ (8.24). The unitary invariance of the causal action implies that the commutator jets satisfy the linearized field equations (see Lemma 8.2.1). Moreover, using that the scalar component of commutator jets vanishes, Proposition 9.3.1 gives for any compact $\Omega \subset M$ the conservation law

$$\gamma^\Omega_\rho(\mathfrak{v}) := \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) = 0\,. \tag{9.40}$$

In order to understand the significance of this conservation law, it is useful to choose $\mathcal{A}$ more specifically as an operator of rank one. More precisely, given a nonzero vector $\psi \in \mathcal{H}$, we form the symmetric linear operator $\mathcal{A} \in \mathrm{L}(\mathcal{H})$ of rank one by

$$\mathcal{A}u := \langle u|\psi\rangle_{\mathcal{H}}\, \psi, \tag{9.41}$$

(thus in bra/ket notation, $\mathcal{A} = |\psi\rangle\langle\psi|$). We now form the corresponding commutator jet (8.24). Varying the vector ψ, we obtain a mapping

$$\mathfrak{j} \,:\, \mathcal{H} \to \mathfrak{J}^{\text{lin}} \,, \qquad \psi \mapsto \mathfrak{v} \,. \tag{9.42}$$

Moreover, we choose Ω again as the past of a Cauchy surface (as shown in Figure 9.2 (c)). We write the corresponding conserved surface layer integral in (9.40) as

$$\mathcal{C}^\Omega_\rho(u) := \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(D_{1,\mathfrak{j}(u)} - D_{2,\mathfrak{j}(u)} \big) \mathcal{L}(x,y) \qquad \text{with } u \in \mathcal{H}\,, \tag{9.43}$$

where for technical simplicity we assume smoothness in order to interchange differentiation with integration. Clearly, the mapping $\mathfrak{j}$ in (9.42), and consequently also the mapping $\mathcal{C}^\Omega_\rho$, are homogeneous of degree two, that is,

$$\mathcal{C}^\Omega_\rho\big(\lambda u\big) = |\lambda|^2\, \mathcal{C}^\Omega_\rho(u) \qquad \text{for all } u \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}\,. \tag{9.44}$$

Therefore, we can use the polarization formula to define a sesquilinear form on the Hilbert space $\mathcal{H}$,

$$\langle u|v\rangle^\Omega_\rho := \frac{1}{4}\, \Big(\mathcal{C}^\Omega_\rho(u+v) - \mathcal{C}^\Omega_\rho(u-v) \Big) - \frac{\mathrm{i}}{4}\, \Big(\mathcal{C}^\Omega_\rho(u+\mathrm{i}v) - \mathcal{C}^\Omega_\rho(u-\mathrm{i}v) \Big)\,. \tag{9.45}$$

This sesquilinear form is referred to as the *commutator inner product* (for details, see [82, Section 3]). In [61, Section 5.2], it is shown that for Dirac systems describing the Minkowski vacuum, the commutator inner product coincides (up to an irrelevant prefactor) with the scalar product on Dirac solutions (1.37). In this way, the conservation law for the commutator inner product gives back the conservation of the Dirac current (1.36). We thus recover current conservation as a special case of a more general conservation law for causal fermion systems. Since in examples of physical interest, the conserved surface layer integral $\mathcal{C}^\Omega_\rho(u,v)$ gives back the Hilbert space scalar product, we give this property a name:

Definition 9.4.1 *Given a critical measure ρ and a subset $\Omega \subset M$, the surface layer integral $\mathcal{C}^\Omega_\rho$ is said to* **represent the scalar product** *on the subspace $\mathcal{H}^{\mathrm{f}} \subset \mathcal{H}$ if there is a nonzero real constant c such that the sesquilinear form $\langle .|.\rangle^\Omega_\rho$ defined by* (9.45) *has the property*

$$\langle u|u\rangle^\Omega_\rho = c\, \|u\|^2_{\mathcal{H}} \qquad \textit{for all } u \in \mathcal{H}^{\mathrm{f}}\,. \tag{9.46}$$

In view of the conservation law of Proposition 9.3.1, this property remains valid if Ω is changed by a compact subset of M. We point out that the representation (9.46) *cannot* hold on the whole Hilbert space, that is, for all $u \in \mathcal{H}$; for details, see Exercise 9.5 and [53, Appendix A].

