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RESEARCH ARTICLE

On the k-volume rigidity of a simplicial complex in $\mathbb{R}^d$

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Abstract
We define a generic rigidity matroid for k-volumes of a simplicial complex in $\mathbb{R}^d$ and prove that for $2 \le k \le d-1$ it has the same rank as the classical generic d-rigidity matroid on the same vertex set (namely, the case $k = 1$). This is in contrast with the $k = d$ case, previously studied by Lubetzky and Peled, which presents a different behavior. We conjecture a characterization for the bases of this matroid in terms of d-rigidity of the 1-skeleton of the complex and a combinatorial Hall condition on incidences of edges in k-faces.

Contents

1. Introduction

A simplicial complex X is a family of subsets of some finite set V, such that if $\sigma \in X$, then $\tau \in X$ for all $\tau \subset \sigma$. The elements of X are called the *simplices* or *faces* of X. The *dimension* of a face $\sigma \in X$ is defined as $|\sigma| - 1$. The dimension of the complex X is the maximum dimension of a simplex in X. For a k-dimensional simplicial complex X and $i = 0, 1, \ldots, k$, we denote by X_i the set of i-dimensional faces of X. The set X_0 of 0-dimensional faces of X is called the *vertex set* of X.

Let X be a k-dimensional simplicial complex, and let $\mathbf{p} \in (\mathbb{R}^d)^{|X_0|}$ be an embedding of its vertex set X_0 in $\mathbb{R}^d$. For $0 \le i \le k$ and $\tau \in X_i$, we denote by $\mathbf{p}(\tau)$ the convex hull of the image of τ under $\mathbf{p}$, and by $\mathrm{vol}_i(\tau)$ its i-dimensional volume.

Let $v_X : (\mathbb{R}^d)^{|X_0|} \to \mathbb{R}^{|X_k|}$ map an embedding of X_0 in $\mathbb{R}^d$ to the vector of squared k-volumes of its k-dimensional simplices. That is, for $\mathbf{p} \in (\mathbb{R}^d)^{|X_0|}$, $v_X(\mathbf{p}) \in \mathbb{R}^{|X_k|}$ is defined by

$$v_X(\mathbf{p})_\sigma = \mathrm{vol}_k(\sigma)^2$$

for all $\sigma \in X_k$. We define the (k,d)*-volume rigidity matrix* of X at an embedding $\mathbf{p} \in (\mathbb{R}^d)^{|X_0|}$ to be

$$\mathcal{B}(X, \mathbf{p}) := J_{v_X}(\mathbf{p}),$$

the Jacobian matrix of v_X at the point $\mathbf{p}$, which is an $|X_k| \times d|X_0|$ matrix.

We index the rows of $\mathcal{B}(X,\mathbf{p})$ by simplices in X_k, and every d consecutive columns of $\mathcal{B}(X,\mathbf{p})$ by the vertices in X_0. Using the fact that for every $\sigma \in X_k$ and $v \in \sigma$, $\mathrm{vol}_k(\sigma) = \mathrm{vol}_{k-1}(\sigma \setminus \{v\}) \cdot h/k$, where h is the altitude of $\mathbf{p}(\sigma)$ with respect to $\mathbf{p}(v)$, it is easy to check that

$$\mathcal{B}(X,\mathbf{p})_{\sigma,v} = \begin{cases} \frac{2\mathrm{vol}_k(\sigma)}{k} \cdot \mathrm{vol}_{k-1}(\sigma \setminus \{v\}) N_{\sigma,v} & v \in \sigma, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

for all $\sigma \in X_k$ and $v \in X_0$, where $N_{\sigma,v} \in \mathbb{R}^d$ stands for the unit vector parallel to the altitude of $\mathbf{p}(\sigma)$ with respect to $\mathbf{p}(v)$ (pointing from the $(k-1)$-flat spanned by $\mathbf{p}(\sigma \setminus \{v\})$ to $\mathbf{p}(v)$).

We define the (k,d)*-volume rigidity matroid* of X, denoted by $\mathcal{M}_{k,d}(X)$, to be the matroid whose elements are the k-simplices of X and its independent sets correspond to linearly independent sets of rows of the matrix $\mathcal{B}(X,\mathbf{p})$, for a generic[1] $\mathbf{p}$. Note that $\mathcal{M}_{1,d}(X)$ is the standard d-rigidity matroid of the graph $G = (X_0, X_1)$; see, for example, [5, 9, 11, 1, 2] for relevant background. The matroid $\mathcal{M}_{d,d}(X)$, corresponding to the case $k = d$, was introduced in ([13], Appendix A) (see also [3]) and further studied in [4].

Let $\Delta_{n,k}$ denote the complete k-dimensional simplicial complex on n vertices. We introduce the following definition of k-volume rigidity in $\mathbb{R}^d$.

