THE CLASS OF KRASNER HYPERFIELDS IS NOT ELEMENTARY

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Abstract. We show that the class of Krasner hyperfields is not elementary. To show this, we determine the rational rank of quotients of multiplicative groups in field extensions. We also discuss some related questions.

§1. Introduction. The notion of a *hyperfield* (*hypercorps*) was introduced first by Marc Krasner in [13] as a tool to study valued fields. In his later paper [14], he introduced the quotient construction of a hyperfield from a given field and a subgroup of its multiplicative group (see Theorem 2.2).

The question whether all hyperfields come from this quotient construction has been an open problem until the example of Massouros, who showed in [18] that it is not the case. Nevertheless, the class of *Krasner hyperfields* (i.e., hyperfields obtained by this quotient construction) contains a lot of known examples of hyperfields. Among them, there are the hyperfields known as RV-sorts, which were studied in the model theory of valued fields under the name of leading term structures (see, e.g., [8, 19]) as a tool to obtain (relative) quantifier elimination for valued fields. Joseph Flenner proved in [8] that RV-sorts are bi-interpretable with *amc*-structures (three sorted structures) introduced by Franz-Viktor Kuhlmann in [15]. Currently, hyperfields in the form of RV-sorts are one of the main objects used to study model theory of valued fields (see, e.g., [3, 16, 23, 24]).

In view of the usefulness of Krasner hyperfields for the model theory of valued fields discussed above, we were motivated to study model-theoretical properties of Krasner hyperfields themselves. Since the definition of Krasner hyperfields is purely algebraical, the first question we faced was: "Is the class of Krasner hyperfields elementary?". Based on the results of Alain Connes and Caterina Consani from [4], we show in this article that this class is *not* elementary.

The article is organized as follows. In Section 2, we collect the necessary facts and results about hyperfields. In Section 3, we use the notion of a rational rank to show that the class of Krasner hyperfields is not elementary (Theorem 3.5). In Section 4, we discuss some model-theoretical problems related with hyperfields and the algebraic methods used in this article (Question 4.1 and Conjecture 4.2) and also answer a question of the referee which fits very nicely to the topic of this

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article (Remark 4.3). In the Appendix, we discuss different approaches to the main algebraic result (Theorem 3.3) needed for the proof of Theorem 3.5.

§2. Preliminaries. In this section, we introduce the necessary notions we are going to use throughout this article (or we cite the necessary sources). Further we will present the results from the paper of Alain Connes and Caterina Consani [4], where (among other things) they studied connections between Krasner hyperfields and projective geometries.

Not everything from this section is directly needed for the arguments in Section 3, e.g., Theorem 2.13, Facts 2.15 and 2.16, or Remark 2.18 will not be used directly. However, we hope that these extra results provide a greater picture and they also show how to avoid possible "wrong paths" in the main argument.

2.1. Hyperfields. The notion of a hyperfield, as one could expect, generalises the one of a field. The twist is that the addition is a multivalued operation (called *hyperaddition*), so instead of an element, it returns a nonempty set.

Definition 2.1. A *hyperfield* is a tuple $(\mathcal{H}, +, \cdot, 0, 1)$, where $(\mathcal{H} \setminus \{0\}, \cdot, 1)$ is an abelian group and

$$+ : \mathcal{H} \times \mathcal{H} \to \mathcal{P}(\mathcal{H}) \setminus \{\emptyset\}$$

satisfies the following axioms, where $x, y, z \in \mathcal{H}$ and $+, \cdot$ are naturally extended to subsets of $\mathcal{H}$:

- $x + y = y + x$ (commutativity),
- $(x + y) + z = x + (y + z)$ (associativity),
- for each $x \in \mathcal{H}$, there is a unique $-x \in \mathcal{H}$ such that $0 \in x + (-x)$ (unique inverse),
- $z \in x + y \Rightarrow y \in z + (-x)$ (reversibility),
- $x + 0 = \{x\}$ (neutral element),
- $z \cdot (x + y) = z \cdot x + z \cdot y$ (distributivity).

Note that every field can be viewed as a hyperfield in the obvious way. For more details and preliminary notions concerning hyperfields (such as homomorphisms, hyperideals, etc.), we direct the reader to [5, 11, 17].

We state now the theorem of Krasner, which was mentioned in the introduction.

