PARTIALLY-ELEMENTARY END EXTENSIONS OF COUNTABLE MODELS OF SET THEORY

ZACHIRI MCKENZIE

Abstract. Let KP denote Kripke–Platek Set Theory and let M be the weak set theory obtained from ZF by removing the collection scheme, restricting separation to Δ_0-formulae and adding an axiom asserting that every set is contained in a transitive set (TCo). A result due to Kaufmann [9] shows that every countable model, $\mathcal{M}$, of KP + Π_n-Collection has a proper Σ_{n+1}-elementary end extension. We show that for all $n \geq 1$, there exists an L_α (where L_α is the α^{th} approximation of the constructible universe L) that satisfies Separation, Powerset and Π_n-Collection, but that has no Σ_{n+1}-elementary end extension satisfying either Π_n-Collection or Π_{n+3}-Foundation. Thus showing that there are limits to the amount of the theory of $\mathcal{M}$ that can be transferred to the end extensions that are guaranteed by Kaufmann's theorem. Using admissible covers and the Barwise Compactness theorem, we show that if $\mathcal{M}$ is a countable model KP + Π_n-Collection + Σ_{n+1}-Foundation and T is a recursive theory that holds in $\mathcal{M}$, then there exists a proper Σ_n-elementary end extension of $\mathcal{M}$ that satisfies T. We use this result to show that the theory M + Π_n-Collection + Π_{n+1}-Foundation proves Σ_{n+1}-Separation.

§1. Introduction. Keisler and Morley [10] prove that every countable model of ZF has a proper elementary end extension. Kaufmann [9] refines this result showing that if $n \geq 1$ and $\mathcal{M}$ is a countable structure in the language of set theory that satisfies KP + Π_n-Collection, then $\mathcal{M}$ has proper Σ_{n+1}-elementary end extension.[1] And, conversely, if $n \geq 1$ and $\mathcal{M}$ is a structure in the language of set theory that satisfies KP + V = L and has a proper Σ_{n+1}-elementary end extension, then $\mathcal{M}$ satisfies Π_n-Collection.[2] Keisler and Morley's result can be proved using the Omitting Types theorem (see [3, Theorem 2.2.18]) and Kaufmann employs a refined version of the Omitting Types theorem in [9]. A natural question to ask is how much of the theory of $\mathcal{M}$ satisfying KP + Π_n-Collection can be made to hold in a proper Σ_{n+1}-elementary end extension whose existence is guaranteed by Kaufmann's result? In particular, is there a proper Σ_{n+1}-elementary end extension of $\mathcal{M}$ that also satisfies KP + Π_n-Collection? Or, if $\mathcal{M}$ is transitive, is there a proper Σ_{n+1}-elementary end extension of $\mathcal{M}$ that satisfies full $\in$-induction for all set-theoretic formulae[3]? In Section 3 we show that the answers to the latter two of these questions is "no".

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[1]This is a slightly strengthened statement of [9, Theorem 1: (ii)⇒(i)] obtained using the well-known equivalence of Π_n-Collection and Σ_{n+1}-Collection over KP.

[2]This is a weakening of [9, Theorem 1: (i)⇒(ii)] which only assumes that $\mathcal{M}$ is a *resolvable* model of a subsystem of KP that does not include any collection or class foundation.

[3]Over the theory KP, Γ-Foundation is equivalent to $\in$-induction for all formulae in $\neg\Gamma = \{\neg\gamma \mid \gamma \in \Gamma\}$.

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For $n \geq 1$, there is an L_α (where L_α is the α^{th} approximation of the constructible universe L) satisfying Separation, Powerset and Π_n-Collection that has no proper Σ_{n+1}-elementary end extension satisfying either Π_n-Collection or Π_{n+3}-Foundation. A key ingredient is a generalisation of a result due to Simpson (see [9, Remark 2]) showing that if $n \geq 1$ and $\mathcal{M}$ is a structure in the language of set theory satisfying $\mathsf{KP} + \mathsf{V} = \mathsf{L}$ that has Σ_n-elementary end extension satisfying enough set theory and with a new ordinal but no least new ordinal, then $\mathcal{M}$ satisfies Π_n-Collection. Here "enough set theory" is either $\mathsf{KP} + \Pi_{n-1}$-Collection or $\mathsf{KP} + \Pi_{n+2}$-Foundation.

In Section 4, we use Barwise's admissible cover machinery to build partially-elementary end extensions that satisfy significant fragments of the theory of the model being extended. In particular, we show that if T is a recursively enumerable theory in the language of set theory that extends $\mathsf{KP} + \Pi_n$-Collection $+$ Σ_{n+1}-Foundation and $\mathcal{M}$ is a structure that satisfies T, then $\mathcal{M}$ has a proper Σ_n-elementary end extension that satisfies T. That is, by settling for less elementarity we can ensure that there exists an end extension that satisfies any recursively enumerable theory that holds in the model being extended. The special case of this result that applies only to countable transitive $\mathcal{M}$ is provable from the Barwise Compactness theorem, and a sketch of this argument is provided as motivation in the introduction of this section.

The end-extension result proved in Section 4 is used in Section 5 to shed light on the relationship between subsystems of ZF that include the Powerset axiom. We use M to denote the set theory that is axiomatised by: Extensionality, Emptyset, Pair, Powerset, TCo, Infinity, Δ_0-Separation, and Set-Foundation. We show that for all $n \geq 1$, $\mathsf{M} + \Pi_n$-Collection $+ \Pi_{n+1}$-Foundation proves Σ_{n+1}-Separation. In particular, for all $n \geq 1$, the theories $\mathsf{M} + \Pi_n$-Collection and M + Strong Π_n-Collection have the same well-founded models, settling a question about heights of minimum models of subsystems of ZF including Powerset left open in Gostanian's paper [8].

§2. Background. Let $\mathcal{L}$ be the language of set theory—the language whose only non-logical symbol is the binary relation $\in$. Let $\mathcal{L}'$ be a language that contains $\mathcal{L}$ and let Γ be a collection of $\mathcal{L}'$-formulae.

- Γ-Separation is the scheme that consists of the sentences

$$\forall \vec{z} \forall w \exists y \forall x (x \in y \iff (x \in w \wedge \phi(x, \vec{z})),$$

  for all formulae $\phi(x, \vec{z})$ in Γ. Separation is the scheme that consists of these sentences for every formula $\phi(x, \vec{z})$ in $\mathcal{L}$.
- Γ-Collection is the scheme that consists of the sentences

$$\forall \vec{z} \forall w ((\forall x \in w) \exists y \phi(x, y, \vec{z}) \Rightarrow \exists c (\forall x \in w)(\exists y \in c) \phi(x, y, \vec{z})),$$

  for all formulae $\phi(x, y, \vec{z})$ in Γ. Collection is the scheme that consists of these sentences for every formula $\phi(x, y, \vec{z})$ in $\mathcal{L}$.
- Strong Γ-Collection is the scheme that consists of the sentences

$$\forall \vec{z} \forall w \exists c (\forall x \in w)(\exists y \phi(x, y, \vec{z}) \Rightarrow (\exists y \in c) \phi(x, y, \vec{z})),$$

  for all formulae $\phi(x, y, \vec{z})$ in Γ. Strong Collection is the scheme that consists of these sentences for every formula $\phi(x, y, \vec{z})$ in $\mathcal{L}$.

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- Γ-Foundation is the scheme that consists of the sentences

$$\forall \vec{z}(\exists x \phi(x, \vec{z}) \Rightarrow \exists y(\phi(y, \vec{z}) \wedge (\forall w \in y) \neg \phi(w, \vec{z}))),$$

for all formulae $\phi(x, \vec{z})$ in Γ. If $\Gamma = \{x \in z\}$, then the resulting axiom is referred to as Set-Foundation. Foundation is the scheme that consists of these sentences for every formula $\phi(x, \vec{z})$ in $\mathcal{L}$.

In addition to the Lévy classes of $\mathcal{L}$-formulae, Δ_0, Σ_1, Π_1, ..., we will also make reference to the class $\Delta_0^{\mathcal{P}}$, introduced by Takahashi [17], that consists of $\mathcal{L}$-formulae whose quantifiers are bounded either by the membership relation ($\in$) or the subset relation ($\subseteq$), and the classes $\Sigma_1^{\mathcal{P}}, \Pi_1^{\mathcal{P}}, \Sigma_2^{\mathcal{P}}$, ...that are defined from $\Delta_0^{\mathcal{P}}$ in the same way that the classes Σ_1, Π_1, Σ_2, ...are defined from Δ_0. Let T be a theory in a language, $\mathcal{L}'$, that includes $\mathcal{L}$. Let Γ be a class of $\mathcal{L}'$-formulae. A formula is Γ in T or Γ^T if it is provably equivalent in T to a formula in Γ. A formula is Δ_n in T or Δ_n^T if it is both Σ_n^T and Π_n^T.

- Δ_n-Separation is the scheme that consists of the sentences

$$\forall \vec{z}(\forall v(\phi(v, \vec{z}) \iff \psi(v, \vec{z})) \Rightarrow \forall w \exists y \forall x(x \in y \iff (x \in w \wedge \phi(x, \vec{z}))))$$

for all Σ_n-formulae $\phi(x, \vec{z})$ and Π_n-formulae $\psi(x, \vec{z})$.
- Δ_n-Foundation is the scheme that consists of the sentences

$$\forall \vec{z}(\forall v(\phi(x, \vec{z}) \iff \psi(x, \vec{z})) \Rightarrow (\exists x \phi(x, \vec{z}) \Rightarrow \exists y(\phi(y, \vec{z}) \wedge (\forall w \in y) \neg \phi(w, \vec{z}))))$$

for all Σ_n-formulae $\phi(x, \vec{z})$ and Π_n-formulae $\psi(x, \vec{z})$.

We use S_1 to denote the $\mathcal{L}$-theory with axioms: Extensionality, Emptyset, Pair, Union, Set Difference, and Powerset. Following [13], we take *Kripke–Platek Set Theory* (KP) to be the theory obtained from S_1 by removing Powerset and adding Δ_0-Separation, Δ_0-Collection and Π_1-Foundation. Note that this differs from [2, 6], which defines Kripke–Platek Set Theory to include Foundation. The theory KPI is obtained from KP by adding the axiom Infinity, which states that a superset of the von Neumann ordinal ω exists. We use M^- to denote the theory that is obtained from KPI by replacing Π_1-Foundation with Set-Foundation and removing Δ_0-Collection, and adding an axiom TCo asserting that every set is contained in a transitive set. The theory M is obtained from M^- by adding Powerset. The theory MOST is obtained from M by adding Strong Δ_0-Collection and the Axiom of Choice (AC). *Zermelo Set Theory* (Z) is obtained from M by removing TCo and adding Separation. The theory $\mathsf{KP}^{\mathcal{P}}$ is obtained from M by adding $\Delta_0^{\mathcal{P}}$-Collection and $\Pi_1^{\mathcal{P}}$-Foundation.

The theory KP proves TCo (see, for example, [2, Theorem I.6.1]). Both KP and M prove that every set x is contained in a least transitive set that is called the *transitive closure* of x, and denoted $\mathsf{TC}(x)$. The following are some important relationships between axiom schemes over the theory M^-:

- In the theory M^-, Γ-Separation implies Γ-Foundation.
- The proof of [2, Theorem I.4.4] generalises to show that, in the theory M^-, Π_n-Collection implies Σ_{n+1}-Collection.
- [7, Lemma 4.13] shows that, over M^-, Π_n-Collection implies Δ_{n+1}-Separation.
- It is noted in [7, Proposition 2.4] that if T is $\mathsf{M}^- + \Pi_n$-Collection, then the classes Σ_{n+1}^T and Π_{n+1}^T are closed under bounded quantification.

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- [12, Lemma 2.4], for example, shows that, over M^-, Strong Π_n-Collection is equivalent to Π_n-Collection + Σ_{n+1}-Separation.

Let $\mathcal{L}'$ be a language that contains $\mathcal{L}$. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}}, \ldots \rangle$ be an $\mathcal{L}'$-structure. If $a \in M$, then we will use a^* to denote the set $\{x \in M \mid \mathcal{M} \models (x \in a)\}$, as long as $\mathcal{M}$ is clear from the context. Let Γ be a collection of $\mathcal{L}'$-formulae. We say $X \subseteq M$ is Γ over $\mathcal{M}$ if there is a formula $\phi(x, \vec{z})$ in Γ and $\vec{a} \in M$ such that $X = \{x \in M \mid \mathcal{M} \models \phi(x, \vec{a})\}$. In the special case that Γ is all $\mathcal{L}'$-formulae, we say that X is a *definable subclass* of $\mathcal{M}$. A set $X \subseteq M$ is Δ_n over $\mathcal{M}$ if it is both Σ_n over $\mathcal{M}$ and Π_n over $\mathcal{M}$.

A structure $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is an *end extension* of $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$, written $\mathcal{M} \subseteq_e \mathcal{N}$, if $\mathcal{M}$ is a substructure of $\mathcal{N}$ and for all $x \in M$ and for all $y \in N$, if $\mathcal{N} \models (y \in x)$, then $y \in M$. An end extension $\mathcal{N}$ of $\mathcal{M}$ is *proper* if $M \neq N$. If $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is an end extension of $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ and for all $x \in M$ and for all $y \in N$, if $\mathcal{N} \models (y \subseteq x)$, then $y \in M$, then we say that $\mathcal{N}$ is a *powerset-preserving end extension* of $\mathcal{M}$ and write $\mathcal{M} \subseteq_e^{\mathcal{P}} \mathcal{N}$. We say that $\mathcal{N}$ is a *Σ_n-elementary end extension* of $\mathcal{M}$, and write $\mathcal{M} \prec_{e,n} \mathcal{N}$, if $\mathcal{M} \subseteq_e \mathcal{N}$ and Σ_n properties are preserved between $\mathcal{M}$ and $\mathcal{N}$.