Definition 1.1. Let X be a k-dimensional simplicial complex on n vertices, and let $d \geq k$. We say that X is *k-volume rigid in $\mathbb{R}^d$* if

$$\mathrm{rank}(\mathcal{M}_{k,d}(X)) = \mathrm{rank}(\mathcal{M}_{k,d}(\Delta_{n,k})).$$

The following is our main result.

Theorem 1.2. *Let $d \geq 2$. Let k, n such that either (i) $k = d-1$ and $n \geq d+2$, or (ii) $1 \leq k \leq d-2$ and $n \geq d+1$. Then,*

$$\mathrm{rank}(\mathcal{M}_{k,d}(\Delta_{n,k})) = dn - \binom{d+1}{2}.$$

The proof of Theorem 1.2 consists of two main steps. First, we apply an inductive argument to reduce the problem to the case $n = d+2$ (when $k = d-1$) or $n = d+1$ (when $1 \leq k \leq d-2$). Then, we analyze these base cases by showing, in each case, that the corresponding (k,d)-volume rigidity matrix (for a specific choice of embedding $\mathbf{p}$) is tightly related to certain "subset inclusion matrix," introduced by Gottlieb in [8] (and independently by Graver and Jurkat in [10], and by Wilson in [20]).

Remarks.

1. The case $k = 1$ is a well-known result about the standard d-rigidity matroid (see, e.g., [11]). The case $k = d$ behaves differently. Indeed, it was shown by Lubetzky and Peled ([13], Appendix A) that, for all $n \geq d+1$,

$$\mathrm{rank}(\mathcal{M}_{d,d}(\Delta_{n,d})) = dn - (d^2 + d - 1).$$

[1] By generic we mean that the $d|X_0|$ entries defining $\mathbf{p}$ are algebraically independent over the field of rationals.

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2. The case $k = d - 1$ and $n = d + 1$, excluded from Theorem 1.2, follows by similar arguments. Indeed, it is easy to show that in this case we have $\mathrm{rank}(\mathcal{M}_{d-1,d}(\Delta_{d+1,d-1})) = d + 1$ (see Proposition 3.3).
3. Let us mention that a related, but different, notion of rigidity for simplicial complexes was studied by Lee in [12] (building on previous unpublished work by Filliman), and further developed by Tay, White, and Whiteley in [15, 16] (see also [17]).

The paper is organized as follows. In Section 2 we prove some preliminary results that will be used later. In Section 3 we present the proof of our main result, Theorem 1.2. In Section 4 we propose a conjecture providing a characterization for the rank of the (k, d)-volume rigidity matroid of a complex in terms of the standard rigidity matroid of its 1-skeleton, and discuss some of its consequences.

2. Preliminary results

2.1. Adding a vertex

Let X be a simplicial complex, and let $v \in X_0$. We denote by $X \setminus v$ the simplicial complex on vertex set $X_0 \setminus \{v\}$ whose simplices are all the simplices of X that do not contain v. We define the *link* of v in X to be the subcomplex of X consisting of all simplices of the form $\sigma \setminus \{v\}$, where $v \in \sigma \in X$.

Lemma 2.1 (Vertex addition lemma). *Let $1 \le k \le d - 1$ and let $n \ge d + 1$. Let X be a k-dimensional simplicial complex on n vertices and let $v \in X_0$ be a vertex whose link in X contains a complete $(k - 1)$-dimensional complex on at least d vertices. Then,*

$$\mathrm{rank}(\mathcal{M}_{k,d}(X)) \ge \mathrm{rank}(\mathcal{M}_{k,d}(X \setminus v)) + d.$$

Proof. Let $\mathbf{p}$ be a generic embedding of X_0 in $\mathbb{R}^d$. Consider the matrices $\mathcal{B} = \mathcal{B}(X, \mathbf{p})$ and $\mathcal{B}' = \mathcal{B}(X \setminus v, \mathbf{p}|_{X_0 \setminus \{v\}})$. Our goal is to show that

$$\mathrm{rank}(\mathcal{B}) \ge \mathrm{rank}(\mathcal{B}') + d.$$

By assumption, the link of v in X contains a complete $(k - 1)$-dimensional complex Y with at least d vertices. Let $v_1, \ldots, v_d \in X_0 \setminus \{v\}$ be distinct vertices in Y. Let

$$\Sigma = \{\eta \cup \{v\} : \eta \subset \{v_1, \ldots, v_d\}, |\eta| = k\}.$$

