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ARITHMETIC PROPERTIES OF COEFFICIENTS OF THE MOCK THETA FUNCTION $B(q)$

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Abstract

We investigate the arithmetic properties of the second-order mock theta function $B(q)$ and establish two identities for the coefficients of this function along arithmetic progressions. As applications, we prove several congruences for these coefficients.

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1. Introduction

In his last letter to Hardy [23, pages xxxi–xxxxii, 354–355], Ramanujan introduced several families of mock theta functions. Since then, properties of these functions have been widely studied. One important direction involves identities between mock theta functions and generalised Lambert series (see [2, 5, 9, 18, 19]). Recently, using the theory of modular forms, Zwegers [27] and Bringmann and Ono [10–12] proved that mock theta functions are the holomorphic parts of certain nonholomorphic modular forms (see [21, 26] for more details).

We study arithmetic properties of the coefficients $f(n)$ of the second-order mock theta function

$$B(q) := \sum_{n\geq 0} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}} =: \sum_{n=0}^{\infty} f(n)q^n.$$

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In the equation above and for the rest of this paper, we use the notation

$$(x_1, x_2, \ldots, x_k)_m = (x_1, x_2, \ldots, x_k; q)_m := \prod_{n=0}^{m-1}(1 - x_1q^n)(1 - x_2q^n)\cdots(1 - x_kq^n),$$

$$(x_1, x_2, \ldots, x_k)_\infty = (x_1, x_2, \ldots, x_k; q)_\infty := \prod_{n=0}^{\infty}(1 - x_1q^n)(1 - x_2q^n)\cdots(1 - x_kq^n),$$

$$[x_1, x_2, \ldots, x_k]_\infty = [x_1, x_2, \ldots, x_k; q]_\infty := (x_1, q/x_1, x_2, q/x_2, \ldots, x_k, q/x_k; q)_\infty,$$

$$j(x; q) := (x)_\infty(q/x)_\infty(q)_\infty,$$

$$J_{a,b} := (q^a, q^{b-a}, q^b; q^b),$$

$$J_b := (q^b; q^b)_\infty,$$

and we require $|q| < 1$ for absolute convergence.

Modular transformation formulas for $B(q)$ were first studied in [1]. Also, in [1, (4.3)], the function $B(q)$ is expressed in terms of a generalised Lambert series: that is,

$$B(q) = \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+2n}}{1 - q^{2n+1}}.$$

Armed with the above identity, Gordon and McIntosh [17] proved that

$$\frac{B(q) + B(-q)}{2} = (q^4; q^4)_\infty(-q^2; q^2)^4_\infty. \tag{1.1}$$

It is easy to see that (1.1) is equivalent to

$$\sum_{n=0}^{\infty} f(2n)q^n = \frac{(q^2; q^2)_\infty}{(q, q; q^2)^2_\infty}. \tag{1.2}$$

In [14], Chan and the author found that

$$\sum_{n=0}^{\infty} f(4n + 2)q^n = 4\frac{(q^4; q^4)^4_\infty}{(q; q)^3_\infty(q, q^3; q^4)^2_\infty} \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} f(4n + 1)q^n = 2\frac{(-q; q^2)^2_\infty(q^4; q^4)^4_\infty}{(q; q)^3_\infty(q, q^3; q^4)^4_\infty(-q^2; q^2)^2_\infty}. \tag{1.4}$$

The first objective of this paper is to give analogues of (1.2), (1.3) and (1.4) modulo 6.

Theorem 1.1.

$$\sum_{n=0}^{\infty} f(6n + 2)q^n = \frac{4(q^6; q^6)_\infty}{[q; q^6]^{10}_\infty[q^3; q^6]^4_\infty} \tag{1.5}$$

and

$$\sum_{n=0}^{\infty} f(6n + 4)q^n = \frac{9(q^6; q^6)_\infty}{[q; q^6]^8_\infty[q^2; q^6]^4_\infty[q^3; q^6]^2_\infty}. \tag{1.6}$$

In particular, $f(6n + 2) \equiv 0 \pmod 4$ *and* $f(6n + 4) \equiv 0 \pmod 9$.