We use Ord to denote the class of ordinals. The construction of Gödel's constructible universe (L) presented in [2, Chapter II] invokes no more than Π_1-Foundation and can therefore be carried out in the theory KP. For all sets X,

$$\mathsf{Def}(X) = \{Y \subseteq X \mid Y \text{ is a definable subclass of } \langle X, \in \rangle\},$$

which can be seen to be a set in the theory KP using a formula for satisfaction in set structures such as the one described in [2, Section III.1]. The levels of L are then defined by the recursion:

$$L_0 = \emptyset \text{ and } L_\alpha = \bigcup_{\beta<\alpha} L_\beta \text{ if } \alpha \text{ is a limit ordinal,}$$
$$L_{\alpha+1} = L_\alpha \cup \mathsf{Def}(L_\alpha), \text{ and}$$
$$L = \bigcup_{\alpha \in \mathsf{Ord}} L_\alpha.$$

The function $\alpha \mapsto L_\alpha$ is total in KP and Δ_1^{KP}. The axiom $\mathsf{V} = \mathsf{L}$ asserts that every set is the member of some L_α. A transitive set M such that $\langle M, \in \rangle$ satisfies KP is said to be an *admissible set*. An ordinal α is said to be an *admissible ordinal* if L_α is an admissible set.

The theory $\mathsf{KP}^{\mathcal{P}}$ proves that the function $\alpha \mapsto V_\alpha$ is total and $\Delta_1^{\mathcal{P}}$. Mathias [13, Proposition Scheme 6.12] refines the relationships between the classes $\Delta_0^{\mathcal{P}}$, $\Sigma_1^{\mathcal{P}}$, $\Pi_1^{\mathcal{P}}$, ..., and the Lévy classes by showing that $\Sigma_1 \subseteq (\Delta_1^{\mathcal{P}})^{\mathsf{MOST}}$ and $\Delta_0^{\mathcal{P}} \subseteq \Delta_2^{\mathsf{S}_1}$. Therefore, the function $\alpha \mapsto V_\alpha$ is $\Delta_2^{\mathsf{KP}^{\mathcal{P}}}$. It also follows from this analysis that $\mathsf{KP}^{\mathcal{P}}$ is a subtheory of $\mathsf{M} + \Pi_1$-Collection + Π_2-Foundation.

Let T be an $\mathcal{L}$-theory. A transitive set M is said to be the *minimum model* of T if $\langle M, \in \rangle \models T$ and for all transitive sets N with $\langle N, \in \rangle \models T$, $M \subseteq N$. For example, $L_{\omega_1^{\mathrm{CK}}}$ is the minimum model of KPI. For an $\mathcal{L}$-theory T to have a minimum model it is sufficient that the conjunction of the following conditions hold:

(I) There exists a transitive set M such that $\langle M, \in \rangle \models T$;
(II) for all transitive M with $\langle M, \in \rangle \models T$, $\langle L^M, \in \rangle \models T$.

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Gostanian [8, Section 1] shows that all sufficiently strong subsystems of ZF and ZF^- obtained by restricting the separation and collection schemes to formulae in the Lévy classes have minimum models. In particular:

THEOREM 2.1 (Gostanian [8]). *Let $n, m \in \omega$.*

(I) *The theory* $\mathsf{KPI} + \Pi_m\text{-}\mathsf{Separation} + \Pi_n\text{-}\mathsf{Collection}$ *has a minimum model. Moreover, the minimum model of this theory satisfies* $\mathsf{V} = \mathsf{L}$.

(II) *If $n \geq 1$ or $m \geq 1$, then the theory* $\mathsf{KPI} + \mathsf{Powerset} + \Pi_m\text{-}\mathsf{Separation} + \Pi_n\text{-}\mathsf{Collection}$ *has a minimum model. Moreover, the minimum model of this theory satisfies* $\mathsf{V} = \mathsf{L}$.

Gostanian's analysis also yields:

THEOREM 2.2. *Let $n \in \omega$. The theory* $\mathsf{Z} + \Pi_n\text{-}\mathsf{Collection}$ *has a minimum model. Moreover, the minimum model of this theory satisfies* $\mathsf{V} = \mathsf{L}$.

The fact that KP is able to define satisfaction in set structures also facilitates the definition of formulae expressing satisfaction, in the universe, for formulae in any given level of the Lévy hierarchy.

DEFINITION 2.1. The formula $\mathsf{Sat}_{\Delta_0}(q, x)$ is defined as

$$\begin{array}{c}(q \in \omega) \wedge (q = \ulcorner \phi(v_1, \ldots, v_m) \urcorner \text{ where } \phi \text{ is } \Delta_0) \wedge (x = \langle x_1, \ldots, x_m \rangle) \wedge \\ \exists N \left(\bigcup N \subseteq N \wedge (x_1, \ldots, x_m \in N) \wedge (\langle N, \in \rangle \models \phi[x_1, \ldots, x_m]) \right)\end{array}.$$

We can now inductively define formulae $\mathsf{Sat}_{\Sigma_n}(q, x)$ and $\mathsf{Sat}_{\Pi_n}(q, x)$ that express satisfaction for formulae in the classes Σ_n and Π_n.

DEFINITION 2.2. The formulae $\mathsf{Sat}_{\Sigma_n}(q, x)$ and $\mathsf{Sat}_{\Pi_n}(q, x)$ are defined recursively for $n > 0$. $\mathsf{Sat}_{\Sigma_{n+1}}(q, x)$ is defined as the formula

$$\exists \vec{y} \exists k \exists b \left(\begin{array}{c} (q = \ulcorner \exists \vec{u} \phi(\vec{u}, v_1, \ldots, v_l) \urcorner \text{ where } \phi \text{ is } \Pi_n) \wedge (x = \langle x_1, \ldots, x_l \rangle) \\ \wedge (b = \langle \vec{y}, x_1, \ldots, x_l \rangle) \wedge (k = \ulcorner \phi(\vec{u}, v_1, \ldots, v_l) \urcorner) \wedge \mathsf{Sat}_{\Pi_n}(k, b) \end{array} \right);$$

and $\mathsf{Sat}_{\Pi_{n+1}}(q, x)$ is defined as the formula

$$\forall \vec{y} \forall k \forall b \left(\begin{array}{c} (q = \ulcorner \forall \vec{u} \phi(\vec{u}, v_1, \ldots, v_l) \urcorner \text{ where } \phi \text{ is } \Sigma_n) \wedge (x = \langle x_1, \ldots, x_l \rangle) \\ \wedge ((b = \langle \vec{y}, x_1, \ldots, x_l \rangle) \wedge (k = \ulcorner \phi(\vec{u}, v_1, \ldots, v_l) \urcorner) \Rightarrow \mathsf{Sat}_{\Sigma_n}(k, b)) \end{array} \right).$$

THEOREM 2.3. *Suppose $n \in \omega$ and $m = \max\{1, n\}$. The formula* $\mathsf{Sat}_{\Sigma_n}(q, x)$ *(respectively* $\mathsf{Sat}_{\Pi_n}(q, x)$*) is* Σ_m^{KP} *(*Π_m^{KP}*, respectively). Moreover,* $\mathsf{Sat}_{\Sigma_n}(q, x)$ *(respectively* $\mathsf{Sat}_{\Pi_n}(q, x)$*) expresses satisfaction for Σ_n-formulae (Π_n-formulae, respectively) in the theory* KP, *i.e., if $\mathcal{M} \models \mathsf{KP}$, $\phi(v_1, \ldots, v_k)$ is a Σ_n-formula, and $x_1, \ldots, x_k$ are in M, then for $q = \ulcorner \phi(v_1, \ldots, v_k) \urcorner$, $\mathcal{M}$ satisfies the universal generalisation of the following formula*:

$$x = \langle x_1, \ldots, x_k \rangle \Rightarrow \left(\phi(x_1, \ldots, x_k) \iff \mathsf{Sat}_{\Sigma_n}(q, x) \right).$$

Kaufmann [9] identifies necessary and sufficient conditions for models of KP to have proper Σ_n-elementary end extensions.

THEOREM 2.4 (Kaufmann [9, Theorem 1]). *Let $n \geq 1$. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be a model of* KP. *Consider*

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(I) *there exists* $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ *such that* $\mathcal{M} \prec_{e,n+1} \mathcal{N}$ *and* $M \neq N$;
(II) $\mathcal{M} \models \Pi_n$-Collection.

If $\mathcal{M} \models \mathsf{V} = \mathsf{L}$, *then* $(I) \Rightarrow (II)$. *If* M *is countable, then* $(II) \Rightarrow (I)$.

It should be noted that Kaufmann proves that (I) implies (II) in the above under the weaker assumption that $\mathcal{M}$ is a resolvable model of M^-. A model $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ of M^- is *resolvable* if there is a function F that is Δ_1 over $\mathcal{M}$ such that for all $x \in M$, there exists $\alpha \in \mathsf{Ord}^{\mathcal{M}}$ such that $x \in F(\alpha)$. The function $\alpha \mapsto L_\alpha$ witnesses the fact that every model of $\mathsf{KP} + \mathsf{V} = \mathsf{L}$ is resolvable.

§3. Limitations of Kaufmann's theorem. In this section we show that there are limitations on the amount of the theory of the base model that can be transferred to the partially-elementary end extension guaranteed by Theorem 2.4. We utilise a generalisation of a result, due to Simpson and that is mentioned in [9, Remark 2], showing that if a $\mathcal{M}$ satisfies $\mathsf{KP} + \mathsf{V} = \mathsf{L}$ and has a Σ_n-elementary end extension that satisfies enough set theory and contains no least new ordinal, then $\mathcal{M}$ must satisfy Π_n-Collection. The proof of this generalisation, Theorem 3.1, is based on Enayat's proof of a refinement of Simpson's result (personal communication) that corresponds to the specific case where $n = 1$ and $\mathcal{M}$ is transitive.

THEOREM 3.1. *Let* $n \geq 1$. *Let* $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ *be a model of* $\mathsf{KP} + \mathsf{V} = \mathsf{L}$. *Suppose* $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ *is such that* $\mathcal{M} \prec_{e,n} \mathcal{N}$, $\mathcal{N} \models \mathsf{KP}$ *and* $\mathsf{Ord}^{\mathcal{N}} \backslash \mathsf{Ord}^{\mathcal{M}}$ *is nonempty and has no least element. If* $\mathcal{N} \models \Pi_{n-1}$-Collection *or* $\mathcal{N} \models \Pi_{n+2}$-Foundation, *then* $\mathcal{M} \models \Pi_n$-Collection.

PROOF. Assume that $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that
(I) $\mathcal{M} \prec_{e,n} \mathcal{N}$;
(II) $\mathcal{N} \models \mathsf{KP}$;
(III) $\mathsf{Ord}^{\mathcal{N}} \backslash \mathsf{Ord}^{\mathcal{M}}$ is nonempty and has no least element.

Note that, since $\mathcal{M} \prec_{e,1} \mathcal{N}$ and $\mathcal{M} \models \mathsf{V} = \mathsf{L}$, for all $\beta \in \mathsf{Ord}^{\mathcal{N}} \backslash \mathsf{Ord}^{\mathcal{M}}$, $M \subseteq (L_\beta^{\mathcal{N}})^*$. We need to show that if either Π_{n-1}-Collection or Π_{n+2}-Foundation hold in $\mathcal{N}$, then $\mathcal{M} \models \Pi_n$-Collection. Let $\phi(x, y, \vec{z})$ be a Π_n-formula. Let $\vec{a}, b \in M$ be such that

$$\mathcal{M} \models (\forall x \in b) \exists y \phi(x, y, \vec{a}).$$

So, for all $x \in b^*$, there exists $y \in M$ such that

$$\mathcal{M} \models \phi(x, y, \vec{a}).$$

Therefore, since $\mathcal{M} \prec_{e,n} \mathcal{N}$, for all $x \in b^*$, there exists $y \in M$ such that

$$\mathcal{N} \models \phi(x, y, \vec{a}).$$

Now, $\phi(x, y, \vec{z})$ can be written as $\forall w \psi(w, x, y, \vec{z})$ where $\psi(w, x, y, \vec{z})$ is Σ_{n-1}. Let $\xi \in \mathsf{Ord}^{\mathcal{N}} \backslash \mathsf{Ord}^{\mathcal{M}}$. So, for all $\beta \in \mathsf{Ord}^{\mathcal{N}} \backslash \mathsf{Ord}^{\mathcal{M}}$ and for all $x \in b^*$, there exists $y \in (L_\beta^{\mathcal{N}})^*$ such that

$$\mathcal{N} \models (\forall w \in L_\xi) \psi(w, x, y, \vec{a}).$$

Therefore, for all $\beta \in \mathsf{Ord}^{\mathcal{N}} \backslash \mathsf{Ord}^{\mathcal{M}}$,

$$\mathcal{N} \models (\forall x \in b)(\exists y \in L_\beta)(\forall w \in L_\xi) \psi(w, x, y, \vec{a}). \tag{1}$$

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Now, define $\theta(\beta, \xi, b, \vec{a})$ to be the formula

$$(\forall x \in b)(\exists y \in L_\beta)(\forall w \in L_\xi)\psi(w, x, y, \vec{a}).$$

If Π_{n-1}-Collection holds in $\mathcal{N}$, then $\theta(\beta, \xi, b, \vec{a})$ is equivalent to a Σ_{n-1}-formula. Without Π_{n-1}-Collection, $\theta(\beta, \xi, b, \vec{a})$ can be written as a Π_{n+2}-formula. Therefore, Π_{n-1}-Collection or Π_{n+2}-Foundation in $\mathcal{N}$ will ensure that there is a least $\beta_0 \in \mathrm{Ord}^{\mathcal{N}}$ such that $\mathcal{N} \models \theta(\beta_0, \xi, b, \vec{a})$. Moreover, by (1), $\beta_0 \in M$. Therefore,

$$\mathcal{N} \models (\forall x \in b)(\exists y \in L_{\beta_0})(\forall w \in L_\xi)\psi(w, x, y, \vec{a}).$$

So, for all $x \in b^*$, there exists $y \in (L_{\beta_0}^{\mathcal{M}})^*$, for all $w \in (L_\xi^{\mathcal{N}})^*$,

$$\mathcal{N} \models \psi(w, x, y, \vec{a}).$$

Which, since $\mathcal{M} \prec_{e,n} \mathcal{N}$, implies that for all $x \in b^*$, there exists $y \in (L_{\beta_0}^{\mathcal{M}})^*$, for all $w \in M$,

$$\mathcal{M} \models \psi(w, x, y, \vec{a}).$$

Therefore, $\mathcal{M} \models (\forall x \in b)(\exists y \in L_{\beta_0})\phi(x, y, \vec{a})$. This shows that Π_n-Collection holds in $\mathcal{M}$. ⊣

Enayat (personal communication) uses a specific case of Theorem 3.1 to show that the $\langle L_{\omega_1^{\mathrm{CK}}}, \in\rangle$ has no proper Σ_1-elementary end extension that satisfies KP. We now turn to generalising this result to show that for all $n \geq 1$, the minimum model of $\mathsf{Z} + \Pi_n$-Collection has no proper Σ_{n+1}-elementary end extension that satisfies either $\mathsf{KP} + \Pi_{n+3}$-Foundation or $\mathsf{KP} + \Pi_n$-Collection. However, by Theorem 2.4, for all $n \geq 1$, the minimum model of $\mathsf{Z} + \Pi_n$-Collection does have a proper Σ_{n+1}-elementary end extension.