Theorem 2.2. *Let K be a field and G a subgroup of $K^\times$. The quotient $K^\times/G$ together with an extra element 0 and $+, \cdot$ defined as:*

- $aG \cdot bG := abG$,
- $aG + bG := \{(x + y)G \mid x \in aG, y \in bG\}$

forms a hyperfield, where $1 = G$.

Notation 2.3. *We will abbreviate $(K^\times/G) \cup \{0\}$ from Theorem 2.2 as K/G.*

Definition 2.4. If a hyperfield $\mathcal{H}$ is isomorphic to K/G (as in Notation 2.3), then we call it a *Krasner hyperfield*.

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2.2. Projective geometries and characteristic one hyperfields. The material of this section comes from [4]. We introduce first the definition of characteristic one hyperfields.

DEFINITION 2.5. We call a hyperfield $(\mathcal{H}, +, \cdot, 0, 1)$ a *characteristic one hyperfield*, if

$$\forall x \in \mathcal{H} \;\; x + x = \{x, 0\}.$$

EXAMPLE 2.6.

(1) The simplest characteristic one hyperfield is *the Krasner hyperfield* **K**, which is a two-element hyperfield $\{0, 1\}$ with the usual multiplication, and the hyperaddition defined as follows:

$$0 + 0 = \{0\}, \quad 0 + 1 = \{1\}, \quad 1 + 1 = \{0, 1\}.$$

(2) Proposition 2.10 below identifies characteristic one hyperfields with certain projective geometries.

(3) Any commutative group H of order at least 4 expands to the multiplicative group of a characteristic one hyperfield, where the hyperaddition is "trivial" (see [4, Proposition 3.6]), that is, for $x, y \in H \cup \{0\}$, we have:

$$x + y = \begin{cases} x & \text{if } y = 0, \\ \{0, x\} & \text{if } y = x, \\ H \setminus \{x, y\} & \text{if } |\{0, x, y\}| = 3. \end{cases}$$

These hyperfields are definable just from their multiplicative structure. Such a phenomenon is impossible for fields and this is the base of our proof of the main result of this article (Theorem 3.5).

(4) There is an interesting example (see [4, Example 4.7]) of a characteristic one hyperfield whose multiplicative group is infinite cyclic and the corresponding projective geometry is two-dimensional and not Desarguesian (see Definition 2.9). This example will be used later in Remark 4.3.

REMARK 2.7. The name "characteristic one hyperfield" was suggested to us by the referee as a more suggestive choice comparing to the one we made in the previous version of this article ("CC-hyperfield"). The name used in [4] is quite technical ("hyperfield extensions of the Krasner hyperfield **K**"), however the characteristic one context is mentioned in the Introduction to [4].

There is the following nice description of Krasner characteristic one hyperfields.

PROPOSITION 2.8 (Proposition 2.7 in [4]). *Let K be a field and G be a subgroup of $K^{\times}$. Assume that $G \neq \{1\}$. Then the hyperfield K/G is a characteristic one hyperfield if and only if $\{0\} \cup G$ is a subfield of K.*

The class of characteristic one hyperfields is also closely related to *projective geometries* in the sense of *incidence geometry*, we give below the necessary definitions coming from [2]. For visualisations of these notions, we refer the reader to the pictures on pages 10 and 11 of [2].

DEFINITION 2.9. Let P be an arbitrary set which will be thought of as the set of *points* and L be a fixed subset of $\mathcal{P}(P)$ which will be thought of as the set of *lines*.

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(1) We call the pair (P, L) a *projective geometry* if the following holds.
   (a) Every line contains at least two points.
   (b) Every pair of distinct points a and b is contained in a unique line $L(a,b)$.
   (c) For any pairwise distinct points a,b,c,d,e such that
$$L(a,b) = L(a,c) \neq L(a,d) = L(a,e),$$
   the set $L(b,d) \cap L(c,e)$ is non-empty.

(2) The projective geometry (P, L) is called *Desarguesian* if (we take this definition verbatim from page 11 of [2]) given ten distinct points,
$$a, b, c, d, e, f, g, h, i, j,$$
such that the following trios are collinear on distinct lines:
$$(a,b,c), (a,d,e), (a,f,g), (b,d,h), (c,e,h), (b,f,z), (c,g,z), (d,f,j), (e,g,j),$$
it follows that h, i, j are collinear.