That is, Σ is the set of k-simplices that are the union of v with a k-subset of $\{v_1, \ldots, v_d\}$. Note that all the elements of Σ are simplices of X. Let $\hat{X}$ be the subcomplex of X defined by $\hat{X}_i = X_i$ for $0 \le i \le k - 1$, and

$$\hat{X}_k := (X \setminus v)_k \cup \Sigma.$$

Let $\hat{\mathcal{B}} = \mathcal{B}(\hat{X}, \mathbf{p})$. As $\hat{X}_k \subset X_k$ and $\hat{X}_0 = X_0$, we obtain $\hat{\mathcal{B}}$ from $\mathcal{B}$ by removing some of its rows. Hence,

$$\mathrm{rank}(\mathcal{B}) \ge \mathrm{rank}(\hat{\mathcal{B}}).$$

Fixing an ordering on the rows of $\hat{\mathcal{B}}$ so that the rows corresponding to the k-simplices in Σ are last, and an ordering on its columns so the d columns corresponding to the vertex v are last, we get that $\hat{\mathcal{B}}$ is a block lower-triangular matrix, consisting of two diagonal blocks: one of them is $\mathcal{B}'$, and the other, which we denote by $\mathcal{B}''$, is the $\binom{d}{k} \times d$ submatrix of $\mathcal{B}$ corresponding to the rows of Σ and to the d columns associated with v. Note that, by our assumption on k, we have that $\binom{d}{k} \ge d$.

We claim that $\mathrm{rank}(\mathcal{B}'') = d$. Indeed, in view of (1.1), the rows of $\mathcal{B}''$ correspond to a scaling of the vectors $N_{\sigma,v}(\mathbf{p})$, for $\sigma \in \Sigma$. For $\sigma \in \Sigma$, let $y_\sigma(\mathbf{p})$ be the foot of the altitude of $\mathbf{p}(\sigma)$ with respect to $\mathbf{p}(v)$. Note that, assuming that $\mathbf{p}(v)$ does not lie in the affine span of $\mathbf{p}(v_1), \ldots, \mathbf{p}(v_d)$, the vectors $\{N_{\sigma,v}(\mathbf{p})\}_{\sigma \in \Sigma}$ linearly span $\mathbb{R}^d$ if and only if the affine span of $\{y_\sigma(\mathbf{p})\}_{\sigma \in \Sigma}$ in $\mathbb{R}^d$ has dimension $d - 1$. Now, consider a special embedding $\mathbf{p}' \in (\mathbb{R}^d)^{|X_0|}$ that maps the vertices $v, v_1, \ldots, v_d$ to the vertices of a

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regular d-simplex in $\mathbb{R}^d$. Observe that for each $\sigma \in \Sigma$, $y_\sigma(\mathbf{p}')$ is the barycenter of $\mathbf{p}'(\sigma \setminus \{v\})$. It is easy to see that the convex hull of $\{y_\sigma(\mathbf{p}')\}_{\sigma\in\Sigma}$ has dimension $d-1$, thus the vectors $\{N_{\sigma,v}(\mathbf{p}')\}_{\sigma\in\Sigma}$ linearly span $\mathbb{R}^d$. Finally, the entries of $\mathcal{B}''$ are algebraic expressions in the entries of the embedding $\mathbf{p}$, and thus, as $\mathbf{p}$ is generic, the vectors $\{N_{\sigma,v}(\mathbf{p})\}_{\sigma\in\Sigma}$ also linearly span $\mathbb{R}^d$, or equivalently, $\operatorname{rank}(\mathcal{B}'') = d$. Thus, we obtain

$$\operatorname{rank}(\mathcal{B}) \geq \operatorname{rank}(\hat{\mathcal{B}}) \geq \operatorname{rank}(\mathcal{B}') + d,$$

as wanted. □

2.2. Cayley–Menger formula

Consider a k-dimensional simplex, σ, with vertices $\{0, 1, \ldots, k\}$, embedded in $\mathbb{R}^d$ (for some $d \geq k$). For $0 \leq i < j \leq k$, let d_{ij} denote the distance between the vertices i and j. Recall that by the Cayley–Menger formula (see, for example, [7, Chapter 1]), we have

$$\operatorname{vol}_k^2(\sigma) = g(d_{01}, d_{02}, \ldots, d_{k-1,k}) := \frac{(-1)^{k+1}}{(k!)^2 2^k} \det \begin{bmatrix} 0 & d_{01}^2 & d_{02}^2 & \cdots & d_{0k}^2 & 1 \\ d_{01}^2 & 0 & d_{12}^2 & \cdots & d_{1k}^2 & 1 \\ d_{02}^2 & d_{12}^2 & 0 & \cdots & d_{2k}^2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{0k}^2 & d_{1k}^2 & d_{2k}^2 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}. \tag{2.1}$$

Lemma 2.2. *Let $f(t) = g(\sqrt{t}, d_{02}, \ldots, d_{k-1,k})$ denote the squared k-volume of a k-dimensional simplex on $\{0, 1, \ldots, k\}$ whose edge lengths $d_{ij} > 0$ are fixed, except for the edge $\{0, 1\}$ whose squared length is a parameter t. Assume that the $(k-2)$-simplex on vertices $\{2, \ldots, k\}$ is nondegenerate. Let $t_0, t_{\pi/2}, t_\pi$ be the values of t for which the angle between the simplices $\{0\} \cup \{2, \ldots, k\}$ and $\{1\} \cup \{2, \ldots, k\}$ is 0, $\pi/2$, and π, respectively. Note that the domain of f is the closed interval $[t_0, t_\pi]$. Then, for $t \in (t_0, t_\pi)$, $f'(t) \neq 0$ unless $t = t_{\pi/2}$.*

