Adv. Appl. Probab. **57,** 1484–1509 (2025)
*doi:*10.1017/apr.2025.17

ASYMPTOTIC BEHAVIORS OF SUBCRITICAL BRANCHING KILLED BROWNIAN MOTION WITH DRIFT

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Abstract

In this paper, we study asymptotic behaviors of a subcritical branching Brownian motion with drift $-\rho$, killed upon exiting $(0, \infty)$, and offspring distribution $\{p_k: k \geq 0\}$. Let $\widetilde{\zeta}^{-\rho}$ be the extinction time of this subcritical branching killed Brownian motion, $\widetilde{M}_t^{-\rho}$ the maximal position of all the particles alive at time t and $\widetilde{M}^{-\rho} := \max_{t\geq 0} \widetilde{M}_t^{-\rho}$ the all-time maximal position. Let $\mathbb{P}_x$ be the law of this subcritical branching killed Brownian motion when the initial particle is located at $x \in (0, \infty)$. Under the assumption $\sum_{k=1}^{\infty} k(\log k)p_k < \infty$, we establish the decay rates of $\mathbb{P}_x(\widetilde{\zeta}^{-\rho} > t)$ and $\mathbb{P}_x(\widetilde{M}^{-\rho} > y)$ as t and y respectively tend to ∞. We also establish the decay rate of $\mathbb{P}_x(\widetilde{M}_t^{-\rho} > z(t, \rho))$ as $t \to \infty$, where $z(t, \rho) = \sqrt{t}z - \rho t$ for $\rho \leq 0$ and $z(t, \rho) = z$ for $\rho > 0$. As a consequence, we obtain a Yaglom-type limit theorem.

Keywords: Branching killed Brownian motion; survival probability; maximal displacement; Feynman–Kac representation.

2020 Mathematics Subject Classification: Primary 60J80
Secondary 60J65; 69G57

1. Introduction

The goal of this paper is to study asymptotic behaviors of subcritical branching killed Brownian motions with drifts. Before we introduce the model of branching killed Brownian motion, we first give some preliminaries on branching Brownian motions without killing and review some related literature about them. Then we introduce branching killed Brownian motions and present our main results.

Received 1 July 2024; accepted 5 March 2025.

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1.1. Branching Brownian motions

A branching Brownian motion with drift $-\rho$ is a continuous-time Markov process defined as follows. At time 0, there is a particle at $x \in \mathbb{R}$ and this particle moves according to a Brownian motion with drift $-\rho \in \mathbb{R}$. After an exponential time with parameter $\beta > 0$, independent of the spatial motion, this particle dies and is replaced by k offspring with probability p_k, $k \geq 0$. The offspring move independently according to Brownian motion with drift $-\rho$ from the place where they are born and obey the same branching mechanism as their parent. This procedure continues. Let N_t be the collection of particles alive at time t. If $u \in N_t$, let $X_u^{-\rho}(t)$ denote the position of the particle u at time t, and for $s \in (0, t)$ we denote by $X_u^{-\rho}(s)$ the position at time s of the ancestor of u. The point process $(Z_t^{-\rho})_{t \geq 0}$ defined by

$$Z_t^{-\rho} := \sum_{u \in N_t} \delta_{X_u^{-\rho}(t)}, \quad t \geq 0,$$

is called a branching Brownian motion with drift $-\rho$. We will use $\mathbb{P}_x$ to denote the law of this process and use $\mathbb{E}_x$ to denote the corresponding expectation. Let

$$\zeta := \inf\{t > 0, N_t = \emptyset\}$$

be the extinction time of $(Z_t^{-\rho})_{t \geq 0}$. Note that the law of ζ does not depend on ρ and is equal to that of the extinction time of the continuous-time Galton–Watson process with the same branching mechanism as the branching Brownian motion. Let $m := \sum_{k=0}^{\infty} k p_k$ be the mean number of offspring and let f be the generating function of the offspring distribution, $f(u) = \sum_{k=0}^{\infty} p_k u^k$, $u \in [0, 1]$. It is well known that the process will become extinct in finite time with probability 1 if and only if $m < 1$ (subcritical) or $m = 1$ and $p_1 \neq 1$ (critical). When $m > 1$ (supercritical), the process survives with positive probability.

For any $t \geq 0$, let

$$M_t^{-\rho} := \max\{X_u^{-\rho}(t) : u \in N_t\}$$

be the maximal position of all the particles alive at time t and let

$$M^{-\rho} := \sup_{t>0} M_t^{-\rho}$$

be the all-time maximal position. In the subcritical and critical cases, $\mathbb{P}_x\left(M^{-\rho} < \infty\right) = 1$ for any x, $\rho \in \mathbb{R}$.