(3) The *dimension* of a projective geometry (P, L) is the smallest number n such that there is a set $p_0, \dots, p_n \in P$ with the property that there is no proper $V \subset P$ containing $p_0, \dots, p_n$ and "closed under lines", that is for any distinct $x, y \in V$, we have $L(x,y) \subseteq V$.

The next result explains the connection between characteristic one hyperfields and projective geometries.

PROPOSITION 2.10 (Proposition 3.5 in [4]). *If $\mathcal{H}$ is a characteristic one hyperfield, then there is a unique projective geometry on $\mathcal{H} \setminus \{0\}$ such that for distinct $x, y \in \mathcal{H} \setminus \{0\}$, the unique line through x and y coincides with $\{x,y\} \cup x + y$ and for any $a \in \mathcal{H} \setminus \{0\}$, the map*
$$\mathcal{H} \setminus \{0\} \ni x \mapsto a \cdot x \in \mathcal{H} \setminus \{0\}$$
takes lines to lines.

Conversely, if G is a commutative group with the structure of a projective geometry such that translations by elements of G preserve lines and each line has at least four points, then G can be expanded to a hyperfield, where the hyperaddition on $G \cup \{0\}$ is defined by the rule:
$$x + y := L(x,y) \setminus \{x,y\}.$$

NOTATION 2.11. *For a characteristic one hyperfield $\mathcal{H}$, we will denote the above projective geometry by $\mathcal{P}_{\mathcal{H}}$.*

REMARK 2.12. If $\mathcal{H}$ is a Krasner characteristic one hyperfield, then the corresponding projective geometry $\mathcal{P}_{\mathcal{H}}$ is the classical one, which we will see below. By Proposition 2.8, $\mathcal{H} = L/K^{\times}$, where K is a subfield of L. Then we can view L as a vector space over K and consider the classical projective geometry associated with this vector space. This geometry happens to be exactly the projective geometry associated with the characteristic one hyperfield $L/K^{\times}$. In particular, such a projective geometry is always Desarguesian (see Definition 2.9(3)) and we have
$$\dim(\mathcal{P}_{\mathcal{H}}) + 1 = [L : K].$$

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We finish this section with a result from [4] which will tell us later that we need to focus on Krasner characteristic one hyperfields of dimension one, where by the dimension of a characteristic one hyperfield, we always mean the dimension of its associated projective geometry.

THEOREM 2.13 (Theorem 3.8 in [4]). *Let $\mathcal{H}$ be a characteristic one hyperfield. Assume that the projective geometry $\mathcal{P}_{\mathcal{H}}$ is Desarguesian and of dimension at least 2. Then there exists a unique pair (L, K), where L is a field, and K is its subfield such that*

$$\mathcal{H} = L/K^{\times}.$$

2.3. Model theory. In this section, we specify the model-theoretical set-up which is needed to work with hyperfields. We also show several reduction results.

We start with specifying the first-order language of hyperfields.

DEFINITION 2.14. Let us set the language of hyperfields as the tuple $(\oplus, \ominus, \odot, ^{-1}, \underline{0}, \underline{1})$, where:

- $\odot$ is a binary function symbol interpreted as a multiplication,
- $^{-1}$ is unary function symbol interpreted as a multiplicative inverse,
- $\oplus$ is a ternary relation symbol encoding the hyperaddition (so, in a hyperfield, we will have: $\oplus(x, y, z)$ if and only if $z \in x + y$),
- $\ominus$ is a unary function symbol encoding the additive inverse (so, $\oplus(x, \ominus x, 0)$ holds in a hyperfield),
- $\underline{1}$ and $\underline{0}$ are constant symbols corresponding to the neutral elements of the multiplication and the hyperaddition, respectively.

Clearly, the class of hyperfields can be first-order axiomatized in the language above. Let us state the following well-known result.

FACT 2.15. *The class $\mathcal{C}$ of structures (in a fixed language) is elementary if and only if $\mathcal{C}$ is closed under elementary equivalence and under ultraproducts.*

As a simple consequence of Łoś's theorem, one obtains the following.