The following result follows from [12, Theorem 4.4].

THEOREM 3.2. *Let $n \geq 1$. The theory* $\mathsf{M} + \Pi_{n+1}$-Collection $+ \Pi_{n+2}$-Foundation *proves that there exists a transitive model of* $\mathsf{Z} + \Pi_n$-Collection.

COROLLARY 3.3. *Let $n \geq 1$. Let M be the minimum model of* $\mathsf{Z} + \Pi_n$-Collection. *Then there is an instance of* Π_{n+1}-Collection *that fails in* $\langle M, \in\rangle$.

THEOREM 3.4. *Let $n \geq 1$. Let M be the minimum model of* $\mathsf{Z} + \Pi_n$-Collection. *Then $\langle M, \in\rangle$ has a proper Σ_{n+1}-elementary end extension $\mathcal{N}$, but if such an end extension satisfies* KP, *then both* Π_{n+3}-Foundation *and* Π_n-Collection *fail in $\mathcal{N}$.*

PROOF. The fact that $\langle M, \in\rangle$ has a proper Σ_{n+1}-elementary end extension follows from Theorem 2.4. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}}\rangle$ be such that $\mathcal{N} \models \mathsf{KP}$, $N \neq M$ and $\langle M, \in\rangle \prec_{e,n+1} \mathcal{N}$. Since M is the minimal model of $\mathsf{Z} + \Pi_n$-Collection, $\langle M, \in\rangle \models \neg\sigma$ where σ is the sentence

$$\exists x(x \text{ is transitive} \wedge \langle x, \in\rangle \models \mathsf{Z} + \Pi_n\text{-Collection}).$$

Since σ is Σ_1^{KP} and $\langle M, \in\rangle \prec_{e,1} \mathcal{N}$, $\mathcal{N} \models \neg\sigma$. Since $\mathcal{N} \models \mathsf{KP}$ and $M \neq N$, $\mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\langle M, \in\rangle}$ is nonempty. If γ is the least element of $\mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\langle M, \in\rangle}$, then

$$\mathcal{N} \models (\langle L_\gamma, \in\rangle \models \mathsf{Z} + \Pi_n\text{-Collection}),$$

which contradicts the fact that $\mathcal{N} \models \neg\sigma$. Therefore, $\mathsf{Ord}^{\mathcal{N}} \backslash \mathsf{Ord}^{\langle M, \in \rangle}$ is nonempty and contains no least element. Therefore, by Theorem 3.1 and Corollary 3.3, there must be both an instance of Π_n-Collection and an instance of Π_{n+3}-Foundation that fails in $\mathcal{N}$. ⊣

We can also obtain an analog of Theorem 3.4 for the minimum models of $\mathsf{KPI} + \Pi_n$-Collection that allow us to recover Enayat's result. [8, Theorem 2.3] yields the following analog of Corollary 3.3.

THEOREM 3.5 (Gostanian). *Let $n \in \omega$. Let M be the minimum model of* $\mathsf{KPI} + \Pi_n$-Collection. *Then there is an instance of* Π_{n+1}-Collection *that fails in* $\langle M, \in \rangle$.

Using Theorems 3.1 and 3.5, and the same argument used in the proof of Theorem 3.4 now yields:

THEOREM 3.6. *Let $n \in \omega$. Let M be the minimum model of* $\mathsf{KPI} + \Pi_n$-Collection. *If $n = 0$, then $\langle M, \in \rangle$ has no proper Σ_1-elementary end extension that satisfies* KP. *If $n > 0$, then $\langle M, \in \rangle$ has a proper Σ_{n+1}-elementary end extension $\mathcal{N}$, but if such an end extension satisfies* KP, *then both* Π_{n+3}-Foundation *and* Π_n-Collection *fail in $\mathcal{N}$.*

§4. Building partially-elementary end extensions. In this section we will show that if $\mathcal{M}$ is a countable model of $\mathsf{KP} + \Pi_n$-Collection $+ \Sigma_{n+1}$-Foundation and T is a recursively enumerable theory that holds in $\mathcal{M}$, then there exists a proper Σ_n-elementary end extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ satisfies T (Theorem 4.15). The special case of this result for $\mathcal{M}$ transitive can be proved using the Barwise Compactness theorem. The more general result is obtained using Barwise's machinery of admissible covers that facilitate the application of Barwise compactness arguments to nonstandard models. In order to motivate the proof of Theorem 4.15, we begin by sketching the proof of the special case that applies only to countable transitive models.

THEOREM 4.1. *Let T be a recursively enumerable $\mathcal{L}$-theory such that*

$$T \vdash \mathsf{KP} + \Pi_n\text{-Collection},$$

and let M be countable and transitive with $\langle M, \in \rangle \models T$. Then there exists $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ such that $\langle M, \in \rangle \prec_{e,n} \mathcal{N} \models T$ and there exists $d \in N$ such that for all $x \in M$, $\mathcal{N} \models (x \in d)$.

PROOF. (Sketch) Let $\mathcal{L}'$ be the language obtained from $\mathcal{L}$ by constant symbols $\bar{a}$ for each $a \in M$ and a constant symbol $\mathbf{c}$. Let $\mathcal{L}_{\mathbf{S}}$ be the language obtained from $\mathcal{L}$ by adding a binary relation symbol $\mathbf{S}$. Fix a sufficiently simple coding in set theory of the infinitary language $\mathcal{L}'_{\infty\omega}$ based on $\mathcal{L}'$ that allows arbitrarily long conjunctions and disjunction but only finite blocks of quantifiers. Let $\mathcal{L}'_M$ be the fragment of $\mathcal{L}'_{\infty\omega}$ that is coded in M. Let $S \subseteq M$ be a satisfaction class for Σ_n-formulae and note that S is Σ_n definable over $\langle M, \in \rangle$. The fact that $\langle M, \in \rangle$ satisfies $\mathsf{KP} + \Pi_n$-Collection ensures that the $\mathcal{L}_{\mathbf{S}}$-structure $\langle M, \in, S \rangle$ is admissible. Now, let Q be that $\mathcal{L}'_M$-theory that contains:

- T;
- for all $a, b \in M$ with $a \in b$, $\bar{a} \in \bar{b}$;

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- for all $a \in M$,

$$\forall x \left(x \in a \iff \bigvee_{b \in a} (x = \bar{b}) \right);$$

- for all $a \in M$, $\bar{a} \in \mathbf{c}$;
- for all Π_n-formulae, $\phi(x_0, \ldots, x_{m-1})$, and for all $a_0, \ldots, a_{m-1} \in M$ such that $\langle M, \in \rangle \models \phi(a_0, \ldots, a_{m-1})$,

$$\phi(\bar{a}_0, \ldots, \bar{a}_{m-1}).$$

Since S is a satisfaction class for Σ_n-formulae of $\langle M, \in \rangle$, Q is $\Sigma_1(\mathcal{L}_\mathsf{S})$-definable over $\langle M, \in, S \rangle$. If $Q_0 \subseteq Q$ is such that (when thought of as a set of codes) $Q_0 \in M$, then the structure $\langle M, \in \rangle$ can be expanded to a model of Q_0. Therefore, by the Barwise Compactness theorem, Q has a model, and the $\mathcal{L}$-reduct of this model is the required end extension of $\langle M, \in \rangle$. ⊣

Barwise [1] and [2, Appendix] introduces the machinery of admissible covers to apply infinitary compactness arguments, such as the one used in the proof sketch of Theorem 4.1, to nonstandard countable models. The proof of [2, Theorem A.4.1] shows that for any countable model $\mathcal{M}$ of $\mathsf{KP} + \mathsf{Foundation}$ and for any recursively enumerable $\mathcal{L}$-theory T that holds in $\mathcal{M}$, $\mathcal{M}$ has proper end extension that satisfies T. By calibrating [2, Appendix], Ressayre [15, Theorem 2.15] shows that this result also holds for countable models of $\mathsf{KP} + \Sigma_1\text{-}\mathsf{Foundation}$.

THEOREM 4.2 (Ressayre). *Let $\mathcal{M} = \langle M, \in^\mathcal{M} \rangle$ be a countable model of* $\mathsf{KP} + \Sigma_1\text{-}\mathsf{Foundation}$. *Let T be a recursively enumerable theory such that $\mathcal{M} \models T$. Then there exists $\mathcal{N} \models T$ such that $\mathcal{M} \subseteq_e \mathcal{N}$ and $M \neq N$.*

In [15, 2.17 Remarks], Ressayre notes, without providing the details, that if $\mathcal{M}$ satisfies $\mathsf{KP} + \Pi_n\text{-}\mathsf{Collection} + \Pi_{n+1} \cup \Sigma_{n+1}\text{-}\mathsf{Foundation}$, then the end extension obtained in Theorem 4.2 can be guaranteed to be Σ_n-elementary. In this section, we work through the details of this result showing that the assumption that the model $\mathcal{M}$ being extended satisfies $\Pi_{n+1}\text{-}\mathsf{Foundation}$ is not necessary. Our main result (Theorem 4.15) can be viewed as a generalisation of [5, Theorem 5.3], where admissible covers are used to build powerset-preserving end extension of countable models of set theory. Here we follow the presentation of admissible covers presented in [5].

In order to present admissible covers of (not necessarily well-founded) models of extensions of KP we need to describe extensions of Kripke–Platek Set Theory that allow structures to appear as *urelements* in the domain of discourse. Let $\mathcal{L}^*$ be obtained from $\mathcal{L}$ by adding a new unary predicate U, binary relation E and unary function symbol F. Let $\mathcal{L}^*_\mathsf{S}$ be obtained from $\mathcal{L}^*$ by adding a new binary predicate S. The intention is that U distinguishes objects that are urelements from objects that are sets, the urelements together with E form an $\mathcal{L}$-structure, and $\in$ is a membership relation between sets or urelments and sets. That is, the $\mathcal{L}^*$- and $\mathcal{L}^*_\mathsf{S}$-structures we will consider will be structures in the form $\mathfrak{A}_\mathcal{M} = \langle \mathcal{M}; A, \in^\mathfrak{A}, \mathsf{F}^\mathfrak{A} \rangle$ or $\mathfrak{A}_\mathcal{M} = \langle \mathcal{M}; A, \in^\mathfrak{A}, \mathsf{F}^\mathfrak{A}, \mathsf{S}^\mathfrak{A} \rangle$, where $\mathcal{M} = \langle M, \mathsf{E}^\mathfrak{A} \rangle$, M is the extension of U, $\mathsf{E}^\mathfrak{A} \subseteq M \times M$, A is the extension of $\neg\mathsf{U}$ and $\in^\mathfrak{A} \subseteq (M \cup A) \times A$.

Following [2] we simplify the presentation of $\mathcal{L}^*$- and $\mathcal{L}^*_{\mathsf{S}}$-formulae by treating these languages as two-sorted instead of one-sorted and using the following conventions:

- The variables $p, q, r, p_1, \ldots$range over elements of the domain that satisfy U;
- the variables $a, b, c, a_1, \ldots$range over elements of the domain that satisfy $\neg\mathsf{U}$;
- the variables $x, y, z, w, x_1, \ldots$range over all elements of the domain.

So, $\forall a(\cdots)$ is an abbreviation of $\forall x(\neg\mathsf{U}(x) \Rightarrow \cdots)$, $\exists p(\cdots)$ is an abbreviation of $\exists x(\mathsf{U}(x) \wedge \cdots)$, etc. These conventions are used in the following $\mathcal{L}^*_{\mathsf{S}}$-axioms and -axiom schemes:

(Extensionality for sets) $\forall a \forall b(a = b \iff \forall x(x \in a \iff x \in b))$.

(Pair) $\forall x \forall y \exists a \forall z(z \in a \iff z = x \vee z = y)$.

(Union) $\forall a \exists b(\forall y \in b)(\forall x \in y)(x \in b)$.

Let Γ be a class of $\mathcal{L}^*_{\mathsf{S}}$-formulae.

(Γ-Separation) For all $\phi(x, \vec{z})$ in Γ,

$$\forall \vec{z} \forall a \exists b \forall x(x \in b \iff (x \in a) \wedge \phi(x, \vec{z})).$$

(Γ-Collection) For all $\phi(x, y, \vec{z})$ in Γ,

$$\forall \vec{z} \forall a((\forall x \in a)\exists y \phi(x, y, \vec{z}) \Rightarrow \exists b(\forall x \in a)(\exists y \in b)\phi(x, y, \vec{z})).$$

(Γ-Foundation) For all $\phi(x, \vec{z})$ in Γ,

$$\forall \vec{z}(\exists x \phi(x, \vec{z}) \Rightarrow \exists y(\phi(y, \vec{z}) \wedge (\forall w \in y)\neg\phi(w, \vec{z}))).$$

The interpretation of the function symbol F will map urelements, p, to sets, a, such that the E-extension of p is equal to the $\in$-extension of a. This is captured by the following axiom:

($\dagger$) $\forall p \exists a(a = \mathsf{F}(p) \wedge \forall x(x \mathsf{E} p \iff x \in a)) \wedge \forall b(\mathsf{F}(b) = \emptyset)$.