At present, there is no general argument why the surface layer integral $\mathcal{C}^\Omega_\rho$ should represent the scalar product on a nontrivial subspace $\mathcal{H}^{\mathrm{f}} \subset \mathcal{H}$. Therefore, in this book, we shall not assume that this property holds. Instead, we make the following weaker assumption. We assume that the sesquilinear form $\mathcal{C}^\Omega_\rho$ is equivalent to the scalar product in the sense that

$$\langle u|v\rangle^\Omega_\rho = \langle u\,|\,\mathcal{C}_\rho\, v\rangle_{\mathcal{H}} \qquad \text{for all } u,v \in \mathcal{H}^{\mathrm{f}}\,, \tag{9.47}$$

where $\mathcal{C}_\rho$ is a bounded linear operator on $\mathcal{H}$ with bounded inverse. Under this assumption, the Hilbert space scalar product can be expressed by

$$\langle u \,|\, v \rangle_{\mathcal{H}} = \langle u \,|\, \mathcal{C}_\rho^{-1}\, v \rangle_\rho^\Omega \qquad \text{for all } u, v \in \mathcal{H}^{\mathrm{f}}\,. \tag{9.48}$$

In this way, the Hilbert space scalar product can be represented by a surface layer integral involving the physical wave functions in spacetime.

We conclude this section with a remark on the connection between the commutator inner product and the scalar product on solutions of the Dirac equation. As already mentioned after (9.45), for Dirac systems describing the Minkowski vacuum, the commutator inner product (9.45) coincides with the scalar product (1.37). Since both inner products are conserved, the same is true for any Dirac system that evolved from the vacuum (e.g., by "turning on" an interaction). The basic shortcoming of this correspondence is that it holds only for the physical wave functions, that is, for all occupied one-particle states of the system. Thus, in the example of the Minkowski vacuum, the connection between (9.45) and (1.37) can be made only for the negative-energy solutions of the Dirac equation. The positive-energy solutions, however, do not correspond to physical wave functions, so that the commutator inner product is undefined. In order to improve the situation, one would like to extend the commutator inner product to more general wave functions, in such a way that it still agrees with (1.37). This construction is carried out in [82, 53]. Current conservation continues to hold for the extension, provided that the wave functions satisfy the so-called *dynamical wave equation*

$$\int_M Q^{\mathrm{dyn}}(x,y)\, \psi(y)\; d\rho(y) = 0\,. \tag{9.49}$$

Here, the integral kernel Q^{dyn} is constructed from first variations of the causal Lagrangian. In this formulation, the commutator inner product takes the form

$$\begin{aligned} \langle \psi | \phi \rangle_\rho^\Omega := -2i \left(\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) - \int_{M\setminus\Omega} d\rho(x) \int_\Omega d\rho(y) \right) \\ \times \prec\!\psi(x) \,|\, Q^{\mathrm{dyn}}(x,y)\, \phi(y)\!\succ_x . \end{aligned} \tag{9.50}$$

For some more details on these connections, see Exercises 9.3 and 9.4.

After these extensions have been made, the dynamical wave equation (9.49) can be regarded as the generalization of the Dirac equation to causal fermion systems. Moreover, the commutator inner product (9.50) generalizes the scalar product on Dirac solutions (1.37), thereby also extending current conservation to dynamical waves.

9.5 The Symplectic Form and the Surface Layer Inner Product

For the applications, the most important surface layer integrals are I_1^Ω (also denoted by γ_ρ^Ω; see Proposition 9.3.1 and Theorem 9.3.2 in the case $k = 0$) and I_2^Ω (see Theorem 9.3.2 in the case $k = 1$). We now have a closer look at the surface layer integral I_2^Ω. It is defined by

$$\begin{aligned} I_2^\Omega = &\int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y) \\ &\times \big(\partial_{1,s} - \partial_{2,s}\big)\big(\partial_{1,t} + \partial_{2,t}\big) f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y)\Big|_{s=t=0}, \end{aligned} \tag{9.51}$$

and satisfies for any compact subset $\Omega \subset M$ the identity

$$I_2^\Omega = \mathfrak{s} \int_\Omega \partial_s \partial_t f_{s,t}(x)\Big|_{s=t=0} \,\mathrm{d}\rho(x)\,. \tag{9.52}$$