Proof. Using the Cayley–Menger formula (2.1) and setting $t = d_{01}^2$, we get

$$f(t) = At^2 + Bt + C,$$

for some constants A, B, C. It is easy to verify that

$$A = \begin{cases} \frac{-1}{16} & k = 2, \\ \frac{-1}{4k^2(k-1)^2} \operatorname{vol}_{k-2}^2(\sigma_{01}) & k > 2, \end{cases}$$

where σ_{01} stands for the $(k-2)$-dimensional simplex spanned by $\{2, \ldots, k\}$. In particular, (by the assumption that σ_{01} is nondegenerate) we have $A \neq 0$.

Thus, $f'(t) = 2At + B$, and we have $f'(t) = 0$ for $t = \frac{-B}{2A}$. In particular, $f(t)$ has at most one local extremum. On the other hand, we know that f has local maximum when the $(k-1)$-dimensional simplices spanned by $\{0\} \cup \{2, \ldots, k\}$ and $\{1\} \cup \{2, \ldots, k\}$ are orthogonal to one another, that is, when $t = t_{\pi/2}$. The lemma then follows. □

Remark. In the proof of Lemma 2.2, it follows that $B = -2A(d_{01}^*)^2$ and $C = A(d_{01}^*)^4 + (v^*)^2$ where d_{01}^* is the edge-length when the volume is maximized (that is, when the two faces $\{0\} \cup \{2, \ldots, k\}$ and $\{1\} \cup \{2, \ldots, k\}$ are orthogonal), and v^* is this maximal volume. However, we do not use these facts.

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Definition 2.3. Let X be a k-dimensional simplicial complex. Let $h_X : \mathbb{R}^{|X_1|} \to \mathbb{R}^{|X_k|}$ map squared edge lengths to squared k-volumes, using the Cayley–Menger formula (2.1). Let

$$\mathcal{C}(X, \mathbf{d}) := J_{h_X}(\mathbf{d})$$

denote the Jacobian matrix of h_X, evaluated for a given vector $\mathbf{d}$ of squared edge lengths, arising from some embedding $\mathbf{p}$ of X_0 in $\mathbb{R}^d$.

Note that $\mathcal{C}(X, \mathbf{d})$ is an $|X_k| \times |X_1|$ matrix, whose (σ, e)-entry, for $\sigma \in X_k$ and $e \in X_1$, is the partial derivative of the square of the k-volume of the simplex σ with respect to the variable $\mathbf{d}_e$, evaluated at the point $\mathbf{d}$. Observe that

$$v_X = h_X \circ f_{X_1},$$

where $f_{X_1} : (\mathbb{R}^d)^{|X_0|} \to \mathbb{R}^{|X_1|}$ is the squared edge length map. That is, f_{X_1} is the map defined by

$$f_{X_1}(\mathbf{p})_e = \|\mathbf{p}(u) - \mathbf{p}(v)\|^2$$

for all $e = \{u, v\} \in X_1$. By the chain rule, we have

$$\mathcal{B}(X, \mathbf{p}) = \mathcal{C}(X, \mathbf{d}) \cdot R(G, \mathbf{p}), \tag{2.2}$$

where $R(G, \mathbf{p}) = J_{f_{X_1}}(\mathbf{p})$ is the standard rigidity matrix of the graph $G = (X_0, X_1)$ and $\mathbf{d} := f_{X_1}(\mathbf{p})$.

3. Proof of main result

Our main result, Theorem 1.2, follows immediately from the next two statements.

Proposition 3.1. *Let $d \geq 2$, $k = d - 1$, and $n \geq d + 2$. Then,*

$$\operatorname{rank}(\mathcal{M}_{k,d}(\Delta_{n,k})) = dn - \binom{d+1}{2}.$$

Proposition 3.2. *Let $d \geq 3$, $1 \leq k \leq d - 2$, and $n \geq d + 1$. Then,*

$$\operatorname{rank}(\mathcal{M}_{k,d}(\Delta_{n,k})) = dn - \binom{d+1}{2}.$$

Proof of Proposition 3.1. First note that, for any $n \geq d + 1$, we have

$$\operatorname{rank}(\mathcal{M}_{k,d}(\Delta_{n,k})) \leq dn - \binom{d+1}{2}. \tag{3.1}$$

Indeed, by (2.2), we have, for any $\mathbf{p}$,

$$\operatorname{rank}(\mathcal{B}(\Delta_{n,k}, \mathbf{p})) \leq \operatorname{rank}(R(\Delta_{n,1}, \mathbf{p})) \leq dn - \binom{d+1}{2},$$

where the last inequality is a standard result on the rigidity of graphs (see, e.g., [11, Lemma 1.2.1]). Hence, the inequality (3.1) follows.