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Congruences for the coefficients of mock theta functions have been studied widely. Using the theory of (mock) modular forms and Sturm's theorem, Waldherr [24] considered the third-order mock theta function

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} =: \sum_{n=0}^{\infty} a_\omega(n) q^n$$

and established the congruences

$$a_\omega(40n+27) \equiv a_\omega(40n+35) \equiv 0 \pmod 5. \tag{1.7}$$

Wang [25] established many more congruences for $a_\omega(n)$ by using identities for the coefficients of $\omega(q)$ in arithmetic progressions. In particular, he proved that

$$\sum_{n=0}^{\infty} a_\omega(8n+3)q^n = 4\frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^9}. \tag{1.8}$$

One can easily deduce (1.7) from (1.8).

Applying identities on the coefficients in arithmetic progressions, we prove similar congruences for $f(n)$. For example, we see that (1.3) and (1.4) give

$$f(4n+2) \equiv 0 \pmod 4$$

and

$$f(4n+1) \equiv 0 \pmod 2. \tag{1.9}$$

Generalising (1.9), Qu, Wang and Yao [22] proved that $f(2n+1) \equiv 0 \pmod 2$. For more on congruences from mock theta functions, see [4, 6, 16, 20]. Based on (1.2), (1.5) and (1.6), we give some further congruences for $f(n)$.

Corollary 1.2.

$$f(10n+6) \equiv f(10n+8) \equiv 0 \pmod 5, \tag{1.10}$$

$$f(12n+8) \equiv 0 \pmod 8, \tag{1.11}$$

$$f(30n+8) \equiv f(30n+26) \equiv 0 \pmod{20}, \tag{1.12}$$

$$f(12n+10) \equiv 0 \pmod{18}. \tag{1.13}$$

2. Proof of Theorem 1.1

We need to find the 3-dissection of $(q^2;q^2)_\infty/(q;q^2)_\infty^4$.

Lemma 2.1 [13, Lemma 2.1]. *We have*

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{J_{18}^{12}}{J_{3,18}^8 J_{6,18}^4 J_{9,18}} + \frac{2qJ_{18}^{12}}{J_{3,18}^7 J_{6,18}^4 J_{9,18}^2} + \frac{4q^2 J_{18}^{12}}{J_{3,18}^6 J_{6,18}^4 J_{9,18}^3}. \tag{2.1}$$

Lemma 2.2. *We have*

$$\frac{(q^2;q^2)_\infty^2}{(q,q;q^2)_\infty} = \frac{J_{9,18}J_{18}^3}{J_{3,18}^2} + 2q\frac{J_{18}^3}{J_{3,18}} + q^2\frac{J_{18}^3}{J_{9,18}}. \tag{2.2}$$

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PROOF. Recall [8, Equation (2.1)] that

$$\frac{(q;q)_\infty^2}{(q/x;q)_\infty(x;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{1-xq^k}. \tag{2.3}$$

Replacing q by q^2 and setting $x = q$ in (2.3),

$$\frac{(q^2;q^2)_\infty^2}{(q,q;q^2)_\infty} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{k(k+1)}}{1-q^{2k+1}} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{k(k+1)}(1+q^{2k+1}+q^{4k+2})}{1-q^{6k+3}}. \tag{2.4}$$

Splitting the infinite sum on the right-hand side of (2.4) into three sums according to the value of k modulo 3, we find that

$$\begin{aligned}\frac{(q^2;q^2)_\infty^2}{(q,q;q^2)_\infty} = \sum_{k=-\infty}^{\infty} &\frac{(-1)^k q^{3k(3k+1)}(1+q^{6k+1}+q^{12k+2})}{1-q^{18k+3}} \\ &- \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(3k+1)(3k+2)}(1+q^{6k+3}+q^{12k+6})}{1-q^{18k+9}} \\ &+ \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(3k+2)(3k+3)}(1+q^{6k+5}+q^{12k+10})}{1-q^{18k+15}}. \end{aligned}\tag{2.5}$$