These formulas simplify considerably if we *anti-symmetrize* in the parameters s and t. Namely, the formula for I_2^Ω reduces to the surface layer integral

$$\int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y) \big(\partial_{1,s}\partial_{2,t} - \partial_{1,s}\partial_{2,t}\big) f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y)\Big|_{s=t=0}\,. \tag{9.53}$$

Since this expression involves only first partial derivatives, it can be rewritten with jet derivatives as

$$\sigma_\rho^\Omega(\mathfrak{u}, \mathfrak{v}) := \int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y) \big(\nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} - \nabla_{1,\mathfrak{v}} \nabla_{2,\mathfrak{u}}\big) \mathcal{L}(x,y)\,, \tag{9.54}$$

where the jets $\mathfrak{u}$ and $\mathfrak{v}$ are the linearized solutions

$$\mathfrak{u} = \partial_s \big(f_{s,t}, F_{s,t}\big)\big|_{s=t=0} \qquad \text{and} \qquad \mathfrak{v} = \partial_t \big(f_{s,t}, F_{s,t}\big)\big|_{s=t=0}\,. \tag{9.55}$$

Moreover, the right-hand side of (9.52) vanishes when anti-symmetrizing in s and t. We conclude that

$$\sigma_\rho^\Omega(\mathfrak{u}, \mathfrak{v}) = 0 \qquad \text{for every compact } \Omega \subset M\,. \tag{9.56}$$

Choosing Ω again as explained in Figure 9.2, we obtain a conservation law for a surface layer integral over a neighborhood of a hypersurface N that extends to spatial infinity. We refer to σ_ρ^Ω as the *symplectic form* (the connection to symplectic geometry will be explained after (9.59)).

Symmetrizing I_2^Ω in the parameters s and t gives the surface layer integral

$$\begin{aligned} &\int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y) \\ &\quad\times \big(\partial_{1,s}\partial_{1,t} - \partial_{2,s}\partial_{2,t}\big) f_{s,t}(x)\, \mathcal{L}\big(F_{s,t}(x), F_{s,t}(y)\big)\, f_{s,t}(y)\Big|_{s=t=0}\,. \end{aligned} \tag{9.57}$$

This expression has a more difficult structure because it involves second partial derivatives. Such second partial derivatives cannot be expressed directly in terms of second jet derivatives because the derivatives of the jets also need to be taken into account. In a differential geometric language, defining second derivatives would make it necessary to introduce a connection on $\mathcal{F}$. As explained before Definition 8.1.2, we here use the simpler method of taking second partial derivatives in distinguished charts (e.g., symmetric wave charts for causal fermion systems; see the remark after Proposition 3.1.3 and [60, Section 6.1] or [67, Section 3]). Then, it is useful to introduce the *surface layer inner product* $(.,.)^\Omega_\rho$ as the contribution to (9.57) involving second derivatives of the Lagrangian, that is,

$$(\mathfrak{u}, \mathfrak{v})^\Omega_\rho := \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \big(\nabla_{1,\mathfrak{u}}\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{u}}\nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y)\,, \tag{9.58}$$

where the jets $\mathfrak{u}$ and $\mathfrak{v}$ are again the linearized solutions (9.55). We point out that, in contrast to the symplectic form, the surface layer inner product does *not* correspond to a conservation law. This has two reasons: First because the right-hand side of (9.52) gives rise to a volume term, and second because the derivatives of the jets $\mathfrak{u}$ and $\mathfrak{v}$ give additional correction terms. For the details and the interpretation of these correction terms, we refer to [63]. Here, we only remark that the significance of the surface layer inner product is that it is an approximate conservation law. In particular, it can be used for estimating solutions of the linearized field equations and for proving existence results. We will come back to these applications in Chapter 14.