Note also that it suffices to prove the theorem for $n = d + 2$. That is, it suffices to prove that

$$\operatorname{rank}(\mathcal{M}_{k,d}(\Delta_{d+2,k})) = d(d+2) - \binom{d+1}{2}. \tag{3.2}$$

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Indeed, given (3.2), we can then repeatedly apply Lemma 2.1 to add the $n-(d+2)$ remaining vertices of $\Delta_{n,k}$ one by one. At each step, when we add a vertex v, the rank of the resulting matrix is increased by at least d, hence

$$\operatorname{rank}(\mathcal{M}_{k,d}(\Delta_{n,k})) \geq d(d+2) - \binom{d+1}{2} + d(n-(d+2)) = dn - \binom{d+1}{2}.$$

Together with the reverse inequality (3.1), this proves the theorem.

Thus, we only need to prove (3.2). Recall that for a matrix whose entries are algebraic expressions in the entries of $\mathbf{p}$, its generic rank is also its maximal one. So, to prove (3.2), it suffices to find an embedding $\mathbf{p}$ for which

$$\operatorname{rank}(\mathcal{B}(\Delta_{d+2,k}, \mathbf{p})) = d(d+2) - \binom{d+1}{2}. \tag{3.3}$$

Write $X = \Delta_{d+2,k}$. For convenience, assume $X_0 = [d+2]$. Let $\mathbf{p}$ be an embedding of X_0 in $\mathbb{R}^d$ for which $\mathbf{p}(2), \ldots, \mathbf{p}(d+2)$ are the vertices of a regular d-simplex, and $\mathbf{p}(1)$ is the centroid of this simplex. We claim that (3.3) holds for this choice of $\mathbf{p}$.

Using (2.2), we may write

$$\mathcal{B}(\Delta_{d+2,k}, \mathbf{p}) = \mathcal{C}(\Delta_{d+2,k}, \mathbf{d}) \cdot R(K_{d+2}, \mathbf{p}),$$

where K_{d+2} stands for the complete graph on $d+2$ vertices, and $\mathbf{d}$ is the vector of squared edge lengths induced by $\mathbf{p}$. It is well-known and easy to check that

$$\operatorname{rank}(R(K_{d+2}, \mathbf{p})) = d(d+2) - \binom{d+1}{2}.$$

Note also that, as $k = d-1$ and $n = d+2$, we have

$$|X_1| = |X_k| = \binom{d+2}{2},$$

and so $\mathcal{C}(\Delta_{d+2,k}, \mathbf{d})$ is a $\binom{d+2}{2} \times \binom{d+2}{2}$ square matrix. Thus, to prove (3.3), it suffices to prove that

$$\mathcal{C} = \mathcal{C}(\Delta_{d+2,k}, \mathbf{d}) \text{ is invertible.} \tag{3.4}$$

Index the rows of $\mathcal{C}$ by the elements of X_k and its columns by the elements of X_1. Observe that, for $e \in X_1$ and $T \in X_k$, the (T, e)-entry of $\mathcal{C}$ is 0 if $e \not\subset T$. Moreover, by symmetry, we have

$$\mathcal{C}_{T,e} = \begin{cases} 0 & e \not\subset T, \\ \alpha & e \subset T, \quad 1 \notin T, \\ \beta & e \subset T, \quad 1 \in T \cap e, \\ \gamma & e \subset T, \quad 1 \in T \setminus e, \end{cases}$$

for some three real numbers α, β, γ. By Lemma 2.2, we have that α, β, and γ are nonzero.

We apply the following row- and column- scaling: for every $T \in X_k$, we multiply the T-row by $\frac{1}{\alpha}$ if $1 \notin T$, and by $\frac{1}{\gamma}$ otherwise. Then, for every $e \in X_1$ such that $1 \in e$, we multiply the e-column by $\frac{\gamma}{\beta}$. The resulting matrix, which we denote by $A^{d+2}_{k+1,2}$, satisfies

$$\left(A^{d+2}_{k+1,2}\right)_{T,e} = \begin{cases} 0 & e \not\subset T, \\ 1 & e \subset T, \end{cases}$$

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for all $e \in X_1$ and $T \in X_k$. That is, $A^{d+2}_{k+1,2}$ is the incidence matrix between $(k+1)$-subsets and 2-subsets of $X_0 = [d+2]$. By Gottlieb [8, Corollary 2] (or alternatively, by [10, 20]), $A^{d+2}_{k+1,2}$ is invertible. Since $A^{d+2}_{k+1,2}$ was obtained from $\mathcal{C}$ by row and column operations, $\mathcal{C}$ is invertible as well. This completes the proof of the proposition. □

Proof of Proposition 3.2. By an argument similar to the one at the beginning of the proof of Proposition 3.1, it suffices to prove the proposition for $n = d+1$. Indeed, for $n > d+1$, we can then add the remaining $n-(d+1)$ vertices one by one, using Lemma 2.1.