FACT 2.16. *The class of Krasner hyperfields is closed under ultraproducts.*

Therefore, we will aim to show that the class of Krasner hyperfields is not closed under elementary equivalence. We see below that we can restrict ourselves to the class of Krasner characteristic one hyperfields.

LEMMA 2.17. *If the class of Krasner hyperfields is elementary, then the class of Krasner characteristic one hyperfields is elementary.*

PROOF. It is obvious, since the condition $(\forall x)(x + x = \{0, x\})$ is clearly definable in the language from Definition 2.14. ⊣

REMARK 2.18. All the assumptions from Theorem 2.13 can be expressed as first-order sentences in the language of hyperfields introduced above (using Definition 2.9 and the explicit definition of the associated projective geometry from Proposition 2.10). Hence, we obtain that the class of Krasner characteristic one hyperfields of dimension at least 2 *is* elementary.

Because of Remark 2.18, we need to focus on one-dimensional Krasner characteristic one hyperfields. For convenience, we give names to the following two classes.

NOTATION 2.19.
(1) *Let $\mathcal{K}$ denote the class of Krasner characteristic one hyperfields of dimension one.*
(2) *Let $\mathcal{K}^\times$ denote the class of groups which are of the form $L^\times/K^\times$, where $K \subseteq L$ is a field extension of degree* 2.

The next observation explains why the class of groups from Notation 2.19(2) is important for us.

FACT 2.20. *Let $\mathcal{H} \in \mathcal{K}$. Then we have the following.*
(1) *$\mathcal{H}$ is isomorphic to $L/K^\times$, where L is a field, K is its subfield, and* $[L : K] = 2$.
(2) *The hyperaddition in $\mathcal{H}$ is the same as in Example* 2.6(3), *so it is definable in the language* $\{\underline{0}\}$ (*just one constant symbol*).

PROOF. Item (1) follows from Remark 2.12, since $\dim(\mathcal{P}_\mathcal{H}) = 1$ if and only if $\mathcal{H}$ comes from a field extension of degree 2.

Item (2) follows again from Remark 2.12 (and Proposition 2.10), since $\dim(\mathcal{P}_\mathcal{H}) = 1$ implies that there is only one line in the projective geometry $\mathcal{P}_\mathcal{H}$ and this line is the whole space. ⊣

We directly obtain the following.

LEMMA 2.21. *If the class of Krasner characteristic one hyperfields is elementary, then the class $\mathcal{K}$ is elementary.*

PROOF. It follows from Definition 2.9(3) that being of dimension one is a definable property. ⊣

The following easy results outline the further connections between the classes $\mathcal{K}$ and $\mathcal{K}^\times$.

LEMMA 2.22. *Let G and H be commutative groups and $\mathcal{H}_G, \mathcal{H}_H$ be the corresponding hyperfields as in Example* 2.6(3). *If G and H are elementarily equivalent* (*as groups*), *then $\mathcal{H}_G$ and $\mathcal{H}_H$ are elementarily equivalent* (*as hyperfields*).

PROOF. Assume that $(G, \cdot) \equiv (H, \cdot)$. By the uniform definition of the monoid operation in $\mathcal{H}_G, \mathcal{H}_H$, we get that $(\mathcal{H}_G, \cdot) \equiv (\mathcal{H}_H, \cdot)$. Since the hyperaddition in $\mathcal{H}_G, \mathcal{H}_H$ is defined by the same formula in the monoid language (we actually only need the extra constant as in Fact 2.20), we get that $(\mathcal{H}_G, +, \cdot) \equiv (\mathcal{H}_H, +, \cdot)$. ⊣

LEMMA 2.23. *If the class $\mathcal{K}$ is elementary, then the class $\mathcal{K}^\times$ is closed under elementary equivalence.*

PROOF. Assume that the class $\mathcal{K}$ is elementary. Let us take $G \in \mathcal{K}^\times$, so there is a hyperfield $\mathcal{H} \in \mathcal{K}$ such that G is the multiplicative group of $\mathcal{H}$. By Example 2.6(3) and Fact 2.20, we get that $\mathcal{H} = \mathcal{H}_G$ (as hyperfields, see the notation from Lemma 2.22). We also take a group H such that $G \equiv H$. By Lemma 2.22, we get that $\mathcal{H}_H \equiv \mathcal{H}_G = \mathcal{H}$. Since the class $\mathcal{K}$ is elementary, we obtain that $\mathcal{H}_H \in \mathcal{K}$. Therefore, $H \in \mathcal{K}^\times$, which we needed to show. ⊣

We finish this section with stating a result of Szmielew about elementary equivalence of commutative groups (see [21]). We will use the formulation from [6, Theorem 1].