In the critical case $m = 1$ and $p_1 \neq 1$, Sawyer and Fleischman [24] proved that if $\beta = 1$ and the offspring distribution has finite third moment, then

$$\lim_{x \to \infty} x^2 \mathbb{P}_0(M^0 \geq x) = \frac{6}{\sigma^2}, \tag{1.1}$$

where σ is the variance of the offspring distribution. For a critical branching random walk with spatial motion having finite $(4+\varepsilon)$th moment, a result similar to (1.1) was proved by Lalley and Shao [14]. It was also proved in [14] that the law of $M_t^0/\sqrt{t}$ under $\mathbb{P}_0(\cdot|\zeta > t)$ converges weakly to some random variable. For related results in the case of critical branching Lévy processes, see [23].

In the subcritical case $m \in (0, 1)$, let

$$\alpha := \beta(1-m) \in (0, \infty).$$

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Define

$$\Phi(u) := \beta\,(f(1-u)-(1-u)) =: (\alpha+\varphi(u))\,u, \quad u\in[0,1], \tag{1.2}$$

where $\varphi(u)=\frac{\Phi(u)-\alpha u}{u}$ for $u\in(0,1]$ and $\varphi(0)=\Phi'(0+)-\alpha=0$. φ is a nonnegative continuous increasing function; see Lemma 7 below. It is well known (see Theorem 2.4 in [1, p. 121]) that the limit

$$\lim_{t\to\infty} \mathrm{e}^{\alpha t}\mathbb{P}_0(\zeta>t) = C_{\text{sub}} \in (0,\infty) \tag{1.3}$$

if and only if

$$\sum_{k=1}^{\infty} k(\log k)p_k < \infty. \tag{1.4}$$

We now give another equivalent form of (1.3). For any $t>0$, define

$$g(t) := \mathbb{P}_0(\zeta>t).$$

It is well known that $g(t)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) = -\Phi(g(t)) = -\,(\alpha+\varphi(g(t)))\,g(t),$$

thus

$$\mathrm{e}^{\alpha t}g(t) = \exp\left\{-\int_0^t \varphi(g(s))\mathrm{d}s\right\}. \tag{1.5}$$

It follows from (1.3) that

$$C_{\text{sub}} = \exp\left\{-\int_0^\infty \varphi(g(s))\mathrm{d}s\right\}. \tag{1.6}$$

Therefore, (1.3) is equivalent to

$$\int_0^\infty \varphi(g(s))\mathrm{d}s < \infty. \tag{1.7}$$

For $M^{-\rho}$, when the underlying motion is a standard Brownian motion and the offspring distribution has finite third moment, it was proved in [24] that, if $\rho=0$,

$$\lim_{x\to\infty} \frac{\mathbb{P}_0(M^0>x)}{(1-m)s(x)\mathrm{e}^{-\sqrt{2\alpha}x}} = 1, \tag{1.8}$$

where $s(x)$ is a bounded positive function. The limit (1.8) was later generalized in [23] to subcritical branching spectrally negative Lévy processes. When specialized to our setting, [23, Theorem 1.1] says that, when $\sum_{k=0}^\infty k^3p_k<\infty$, there exists a constant $\kappa\in(0,\infty)$ such that

$$\lim_{x\to\infty} \mathrm{e}^{\left(\rho+\sqrt{2\alpha+\rho^2}\right)x}\mathbb{P}_0\left(M^{-\rho}\ge x\right) = \kappa. \tag{1.9}$$

In the case of subcritical branching random walks, it was proved in [21, Theorem 1.2] that when the random walk has finite range and is nearly right-continuous in the sense of [21], a result similar to (1.8) holds. In [21], the authors also gave some estimates for the limit behavior of $\mathbb{P}_0(M^0\ge x)$ in the case of general subcritical branching random walks. For related results about near-critical branching random walks, see [22].