Write $X = \Delta_{d+1,k}$, and assume for convenience $X_0 = [d+1]$. Let $\mathbf{p}$ be an embedding of X_0 in $\mathbb{R}^d$ for which $\mathbf{p}(1), \dots, \mathbf{p}(d+1)$ are the vertices of a regular d-simplex. We claim that, for $1 \le k \le d-2$, one has

$$\operatorname{rank}(\mathcal{B}(\Delta_{d+1,k}, \mathbf{p})) = d(d+1) - \binom{d+1}{2}. \tag{3.5}$$

Using (2.2), we may write

$$\mathcal{B}(\Delta_{d+1,k}, \mathbf{p}) = \mathcal{C}(\Delta_{d+1,k}, \mathbf{d}) \cdot R(K_{d+1}, \mathbf{p}), \tag{3.6}$$

where K_{d+1} stands for the complete graph on $d+1$ vertices, and $\mathbf{d}$ is the vector of squared edge lengths induced by $\mathbf{p}$. Namely, $\mathbf{d}$ is the all-ones vector. Note that

$$\operatorname{rank}(R(K_{d+1}, \mathbf{p})) = d(d+1) - \binom{d+1}{2} = \binom{d+1}{2}.$$

Since the number of rows of $R(K_{d+1}, \mathbf{p})$ is exactly $\binom{d+1}{2}$, $R(K_{d+1}, \mathbf{p})$ has a full rank, and its image is all of $\mathbb{R}^{\binom{d+1}{2}}$. In view of (3.6), this implies that

$$\operatorname{rank}(\mathcal{B}(\Delta_{d+1,k}, \mathbf{p})) = \operatorname{rank}(\mathcal{C}(\Delta_{d+1,k}, \mathbf{d})).$$

So, in order to prove (3.5), we need to show that

$$\operatorname{rank}(\mathcal{C}(\Delta_{d+1,k}, \mathbf{d})) = \binom{d+1}{2}. \tag{3.7}$$

Write $\mathcal{C} = \mathcal{C}(\Delta_{d+1,k}, \mathbf{d})$. We index the rows of $\mathcal{C}$ by the elements of X_k and the columns of $\mathcal{C}$ by the elements of X_1. Observe that for $e \in X_1$ and $T \in X_k$, the (T, e)-entry of $\mathcal{C}$ is 0 if $e \not\subset T$. Moreover, by symmetry, we have

$$\mathcal{C}_{T,e} = \begin{cases} 0 & e \not\subset T, \\ \alpha & e \subset T, \end{cases}$$

for some real number α. By Lemma 2.2, we have $\alpha \neq 0$.

Thus $\mathcal{C} = \alpha A^{d+1}_{k+1,2}$, where $A^{d+1}_{k+1,2}$ is the incidence matrix between $(k+1)$-subsets and 2-subsets of $X_0 = [d+1]$. By [8, Corollary 2], the matrix $A^{d+1}_{k+1,2}$ has maximal rank. Since $1 \le k \le d-2$, we get

$$\operatorname{rank}(\mathcal{C}) = \operatorname{rank}(A^{d+1}_{k+1,2}) = \binom{d+1}{2},$$

as needed. This proves (3.7), and hence (3.5), and thus completes the proof of the proposition. □

Finally, let us note that the case $k = d-1$ and $n = d+1$, excluded from Theorem 1.2, follows easily by similar arguments, as detailed next.

Proposition 3.3. *Let $d \geq 2$. Then,*

$$\mathrm{rank}(\mathcal{M}_{d-1,d}(\Delta_{d+1,d-1})) = d+1.$$

Proof. Let $\mathbf{p}$ map $(\Delta_{d+1,d-1})_0$ to the vertices of a regular d-simplex in $\mathbb{R}^d$. Then, following the same arguments as in Proposition 3.2, we obtain

$$\mathrm{rank}(\mathcal{B}(\Delta_{d+1,d-1}, \mathbf{p})) = \mathrm{rank}(A_{d,2}^{d+1}),$$

where $A_{d,2}^{d+1}$ is the incidence matrix between d-subsets and 2-subsets of the set $[d+1]$. By [8, Corollary 2], this matrix has maximal rank, namely $\mathrm{rank}(A_{d,2}^{d+1}) = d+1$. Therefore,

$$\mathrm{rank}(\mathcal{M}_{d-1,d}(\Delta_{d+1,d-1})) \geq \mathrm{rank}(\mathcal{B}(\Delta_{d+1,d-1}, \mathbf{p})) = d+1.$$

On the other hand, since $\mathcal{M}_{d-1,d}$ has exactly $\binom{d+1}{d} = d+1$ elements, we must have

$$\mathrm{rank}(\mathcal{M}_{d-1,d}(\Delta_{d+1,d-1})) = d+1,$$

as wanted. □

4. Discussion

For a complex X, we call the graph $G = (X_0, X_1)$ the 1-*skeleton* of X. Combining (2.2) with Theorem 1.2, we obtain that if a simplicial complex on $n \geq d+2$ vertices is k-volume rigid in $\mathbb{R}^d$, for some $d > k \geq 1$, then its 1-skeleton must be d-rigid as a graph. The converse does not hold (see Example 4.3 below). However, we propose the following conjecture, which gives a characterization for the rank of the (k,d)-volume rigidity matroid of a complex in terms of the standard d-rigidity matroid of its 1-skeleton.