Rewrite (2.5) as

$$\frac{(q^2;q^2)_\infty^2}{(q,q;q^2)_\infty} = S_0 + qS_1 + q^2S_2,$$

where we define

$$\begin{aligned} S_0 &:= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+3k}}{1-q^{18k+3}} + \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+15k+6}}{1-q^{18k+15}}, \\ S_1 &:= \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+9k}}{1-q^{18k+3}} + \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+27k+15}}{1-q^{18k+15}}, \end{aligned}\tag{2.6}$$

$$\begin{aligned} S_2 := &\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+15k}}{1-q^{18k+3}} + \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+21k+9}}{1-q^{18k+15}} - \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+9k}}{1-q^{18k+9}} \\ &- \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+21k+6}}{1-q^{18k+9}} - \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+15k+3}}{1-q^{18k+9}}. \end{aligned}\tag{2.7}$$

Hence, it suffices to show that

$$S_0 = \frac{J_{9,18}J_{18}^3}{J_{3,18}^2}, \tag{2.8}$$

$$S_1 = 2\frac{J_{18}^3}{J_{3,18}}, \tag{2.9}$$

$$S_2 = \frac{J_{18}^3}{J_{9,18}}. \tag{2.10}$$

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Recall the following special case with $s = 2, r = 1$ of [13, Theorem 2.1]: that is,

$$\frac{[a]_\infty (q)_\infty^2}{[b_1, b_2]_\infty} = \frac{[a/b_1]_\infty}{[b_2/b_1]_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{1 - b_1 q^k} \left(\frac{a}{b_2}\right)^k + \frac{[a/b_2]_\infty}{[b_1/b_2]_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{1 - b_2 q^k} \left(\frac{a}{b_1}\right)^k. \tag{2.11}$$

Replacing q by q^{18} and setting $a = q^9, b_1 = q^3, b_2 = q^{15}$ in (2.11),

$$\frac{[q^9; q^{18}]_\infty (q^{18}; q^{18})_\infty^2}{[q^3; q^{18}]_\infty^2} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+3k}}{1 - q^{18k+3}} + \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+15k+6}}{1 - q^{18k+15}} = S_0,$$

which gives (2.8). Replacing the summation index k by $-k$ in the second sum on the right-hand side of (2.9) and simplifying gives

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+27k+15}}{1 - q^{18k+15}} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+9k}}{1 - q^{18k+3}}.$$

Thus, by (2.6),

$$S_1 = 2 \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+9k}}{1 - q^{18k+3}}. \tag{2.12}$$

Replacing q by q^{18} and setting $x = q^3$ in (2.3),

$$\frac{(q^{18}; q^{18})_\infty^2}{[q^3; q^{18}]_\infty} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+9k}}{1 - q^{18k+3}}. \tag{2.13}$$

Equations (2.12) and (2.13) imply (2.9). Proceeding as in the proof of (2.8) and (2.9), after applying (2.3) and (2.11),

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+15k}}{1 - q^{18k+3}} - \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+15k+3}}{1 - q^{18k+9}} = \frac{(q^{18}; q^{18})_\infty^2}{[q^9; q^{18}]_\infty}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+21k+9}}{1 - q^{18k+15}} - \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+21k+6}}{1 - q^{18k+9}} = \frac{(q^{18}; q^{18})_\infty^2}{[q^9; q^{18}]_\infty}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{9k^2+9k}}{1 - q^{18k+9}} = \frac{(q^{18}; q^{18})_\infty^2}{[q^9; q^{18}]_\infty}.$$

Substituting these three equations into (2.7) gives (2.10). □

Now we are in a position to prove (1.5) and (1.6). Note that

$$\sum_{n=0}^{\infty} f(2n) q^n = \frac{(q^2; q^2)_\infty}{(q, q; q^2)_\infty^2} = \frac{(q^2; q^2)_\infty^2}{(q, q; q^2)_\infty} \frac{(-q; q)_\infty}{(q; q)_\infty}. \tag{2.14}$$