We finally comment on the name *symplectic form*. Clearly, this name is taken from symplectic geometry, where it refers to a closed and nondegenerate two-form σ on a manifold which we denote by $\mathcal{B}$. The connection to the surface layer integral (9.54) is obtained if we assume that the set of all critical measures of the form (8.4) forms a smooth manifold $\mathcal{B}$ (which may be an infinite-dimensional Banach manifold). In this case, a jet $\mathfrak{v}$ describing the first variations of a measure (8.11) is a tangent vector in $T_\rho\mathcal{B}$. Consequently, the jet space $\mathfrak{J}$ can be identified with the tangent space $T_\rho\mathcal{B}$. The surface layer integral (9.54) can be regarded as a mapping

$$\sigma^\Omega_\rho \;:\; T_\rho\mathcal{B} \times T_\rho\mathcal{B} \to \mathbb{R}\,. \tag{9.59}$$

Being anti-symmetric, it can be regarded as a two-form. Similarly, the conserved surface layer integral γ^Ω_ρ in (9.19) can be regarded as a one-form. Moreover, the t-derivative in (9.51) can be regarded as a directional derivative acting on $I^\Omega_1 = \gamma^\Omega_\rho$. Anti-symmetrizing in s and t corresponds to taking the outer derivative. We thus obtain

$$\sigma^\Omega_\rho = d\gamma^\Omega_\rho\,, \tag{9.60}$$

which also shows again that σ^Ω_ρ is closed. Thus, exactly as in symplectic geometry, the symplectic form defined as the surface layer integral (9.59) is a closed two-form. In contrast to symplectic geometry, it does not need to be nondegenerate. But this can be arranged by restricting attention to a more specific class of measures of the form (9.23). We refer to [62] for a more general discussion of this point.

We finally note that the relation (9.60) resembles the representation of the symplectic potential as the derivative of the *symplectic potential* (sometimes also referred to as the tautological one-form or canonical one-form). It is a major difference to classical mechanics and classical field theory that, in the setting of causal variational principles, the one-form γ^Ω_ρ is canonically defined and represented by a conserved surface layer integral in spacetime.

9.6 The Nonlinear Surface Layer Integral

We now introduce a different type of surface layer integral, which can be regarded as a generalization of the surface layer integrals considered so far. In order to explain the basic concept, we return to the general structure of a surface layer integral (9.2). The differential operator $(\cdots)$ in the integrand can be regarded as describing the first or second variations of the measure ρ. As we saw in the preceding Sections 9.4 and 9.6, the resulting surface layer integrals give rise to conserved currents, the symplectic form and inner products. Instead of considering the first or second variations of a measure ρ, we now consider an additional measure $\tilde{\rho}$ that can be thought of as a finite perturbation of the measure ρ. Consequently, we also have two spacetimes

$$M := \operatorname{supp} \rho \qquad \text{and} \qquad \tilde{M} := \operatorname{supp} \tilde{\rho}\,. \tag{9.61}$$

Choosing two compact subsets $\Omega \subset M$ and $\tilde{\Omega} \subset \tilde{M}$ of the corresponding spacetimes, we form the *nonlinear surface layer integral* by

$$\begin{aligned} \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\rho) := &\int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\ &- \int_{\Omega} \mathrm{d}\rho(x) \int_{\tilde{M} \setminus \tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(x,y)\,. \end{aligned} \tag{9.62}$$

Note that one argument of the Lagrangian is in M, whereas the other is in $\tilde{M}$. Moreover, one argument lies inside the set Ω respectively $\tilde{\Omega}$, whereas the other argument lies outside. In this way, the nonlinear surface layer integral "compares" the two spacetimes near the boundaries of Ω and $\tilde{\Omega}$, as is illustrated in Figure 9.3.

If $\tilde{\rho}$ is a first or second variation of ρ, one recovers surface layer integrals of the form (9.2). In this way, the nonlinear surface layer integral can be regarded as a generating functional for the previous surface layer integrals. Moreover, it has the advantage that it does not rely on continuous variations or a perturbative treatment. Instead, it can be used for comparing two arbitrary measures ρ and $\tilde{\rho}$. This nonlinear surface layer integral was introduced in [57]. It plays a central role in getting the connection to quantum field theory (as will be outlined in Chapter 22).