The following theory is the analog of KP in the language $\mathcal{L}^*$:

- $\mathsf{KPU}_{\mathbb{C}\mathsf{ov}}$ is the $\mathcal{L}^*$-theory with axioms: $\exists a(a = a)$, $\forall p \forall x(x \notin p)$, Extensionality for sets, Pair, Union, $\Delta_0(\mathcal{L}^*)$-Separation, $\Delta_0(\mathcal{L}^*)$-Collection, $\Pi_1(\mathcal{L}^*)$-Foundation and ($\dagger$).

An order pair $\langle x, y\rangle$ is coded in $\mathsf{KPU}_{\mathbb{C}\mathsf{ov}}$ by the set $\{\{x\}, \{x, y\}\}$, and we write $\mathsf{OP}(x)$ for the usual Δ_0-formula that says that z is an order pair and that also works in this theory. We write fst for the function $\langle x, y\rangle \mapsto x$ and snd for the function $\langle x, y\rangle \mapsto y$. The usual Δ_0 definitions of the graphs of these functions also work in $\mathsf{KPU}_{\mathbb{C}\mathsf{ov}}$. The *rank function*, ρ, and *support function*, sp, are defined in $\mathsf{KPU}_{\mathbb{C}\mathsf{ov}}$ by recursion:

$$\rho(p) = 0 \text{ for all urelements } p, \text{ and } \rho(a) = \sup\{\rho(x) + 1 \mid x \in a\} \text{ for all sets } a;$$

$$\mathsf{sp}(p) = \{p\} \text{ for all urelements } p, \text{ and } \mathsf{sp}(a) = \bigcup_{x \in a} \mathsf{sp}(x) \text{ for all sets } a.$$

The theory $\mathsf{KPU}_{\mathbb{C}\mathsf{ov}}$ proves that both sp and ρ are total functions and their graphs are $\Delta_1(\mathcal{L}^*)$. We say that x is a *pure set* if $\mathsf{sp}(x) = \emptyset$. The following $\Delta_0(\mathcal{L}^*)$-formulae assert that 'x is transitive' and 'x is an ordinal (a hereditarily transitive pure set)':

$$\mathsf{Transitive}(x) \iff \neg\mathsf{U}(x) \wedge (\forall y \in x)(\forall z \in y)(z \in x);$$

$$\mathsf{Ord}(x) \iff (\mathsf{Transitive}(x) \wedge (\forall y \in x)\mathsf{Transitive}(y)).$$

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We will consider $\mathcal{L}_{\mathsf{S}}^*$-structures in which the predicate S is a satisfaction class for the Σ_n-formulae of the $\mathcal{L}$-structure $\mathcal{M}$. Let $\mathsf{KPU}'_{\mathbb{C}\mathrm{ov}}$ be obtained from $\mathsf{KPU}_{\mathbb{C}\mathrm{ov}}$ by adding axioms asserting that the $\mathcal{L}$-structure formed by the urelements and the binary relation E satisfies KP. For $n \in \omega$, define

(n-Sat) $\mathsf{S}(m, x)$ if and only if $\mathsf{U}(m)$ and $\mathsf{U}(x)$ and $\mathsf{Sat}_{\Sigma_n}(m, x)$ holds in the $\mathcal{L}$-structure defined by U and E.

We can now define a family of $\mathcal{L}_{\mathsf{S}}^*$-theories extending $\mathsf{KPU}_{\mathbb{C}\mathrm{ov}}$ that assert that the $\mathcal{L}$-structure defined by U and E satisfies KP and S is a satisfaction class on this structure for Σ_n-formulae, and S can be used in the separation, collection and foundation schemes.

- For all $n \in \omega$, define $\mathsf{KPU}^n_{\mathbb{C}\mathrm{ov}}$ to be the $\mathcal{L}_{\mathsf{S}}^*$-theory extending $\mathsf{KPU}'_{\mathbb{C}\mathrm{ov}}$ with the axiom n-Sat and the schemes $\Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-Separation, $\Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-Collection and $\Pi_1(\mathcal{L}_{\mathsf{S}}^*)$-Foundation.

The arguments used in [2, Theorems I.4.4 and I.4.5] show that $\mathsf{KPU}_{\mathbb{C}\mathrm{ov}}$ proves the schemes of $\Sigma_1(\mathcal{L}^*)$-Collection and $\Delta_1(\mathcal{L}^*)$-Separation, and for all $n \in \omega$, $\mathsf{KPU}^n_{\mathbb{C}\mathrm{ov}}$ proves the schemes of $\Sigma_1(\mathcal{L}_{\mathsf{S}}^*)$-Collection and $\Delta_1(\mathcal{L}_{\mathsf{S}}^*)$-Separation.

DEFINITION 4.1. Let $\mathcal{M} = \langle M, \mathsf{E}^{\mathcal{M}} \rangle$ be an $\mathcal{L}$-structure. An *admissible set covering* $\mathcal{M}$ is an $\mathcal{L}^*$-structure

$$\mathfrak{A}_{\mathcal{M}} = \langle \mathcal{M}; A, \in^{\mathfrak{A}}, \mathsf{F}^{\mathfrak{A}} \rangle \models \mathsf{KPU}_{\mathbb{C}\mathrm{ov}}$$

such that $\in^{\mathfrak{A}}$ is well-founded. If $\mathcal{M} \models \mathsf{KP}$ and $n \in \omega$, then an *n-admissible set covering* $\mathcal{M}$ is an $\mathcal{L}_{\mathsf{S}}^*$-structure

$$\mathfrak{A}_{\mathcal{M}} = \langle \mathcal{M}; A, \in^{\mathfrak{A}}, \mathsf{F}^{\mathfrak{A}}, \mathsf{S}^{\mathfrak{A}} \rangle \models \mathsf{KPU}^n_{\mathbb{C}\mathrm{ov}}$$

such that $\in^{\mathfrak{A}}$ is well-founded. Note that if $\mathfrak{A}_{\mathcal{M}} = \langle \mathcal{M}; A, \in^{\mathfrak{A}}, \mathsf{F}^{\mathfrak{A}}, \ldots \rangle$ is an (n-)admissible set covering $\mathcal{M}$, then $\mathfrak{A}_{\mathcal{M}}$ is isomorphic to a structure whose membership relation ($\in$) is the membership relation of the metatheory. The *admissible cover* of $\mathcal{M}$, denoted $\mathbb{C}\mathrm{ov}_{\mathcal{M}} = \langle \mathcal{M}; A_{\mathcal{M}}, \in, \mathsf{F}_{\mathcal{M}} \rangle$, is the smallest admissible set covering $\mathcal{M}$ whose membership relation ($\in$) coincides with the membership relation of the metatheory. If $\mathcal{M} \models \mathsf{KP}$ and $n \in \omega$, the *n-admissible cover* of $\mathcal{M}$, denoted $\mathbb{C}\mathrm{ov}^n_{\mathcal{M}} = \langle \mathcal{M}; A_{\mathcal{M}}, \in, \mathsf{F}_{\mathcal{M}}, \mathsf{S}_{\mathcal{M}} \rangle$, is the smallest n-admissible set covering $\mathcal{M}$ whose membership relation ($\in$) coincides with the membership relation of the metatheory.

DEFINITION 4.2. Let $\mathcal{M} = \langle M, \mathsf{E}^{\mathcal{M}} \rangle$ be an $\mathcal{L}$-structure, and let $\mathfrak{A}_{\mathcal{M}} = \langle \mathcal{M}; A, \in, \mathsf{F}^{\mathfrak{A}}, \ldots \rangle$ be an $\mathcal{L}^*$- or $\mathcal{L}_{\mathsf{S}}^*$-structure. We use $\mathrm{WF}(A)$ to denote the largest $B \subseteq A$ such that $\langle B, \in^{\mathfrak{A}} \rangle \subseteq_e \langle A, \in^{\mathfrak{A}} \rangle$ and $\langle B, \in^{\mathfrak{A}} \rangle$ is well-founded. The *well-founded part* of $\mathfrak{A}_{\mathcal{M}}$ is the $\mathcal{L}^*$- or $\mathcal{L}_{\mathsf{S}}^*$-structure

$$\mathrm{WF}(\mathfrak{A}_{\mathcal{M}}) = \langle \mathcal{M}; \mathrm{WF}(A), \in^{\mathfrak{A}}, \mathsf{F}^{\mathfrak{A}}, \ldots \rangle.$$

Note that $\mathrm{WF}(\mathfrak{A}_{\mathcal{M}})$ is always isomorphic to a structure whose membership relation $\in$ coincides with the membership relation of the metatheory.

Let $\mathcal{M} = \langle M, \mathsf{E}^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{KP}$. Let $\mathcal{L}^{\mathsf{ee}}$ be the language obtained from $\mathcal{L}$ by adding new constant symbols $\bar{a}$ for each $a \in M$ and a new constant symbol $\mathbf{c}$. Let $\mathfrak{A}_{\mathcal{M}} = \langle \mathcal{M}; A, \in, \mathsf{F}^{\mathfrak{A}}, \mathsf{S}^{\mathfrak{A}} \rangle$ be an n-admissible set covering $\mathcal{M}$. There is a coding $\ulcorner \cdot \urcorner$ of a fragment of the infinitary language $\mathcal{L}^{\mathsf{ee}}_{\infty\omega}$ in $\mathfrak{A}_{\mathcal{M}}$ with the property that

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the classes of codes of *atomic formulae*, *variables*, *constants*, *well-formed formulae*, *sentences*, etc. are all $\Delta_1(\mathcal{L}^*)$-definable over $\mathfrak{A}_\mathcal{M}$ (see [5, p. 9] for an explicit definition of such a coding). We write $\mathcal{L}^{\mathsf{ee}}_{\mathfrak{A}_\mathcal{M}}$ for the fragment of $\mathcal{L}^{\mathsf{ee}}_{\infty\omega}$ whose codes appear in $\mathfrak{A}_\mathcal{M}$. In order to apply compactness arguments to $\mathcal{L}^{\mathsf{ee}}_{\mathfrak{A}_\mathcal{M}}$-theories where $\mathfrak{A}_\mathcal{M}$ is an n-admissible set, we will use the following specific version of the Barwise Compactness theorem ([2, Theorem III.5.6]):

THEOREM 4.3 (Barwise Compactness theorem). *Let $\mathfrak{A}_\mathcal{M} = \langle \mathcal{M}; A, \in \mathsf{F}^\mathfrak{A}, \mathsf{S}^\mathfrak{A}\rangle$ be an n-admissible set covering $\mathcal{M}$. Let T be an $\mathcal{L}^{\mathsf{ee}}_{\mathfrak{A}_\mathcal{M}}$-theory that is $\Sigma_1(\mathcal{L}^*_\mathsf{S})$-definable over $\mathfrak{A}_\mathcal{M}$ and such that for all $T_0 \subseteq T$, if $T_0 \in A$, then T_0 has a model. Then T has a model.*

The work in [2, Appendix] and [15, Chapter 2] shows that if $\mathcal{M}$ satisfies $\mathsf{KP}+\Sigma_1$-Foundation, then $\mathbb{C}\mathrm{ov}_\mathcal{M}$ exists. In particular, $\mathbb{C}\mathrm{ov}_\mathcal{M}$ can be obtained from $\mathcal{M}$ by first defining a model of $\mathsf{KPU}_{\mathbb{C}\mathrm{ov}}$ inside $\mathcal{M}$ and then considering the well-founded part of this model. We now turn to reviewing the construction of $\mathbb{C}\mathrm{ov}_\mathcal{M}$ from $\mathcal{M}$ and showing that if $\mathcal{M}$ satisfies $\mathsf{KP}+\Pi_n$-Collection $+\Sigma_{n+1}$-Foundation, then $\mathbb{C}\mathrm{ov}_\mathcal{M}$ can be expanded to an $\mathcal{L}^*_\mathsf{S}$-structure corresponding to $\mathbb{C}\mathrm{ov}^n_\mathcal{M}$.