1.2. Branching killed Brownian motions

We are interested in asymptotic behaviors of branching killed Brownian motions with drift $-\rho$, in which particles are killed (along with their descendants) upon exiting $(0, \infty)$. The point process $(\widetilde{Z}_t^{-\rho})_{t\geq 0}$ defined by

$$\widetilde{Z}_t^{-\rho} := \sum_{u\in N_t} 1_{\{\min_{s\leq t} X_u^{-\rho}(s)>0\}} \delta_{X_u^{-\rho}(t)}$$

is called a branching killed Brownian motion with drift $-\rho$. Let

$$\widetilde{\zeta}^{-\rho} := \inf\left\{t \geq 0 : \widetilde{Z}_t^{-\rho}((0, \infty)) = 0\right\}$$

be the extinction time of $(\widetilde{Z}_t^{-\rho})_{t\geq 0}$. We define the maximal position at time t and the all-time maximal position of $(\widetilde{Z}_t^{-\rho})_{t\geq 0}$ by

$$\widetilde{M}_t^{-\rho} := \max_{u\in N_t :\, \min_{s\leq t} X_u^{-\rho}(s)>0} X_u^{-\rho}(t) \quad \text{and} \quad \widetilde{M}^{-\rho} := \max_{t\geq 0} \widetilde{M}_t^{-\rho}.$$

In the critical case ($m = 1$ and $p_1 \neq 1$), Lalley and Zheng [15, Theorem 6.1] proved that, if $\sum_{k=0}^{\infty} k^3 p_k < \infty$, then

$$\lim_{y\to\infty} y^3 \mathbb{P}_x(\widetilde{M}^0 \geq y) = C_1 x,$$

where $C_1 \in (0, \infty)$ is a constant independent of x. It was also shown in [15, Theorem 6.1] that, for any $s \in (0, 1)$,

$$\lim_{y\to\infty} y^2 \mathbb{P}_{sy}(\widetilde{M}^0 \geq y) = C_2(s) \in (0, \infty).$$

Recently, Hou et al. [12] studied the asymptotic behaviors of the tails of the extinction time and the maximal displacement of critical branching killed Lévy processes under some assumptions on the spatial motion and the assumption that the offspring distribution belongs to the domain of attraction of an α-stable distribution for $\alpha \in (1, 2]$.

There are also quite a few papers in the literature studying the asymptotic behaviors of supercritical (i.e. $m \in (1, \infty)$) branching killed Brownian motions with drift $-\rho$. Kesten [13] proved that, when $\rho > \sqrt{2\beta(m-1)}$, the process will become extinct almost surely and Harris and Harris [10, Theorem 1] obtained the asymptotic behavior of the survival probability. In the case $\rho < \sqrt{2\beta(m-1)}$, Harris, Harris and Kyprianou [11] investigated the large deviation probability of the maximal position. The papers [3, 7, 18] studied the tail of the total number of killed particles under different drift conditions. [3, Proposition 4] studied the asymptotic behavior of the probability that the all-time minimum of a dyadic branching Brownian motion with drift $-\rho > \sqrt{2\beta}$ starting from 0 does not fall below $-x$, which is similar in spirit to our Theorem 2 below. For related results in the critical case $\rho = \sqrt{2\beta(m-1)}$, see [2, 13, 19, 20].

The main focus of this paper is on the asymptotic behaviors of subcritical branching killed Brownian motions with drift. More precisely, we will study the asymptotic behaviors of $\mathbb{P}_x\big(\widetilde{\zeta}^{-\rho} > t\big)$ and $\mathbb{P}_x\big(\widetilde{M}^{-\rho} > y\big)$ as t and y tend to ∞, respectively. Define

$$z(t, \rho) = \begin{cases} \sqrt{t}z - \rho t, & \text{if } \rho \leq 0, \\ z, & \text{if } \rho > 0. \end{cases} \tag{1.10}$$

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We will also study the decay rate of $\mathbb{P}_x\big(\widetilde{M}_t^{-\rho} > z(t, \rho)\big)$ as t tends to ∞.

Our first main result is as follows. Recall that C_{sub} is given in (1.3). Also, the notation $f(t) \sim g(t)$ as $t \to a$ means that $\lim_{t\to a} f(t)/g(t) = 1$.

Theorem 1. *Assume $m \in (0, 1)$ and (1.4). Let $x > 0$.*

(i) *If $\rho = 0$, then*

$$\lim_{t\to\infty} \sqrt{t}\mathrm{e}^{\alpha t}\mathbb{P}_x\left(\widetilde{\zeta}^{-\rho} > t\right) = \sqrt{\frac{2}{\pi}}C_{\mathrm{sub}}x.$$

(ii) *If $\rho < 0$, then*

$$\lim_{t\to\infty} \mathrm{e}^{\alpha t}\mathbb{P}_x\big(\widetilde{\zeta}^{-\rho} > t\big) = C_{\mathrm{sub}}(1 - \mathrm{e}^{2\rho x}).$$

(iii) *If $\rho > 0$, then*

$$\lim_{t\to\infty} t^{\frac{3}{2}}\mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)t}\mathbb{P}_x\left(\widetilde{\zeta}^{-\rho} > t\right) = \sqrt{\frac{2}{\pi}}C_0(\rho)x\mathrm{e}^{\rho x},$$

where $C_0(\rho) := \lim_{N\to\infty} \mathrm{e}^{(\alpha+\frac{\rho^2}{2})N} \int_0^\infty y\mathrm{e}^{-\rho y}\mathbb{P}_y\big(\widetilde{\zeta}^{-\rho} > N\big)\mathrm{d}y \in (0, \infty)$.