The nonlinear surface layer integral comes with a corresponding conservation law, as we now explain. For technical simplicity, we assume that the measure $\tilde{\rho}$ can be obtained from ρ by multiplication with a weight function and a push-forward, that is,

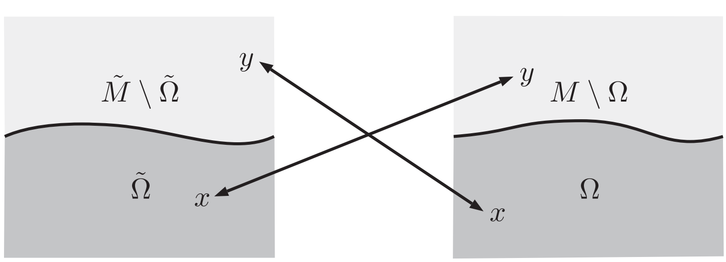


Figure 9.3 The nonlinear surface layer integral. From [58].

$$\tilde{\rho} = F_*(f\rho), \tag{9.63}$$

with smooth functions $f \in C^\infty(M, \mathbb{R}^+)$ and $F \in C^\infty(M, \mathcal{F})$. We use the mapping F in order to identify M with $\tilde{M}$. In particular, we choose

$$\tilde{\Omega} = F(\Omega)\,. \tag{9.64}$$

Then, using the definition of the push-forward measure, the nonlinear surface layer integral can be written alternatively as

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \rho) = \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \Big(f(x)\, \mathcal{L}\big(F(x), y\big) - \mathcal{L}\big(x, F(y)\big)\, f(y) \Big)\,. \tag{9.65}$$

As explained in Section 9.2 in the connection of Noether-like theorems, by a "conservation law," we mean that the nonlinear surface layer integral should vanish for all compact Ω. In preparation for analyzing how to satisfy this condition, we rewrite the nonlinear surface layer integral as a volume integral by using the anti-symmetry of the integrand in (9.65),

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \rho) = \int_\Omega \mathrm{d}\rho(x) \int_{M} \mathrm{d}\rho(y)\, \Big(f(x)\, \mathcal{L}\big(F(x), y\big) - \mathcal{L}\big(x, F(y)\big)\, f(y) \Big)\,. \tag{9.66}$$

In order to write this equation in a simpler form, we introduce a measure ν on M and a measure $\tilde{\nu}$ on $\tilde{M}$ by

$$\mathrm{d}\nu(x) := \bigg(\int_{\tilde{M}} \mathcal{L}(x,y)\; \mathrm{d}\tilde{\rho}(y) \bigg)\; \mathrm{d}\rho(x)\,, \tag{9.67}$$

$$\mathrm{d}\tilde{\nu}(x) := \bigg(\int_{M} \mathcal{L}(x,y)\; \mathrm{d}\rho(y) \bigg)\; \mathrm{d}\tilde{\rho}(x)\,. \tag{9.68}$$

Intuitively speaking, these measures describe how the measures ρ and $\tilde{\rho}$ are connected to each other by the Lagrangian. We refer to them as the *correlation measures*. Then, we can rewrite (9.66) as

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \rho) = \tilde{\nu}\big(F(\Omega)\big) - \nu(\Omega)\,. \tag{9.69}$$

In order to obtain a conservation law, this expression should vanish for all compact Ω. In other words, the measure ν should be the push-forward of the measure $\tilde{\nu}$ under the mapping F,

$$\nu = F_* \tilde{\nu}\,. \tag{9.70}$$

In this way, the task of finding a conservation law is reduced to the following abstract problem: Given two measures ν on M and $\tilde{\nu}$ on $\tilde{M}$, under which assumptions can one measure be realized as the push-forward of the other? If ν and $\tilde{\nu}$ are volume forms on compact manifolds, such a push-forward mapping is obtained from a classical theorem of Jürgen Moser (see, e.g., [113, Section XVIII, §2]). In the non-compact case, the existence of F has been proven under general assumptions in [97]. In this way, the conservation law for the nonlinear surface layer integral can be arranged by adjusting the identification of the spacetimes M and $\tilde{M}$.