Let $n \geq 1$. Fix a model $\mathcal{M} = \langle M, \mathsf{E}^\mathcal{M}\rangle$ that satisfies $\mathsf{KP}+\Pi_n$-Collection $+$ Σ_{n+1}-Foundation. Working inside $\mathcal{M}$, define unary relations N and Set, binary relations E', $\mathcal{E}$ and $\bar{\mathsf{S}}$, and unary function $\bar{\mathsf{F}}$ by:

$$\mathsf{N}(x) \text{ iff } \exists y(x = \langle 0, y\rangle);$$

$$x\mathsf{E}'y \text{ iff } \exists w\exists z(x = \langle 0, w\rangle \wedge y = \langle 0, z\rangle \wedge w \in z);$$

$$\mathsf{Set}(x) = \exists y(x = \langle 1, y\rangle \wedge (\forall z \in y)(\mathsf{N}(z) \vee \mathsf{Set}(z)));$$

$$x\mathcal{E}y \text{ iff } \exists z(y = \langle 1, z\rangle \wedge x \in z);$$

$$\bar{\mathsf{F}}(x) = \langle 1, X\rangle \text{ where } X = \{\langle 0, y\rangle \mid \exists w(x = \langle 0, w\rangle \wedge y \in w)\};$$

$$\bar{\mathsf{S}}(x, y) \text{ iff } \exists z\exists w(x = \langle 0, w\rangle \wedge y = \langle 0, z\rangle \wedge \mathsf{Sat}_{\Sigma_n}(w, z)).$$

It is noted in [2, Appendix Section 3] that N, E', $\mathcal{E}$ and $\bar{\mathsf{F}}$ are defined by Δ_0-formulae in $\mathcal{M}$. The Second Recursion theorem ([2, Theorem V.2.3]), provable in $\mathsf{KP}+\Sigma_1$-Foundation as note in [15], ensures that Set can be expressed as a Σ_1-formula in $\mathcal{M}$. Theorem 2.3 implies that $\bar{\mathsf{S}}$ is defined by a Σ_n-formula in $\mathcal{M}$. These definitions yield an interpretation, $\mathcal{I}$, of an $\mathcal{L}^*_\mathsf{S}$-structure $\mathfrak{A}_\mathcal{N} = \langle \mathcal{N}; \mathsf{Set}^\mathcal{M}, \mathcal{E}^\mathcal{M}, \bar{\mathsf{F}}^\mathcal{M}, \bar{\mathsf{S}}^\mathcal{M}\rangle$, where $\mathcal{N} = \langle \mathsf{N}^\mathcal{M}, (\mathsf{E}')^\mathcal{M}\rangle$. Table 1 extends the table on [2, p. 373] and summarises the interpretation $\mathcal{I}$:

If ϕ is an $\mathcal{L}^*_\mathsf{S}$-formula, then we write $\phi^\mathcal{I}$ for the translation of ϕ into an $\mathcal{L}$-formula described in Table 1. By ignoring the interpretation $\bar{\mathsf{S}}$ of S we obtain, instead, an interpretation, $\mathcal{I}^-$, of an $\mathcal{L}^*$-structure in $\mathcal{M}$ and we write $\mathfrak{A}^-_\mathcal{N}$ for this reduct. Note that the map $x \mapsto \langle 0, x\rangle$ defines an isomorphism between $\mathcal{M}$ and $\mathcal{N} = \langle \mathsf{N}^\mathcal{M}, (\mathsf{E}')^\mathcal{M}\rangle$. Ressayre, refining [2, Appendix Lemma 3.2], shows that if $\mathcal{M}$ satisfies $\mathsf{KP}+\Sigma_1$-Foundation, then interpretation $\mathcal{I}^-$ yields a structure satisfying $\mathsf{KPU}_{\mathbb{C}\mathrm{ov}}$.

THEOREM 4.4. $\mathfrak{A}^-_\mathcal{N} \models \mathsf{KPU}_{\mathbb{C}\mathrm{ov}}$.

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$\mathcal{L}_{\mathsf{S}}^*$ Symbol	$\mathcal{L}$ expression under $\mathcal{I}$
$\forall x$	$\forall x(\mathsf{N}(x) \vee \mathsf{Set}(x) \Rightarrow \cdots)$
$=$	$=$
$\mathsf{U}(x)$	$\mathsf{N}(x)$
$x\mathsf{E}y$	$x\mathsf{E}'y$
$x \in y$	$x\mathcal{E}y$
$\mathsf{F}(x)$	$\bar{\mathsf{F}}(x)$
$\mathsf{S}(x, y)$	$\bar{\mathsf{S}}(x, y)$

TABLE 1. The interpretation $\mathcal{I}$.

LEMMA 4.5. *Let $\phi(\vec{x})$ be a $\Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-formula. Then $\phi^{\mathcal{I}}(\vec{x})$ is equivalent to a Δ_{n+1}-formula in $\mathcal{M}$.*

PROOF. We prove this result by induction on the complexity of ϕ. Above, we observed that $\mathsf{N}(x)$, $x\mathsf{E}'y$, $x\mathcal{E}y$ and $y = \bar{\mathsf{F}}(x)$ can be written as Δ_0-formulae. And $\bar{\mathsf{S}}(x, y)$ can be written as a Σ_n-formula. Now, $y\mathcal{E}\bar{F}(x)$ if and only if

$$\mathsf{fst}(y) = 0 \wedge \mathsf{snd}(y) \in \mathsf{snd}(x),$$

which is Δ_0. Therefore, if $\phi(\vec{x})$ is a quantifier-free $\mathcal{L}_{\mathsf{S}}^*$-formula, then $\phi^{\mathcal{I}}(\vec{x})$ is equivalent to a Δ_{n+1}-formula in $\mathcal{M}$. Now, suppose that $\phi(x_0, \ldots, x_{m-1})$ is in the form $(\exists y \in x_0)\psi(x_0, \ldots, x_{m-1}, y)$ where $\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y)$ is equivalent to a Δ_{n+1}-formula in $\mathcal{M}$. Therefore, $\phi^{\mathcal{I}}(x_0, \ldots, x_{m-1}) = (\exists y\mathcal{E}x_0)\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y)$, and $(\exists y\mathcal{E}x_0)\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y)$ iff

$$(\exists y \in \mathsf{snd}(x_0))\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y).$$

So, since $\mathcal{M}$ satisfies Π_n-Collection, $\phi^{\mathcal{I}}(x_0, \ldots, x_{m-1})$ is equivalent to a Δ_{n+1}-formula in $\mathcal{M}$. Finally, suppose that $\phi(x_0, \ldots, x_{m-1})$ is in the form $(\exists y \in \mathsf{F}(x_0))\psi(x_0, \ldots, x_{m-1}, y)$ where $\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y)$ is equivalent to a Δ_{n+1}-formula in $\mathcal{M}$. Therefore, $\phi^{\mathcal{I}}(x_0, \ldots, x_{m-1}) = (\exists y\mathcal{E}\bar{\mathsf{F}}(x_0))\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y)$, and $(\exists y\mathcal{E}\bar{\mathsf{F}}(x_0))\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y)$ iff

$$\exists z(z = \bar{\mathsf{F}}(x_0) \wedge (\exists y \in \mathsf{snd}(z))\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y))$$

$$\text{iff } \forall z(z = \bar{\mathsf{F}}(x_0) \Rightarrow (\exists y \in \mathsf{snd}(z))\psi^{\mathcal{I}}(x_0, \ldots, x_{m-1}, y)).$$

Therefore, since $\mathcal{M}$ satisfies Π_n-Collection, $\phi^{\mathcal{I}}(x_0, \ldots, x_{m-1})$ is equivalent to a Δ_{n+1}-formula in $\mathcal{M}$. The lemma now follows by induction. ⊣

LEMMA 4.6. $\mathfrak{A}_{\mathcal{N}} \models \Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-Separation.

PROOF. Let $\phi(x, \vec{z})$ be a $\Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-formula. Let $\vec{v}$ be a finite sequence of sets and/or urelements of $\mathfrak{A}_{\mathcal{N}}$ and a a set of $\mathfrak{A}_{\mathcal{N}}$. Work inside $\mathcal{M}$. Now, $a = \langle 1, a_0 \rangle$. Let

$$b_0 = \{x \in a_0 \mid \phi^{\mathcal{I}}(x, \vec{v})\},$$

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which is a set by Δ_{n+1}-Separation. Let $b = \langle 1, b_0 \rangle$. Therefore, for all x such that $\mathsf{Set}(x)$,

$$x \mathcal{E} b \text{ if and only if } x \mathcal{E} a \wedge \phi^{\mathcal{I}}(x, \vec{v}).$$

This shows that $\mathfrak{A}_{\mathcal{N}}$ satisfies $\Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-Separation. ⊣

LEMMA 4.7. $\mathfrak{A}_{\mathcal{N}} \models \Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-Collection.

PROOF. Let $\phi(x, y, \vec{z})$ be a $\Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-formula. Let $\vec{v}$ be a finite sequence of sets and/or urelements of $\mathfrak{A}_{\mathcal{N}}$ and let a be a set of $\mathfrak{A}_{\mathcal{N}}$ such that

$$\mathfrak{A}_{\mathcal{N}} \models (\forall x \in a)\exists y \phi(x, y, \vec{v}).$$

Work inside $\mathcal{M}$. Now, $a = \langle 1, a_0 \rangle$. And,

$$(\forall x \mathcal{E} a)\exists y((\mathsf{N}(y) \vee \mathsf{Set}(y)) \wedge \phi^{\mathcal{I}}(x, y, \vec{v})).$$

So,

$$(\forall x \in a_0)\exists y((\mathsf{N}(y) \vee \mathsf{Set}(y)) \wedge \phi^{\mathcal{I}}(x, y, \vec{v})).$$

Since $(\mathsf{N}(y) \vee \mathsf{Set}(y)) \wedge \phi^{\mathcal{I}}(x, y, \vec{v})$ is equivalent to a Σ_{n+1}-formula, we can use Π_n-Collection to find b_0 such that

$$(\forall x \in a_0)(\exists y \in b_0)((\mathsf{N}(y) \vee \mathsf{Set}(y)) \wedge \phi^{\mathcal{I}}(x, y, \vec{v})).$$

Let $b_1 = \{y \in b_0 \mid \mathsf{N}(y) \vee \mathsf{Set}(y)\}$, which is a set by Σ_1-Separation. Let $b = \langle 1, b_1 \rangle$. Therefore, $\mathsf{Set}(b)$ and

$$(\forall x \mathcal{E} a)(\exists y \mathcal{E} b)\phi^{\mathcal{I}}(x, y, \vec{v}).$$

So,

$$\mathfrak{A}_{\mathcal{N}} \models (\forall x \in a)(\exists y \in b)\phi(x, y, \vec{v}).$$

This shows that $\mathfrak{A}_{\mathcal{N}}$ satisfies $\Delta_0(\mathcal{L}_{\mathsf{S}}^*)$-Collection. ⊣

LEMMA 4.8. $\mathfrak{A}_{\mathcal{N}} \models \Sigma_1(\mathcal{L}_{\mathsf{S}}^*)$-Foundation.

PROOF. Let $\phi(x, \vec{z})$ be a $\Sigma_1(\mathcal{L}_{\mathsf{S}}^*)$-formula. Let $\vec{v}$ be a sequence of sets and/or urelements such that

$$\{x \in \mathfrak{A}_{\mathcal{N}} \mid \mathfrak{A}_{\mathcal{N}} \models \phi(x, \vec{v})\} \text{ is nonempty.}$$

Work inside $\mathcal{M}$. Consider $\theta(\alpha, \vec{z})$ defined by

$$(\alpha \text{ is an ordinal}) \wedge \exists x((\mathsf{N}(x) \vee \mathsf{Set}(x)) \wedge \rho(x) = \alpha \wedge \phi^{\mathcal{I}}(x, \vec{z})).$$

Note that $\theta(\alpha, \vec{z})$ is equivalent to a Σ_{n+1}-formula and $\exists \alpha \theta(\alpha, \vec{v})$. Therefore, using Σ_{n+1}-Foundation, let β be a $\in$-least element of

$$\{\alpha \in M \mid \mathcal{M} \models \theta(\alpha, \vec{v})\}.$$

Let y be such that $(\mathsf{N}(y) \vee \mathsf{Set}(y))$, $\rho(y) = \beta$ and $\phi^{\mathcal{I}}(y, \vec{v})$. Note that if $x \mathcal{E} y$, then $\rho(x) < \rho(y)$. Therefore y is an $\mathcal{E}$-least element of

$$\{x \in \mathfrak{A}_{\mathcal{N}} \mid \mathfrak{A}_{\mathcal{N}} \models \phi(x, \vec{v})\}.$$ ⊣

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The results of [2, Appendix Section 3] show that $\mathbb{C}\mathrm{ov}_{\mathcal{M}}$ is the $\mathcal{L}^*$-reduct of the well-founded part of $\mathfrak{A}_{\mathcal{N}}$.

THEOREM 4.9 (Barwise). *The $\mathcal{L}^*$-reduct of* $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$, $\mathrm{WF}^-(\mathfrak{A}_{\mathcal{N}}) = \langle \mathcal{N}; \mathrm{WF}(\mathsf{Set}^{\mathcal{M}}), \mathcal{E}^{\mathcal{M}}, \bar{\mathsf{F}}^{\mathcal{M}} \rangle$, *is an admissible set covering $\mathcal{N}$ that is isomorphic to* $\mathbb{C}\mathrm{ov}_{\mathcal{M}}$.

We can extend this result to show that $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ is an n-admissible cover of $\mathcal{N}$ and, therefore, isomorphic to $\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$.

THEOREM 4.10. *The structure* $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}}) = \langle \mathcal{N}; \mathrm{WF}(\mathsf{Set}^{\mathcal{M}}), \mathcal{E}^{\mathcal{M}}, \bar{\mathsf{F}}^{\mathcal{M}}, \bar{\mathsf{S}}^{\mathcal{M}} \rangle$ *is an n-admissible set covering $\mathcal{N}$. Moreover,* $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ *is isomorphic to* $\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$.