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Substituting (2.1) and (2.2) into (2.14),

$$\sum_{n=0}^{\infty} f(2n)q^n = \left\{\frac{J_{9,18}J_{18}^3}{J_{3,18}^2} + 2q\frac{J_{18}^3}{J_{3,18}} + q^2\frac{J_{18}^3}{J_{9,18}}\right\} \times \left\{\frac{J_{18}^{12}}{J_{3,18}^8 J_{6,18}^4 J_{9,18}} + \frac{2qJ_{18}^{12}}{J_{3,18}^7 J_{6,18}^4 J_{9,18}^2} + \frac{4q^2J_{18}^{12}}{J_{3,18}^6 J_{6,18}^4 J_{9,18}^3}\right\}. \tag{2.15}$$

Extracting the terms in q^n where $n \equiv 2 \pmod 3$, we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} f(6n+4)q^{3n+2} &= \frac{J_{9,18}J_{18}^3}{J_{3,18}^2}\cdot\frac{4q^2J_{18}^{12}}{J_{3,18}^6 J_{6,18}^4 J_{9,18}^3} \\ &\quad + 2q\frac{J_{18}^3}{J_{3,18}}\cdot\frac{2qJ_{18}^{12}}{J_{3,18}^7 J_{6,18}^4 J_{9,18}^2} + q^2\frac{J_{18}^3}{J_{9,18}}\cdot\frac{J_{18}^{12}}{J_{3,18}^8 J_{6,18}^4 J_{9,18}} \\ &= 9q^2\frac{J_{18}^{15}}{J_{3,18}^8 J_{6,18}^4 J_{9,18}^2},\end{aligned}$$

which gives (1.6). Extracting terms with q^n where $n \equiv 1 \pmod 3$ in (2.15),

$$\begin{aligned}\sum_{n=0}^{\infty} f(6n+2)q^{3n+1} &= \frac{J_{9,18}J_{18}^3}{J_{3,18}^2}\cdot\frac{2qJ_{18}^{12}}{J_{3,18}^7 J_{6,18}^4 J_{9,18}^2} \\ &\quad + 2q\frac{J_{18}^3}{J_{3,18}}\cdot\frac{J_{18}^{12}}{J_{3,18}^8 J_{6,18}^4 J_{9,18}} + q^2\frac{J_{18}^3}{J_{9,18}}\cdot\frac{4q^2J_{18}^{12}}{J_{3,18}^6 J_{6,18}^4 J_{9,18}^3} \\ &= 4q\frac{J_{18}^{15}}{J_{3,18}^9 J_{6,18}^4 J_{9,18}} + 4q^4\frac{J_{18}^{15}}{J_{3,18}^6 J_{6,18}^4 J_{9,18}^4}.\end{aligned} \tag{2.16}$$

From the identity [15, Equation (3.1)], namely,

$$[A/b, A/c, A/d, A/e; q]_\infty - [b, c, d, e; q]_\infty = b[A, A/bc, A/bd, A/be; q]_\infty,$$

with q replaced by q^{18} and $(A, b, c, d, e) = (q^{15}, q^3, q^9, q^9, q^9)$, we find that

$$[q^6; q^{18}]_\infty^4 - [q^3; q^{18}]_\infty [q^9; q^{18}]_\infty^3 = q^3[q^3; q^{18}]_\infty^4.$$

Armed with the above equation, one can easily check that

$$\frac{1}{J_{3,18}^3} + \frac{q^3}{J_{9,18}^3} = \frac{J_{6,18}^4}{J_{3,18}^4 J_{9,18}^3},$$

which gives

$$4q\frac{J_{18}^{15}}{J_{3,18}^9 J_{6,18}^4 J_{9,18}} + 4q^4\frac{J_{18}^{15}}{J_{3,18}^6 J_{6,18}^4 J_{9,18}^4} = \frac{4qJ_{18}^{15}}{J_{3,18}^{10} J_{9,18}^4}.$$

This, together with (2.16), implies (1.5).