We finally remark how the nonlinear surface layer integral can be used to "compare" two causal fermion systems $(\mathcal{H}, \mathcal{F}, \rho)$ and $(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\rho})$. In this setting, one must keep in mind that the causal fermion systems are defined on different Hilbert spaces. Therefore, before forming the nonlinear surface layer integral, we must identify the Hilbert space $\mathcal{H}$ and $\tilde{\mathcal{H}}$ by a unitary transformation $V : \mathcal{H} \to \tilde{H}$. Since this identification is not unique, we are left with the freedom to transform V according to

$$V \to V\mathcal{U} \qquad \text{with} \qquad \mathcal{U} \in \mathrm{L}(\mathcal{H}) \text{ unitary}\,. \tag{9.71}$$

A possible strategy for getting information independent of this freedom is to integrate over the unitary group. For example, this leads to the so-called *partition function*

$$Z^{\tilde{\Omega},\Omega}(\tilde{\rho},\rho) := \int_{\mathcal{G}} \mathrm{e}^{\beta \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho)}\, \mathrm{d}\mu_{\mathcal{G}}(\mathcal{U})\,, \tag{9.72}$$

where β is a real parameter, and $\mathcal{G}$ is a compact subgroup of the unitary group on $\mathcal{H}$ with Haar measure $d\mu_{\mathcal{G}}$. Here, the name "partition function" stems from an analogy to the path integral formulation of quantum field theory. For more details, we refer to Chapter 22 or the research papers [58, 84].

9.7 Two-Dimensional Surface Layer Integrals

The surface layer integrals considered so far were intended to generalize integrals over hypersurfaces. We now explain how lower-dimensional integrals can be described by surface layer integrals. We restrict attention to two-dimensional integrals, noting that the methods can be applied similarly to one-dimensional integrals (i.e., integrals along a curve). It is most convenient to describe a two-dimensional surface $S \subset M$ as

$$S = \partial\Omega \cap \partial V\,, \tag{9.73}$$

where Ω can be thought of as being the past of a Cauchy surface, and V describes a spacetime cylinder. This description has the advantage that the resulting surface layer integrals will be well defined even in cases when spacetime is singular or discrete, in which case the boundaries $\partial\Omega$ and ∂V are no longer a sensible concept. The most obvious way of introducing a surface layer integral localized in a neighborhood of S is a double integral of the form

$$\int_{\Omega\cap V} \bigg(\int_{M\setminus(\Omega\cup V)} (\cdots)\, \mathcal{L}(x,y)\, \mathrm{d}\rho(y) \bigg)\, \mathrm{d}\rho(x), \tag{9.74}$$

(where $(\cdots)$ again stands for a differential operator acting on the Lagrangian). If the Lagrangian has a short range, we only get contributions to this surface layer integral if both x and y are close to the two-dimensional surface S (see Figure 9.4).

The disadvantage of this method is that the surface layer integral (9.74) does not seem to fit together with the EL equations and the linearized field equations. Therefore, at present there is no corresponding conservation law. If one considers the flows of two surfaces, it seems preferable to use the following method introduced in [21]. We need to assume that M has a smooth manifold structure and is

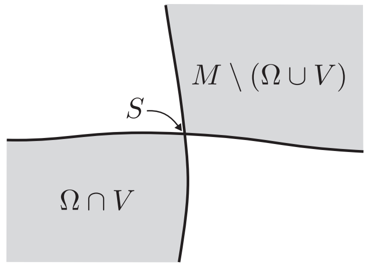


Figure 9.4 A two-dimensional surface layer integral.

four-dimensional (see Definition 8.3.1) and that v is a vector field that is transverse to the hypersurface $\partial\Omega$ and tangential to ∂V. Following Definition 9.3.3, the inner solution corresponding to v gives rise to a volume measure μ on $\partial\Omega$. Thus, we can introduce a two-dimensional surface layer integral by

$$A := \int_{\partial\Omega\cap V} d\mu(\mathfrak{v}, x) \int_{M\setminus V} d\rho(y)\, (\cdots)\mathcal{L}(x, y)\,. \tag{9.75}$$