PROOF. Theorem 4.9, the fact that $\mathcal{M} \models \mathsf{KP}$, and the fact that $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ is well-founded imply that $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ satisfies $\mathsf{KPU}'_{\mathbb{C}\mathrm{ov}} + \mathcal{L}^*_{\mathsf{S}}$-Foundation. The definition of $\bar{\mathsf{S}}$ in $\mathcal{M}$ ensures that $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ satisfies n-Sat. If a is a set $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ and b is a set in $\mathfrak{A}_{\mathcal{N}}$ with $\mathfrak{A}_{\mathcal{N}} \models (b \subseteq a)$, then $b \in \mathrm{WF}(\mathsf{Set}^{\mathcal{M}})$. Therefore, since $\Delta_0(\mathcal{L}^*_{\mathsf{S}})$-formulae are absolute between $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ and $\mathfrak{A}_{\mathcal{N}}$, $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ satisfies $\Delta_0(\mathcal{L}^*_{\mathsf{S}})$-Separation. To show that $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ satisfies $\Delta_0(\mathcal{L}^*_{\mathsf{S}})$-Collection, let $\phi(x, y, \vec{z})$ be a $\Delta_0(\mathcal{L}^*_{\mathsf{S}})$-formula. Let $\vec{v}$ be sets and/or urelements in $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ and let a be a set of $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ such that

$$\mathrm{WF}(\mathfrak{A}_{\mathcal{N}}) \models (\forall x \in a)\exists y \phi(x, y, \vec{v}).$$

Consider the formula $\theta(\beta, \vec{z})$ defined by

$$(\beta \text{ is an ordinal}) \wedge (\forall x \in a)(\exists \alpha \in \beta)\exists y(\rho(y) = \alpha \wedge \phi(x, y, \vec{z})).$$

Note that if β is a nonstandard ordinal of $\mathfrak{A}_{\mathcal{N}}$, then $\mathfrak{A}_{\mathcal{N}} \models \theta(\beta, \vec{v})$. Using $\Delta_0(\mathcal{L}^*_{\mathsf{S}})$-Collection, $\theta(\beta, \vec{z})$ is equivalent to a $\Sigma_1(\mathcal{L}^*_{\mathsf{S}})$-formula in $\mathfrak{A}_{\mathcal{N}}$. Therefore, by $\Sigma_1(\mathcal{L}^*_{\mathsf{S}})$-Foundation in $\mathfrak{A}_{\mathcal{N}}$, $\{\beta \mid \mathfrak{A}_{\mathcal{N}} \models \theta(\beta, \vec{v})\}$ has a least element γ. Note that γ must be an ordinal in $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$. Consider the formula $\psi(x, y, \vec{z}, \gamma)$ defined by $\phi(x, y, \vec{z}) \wedge (\rho(y) < \gamma)$. Then,

$$\mathfrak{A}_{\mathcal{N}} \models (\forall x \in a)\exists y \psi(x, y, \vec{v}, \gamma).$$

By $\Delta_0(\mathcal{L}^*_{\mathsf{S}})$-Collection in $\mathfrak{A}_{\mathcal{N}}$, there is a set b of $\mathfrak{A}_{\mathcal{N}}$ such that

$$\mathfrak{A}_{\mathcal{N}} \models (\forall x \in a)(\exists y \in b)\psi(x, y, \vec{v}, \gamma).$$

Let $c = \{y \in b \mid \rho(y) < \gamma\}$, which is a set in $\mathfrak{A}_{\mathcal{N}}$ by $\Delta_1(\mathcal{L}^*_{\mathsf{S}})$-Separation. Now, c is a set of $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ and

$$\mathrm{WF}(\mathfrak{A}_{\mathcal{N}}) \models (\forall x \in a)(\exists y \in c)\phi(x, y, \vec{v}).$$

Therefore, $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ satisfies $\Delta_0(\mathcal{L}^*_{\mathsf{S}})$-Collection, and so is an n-admissible set covering $\mathcal{N}$. Since the $\mathcal{L}^*$-reduct of $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ is isomorphic to $\mathbb{C}\mathrm{ov}_{\mathcal{M}}$, $\mathrm{WF}(\mathfrak{A}_{\mathcal{N}})$ is isomorphic to $\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$. ⊣

To summarise, we have proved the following.

THEOREM 4.11. *If* $\mathcal{M} \models \mathsf{KP} + \Pi_n\text{-}\mathsf{Collection} + \Sigma_{n+1}\text{-}\mathsf{Foundation}$, *then then there is an interpretation of* S *in* $\mathbb{C}\mathrm{ov}_{\mathcal{M}}$ *that yields the n-admissible cover* $\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$.

Our analysis also yields the following version of [2, Appendix Corollary 2.4], which plays an important role on compactness arguments.

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THEOREM 4.12. *Let $\mathcal{M} = \langle M, \mathsf{E}^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{KP} + \Pi_n\text{-}\mathsf{Collection} + \Sigma_{n+1}\text{-}\mathsf{Foundation}$. For all $A \subseteq M$, there exists $a \in M$ such that $a^* = A$ if and only if $A \in \mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$.*

In particular, we obtain:

LEMMA 4.13. *Let $\mathcal{M} = \langle M, \mathsf{E}^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{KP} + \Pi_n\text{-}\mathsf{Collection} + \Sigma_{n+1}\text{-}\mathsf{Foundation}$. Let T_0 be an $\mathcal{L}^{\mathsf{ee}}_{\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}}$-theory. If $T_0 \in \mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$, then there exists $b \in M$ such that*

$$b^* = \{a \in M \mid \bar{a} \text{ is mentioned in } T_0\}.$$

The next result connects definability in $\mathcal{M}$ with definability in $\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$.

LEMMA 4.14. *Let $\mathcal{M} = \langle M, \mathsf{E}^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{KP} + \Pi_n\text{-}\mathsf{Collection} + \Sigma_{n+1}\text{-}\mathsf{Foundation}$. Let $\phi(\vec{z})$ be a Σ_{n+1}-formula. Then there exists a $\Sigma_1(\mathcal{L}^*_{\mathsf{S}})$-formula $\hat{\phi}(\vec{z})$ such that for all $\vec{z} \in M$,*

$$\mathcal{M} \models \phi(\vec{z}) \text{ if and only if } \mathbb{C}\mathrm{ov}^n_{\mathcal{M}} \models \hat{\phi}(\vec{z}).$$

PROOF. Let $\theta(x, \vec{z})$ be Π_n such that $\phi(\vec{z})$ is $\exists x \theta(x, \vec{z})$. Let $q \in \omega$ be such that $q = \ulcorner \neg\theta(\vec{z}) \urcorner$. Let $z_0, \dots, z_{m-1} \in M$. Then

$$\mathcal{M} \models \phi(z_0, \dots, z_{m-1}) \text{ if and only if } \mathbb{C}\mathrm{ov}^n_{\mathcal{M}} \models \exists x \exists z (z = \langle x, z_0, \dots, z_{m-1} \rangle \wedge \neg \mathsf{S}(q, z)).$$

⊣

THEOREM 4.15. *Let S be a recursively enumerable $\mathcal{L}$-theory such that*

$$S \vdash \mathsf{KP} + \Pi_n\text{-}\mathsf{Collection} + \Sigma_{n+1}\text{-}\mathsf{Foundation},$$

and let $\mathcal{M} = \langle M, \mathsf{E}^{\mathcal{M}} \rangle$ be a countable model of S. Then there exists an $\mathcal{L}$-structure $\mathcal{N} = \langle N, \mathsf{E}^{\mathcal{N}} \rangle$ such that $\mathcal{M} \prec_{e,n} \mathcal{N} \models S$ and there exists $d \in N$ such that for all $x \in M$, $\mathcal{N} \models (x \in d)$.

PROOF. Let T be the $\mathcal{L}^{\mathsf{ee}}_{\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}}$-theory that contains:

- S;
- for all $a, b \in M$ with $\mathcal{M} \models (a \in b)$, $\bar{a} \in \bar{b}$;
- for all $a \in M$,

$$\forall x \left(x \in a \iff \bigvee_{b \in a} (x = \bar{b}) \right);$$

- for all $a \in M$, $\bar{a} \in \mathbf{c}$;
- for all Π_n-formulae, $\phi(x_0, \dots, x_{m-1})$, and for all $a_0, \dots, a_{m-1} \in M$ such that $\mathcal{M} \models \phi(a_0, \dots, a_{m-1})$,

$$\phi(\bar{a}_0, \dots, \bar{a}_{m-1}).$$

Since S is a satisfaction class for Σ_n-formulae (and hence Π_n-formula) of $\mathcal{M}$ in $\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$, $T \subseteq \mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$ is $\Sigma_1(\mathcal{L}^*_{\mathsf{S}})$ over $\mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$. Let $T_0 \subseteq T$ be such that $T_0 \in \mathbb{C}\mathrm{ov}^n_{\mathcal{M}}$. Using Lemma 4.13, let $c \in M$ be such that

$$c^* = \{a \in M \mid \bar{a} \text{ is mentioned in } T_0\}.$$

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Interpreting each $\bar{a}$ that is mentioned in T_0 by $a \in M$ and interpreting $\mathbf{c}$ by c, we expand $\mathcal{M}$ to a model of T_0. Therefore, by the Barwise Compactness theorem, there exists $\mathcal{N} \models T$. The $\mathcal{L}$-reduct of $\mathcal{N}$ is the desired extension of $\mathcal{M}$. ⊣

§5. Well-founded models of collection. In this section we use Theorem 4.15 to show that for all $n \geq 1$, $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \Pi_{n+1}\text{-}\mathsf{Foundation}$ proves $\Sigma_{n+1}\text{-}\mathsf{Separation}$. In particular, the theories $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection}$ and $\mathsf{M} +$ Strong $\Pi_n\text{-}\mathsf{Collection}$ have the same well-founded models.

In order to be able to apply Theorem 4.15 to countable models of $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \Pi_{n+1}\text{-}\mathsf{Foundation}$, we first need to show that $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \Pi_{n+1}\text{-}\mathsf{Foundation}$ proves $\Sigma_{n+1}\text{-}\mathsf{Foundation}$. The proof presented here generalises the argument presented in [5, Section 3] showing that $\mathsf{KP}^{\mathcal{P}}$ proves $\Sigma_1^{\mathcal{P}}\text{-}\mathsf{Foundation}$.

DEFINITION 5.1. Let $\phi(x, y, \vec{z})$ be an $\mathcal{L}$-formula. Define $\delta^\phi(a, b, f)$ to be the $\mathcal{L}$-formula:

$$(a \in \omega) \wedge (f \text{ is a function}) \wedge \mathsf{dom}(f) = a + 1 \wedge f(0) = \{b\} \wedge \\ (\forall u \in \omega)\begin{pmatrix}(\forall x \in f(u))(\exists y \in f(u+1))\phi(x, y, \vec{z}) \\ (\forall y \in f(u+1))(\exists x \in f(u))\phi(x, y, \vec{z})\end{pmatrix}.$$

Define $\delta_\omega^\phi(b, f, \vec{z})$ to be the $\mathcal{L}$-formula:

$$(f \text{ is a function}) \wedge \mathsf{dom}(f) = \omega \wedge f(0) = \{b\} \wedge \\ (\forall u \in \omega)\begin{pmatrix}(\forall x \in f(u))(\exists y \in f(u+1))\phi(x, y, \vec{z}) \\ (\forall y \in f(u+1))(\exists x \in f(u))\phi(x, y, \vec{z})\end{pmatrix}.$$

Viewing $\vec{z}$ as parameters and letting $a \in \omega$, $\delta^\phi(a, b, f, \vec{z})$ says that f describes a family of directed paths of length $a + 1$ starting at b through the directed graph defined by $\phi(x, y, \vec{z})$. Similarly, viewing $\vec{z}$ as parameters, $\delta_\omega^\phi(b, f, \vec{z})$ says that f describes a family of directed paths of length ω starting at b through the directed graph defined by $\phi(x, y, \vec{z})$. Note that if $\phi(x, y, \vec{z})$ is Δ_0, then, in the theory M^-, both $\delta^\phi(a, b, f\vec{z})$ and $\delta_\omega^\phi(b, f, \vec{z})$ can be written as a Δ_0-formulae with parameter ω. Moreover, if $n \geq 1$ and $\phi(x, y, \vec{z})$ is a Σ_n-formula (Π_n-formula), then, in the theory $\mathsf{M}^- + \Pi_{n-1}\text{-}\mathsf{Collection}$, both $\delta^\phi(a, b, f\vec{z})$ and $\delta_\omega^\phi(b, f, \vec{z})$ can be written as a Σ_n-formulae (Π_n-formulae, respectively) with parameter ω.

The following generalises Rathjen's Δ_0-weak dependent choices scheme from [14]:

($\Delta_0\text{-}\mathsf{WDC}_\omega$) For all Δ_0-formulae, $\phi(x, y, \vec{z})$,

$$\forall \vec{z}(\forall x \exists y \phi(x, y, \vec{z}) \Rightarrow \forall w \exists f \delta_\omega^\phi(w, f, \vec{z}));$$

and for all $n \geq 1$,

($\Delta_n\text{-}\mathsf{WDC}_\omega$) for all Π_n-formulae, $\phi(x, y, \vec{z})$, and for all Σ_n-formulae, $\psi(x, y, \vec{z})$,

$$\forall \vec{z}(\forall x \forall y(\phi(x, y, \vec{z}) \iff \psi(x, y, \vec{z})) \Rightarrow (\forall x \exists y \phi(x, y, \vec{z}) \Rightarrow \forall w \exists f \delta_\omega^\phi(w, f, \vec{z}))).$$

The following is based on the proof of [14, Proposition 3.2].

THEOREM 5.1. *Let $n \in \omega$ with $n \geq 1$. The theory* $\mathsf{KP} + \Pi_{n-1}\text{-}\mathsf{Collection} + \Sigma_n\text{-}\mathsf{Foundation} + \Delta_{n+1}\text{-}\mathsf{WDC}_\omega$ *proves* $\Sigma_{n+1}\text{-}\mathsf{Foundation}$.

PROOF. Let T be the theory $\mathsf{KP} + \Pi_{n-1}\text{-}\mathsf{Collection} + \Sigma_n\text{-}\mathsf{Foundation} + \Delta_{n+1}\text{-}\mathsf{WDC}_\omega$. Assume, for a contradiction, that $\mathcal{M} = \langle M, \in^\mathcal{M}\rangle$ is such that $\mathcal{M} \models T$ and there is an instance of Σ_{n+1}-Foundation that is false in $\mathcal{M}$. Let $\phi(x, y, \vec{z})$ be a Π_n-formula and let $\vec{a} \in M$ be such that

$$\{x \mid \mathcal{M} \models \exists y \phi(x, y, \vec{a})\}$$

is nonempty and has no $\in$-minimal element. Let $b, d \in M$ be such that $\mathcal{M} \models \phi(b, d, \vec{a})$. Now,

$$\mathcal{M} \models \forall x \forall u \exists y \exists v(\phi(x, u, \vec{a}) \Rightarrow (y \in x) \wedge \phi(y, v, \vec{a})).$$

Therefore, $\mathcal{M} \models \forall x \exists y \theta(x, y, \vec{a})$ where $\theta(x, y, \vec{a})$ is the formula

$$x = \langle x_0, x_1 \rangle \wedge y = \langle y_0, y_1 \rangle \wedge (\phi(x_0, x_1, \vec{a}) \Rightarrow (y_0 \in x_0) \wedge \phi(y_0, y_1, \vec{a})).$$

So, $\theta(x, y, \vec{a})$ is Δ_{n+1}^T. Work inside $\mathcal{M}$. Using Δ_{n+1}-WDC_ω, let f be such that $\delta_\omega^\theta(\langle b, d \rangle, f, \vec{a})$. Note that Σ_n-Foundation implies that for all $n \in \omega$,

(i) $f(n) \neq \emptyset$;
(ii) for all $x \in f(n)$, $x = \langle x_0, x_1 \rangle$ and $\phi(x_0, x_1, \vec{a})$.