THEOREM 2.24 (Szmielew [21]). *If A and B are abelian groups, then A is elementarily equivalent to B if and only if*

$$A \text{ is of finite exponent} \quad \Leftrightarrow \quad B \text{ is of finite exponent};$$

and for each prime p and integer $n \geqslant 0$:

$$\dim_{\mathbb{F}_p}\left(p^n A[p]/p^{n+1}A[p]\right) = \dim_{\mathbb{F}_p}\left(p^n B[p]/p^{n+1}B[p]\right),$$

$$\lim_{n\to\infty} \dim_{\mathbb{F}_p}\left(p^n A/p^{n+1}A\right) = \lim_{n\to\infty} \dim_{\mathbb{F}_p}\left(p^n B/p^{n+1}B\right),$$

$$\lim_{n\to\infty} \dim_{\mathbb{F}_p}\left(p^n A[p]\right) = \lim_{n\to\infty} \dim_{\mathbb{F}_p}\left(p^n B[p]\right),$$

where

$$p^n G = \{p^n x \mid x \in G\}, \qquad G[p] = \{x \in G \mid px = 0\}.$$

§3. Main result. In this section, we prove the main result of this article (Theorem 3.5). We need the following notion (see [7, Section 3.4]).

DEFINITION 3.1. The *rational rank* of a commutative group A is the cardinality of a maximal $\mathbb{Z}$-linearly independent subset of A. Following [7], we denote it by $\mathrm{rr}(A)$.

REMARK 3.2. Let A be a commutative group.

(1) It is easy to see that we have (see [7, Section 3.4]):

$$\mathrm{rr}(A) = \dim_{\mathbb{Q}}\left(A \otimes_{\mathbb{Z}} \mathbb{Q}\right).$$

(2) If $A_0 \leqslant A$, then we have (see [7, Section 3.4]):

$$\mathrm{rr}(A) = \mathrm{rr}(A_0) + \mathrm{rr}\left(A/A_0\right).$$

(3) Other names as “rank” or “Prüfer rank” or “torsion-free rank” are sometimes used in this context as well.

We will use the following algebraic result (see Notation 2.19).

THEOREM 3.3. *The rational rank of any $A \in \mathcal{K}^{\times}$ is either 0 or infinite.*

The proof and history of this result will be discussed in the Appendix. We still include one example here which may be used as a basic illustration of the methods needed to show Theorem 3.3.

EXAMPLE 3.4. We will show that

$$\mathrm{rr}(\mathbb{Q}[i]^{\times}/\mathbb{Q}^{\times}) = \aleph_0.$$

Let us recall that a prime number $p \in \mathbb{Z}$ splits (equivalently in this case: splits completely) in $\mathbb{Z}[i]$ if and only if $p \equiv 1 (\mathrm{mod}\ 4)$ and that there are infinitely many such primes, which is a very special case of Chebotarev's density theorem (see

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[9, Theorem 6.3.1]). Let us take an infinite sequence $p_1, p_2, \ldots$ of prime numbers which split in $\mathbb{Z}[i]$.

We have $p_i = r_i\overline{r_i}$, where r_i is a prime element of $\mathbb{Z}[i]$ and $\overline{r_i}$ is the complex conjugate of r_i. Then $r_1, \overline{r_1}, r_2, \overline{r_2}, \ldots$ is a sequence of pairwise non-associated prime elements of $\mathbb{Z}[i]$. We will show that the cosets $r_1\mathbb{Q}^\times, r_2\mathbb{Q}^\times, \ldots$ are $\mathbb{Z}$-independent in $\mathbb{Q}[i]^\times/\mathbb{Q}^\times$.

Assume not, so there is a non-zero tuple $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ such that $r_1^{n_1} \ldots r_k^{n_k} \in \mathbb{Q}^\times$ (witnessing that $r_1\mathbb{Q}^\times, \ldots, r_k\mathbb{Q}^\times$ are not $\mathbb{Z}$-independent in $\mathbb{Q}[i]^\times/\mathbb{Q}^\times$). We have:

$$r_1^{n_1} \ldots r_k^{n_k} = \overline{r_1}^{n_1} \ldots \overline{r_k}^{n_k},$$

which contradicts the unique factorization in $\mathbb{Z}[i]$.