For two disjoint families of sets $\mathcal{A}, \mathcal{B}$, let $H_{\mathcal{A},\mathcal{B}}$ be the bipartite graph on vertex set $\mathcal{A} \cup \mathcal{B}$ with edge set $\{\{A,B\} : A \in \mathcal{A}, B \in \mathcal{B}, A \subset B\}$. For a graph $G = (V,E)$, let $\nu(G)$ be its matching number, that is, the size of a maximum matching in G.

Conjecture 4.1. *Let $d \geq 3$, $1 \leq k \leq d-1$, and let X be a k-dimensional simplicial complex on $n \geq d+2$ vertices. Then,*

$$\mathrm{rank}(\mathcal{M}_{k,d}(X)) = \max_{E} \nu(H_{E,X_k}),$$

where the maximum is taken over all $E \subset X_1$ that are independent in the standard d-rigidity matroid.

In particular, if $|X_k| = dn - \binom{d+1}{2}$, then Conjecture 4.1 would imply that X is k-volume rigid in $\mathbb{R}^d$ if and only if there exists $E \subset X_1$ of size $dn - \binom{d+1}{2}$ such that $G = (X_0, E)$ is minimally rigid in $\mathbb{R}^d$, and there exists a perfect matching between E and X_k in the bipartite incidence graph H_{E,X_k}. By a classical result of Rado ([14]; see also [18, 19]), this is equivalent to the following statement.

Conjecture 4.2. *Let $d \geq 3$, $1 \leq k \leq d-1$, and let X be a k-dimensional simplicial complex on $n \geq d+2$ vertices. Assume that $|X_k| = dn - \binom{d+1}{2}$. Then, X is k-volume rigid in $\mathbb{R}^d$ if and only if for every $S \subset X_k$, the 1-skeleton of the restriction $X[S] = \{\tau \in X : \tau \subset \sigma \text{ for some } \sigma \in S\}$ has rank at least $|S|$ in the standard d-rigidity matroid.*

Note that the "only if" direction of the conjecture holds. Indeed, for a subset $S \subset X_k$ and a generic $\mathbf{p}$, consider the $|S| \times d|X_0|$ submatrix Q of $\mathcal{B}(X,\mathbf{p})$ corresponding to the rows of S. On the one hand, $\big(\mathcal{C}(X, f_{X_1}(\mathbf{p}))\big)_{\sigma,e} = 0$ for every edge $e \notin X[S]$ and every k-simplex $\sigma \in S$. Therefore, using (2.2), we find that Q is a product of a submatrix of $\mathcal{C}(X, f_{X_1}(\mathbf{p}))$ and the standard d-rigidity matrix of the 1-skeleton of $X[S]$. On the other hand, assuming that $|X_k| = dn - \binom{d+1}{2}$ and X is k-volume rigid in $\mathbb{R}^d$, we have $\operatorname{rank}(Q) = |S|$. The "only if" direction follows since matrix multiplication does not increase the rank.

Example 4.3. Let Y be the simplicial complex obtained from the complete two-dimensional complex on five vertices by removing a single triangle. Let Z be obtained by gluing together two copies of Y along an edge. Any such Z has eight vertices and 18 triangles. Note that the graph $G = (Z_0, Z_1)$ is not generically rigid in $\mathbb{R}^3$, as one can fix one copy of Y and rotate the other copy along the common edge. The same motion also shows that Z is not 2-volume rigid in $\mathbb{R}^3$. Let a, b, c be the vertices in Z unique to the first copy of Y, and let a', b', c' be the vertices in Z unique to the second copy. Let v be a new vertex not in Z, and let X be the union of Z and the three triangles $\{a, a', v\}, \{b, b', v\}$, and $\{c, c', v\}$. Let $\mathbf{p}$ be a generic embedding of X_0 in $\mathbb{R}^3$, and let $G' = (X_0, X_1)$. Then, it is not hard to verify that $\operatorname{rank}(R(G', \mathbf{p})) = 21 = \operatorname{rank}(\mathcal{C}(X, f_{X_1}(\mathbf{p})))$, but $\operatorname{rank}(\mathcal{B}(X, \mathbf{p})) = 20$, so while the 1-skeleton of X is generically rigid in $\mathbb{R}^3$, X is not 2-volume rigid in $\mathbb{R}^3$. The Hall condition mentioned above indeed fails here: letting S be the collection of triangles in X not containing v, it is easy to check that the 1-skeleton of $X[S]$, which is the graph $G = (Z_0, Z_1)$, has rank 17 in the standard 3-rigidity matroid, but $|S| = 18 > 17$.