Furthermore, for any $\rho \in \mathbb{R}$, as $t \to \infty$,

$$\mathbb{P}_x\big(\widetilde{\zeta}^{-\rho} > t\big) \sim \Gamma_\rho \mathbb{E}_x\big(\widetilde{Z}_t^{-\rho}((0, \infty))\big),$$

where $\Gamma_\rho = C_{\mathrm{sub}}$ when $\rho \le 0$ and $\Gamma_\rho = \rho^2 C_0(\rho)$ when $\rho > 0$.

Let $(B_t, \mathbf{P}_x)$ be a standard Brownian motion starting from x. For any $\rho \in \mathbb{R}$, it is known that $\{\mathrm{e}^{-\rho(B_t-x)-\frac{\rho^2}{2}t}, t \ge 0\}$ is a positive $\mathbf{P}_x$-martingale with mean 1. Define $\mathcal{F}_t := \sigma(B_s : s \le t)$ and

$$\left.\frac{\mathrm{d}\mathbf{P}_x^{-\rho}}{\mathrm{d}\mathbf{P}_x}\right|_{\mathcal{F}_t} := \mathrm{e}^{-\rho(B_t-x)-\frac{\rho^2}{2}t}. \tag{1.11}$$

Then under $\mathbf{P}_x^{-\rho}$, $\{B_t, t \ge 0\}$ is a Brownian motion with drift $-\rho$ starting from x.

Remark 1. Combining Theorem 1 with the asymptotic behavior of $\mathbf{P}_x^{-\rho}(\tau_0 > t)$ (where, for any $y \in \mathbb{R}$, τ_y is the first hitting time of y) in Lemma 1, we see that, when $\rho \le 0$, $\mathbb{P}_x\big(\widetilde{\zeta}^{-\rho} > t\big) \sim \mathbb{P}_x(\zeta > t)\, \mathbf{P}_x^{-\rho}(\tau_0 > t)$, i.e., the branching and the spatial motion are nearly independent. For $\rho > 0$, the constant $C_0(\rho)$ is related to the existence of a quasi-stationary distribution. Moreover, one can show that in this case,

$$\lim_{t\to\infty} \frac{\mathbb{P}_x\big(\widetilde{\zeta}^{-\rho} > t\big)}{\mathbb{P}_x(\zeta > t)\, \mathbf{P}_x^{-\rho}(\tau_0 > t)} > 1; \tag{1.12}$$

see (3.26) below.

Our second main result is on the tail probability $\mathbb{P}_x(\widetilde{M}^{-\rho} > y)$. In the case when there is no killing, the results (1.8) and (1.9) were proved under the assumption that the offspring

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distribution has finite third moment. Our assumption (1.4) on the offspring distribution is much weaker.

Theorem 2. *Assume $m \in (0, 1)$ and (1.4). Then for any $\rho \in \mathbb{R}$, there exists a constant $C_*(\rho) \in (0, \infty)$ such that, for any $x > 0$,*

$$\lim_{y\to\infty} \mathrm{e}^{(\rho+\sqrt{2\alpha+\rho^2})y}\mathbb{P}_x(\widetilde{M}^{-\rho} > y) = 2C_*(\rho)\mathrm{e}^{\rho x}\sinh(x\sqrt{2\alpha+\rho^2}).$$

Remark 2. On $\{\widetilde{M}^{-\rho} > y\}$, there is at least one particle which achieves the level y before hitting 0. The reason for the appearance of the sinh function in the theorem above is that this function is related to the Laplace transformation of τ_y on the event $\{\tau_y < \tau_0\}$ and this event gives the main contribution to the tail probability of $\{\widetilde{M}^{-\rho} > y\}$.

Our third main result is on the limit behavior of the maximal position at time t.