Our main model-theoretic result is below.

THEOREM 3.5. *The class of Krasner hyperfields is not elementary.*

PROOF. If the class of Krasner hyperfields is elementary, then the class $\mathcal{K}^\times$ of groups (see Notation 2.19(2)) is closed under elementary equivalence by Lemmas 2.17, 2.21, and 2.23. We will show that this is not the case.

Since $\mathbb{C}^\times$ is divisible, we have:

$$\mathcal{K}^\times \ni \mathbb{C}^\times/\mathbb{R}^\times \cong A \oplus \bigoplus_p C_{p^\infty} \equiv \mathbb{Q} \oplus \bigoplus_p C_{p^\infty},$$

where C_{p^∞} is the Prüfer p-group and A is a vector space over $\mathbb{Q}$ of dimension continuum. The isomorphism above follows from the classification of divisible commutative groups (see [12, Theorem 5 in Section 4]) and the elementary equivalence follows from Theorem 2.24, since for any positive integer n, we have

$$\mathbb{Q}[n] = \{0\}, \quad n\mathbb{Q} = \mathbb{Q}.$$

However, the rational rank of $\mathbb{Q} \oplus \bigoplus_p C_{p^\infty}$ is 1, so this group does not belong to $\mathcal{K}^\times$ by Theorem 3.3. ⊣

§4. Related questions and conjectures. In this section, we discuss some model-theoretical problems related with hyperfields. By Theorem A.3 (as in the proof of Theorem 3.5), the following class of groups:

$$\{K^\times \mid K \text{ is a field}\}$$

is *not* elementary. Interestingly, a similar phenomenon appeared in [10] where the authors consider model completeness of groups of rational points of algebraic groups. One can ask the following.

QUESTION 4.1. *Let $\mathbb{G}$ be a group scheme over $\mathbb{Z}$. Are the following two conditions on $\mathbb{G}$ equivalent?*

(1) *The class*

$$\{\mathbb{G}(K) \mid K \textit{ is a field}\}$$

is elementary.

(2) *If K is a model complete field, then $\mathbb{G}(K)$ is a model complete group.*

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The multiplicative group scheme $\mathbb{G}_{\mathrm{m}}$ fails both items (1) and (2) above. On the other hand, semisimple or unipotent algebraic groups seem to satisfy both these items, which is work in progress related to [10]. Therefore, we do not have counterexamples to the equivalence in Question 4.1. Actually, if item (1) holds, then (as in [10]) it is usually an important step for proving that item (2) holds. The fact that item (1) holds for certain simple algebraic groups follows from [20, 22].

While trying to understand hyperfields (or any other structures) model-theoretically, it is natural to ask first what are the "model-theoretically simplest", that is *strongly minimal*, hyperfields. We propose the following.

CONJECTURE 4.2. A hyperfield is strongly minimal if and only if it is either a strongly minimal field (i.e., an algebraically closed field) or a hyperfield, where the hyperaddition is definable in the structure of its multiplicative group, which is strongly minimal.

Since any infinite commutative group can be expanded to a hyperfield where the hyperaddition is definable just from one constant symbol (see Example 2.6) there are plenty of hyperfields as after "or" in the conjecture above.

REMARK 4.3. The referee asked an interesting question whether there is an elementary statement which is true of all Krasner hyperfields but not of all hyperfields. We answer (in the affirmative) and discuss this question below.

(1) By Fact 2.16 and basic model theory, the referee's question is equivalent to asking whether each hyperfield is elementarily equivalent to a Krasner hyperfield.
(2) It is rather easy to answer this question in the finite case, however our answer is still not so obvious. By Theorem A.1, any finite Krasner hyperfield of characteristic one has cyclic multiplicative group, but this is not true for arbitrary finite hyperfields of characteristic one using Example 2.6(3). So, for example, we can use the following elementary statement to answer the referee's question in the affirmative:

    "if the multiplicative group of a characteristic one hyperfield has four elements, then this group is cyclic".
(3) In the infinite case, we use the fact that the projective geometry of any Krasner hyperfield of characteristic one is Desarguesian (being the classical projective geometry, see Remark 2.12), but there are hyperfields of characteristic one with a non-Desarguesian projective geometry (see Example 2.6(4)). One should also notice that being Desarguesian is a first-order property by Definition 2.9(3). Having all this, one can produce a (rather long) sentence which is true of all infinite Krasner hyperfields but not of all infinite hyperfields.