Applying the Gauss divergence theorem, this surface layer integral can also be written in the usual way as a double spacetime integral involving jet derivatives of the inner solution,

$$\begin{aligned} A &= \int_{\Omega\cap V} d\rho(x)\, \nabla_{\mathfrak{v}} \int_{M\setminus V} d\rho(y)\, (\cdots)\mathcal{L}(x, y) \\ &= \int_{\Omega\cap V} d\rho(x) \int_{M\setminus V} d\rho(y)\, \big(\nabla_{1,\mathfrak{v}} \pm \nabla_{2,\mathfrak{v}}\big)(\cdots)\mathcal{L}(x, y)\,, \end{aligned} \tag{9.76}$$

where the notation $\pm$ means that the formula holds for either choice of the sign (this is because the corresponding term vanishes, as one sees after integrating by parts as in the proof of Lemma 8.3.3 and using that v is tangential to ∂V). The obtained surface layer integral (9.76) harmonizes with the structures of the EL equations and the linearized field equations, as is exemplified in [21] by a simple connection between area change and matter flux.

9.8 Exercises

Exercise 9.1 *(Noether-like theorems)* The goal of this exercise is to illustrate the Noether-like theorems. In order to simplify the problem as far as possible, we consider the compact setting and assume furthermore that the Lagrangian is smooth, that is, $\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0)$. Let ρ be a minimizer of the action under variations of ρ in the class of (positive) normalized regular Borel measures. Let $u \in T\mathcal{F}$ be a vector field on $\mathcal{F}$. Assume that u is a symmetry of the Lagrangian in the sense that

$$\left(u(x)^j \frac{\partial}{\partial x^j} + u(y)^j \frac{\partial}{\partial y^j} \right) \mathcal{L}(x, y) = 0 \quad \text{for all } x, y \in \mathcal{F}. \tag{9.77}$$

Prove that for any measurable set $\Omega \subset \mathcal{F}$,

$$\int_\Omega d\rho(x) \int_{\mathcal{F}\setminus\Omega} d\rho(y)\, u(x)^j \frac{\partial}{\partial x^j} \mathcal{L}(x,y) = 0. \tag{9.78}$$

Hint: Integrate (9.77) over $\Omega \times \Omega$. Transform the integral using the symmetry $\mathcal{L}(x,y) = \mathcal{L}(y,x)$. Finally make use of the Euler–Lagrange equations.

Exercise 9.2 *(Commutator jets and conserved surface layer integrals)*
Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system on a finite-dimensional Hilbert space. For any symmetric operator $S \in L(\mathcal{H})$, we define the corresponding *commutator jet* by

$$\mathfrak{C}_S := (0, \mathcal{C}_S), \quad \text{with} \quad \mathcal{C}_S(x) := \mathrm{i}[S,x] \quad \text{for all } x \in \mathcal{F}. \tag{9.79}$$

Prove the following identity between the conserved one-form and the conserved symplectic form:

$$\gamma_\rho^\Omega\big((0, [\mathcal{C}_A, \mathcal{C}_B])\big) = -\frac{1}{2}\, \sigma_\rho^\Omega(\mathfrak{C}_A, \mathfrak{C}_B), \tag{9.80}$$

where $[\mathcal{C}_A, \mathcal{C}_B]$ denotes the commutator of vector fields on $\mathcal{F}$.

Exercise 9.3 *(Representation of the commutator inner product)* The goal of this exercise is to represent the commutator inner product in a form similar to (9.50).

(a) Show that the first variations of the Lagrangian can be written as

$$\delta\mathcal{L}(x,y) = 2\,\mathrm{Re}\,\mathrm{Tr}_{S_xM}\big(Q(x,y)\,\delta P(y,x)\big), \tag{9.81}$$

with a suitable kernel $Q(x,y) : S_y \to S_x$. Show that this kernel can be chosen to be symmetric, that is, $Q(x,y)^* = Q(y,x)$.