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3. Proof of Corollary 1.2

3.1. Proof of (1.10). By the binomial theorem,

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^4} = \frac{(q;q)_\infty}{(q;q^2)_\infty^5} \equiv \frac{(q;q)_\infty}{(q^5;q^{10})_\infty} = \frac{\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)/2}}{(q^5;q^{10})_\infty} \pmod 5,$$

where the last equality follows from Euler's pentagonal number theorem [3, page 11]: that is,

$$(q;q)_\infty = \sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)/2}.$$

Since $j(3j-1)/2 \equiv 0, 1, 2 \pmod 5$, the coefficients of q^{5n+3} and q^{5n+4} in the q-expansion of $(q^2;q^2)_\infty/(q;q^2)_\infty^4$ are multiples of 5. This, together with (1.2), gives (1.10).

3.2. Proof of (1.11) and (1.12). By the binomial theorem,

$$\frac{(q^6;q^6)_\infty}{[q;q^6]_\infty^{10}[q^3;q^6]_\infty^4} \equiv \frac{(q^6;q^6)_\infty}{[q^2;q^{12}]_\infty^5[q^6;q^{12}]_\infty^2} \pmod 2 \tag{3.1}$$

and

$$\frac{(q^6;q^6)_\infty}{[q;q^6]_\infty^{10}[q^3;q^6]_\infty^4} \equiv \frac{(q^6;q^6)_\infty[q^3;q^6]_\infty}{[q^{10};q^{60}]_\infty[q^{15};q^{30}]_\infty} \pmod 5. \tag{3.2}$$

By (3.1), the coefficients of q^{2n+1} in the q-expansion of $(q^6;q^6)_\infty/[q;q^6]_\infty^{10}[q^3;q^6]_\infty^4$ are all even integers. This, together with (1.5), gives (1.11).

Applying Jacobi's triple product identity [7, pages 33–36]

$$(-qz, -q/z, q^2;q^2)_\infty = \sum_{j=-\infty}^{\infty} z^j q^{j^2},$$

we find that

$$(q^3, q^3, q^6;q^6)_\infty = \sum_{j=-\infty}^{\infty}(-1)^j q^{3j^2}.$$

Thus, by (3.2),

$$\frac{(q^6;q^6)_\infty}{[q;q^6]_\infty^{10}[q^3;q^6]_\infty^4} \equiv \frac{\sum_{j=-\infty}^{\infty}(-1)^j q^{3j^2}}{[q^{10};q^{60}]_\infty[q^{15};q^{30}]_\infty} \pmod 5. \tag{3.3}$$

Since $3j^2 \equiv 0, 2$ or $3 \pmod 5$, the congruence (3.3) implies that the coefficients of q^n with $n \equiv 1, 4 \pmod 5$ in the q-expansion of $(q^6;q^6)_\infty/[q;q^6]_\infty^{10}[q^3;q^6]_\infty^4$ are multiples of 5. This, together with (1.5), proves (1.12).

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3.3. Proof of (1.13). From

$$\frac{(q^6;q^6)_\infty}{[q;q^6]_\infty^8[q^2;q^6]_\infty^4[q^3;q^6]_\infty^2} \equiv \frac{(q^6;q^6)_\infty}{[q^2;q^{12}]_\infty^4[q^4;q^{12}]_\infty^2[q^6;q^{12}]_\infty} \pmod 2,$$

the coefficients of q^{2n+1} in the q-expansion of $(q^6;q^6)_\infty/[q;q^6]_\infty^8[q^2;q^6]_\infty^4[q^3;q^6]_\infty^2$ are all even integers. This, together with (1.6), gives (1.13).

4. Concluding remarks

The referee pointed out that

$$\sum_{n=0}^{\infty} f(4n)q^n = \frac{(q^2;q^2)_\infty^{14}}{(q;q)_\infty^9(q^4;q^4)_\infty^4}. \tag{4.1}$$

Proceeding as in the proof of Theorem 1.1, one can prove (4.1) by making a 2-dissection of the infinite product in (1.2).

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