Therefore, for all $n \in \omega$,

$$\begin{array}{c}(\forall x \in f(n))(\exists y \in f(n+1))(x = \langle x_0, x_1 \rangle \wedge y = \langle y_0, y_1 \rangle \wedge y_0 \in x_0) \wedge \\ (\forall y \in f(n+1))(\exists x \in f(n))(x = \langle x_0, x_1 \rangle \wedge y = \langle y_0, y_1 \rangle \wedge y_0 \in x_0)\end{array}.$$

Let $B = \mathsf{TC}(\{b\})$. Set-Foundation implies that for all $n \in \omega$,

$$(\forall x \in f(n))(x = \langle x_0, x_1 \rangle \wedge x_0 \in B).$$

Let

$$A = \left\{ x \in B \,\middle|\, (\exists n \in \omega)(\exists z \in f(n)) \left(\exists y \in \bigcup z \right) (z = \langle x, y \rangle) \right\},$$

which is a set by Δ_0-Separation. Now, let $x \in A$. Therefore, there exists $n \in \omega$ and $z \in f(n)$ such that $z = \langle x, x_0 \rangle$. And, there exists $w \in f(n+1)$ such that $w = \langle y, y_0 \rangle$ and $y \in x$. But $y \in A$. So A is a set with no $\in$-minimal element, which is the desired contradiction. ⊣

The following refinement of Definition 5.1 will allow us to show that for $n \geq 1$, $\mathsf{M} + \Pi_n$-Collection $+ \Pi_{n+1}$-Foundation proves Δ_{n+1}-WDC_ω.

Definition 5.2. Let $\phi(x, y, \vec{z})$ be an $\mathcal{L}$-formula. Define $\eta^\phi(a, b, f, \vec{z})$ to the $\mathcal{L}$-formula:

$$(\forall u \in a) \exists \alpha \exists X \left(\begin{array}{c} \delta^\phi(a, b, f, \vec{z}) \wedge \\ (\alpha \text{ is an ordinal}) \wedge (X = V_\alpha) \wedge \\ (\forall x \in f(u+1))(x \in X) \\ (\forall y \in X)(\forall x \in f(u))(\phi(x, y, \vec{z}) \Rightarrow y \in f(u+1)) \wedge \\ (\forall \beta \in \alpha)(\forall Y \in X) \left(\begin{array}{c} Y = V_\beta \Rightarrow \\ (\exists x \in f(u))(\forall y \in Y) \neg \phi(x, y, \vec{z}) \end{array} \right) \end{array} \right).$$

The formula $\eta^\phi(a, b, f, \vec{z})$ says that f is a function with domain $a+1$ and for all $u \in a$, $f(u+1)$ is the set of $y \in V_\alpha$ such that there exists $x \in f(u)$ with $\phi(x, y, \vec{z})$ and α is least such that for all $x \in f(u)$, there exists $y \in V_\alpha$ such that $\phi(x, y, \vec{z})$. In the theory $\mathsf{M} + \Pi_1$-Collection $+ \Pi_2$-Foundation, '$X = V_\alpha$' can be expressed as both a Σ_2-formula and a Π_2-formula. If $n \geq 1$ and, for given parameters $\vec{c}$, $\phi(x, y, \vec{c})$ is

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equivalent to both a Σ_{n+1}-formula and a Π_{n+1}-formula, then, in the theory $\mathsf{M}+\Pi_n$-Collection + Π_2-Foundation, $\eta^\phi(a,b,f,\vec{z})$ is equivalent to a Σ_{n+1}-formula.

THEOREM 5.2. *Let $n \in \omega$ with $n \geq 1$. The theory* $\mathsf{M}+\Pi_n$-Collection + Π_{n+1}-Foundation *proves* Δ_{n+1}-WDC_ω.

PROOF. Work in the theory $\mathsf{M}+\Pi_n$-Collection + Π_{n+1}-Foundation. Let $\phi(x,y,\vec{z})$ be a Π_{n+1}-formula. Let $\vec{a}, b$ be sets and let $\theta(x,y,\vec{z})$ be a Σ_{n+1}-formula such that

$$\forall x \forall y(\phi(x,y,\vec{a}) \iff \theta(x,y,\vec{a})).$$

We begin by claiming that for all $m \in \omega$, $\exists f \eta^\phi(m,b,f,\vec{a})$. Assume, for a contradiction, that this does not hold. Using Π_{n+1}-Foundation, let $k \in \omega$ be least such that $\neg\exists f \eta^\phi(k,b,f,\vec{a})$. Since $k \neq 0$, there exists a function g with $\mathsf{dom}(g) = k$ and $\eta^\phi(k-1,b,g,\vec{a})$. Consider the class

$$\begin{aligned} A &= \{\alpha \in \mathsf{Ord} \mid \forall X(X = V_\alpha \Rightarrow (\forall x \in g(k-1))(\exists y \in X)\phi(x,y,\vec{a}))\} \\ &= \{\alpha \in \mathsf{Ord} \mid \exists X(X = V_\alpha \wedge (\forall x \in g(k-1))(\exists y \in X)\theta(x,y,\vec{a}))\}. \end{aligned}$$

Applying Σ_{n+1}-Collection to the formula $\theta(x,y,\vec{a})$ shows that A is nonempty. Moreover, Δ_{n+1}-Foundation ensures that there is a least element $\beta \in A$. Now, let

$$C = \{y \in V_\beta \mid (\exists x \in g(k-1))\phi(x,y,\vec{a})\},$$

which is a set by Δ_{n+1}-Separation. Let $f = g \cup \{\langle k, C\rangle\}$. Then f is such that $\eta^\phi(k,b,f,\vec{a})$, which contradicts our assumption that no such f exists. Therefore, for all $m \in \omega$, $\exists f \eta^\phi(m,b,f,\vec{a})$. Using Σ_{n+1}-Collection, let D be such that $(\forall m \in \omega)(\exists f \in D)\eta^\phi(m,b,f,\vec{a})$. Note that for all $m \in \omega$ and for all functions f and g, if $\eta^\phi(m,b,f,\vec{a})$ and $\eta^\phi(m,b,g,\vec{a})$, then $f = g$. Now, let

$$h = \{\langle m, X\rangle \in \omega \times \mathsf{TC}(D) \mid (\exists f \in D)(\eta^\phi(m,b,f,\vec{a}) \wedge f(m) = X)\}.$$

Since

$$h = \{\langle m, X\rangle \in \omega \times \mathsf{TC}(D) \mid (\forall f \in D)(\eta^\phi(m,b,f,\vec{a}) \Rightarrow f(m) = X)\},$$

h is a set by Δ_{n+1}-Separation. Now, h is the function required by Δ_{n+1}-WDC_ω. ⊣

Note Π_{n+1}-Foundation is only used in the proof of Theorem 5.2 to find the least element of a Π_{n+1}-definable subclass of naturals numbers. Therefore, the proof of Theorem 5.2 also yields the following result.

THEOREM 5.3. *Let $n \in \omega$ with $n \geq 1$. Let $\mathcal{M}$ be an ω-standard model of* $\mathsf{M}+\Pi_n$-Collection + Π_2-Foundation. *Then*

$$\mathcal{M} \models \Delta_{n+1}\text{-}\mathsf{WDC}_\omega.$$

Note that Π_2-Foundation coupled with Π_1-Collection ensures that the function $\alpha \mapsto V_\alpha$ is total.

Combining Theorem 5.1 with Theorems 5.2 and 5.3 yields:

COROLLARY 5.4. *Let $n \in \omega$ with $n \geq 1$. The theory* $\mathsf{M}+\Pi_n$-Collection + Π_{n+1}-Foundation *proves* Σ_{n+1}-Foundation.

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COROLLARY 5.5. *Let $n \in \omega$ with $n \geq 2$. Let $\mathcal{M}$ be an ω-standard model of* $\mathsf{M} + \Pi_n\text{-Collection}$. *Then*

$$\mathcal{M} \models \Sigma_{n+1}\text{-Foundation}.$$

The proof of [5, Theorem 3.11] shows how the use of the cumulative hierarchy can be avoided in the argument used in the proof of Theorem 5.2. The following is [5, Corollary 3.12] combined with [13, Proposition Scheme 6.12] and provides a version of Corollary 5.5 when $n = 1$.

THEOREM 5.6. *Let $\mathcal{M}$ be an ω-standard model of* $\mathsf{MOST} + \Pi_1\text{-Collection}$. *Then*

$$\mathcal{M} \models \Sigma_2\text{-Foundation}.$$

Equipped with these results, we are now able to show that, in the theory $\mathsf{M} + \Pi_n\text{-Collection}$, $\Pi_{n+1}\text{-Foundation}$ implies $\Sigma_{n+1}\text{-Separation}$.

LEMMA 5.7. *Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ and $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M}, \mathcal{N} \models \mathsf{M}$. If $\mathcal{M} \prec_{e,1} \mathcal{N}$, then $\mathcal{M} \subseteq_e^{\mathcal{P}} \mathcal{N}$.*

PROOF. Assume that $\mathcal{M} \prec_{e,1} \mathcal{N}$. Let $x \in M$ and let $y \in N$ with $\mathcal{N} \models (y \subseteq x)$. We need to show that $y \in M$. Let $a \in M$ be such that $\mathcal{M} \models (a = \mathcal{P}(x))$. Therefore, $\mathcal{M} \models \theta(x, a)$ where $\theta(x, a)$ is the Π_1-formula

$$\forall z(z \subseteq x \iff z \in a).$$

So, $\mathcal{N} \models \theta(x, a)$. Therefore, $\mathcal{N} \models (y \in a)$ and so $y \in M$. ⊣

As alluded to in [13, Remark 3.21], the theory $\mathsf{KP} + \Sigma_1\text{-Separation}$ is capable of endowing any well-founded partial order with a ranking function.

LEMMA 5.8. *The theory* $\mathsf{KP} + \Sigma_1\text{-Separation}$ *proves that if $\langle X, R \rangle$ is a well-founded strict partial order, then there exists an ordinal γ and a function $h : X \longrightarrow \gamma$ such that for all $x, y \in X$, if $\langle x, y \rangle \in R$, then $h(x) < h(y)$.*

PROOF. Work in the theory $\mathsf{KP} + \Sigma_1\text{-Separation}$. Let X be a set and $R \subseteq X \times X$ be such that $\langle X, R \rangle$ is a well-founded strict partial order. Let $\theta(x, g, X, R)$ be the conjunction of the following clauses:

(i) g is a function;
(ii) $\mathsf{rng}(g)$ is a set of ordinals;
(iii) $\mathsf{dom}(g) = \{y \in X \mid \langle y, x \rangle \in R \vee y = x\}$;
(iv) $(\forall y, z \in \mathsf{dom}(g))(\langle y, z \rangle \in R \Rightarrow g(y) < g(z))$;
(v) $(\forall y \in \mathsf{dom}(g))(\forall \alpha \in g(y))(\exists z \in X)(\langle z, y \rangle \in R \wedge g(z) \geq \alpha)$.

Note that $\theta(x, g, X, R)$ can be written as a Δ_0-formula. Moreover, for all $x \in X$ and functions g_0 and g_1, if $\theta(x, g_0, X, R)$ and $\theta(x, g_1, X, R)$, then $g_0 = g_1$. And, if $x, y \in X$ with $\langle x, y \rangle \in R$ and g_0 and g_1 are functions with $\theta(y, g_0, X, R)$ and $\theta(x, g_1, X, R)$, then $g_0 = g_1 \upharpoonright \mathsf{dom}(g_0)$. Now, consider

$$A = \{x \in X \mid \neg \exists g \theta(x, g, X, R)\},$$

which is a set by Π_1-Separation. Assume, for a contradiction, that $A \neq \emptyset$. Let $x_0 \in A$ be R-minimal. Let $B = \{y \in X \mid \langle y, x_0 \rangle \in R\}$. Using Δ_0-Collection, let C_0 be such that $(\forall y \in B)(\exists g \in C_0)\theta(y, g, X, R)$. Let

$$D = \{g \in C_0 \mid (\exists y \in B)\theta(y, g, X, R)\}.$$

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Let

$$\beta = \sup\{g(y)+1 \mid y \in B \text{ and } g \in D \text{ with } y \in \mathsf{dom}(g)\}.$$

Then $f = \bigcup D \cup \{\langle x_0, \beta\rangle\}$ is such that $\theta(x_0, f, X, R)$, which contradicts the fact that $x_0 \in A$. Therefore, $A = \emptyset$. Using Δ_0-Collection, let C_1 be such that $(\forall x \in X)(\exists g \in C_1)\theta(x, g, X, R)$. Let

$$F = \{g \in C_1 \mid (\exists x \in X)\theta(x, g, X, R)\}.$$

Then $h = \bigcup F$ is the function we require. ⊣

THEOREM 5.9. *Let $n \in \omega$ with $n \geq 1$. The theory* $\mathsf{M} + \Pi_n$-Collection $+ \Pi_{n+1}$-Foundation *proves* Σ_{n+1}-Separation.