It is natural to wonder about the rank of the matrix $\mathcal{C}(X, \mathbf{d})$ for a generic vector $\mathbf{d} \in \mathbb{R}^{|X_1|}$. The following is a special case of Conjecture 4.1.

Conjecture 4.4. *Let $2 \leq k \leq n-2$, and let X be a k-dimensional simplicial complex on n vertices. Then, for generic $\mathbf{d} \in \mathbb{R}^{|X_1|}$,*

$$\operatorname{rank}(\mathcal{C}(X, \mathbf{d})) = \nu(H_{X_1, X_k}).$$

To see that this is indeed a special case, suppose that Conjecture 4.1 is true for $d = n - 1$. Let $G = (X_0, X_1)$ be the 1-skeleton of X. Then, for a generic embedding $\mathbf{p}$ of X_0 in $\mathbb{R}^{n-1}$, we have $\operatorname{rank}(R(G, \mathbf{p})) = |X_1|$, and therefore the image of $R(G, \mathbf{p})$ is $\mathbb{R}^{|X_1|}$. Combined with (2.2), we see that in this case

$$\operatorname{rank}(\mathcal{B}(X, \mathbf{p})) = \operatorname{rank}(\mathcal{C}(X, \mathbf{d})),$$

where $\mathbf{d} = f_{X_1}(\mathbf{p})$. Note that $f_{X_1}((\mathbb{R}^d)^n)$ is an open subset of $\mathbb{R}^{|X_1|}$, and so in fact we have

$$\max_{\mathbf{p} \in (\mathbb{R}^d)^n} \operatorname{rank}(\mathcal{B}(X, \mathbf{p})) = \max_{\mathbf{d} \in \mathbb{R}^{|X_1|}} \operatorname{rank}(\mathcal{C}(X, \mathbf{d})).$$

In other words, the generic rank of $\mathcal{C}(X, \mathbf{d})$ is equal to the generic rank of $\mathcal{B}(X, \mathbf{p})$. Finally, the claim follows from Conjecture 4.1, noting again that, since $d = n - 1$, every $E \subset X_1$ is independent in the standard d-rigidity matroid.

Lastly, let us mention a relation between the (k, d)-volume rigidity matrix $\mathcal{B}(X, \mathbf{p})$ studied here and a similar matrix studied by Lee in [12]. Let $L(X, \mathbf{p})$ be the $|X_k| \times d|X_{k-1}|$ matrix defined by[2]

$$L(X, \mathbf{p})_{\sigma,\tau} = \begin{cases} h_{\sigma,\tau} & \tau \subset \sigma, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

for every $\sigma \in X_k$ and $\tau \in X_{k-1}$, where for $\tau \subset \sigma$, denoting the unique vertex in $\sigma \setminus \tau$ by v, $h_{\sigma,\tau} \in \mathbb{R}^d$ is the altitude vector of the simplex $\mathbf{p}(\sigma)$ with respect to $\mathbf{p}(v)$ (pointing from $\mathbf{p}(\tau)$ to $\mathbf{p}(v)$). Note that, for $\tau = \sigma \setminus \{v\}$, $h_{\sigma,\tau} = (\operatorname{vol}_k(\sigma) k / \operatorname{vol}_{k-1}(\tau)) N_{\sigma,v}$. It is then not hard to check, using (4.1), (1.1), and the fact that for every k-simplex σ we have

$$\sum_{v \in \sigma} \operatorname{vol}_{k-1}(\sigma \setminus \{v\}) N_{\sigma,v} = 0,$$

[2]The matrix $L(X, \mathbf{p})$ is denoted in [12] by R (see [12, p. 405]).

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that

$$\mathcal{B}(X, \mathbf{p}) = -\frac{2}{k^2} L(X, \mathbf{p}) \cdot D(X, \mathbf{p}) \cdot P(X),$$

where $D(X, p) \in \mathbb{R}^{d|X_{k-1}| \times d|X_{k-1}|}$ is the diagonal matrix defined by $D(X, \mathbf{p})_{\tau,\tau} = \mathrm{vol}_{k-1}(\tau)^2 I_d$ for all $\tau \in X_{k-1}$ (where I_d is the $d \times d$ identity matrix), and $P(X)$ is the $d|X_{k-1}| \times d|X_0|$ matrix where the $d \times d$ block indexed by (τ, v) equals I_d if $v \in \tau$ and 0 otherwise, for every $\tau \in X_{k-1}$ and $v \in X_0$. We do not pursue this relation further in this work.

Remark. Let us mention that the notion of (k, d)-volume rigidity was recently and independently introduced by James Cruickshank, Bill Jackson and Shin-ichi Tanigawa in [6]. In particular, our Proposition 3.2 was independently proven in [6, Theorem 5], by different techniques. We thank Bill, James, and Shin-ichi for sharing a draft of their preprint just before both articles were submitted to arXiv, and for suggesting a better name for Lemma 3.

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