    We would like to point out that the existence of such non-Desarguesian hyperfields of characteristic one which are moreover *finite* is an open problem related with the open problem of the existence of primes of the form $p = n^2 + n + 1$ (see [4, Remark 3.12]).

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Appendix. In the previous version of this article, we showed Theorem 3.3 and then sketched a proof of its generalization to arbitrary field extensions. However, the referee pointed out to us the following result from 1960s (see [1]).

THEOREM A.1. *If K is an infinite field and $K \subset L$ is a proper extension of fields, then the group $L^\times/K^\times$ is not finitely generated.*

REMARK A.2. We discuss here how Theorem 3.3 is related with Theorem A.1.

(1) On the very formal level, neither result implies the other one, since not being finitely generated does not imply having infinite rational rank, and there are groups of rational rank zero which are finitely generated.
(2) However, it is rather clear that the *proof* of Theorem A.1 also gives the statement of Theorem A.3 below (which clearly generalizes Theorem 3.3).
(3) As the referee pointed out, our proof from the previous version of this article was almost identical to the proof of Theorem A.1.

To give the reader an idea of the proof, we include below a shortened sketch of the argument from the previous version of this article. A reader interested in the full argument is referred to the previous version of this article, which is available on ArXiv or to the aforementioned article [1] (in German).

THEOREM A.3. *Let $F \subseteq K$ be an arbitrary field extension. Then both* $\mathrm{rr}(F^\times)$ *and* $\mathrm{rr}(K^\times/F^\times)$ *are* 0 *or infinite.*

SKETCH OF PROOF. We consider the more difficult case of $\mathrm{rr}(K^\times/F^\times)$ only. If the extension $F \subseteq K$ is not algebraic, we take a transcendental $t \in K$ and then $\mathrm{rr}(F(t)^\times/F^\times)$ is infinite by a similar argument as in Example 3.4.

If the extension $F \subseteq K$ is purely inseparable or K is contained in the algebraic closure of a finite field, then it is easy to see that $\mathrm{rr}(K^\times/F^\times) = 0$.

Therefore, we can assume that $F \subseteq K$ is a finite extension which is not contained in the algebraic closure of a finite field and which is also not purely inseparable. We aim to show that $\mathrm{rr}(K^\times/F^\times)$ is infinite. Let us take the field tower $F \subseteq K_0 \subseteq K$, where the first extension is separable and non-trivial and the second one is purely inseparable. We have the following exact sequence:

$$1 \to K_0^\times/F^\times \to K^\times/F^\times \to K^\times/K_0^\times \to 1$$

and we know that $\mathrm{rr}(K^\times/K_0^\times) = 0$. Using Remark 3.2(2), we obtain that

$$\mathrm{rr}(K_0^\times/F^\times) = \mathrm{rr}(K^\times/F^\times),$$

so we can moreover assume that $F \subseteq K$ is finite, separable, and $F = \mathbb{F}_p(X)$ or F is a number field. Let $F \subseteq L$ be the normal closure of $F \subseteq K$ and $n := [K : F] > 1$. By Chebotarev's density theorem (see [9, Theorem 6.3.1]), there are infinitely many prime ideals $P_1, P_2, \ldots$ of $\mathcal{O}_F$, which split completely in $\mathcal{O}_L$. Therefore, for each i, we also have

$$P_i\mathcal{O}_K = Q_{i1} \ldots Q_{in},$$

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where Q_{ij}'s are maximal ideals in $\mathcal{O}_K$. We take $a_1, a_2, \ldots \in \mathcal{O}_K$ such that for each i, we have:

$$a_i \in Q_{i1} \setminus \left(\bigcup_{j=1}^{i-1} Q_{j1} \cup \bigcup_{j=1}^{i} \bigcup_{k=2}^{l} Q_{jk} \right).$$

Then, we can finish in a similar way as in Example 3.4. ⊣

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