PROOF. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}}\rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_n$-Collection $+ \Pi_{n+1}$-Foundation. Let $\theta(x, y, \vec{z})$ be a Π_n-formula and let $b, \vec{a} \in M$. We need to show that $A = \{x \in b \mid \exists y\theta(x, y, \vec{a})\}$ is a set in $\mathcal{M}$. By Corollary 5.4, $\mathcal{M} \models \Sigma_{n+1}$-Foundation. Using Theorem 4.15, let $\mathcal{N} = \langle N, \in^{\mathcal{N}}\rangle$ be such that $\mathcal{M} \prec_{e,n} \mathcal{N}$, $\mathcal{N} \models \mathsf{M} + \Pi_n$-Collection $+ \Pi_{n+1}$-Foundation and there exists $d \in N$ such that for all $x \in M$, $\mathcal{N} \models (x \in d)$. Let $\alpha \in \mathsf{Ord}^{\mathcal{N}}$ be such that for all $x \in M$, $\mathcal{M} \models (x \in V_\alpha)$.

Work inside $\mathcal{N}$. Let

$$D = \{x \in b \mid (\exists y \in V_\alpha)\theta(x, y, \vec{a})\},$$

which is a set by Π_n-Separation. Let

$$g = \left\{\langle x, \beta\rangle \in D \times \alpha \;\middle|\; \begin{array}{l}(\exists y \in V_\alpha)(\rho(y) = \beta \wedge \theta(x, y, \vec{a}))\wedge \\ (\forall z \in V_\alpha)(\phi(x, z, \vec{a}) \Rightarrow \beta \leq \rho(z))\end{array}\right\},$$

which is a set by Δ_{n+1}-Separation. Moreover, g is a function. Let $\lhd = \{\langle x_0, x_1\rangle \in D \times D \mid g(x_0) < g(x_1)\}$. Note that $\lhd$ is a well-founded strict partial order on D.

Since $\mathcal{M} \subseteq_e^{\mathcal{P}} \mathcal{N}$, $D, \lhd \in M$. Moreover,

$$\mathcal{M} \models (\lhd \text{ is a well-founded strict partial order on } D).$$

Work inside $\mathcal{M}$. Since $\mathcal{M} \prec_{e,n} \mathcal{N}$, for all $x \in b$, if $\exists y\theta(x, y, \vec{a})$, then $x \in D$. And, for all $x_0, x_1 \in D$, if $\exists y\theta(x_0, y, \vec{a})$ and $\neg\exists y\theta(x_1, y, \vec{a})$, then $x_0 \lhd x_1$. Using Lemma 5.8, let γ be an ordinal and let $h : D \longrightarrow \gamma$ be such that for all $x_0, x_1 \in D$, if $\langle x_0, x_1\rangle \in D$, then $h(x_0) < h(x_1)$. Consider the class

$$B = \{\beta \in \gamma \mid (\exists x \in D)(h(x) = \beta \wedge \neg\exists y\theta(x, y, \vec{a}))\}.$$

If B is empty, then $D = \{x \in b \mid \exists y\phi(x, y, \vec{a})\}$ and we are done. Therefore, assume that B is nonempty. So, by Π_{n+1}-Foundation, B has a least element ξ. Let $D_\xi = \{x \in D \mid h(x) < \xi\}$. Let $x \in D_\xi$. Since ξ is the least element of B and $h(x) < \xi$, $\exists y\theta(x, y, \vec{a})$. Conversely, let $x \in b$ be such that $\exists y\theta(x, y, \vec{a})$. Let $x_0 \in D$ be such $h(x_0) = \xi$ and $\neg\exists y\theta(x_0, y, \vec{a})$. Since $\exists y\theta(x, y, \vec{a})$, it must be the case that $h(x) < h(x_0) = \xi$. So, $x \in D_\xi$. This shows that $D_\xi = \{x \in b \mid \exists y\theta(x, y, \vec{a})\}$. Therefore, Σ_{n+1}-Separation holds in $\mathcal{M}$. ⊣

Gostanian [8] notes that the techniques he uses to compare the heights of minimum models of subsystems of ZF without the powerset axiom do not apply to subsystems that include the powerset axiom. Theorem 5.9 settles the relationship

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between the heights of the minimum models of the theories $\mathsf{M} + \Pi_n$-Collection and M + Strong Π_n-Collection for all $n \geq 1$.

COROLLARY 5.10. *Let $n \in \omega$ with $n \geq 1$. The theories* $\mathsf{M} + \Pi_n$-Collection *and* M + Strong Π_n-Collection *have the same transitive models. In particular, the minimum models* $\mathsf{M} + \Pi_n$-Collection *and* M + Strong Π_n-Collection *coincide.*

The results of [12] show that for all $n \geq 1$, M + Strong Π_n-Collection proves the consistency of $\mathsf{M} + \Pi_n$-Collection. Theorem 5.9 yields the following.

COROLLARY 5.11. *Let $n \in \omega$ with $n \geq 1$. The theory* M + Strong Π_n-Collection *does not prove the existence of a transitive model of* $\mathsf{M} + \Pi_n$-Collection.

The following example shows that the statement of Theorem 5.9 with $n = 0$ does not hold.

EXAMPLE 5.1. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be an ω-standard model of ZFC in which there is a countable ordinal that is nonstandard. Note that such a model can built from a transitive model of ZFC using, for example, [10, Theorem 2.4], or using the Barwise Compactness theorem as in [11, Lemma 7.2]. Let W be the transitive set that is isomorphic to the well-founded part of $\mathcal{M}$. Then, by [6, Theorem 2.3], $\langle W, \in \rangle$ satisfies $\mathsf{KP}^{\mathcal{P}}$ + Foundation. However, there are well-orderings of ω in $\langle W, \in \rangle$ that are not isormorphic to any ordinal in $\langle W, \in \rangle$, so $\langle W, \in \rangle$ does not satisfy Σ_1-Separation.

The following is a consequence of [8, Theorems 2.1 and 2.2] and shows that the presence of Powerset is essential in Theorem 5.9.

THEOREM 5.12 (Gostanian). *Let $n \in \omega$. Let α be the least ordinal such that $\langle L_\alpha, \in \rangle \models \mathsf{KP} + \Pi_n$-Collection. Then $\langle L_\alpha, \in \rangle$ does not satisfy Σ_{n+1}-Separation.*

In [15, Theorem 4.6] (see also [7, Theorem 4.15]), Ressayre shows that for all $n \in \omega$, the theory $\mathsf{KP} + \mathsf{V} = \mathsf{L} + \Pi_n$-Collection $+ \Sigma_{n+1}$-Foundation does not prove Π_{n+1}-Foundation. Ressayre's construction can be adapted (as noted in [15, Theorem 4.15]) to show that for all $n \geq 1$, $\mathsf{M} + \Pi_n$-Collection $+ \Sigma_{n+1}$-Foundation does not prove Π_{n+1}-Foundation. Since $\mathsf{M} + \Sigma_{n+1}$-Separation proves, Π_{n+1}-Foundation, this shows that $\mathsf{M} + \Pi_n$-Collection $+ \Sigma_{n+1}$-Foundation does not prove Σ_{n+1}-Separation.

THEOREM 5.13 (Ressayre). *Let $n \in \omega$ with $n \geq 1$. The theory* $\mathsf{M} + \Pi_n$-Collection $+ \Sigma_{n+1}$-Foundation *does not prove* Π_{n+1}-Foundation.

PROOF. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be a nonstandard ω-standard model of $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$. Let $\delta \in \mathrm{Ord}^{\mathcal{M}}$ be nonstandard. Let $I \subseteq (\delta + \delta)^*$ be an initial segment of $(\delta + \delta)^*$ such that $\delta \in I$ and $(\delta + \delta)^* \backslash I$ has no least element.

Work inside $\mathcal{M}$. Define a function f with domain $\delta + \delta$ such that

$f(0) = V_\gamma$	where γ is least such that V_γ is a Σ_n-elementary substructure of the universe;
$f(\alpha + 1) = V_\gamma$	where γ is least such that $f(\alpha) \in V_\gamma$ and V_γ is a Σ_n-elementary substructure of the universe;
$f(\beta) = \bigcup_{\alpha \in \beta} f(\alpha)$	if β is a limit ordinal.

Now, working in the metatheory again, define $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ by:

$$N = \bigcup_{\alpha \in I} f(\alpha)^* \text{ and } \in^{\mathcal{N}} \text{ is the restriction of } \in^{\mathcal{M}} \text{ to } N.$$

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Therefore, $\mathcal{N} \prec_{e,n} \mathcal{M}$ and $\mathsf{Ord}^{\mathcal{M}} \backslash \mathsf{Ord}^{\mathcal{N}}$ has no least element. It is clear that $\mathcal{N}$ is ω-standard and satisfies $\mathsf{M} + \mathsf{AC}$. We claim that $\mathcal{N}$ satisfies Strong Δ_0-Collection. Let $\phi(x, y, \vec{z})$ be a Δ_0-formula, and let $b, \vec{a} \in N$. Let $\alpha \in \mathsf{Ord}^{\mathcal{N}}$ be such that $V_\alpha^{\mathcal{M}} \in N$, $b, \vec{a} \in (V_\alpha^{\mathcal{M}})^*$ and $\langle (V_\alpha^{\mathcal{M}})^*, \in^{\mathcal{N}} \rangle \prec_{e,1} \mathcal{N}$. But then

$$\mathcal{N} \models (\forall x \in b)(\exists y \phi(x, y, \vec{a}) \Rightarrow (\exists y \in V_\alpha)\phi(x, y, \vec{a})).$$

This shows that $\mathcal{N}$ satisfies Strong Δ_0-Collection. So, $\mathcal{N} \models \mathsf{MOST} + \mathsf{V} = \mathsf{L}$. Therefore, by Theorem 3.1,

$$\mathcal{N} \models \mathsf{MOST} + \Pi_n\text{-}\mathsf{Collection}.$$

And, by Theorem 5.6 ($n = 1$) and Corollary 5.5 ($n > 1$),

$$\mathcal{N} \models \Sigma_{n+1}\text{-}\mathsf{Foundation}.$$

Note that 'X is Σ_n-elementary submodel of the universe', which we abbreviate $X \prec_n \mathbb{V}$, can be expressed as

$$(\forall x \in X^{<\omega})(\forall m \in \omega)(\mathsf{Sat}_{\Sigma_n}(m, x) \Rightarrow \langle X, \in \rangle \models \mathsf{Sat}_{\Sigma_n}(m, x)),$$

and is equivalent to a Π_n-formula. Now, consider the formula $\theta(\alpha)$ defined by

$$\exists f \left(\begin{array}{c} (f \text{ is a function}) \wedge \mathsf{dom}(f) = \alpha \wedge \\ \exists X \exists \beta (X = V_\beta \wedge X \prec_n \mathbb{V} \wedge f(0) = X \wedge (\forall Y, \gamma \in X)(Y = V_\gamma \Rightarrow \neg(Y \prec_n \mathbb{V}))) \wedge \\ (\forall \eta \in \alpha) \left(\eta = \xi + 1 \Rightarrow \exists X \exists \beta \left(\begin{array}{c} X = V_\beta \wedge X \prec_n \mathbb{V} \wedge f(\eta) = X \wedge f(\xi) \in X \wedge \\ (\forall Y, \gamma \in X)(Y \neq V_\gamma \vee \neg(Y \prec_n \mathbb{V}) \vee f(\xi) \notin Y) \end{array} \right) \right) \\ \wedge (\forall \eta \in \alpha) \left((\eta \text{ is a limit ordinal}) \Rightarrow f(\eta) = \bigcup_{\xi \in \eta} f(\xi) \right) \end{array} \right).$$

Note that $\theta(\alpha)$ can be expressed as a Σ_{n+1}-formula and says that there exists a function that enumerates the first α levels of the cumulative hierarchy that are Σ_n-elementary submodels of the universe. Since $\delta \in I$ and $I \subseteq (\delta + \delta)^*$, $\mathsf{Ord}^{\mathcal{N}} \neq I$. Therefore, the class

$$A = \{\alpha \in \mathsf{Ord}^{\mathcal{N}} \mid \neg\theta(\alpha)\} = \mathsf{Ord}^{\mathcal{N}} \backslash I$$

is nonempty and has no least element, so Π_{n+1}-Foundation fails in $\mathcal{N}$. ⊣

§6. Questions. The use of Theorem 4.15 to prove Theorem 5.9 raises the following.

QUESTION 6.1. *Is there a direct argument that* $\mathsf{M} + \Pi_n$-Collection $+ \Pi_{n+1}$-Foundation *proves* Σ_{n+1}-Separation *that does not go via an end extensions*?

Kaufmann [9, p. 102] asks:

QUESTION 6.2. *If L_α has a Σ_2-elementary end extension, does it necessarily have a Σ_2-elementary end extension that satisfies* Δ_0-Collection?

A more general form of Question 6.2 is asked by Clote [4, p. 39] in the context of arithmetic. The following is the set-theoretic analog of Clote's question:

QUESTION 6.3. *Let $n \geq 1$. Does every countable model of* $\mathsf{KP} + \Pi_n$-Collection *have a Σ_{n+1}-elementary end extension that satisfies* $\mathsf{KP} + \Pi_{n-1}$-Collection?

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Sun [16] has recently provided a positive answer to Clote's original question about end extensions of subsystems of arithmetic.

One wonders if the requirement that $\mathcal{M}$ satisfies Σ_{n+1}-Foundation in Theorem 4.15 is necessary. In particular:

QUESTION 6.4. *Let* $n \geq 1$. *Does every countable model of* $\mathsf{KP} + \Pi_n$-Collection *have a* Σ_n*-elementary end extension that satisfies* $\mathsf{KP} + \Pi_n$-Collection?

And, if Question 6.4 has a negative answer, then:

QUESTION 6.5. *Let* $n \geq 1$. *Does every countable model of* $\mathsf{M} + \Pi_n$-Collection *have a* Σ_n*-elementary end extension that satisfies* $\mathsf{M} + \Pi_n$-Collection?

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHESTER
CHESTER CH1 4BJ
UNITED KINGDOM
E-mail: z.mckenzie@chester.ac.uk

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