(b) Show that the variation described by the commutator jet in (9.42) and (9.41) corresponds to the variation of the integrand in (9.43)

$$\begin{aligned} &\big(D_{1,\mathrm{j}(u)} - D_{2,\mathrm{j}(u)}\big)\mathcal{L}(x,y) \\ &\quad = -2\mathrm{i}\big(\prec\!\psi(x) \,|\, Q(x,y)\,\psi(y)\!\succ_x - \prec\!\psi(y) \,|\, Q(y,x)\,\psi(x)\!\succ_y\big)\,. \end{aligned} \tag{9.82}$$

(c) Use the polarization formula (9.45) to conclude that $\langle u|v\rangle_\rho^\Omega$ has the representation (9.50) with $\psi = \psi^u$ and $\phi = \psi^v$.

Hint: Details on this construction can be found in [82, Section 3].

Exercise 9.4 *(Extending the commutator inner product)* The goal of this exercise is to illustrate how the commutator inner product can be extended to more general wave functions. To this end, assume that we are given a space of wave function $\mathcal{W}$ which all satisfy the dynamical wave equation (9.49) with a suitable kernel $Q^{\mathrm{dyn}}(x,y)$. Prove that, under these assumptions, the inner product (9.50) is conserved for any $\psi, \phi \in \mathcal{W}$.
Hint: In a first step, it seems a good idea to choose $\Omega = \Omega_t$ as the past of an equal time hypersurface and to differentiate with respect to t. More generally, one can consider the difference of (9.50) for two sets Ω and Ω', which differ by a compact set.

Exercise 9.5 *(Representing the Hilbert space scalar product in a surface layer)* The goal of this exercise is to explain why the sesquilinear form $\langle .|.\rangle_\rho^\Omega$ *cannot* represent the scalar product on the whole Hilbert space. To this end, let us assume conversely that

$$\langle u|u\rangle_\rho^\Omega = c\,\langle u|u\rangle_{\mathcal{H}} \qquad \text{for all } u \in \mathcal{H} \text{ and } c \neq 0 \tag{9.83}$$

and derive a contradiction. For technical simplicity, we assume that $\mathcal{H}$ is finite-dimensional and disregard all issues of convergence of integrals.

(a) Show that the surface layer integral can be written as

$$\langle u|u\rangle_\rho^\Omega = \mathrm{i} \int_M \big\langle u \,\big|\, [x, B(x)]\, u \big\rangle_{\mathcal{H}}\; \mathrm{d}\rho(x), \tag{9.84}$$

with $B(x)$ a suitable family of operators on the Hilbert space.

(b) Carry out the x-integral formally to obtain the representation

$$\langle u|u\rangle_\rho^\Omega = \langle u \,|\, Cu\rangle_{\mathcal{H}}\; \mathrm{d}\rho(x), \tag{9.85}$$

with a trace-free operator C. *Hint:* Make use of the commutator structure of the integrand in (9.84).

(c) Conclude from (9.83) that C is a multiple of the identity operator. Why is this a contradiction?

Hint: More details on this argument can be found in [53, Appendix A].

Exercise 9.6 *(On the surface layer inner product)* The goal of this exercise is to show that, under a suitable restriction of the jet space, the surface-layer inner product is indeed positive. On $\mathcal{F} = \mathbb{R}^2$, we define the Lagrangian

$$\mathcal{L}(x,y) = \frac{1}{2}\,\eta(x_1 - y_1)\,(x_2 - y_2)^2, \quad \text{where } \ \eta \in C_0^\infty(\mathbb{R}, \mathbb{R}^+). \tag{9.86}$$

Let $M = \mathbb{R} \subset \mathcal{F}$ equipped with the canonical one-dimensional Lebesgue measure and consider the set of jets

$$\mathfrak{J} := \Big\{ (0,u) \;\Big|\; u = \sum_{i=1}^2 u_i \partial_i \in T\mathcal{F} \quad \text{with } u_1(t,0) = 0 \text{ and } \partial_1 u_2(t,0) \leq 0 \text{ for all } t \in \mathbb{R} \Big\}. \tag{9.87}$$

Let $\Omega_t := (-\infty, t) \subset M$. Show that the corresponding surface-layer inner product $(\,\cdot\,,\,\cdot\,)^{\Omega_t}|_{\mathfrak{J}\times\mathfrak{J}}$ is positive semi-definite. *Hint:* Remember that jets are never differentiated in expressions like $\nabla_{i,\mathfrak{v}} \nabla_{j,\mathfrak{u}}$.