Theorem 3. *Assume $m \in (0, 1)$ and (1.4). Let $x > 0$.*

(i) *For $\rho = 0$ and $z \geq 0$,*

$$\lim_{t\to\infty} \sqrt{t}\mathrm{e}^{\alpha t}\mathbb{P}_x\left(\widetilde{M}_t^{-\rho} > \sqrt{t}z\right) = \sqrt{\frac{2}{\pi}}C_{\mathrm{sub}}x\mathrm{e}^{-z^2/2},$$

or equivalently, as $t \to \infty$,

$$\mathbb{P}_x\left(\widetilde{M}_t^{-\rho} > \sqrt{t}z\right) \sim C_{\mathrm{sub}}\mathbb{E}_x\left(\widetilde{Z}_t^{-\rho}\left((\sqrt{t}z, \infty)\right)\right).$$

(ii) *For $\rho < 0$ and $z \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathrm{e}^{\alpha t}\mathbb{P}_x\left(\widetilde{M}_t^{-\rho} + \rho t > \sqrt{t}z\right) = \frac{C_{\mathrm{sub}}(1-\mathrm{e}^{2\rho x})}{\sqrt{2\pi}}\int_z^\infty \mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y,$$

or equivalently, as $t \to \infty$,

$$\mathbb{P}_x\left(\widetilde{M}_t^{-\rho} + \rho t > \sqrt{t}z\right) \sim C_{\mathrm{sub}}\mathbb{E}_x\left(\widetilde{Z}_t^{-\rho}\left((\sqrt{t}z - \rho t, \infty)\right)\right).$$

(iii) *For $\rho > 0$ and $z \geq 0$,*

$$\lim_{t\to\infty} t^{\frac{3}{2}}\mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)t}\mathbb{P}_x\left(\widetilde{M}_t^{-\rho} > z\right) = \sqrt{\frac{2}{\pi}}C_z(\rho)x\mathrm{e}^{\rho x},$$

where $C_z(\rho) := \lim_{N\to\infty} \mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)N}\int_0^\infty y\mathrm{e}^{-\rho y}\mathbb{P}_y\left(\widetilde{M}_N^{-\rho} > z\right)\mathrm{d}y \in (0, \infty)$ is a function of z independent of x. Or equivalently, as $t \to \infty$,

$$\mathbb{P}_x\left(\widetilde{M}_t^{-\rho} > z\right) \sim \frac{\rho^2 C_z(\rho)\mathrm{e}^{\rho z}}{\rho z + 1}\mathbb{E}_x\left(\widetilde{Z}_t^{-\rho}\left((z, \infty)\right)\right). \tag{1.13}$$

Remark 3. Actually, combining inequalities (3.22) and (3.24) in the proof of Theorem 3, we can get that when $\rho > 0$, for any $z \geq 0$,

$$C_{\mathrm{sub}}\int_z^\infty y\mathrm{e}^{-\rho y}\mathrm{d}y \leq C_z(\rho) \leq \int_z^\infty y\mathrm{e}^{-\rho y}\mathrm{d}y.$$

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We mention here that we do not know the exact expression for $C_z(\rho)$. By (1.12), we know that $C_z(\rho)$ is not equal to $C_{\text{sub}} \int_z^\infty y\mathrm{e}^{-\rho y}\mathrm{d}y$.

Combining Theorems 1 and 3, we get the following Yaglom-type theorem.

Corollary 1. *Assume $m \in (0, 1)$ and (1.4). Let $x > 0$.*

(i) *If $\rho = 0$, then*

$$\mathbb{P}_x\left(\frac{\widetilde{M}_t^{-\rho}}{\sqrt{t}} \in \cdot \Big| \widetilde{\zeta}^{-\rho} > t\right) \quad \stackrel{\mathrm{d}}{\Longrightarrow} \quad \mathbb{P}(R \in \cdot),$$

where $(R, \mathbb{P})$ is a Rayleigh random variable with density $z\mathrm{e}^{-z^2/2}1_{\{z>0\}}$.

(ii) *If $\rho < 0$, then*

$$\mathbb{P}_x\left(\frac{\widetilde{M}_t^{-\rho} + \rho t}{\sqrt{t}} \in \cdot \Big| \widetilde{\zeta}^{-\rho} > t\right) \quad \stackrel{\mathrm{d}}{\Longrightarrow} \quad \mathbf{P}_0(B_1 \in \cdot),$$

where $(B_1, \mathbf{P}_0)$ is a standard normal random variable.

(iii) *If $\rho > 0$, then there exists a random variable $(X, \mathbb{P})$ whose law is independent of x such that*

$$\mathbb{P}_x\left(\widetilde{M}_t^{-\rho} \in \cdot \Big| \widetilde{\zeta}^{-\rho} > t\right) \quad \stackrel{\mathrm{d}}{\Longrightarrow} \quad \mathbb{P}(X \in \cdot).$$

Remark 4. Note that the Rayleigh distribution is the law of a Brownian meander (starting from 0) at time 1, so it is not surprising that it appears in Corollary 1(i) above. Furthermore, compared with [14, Theorem 3] in the case of critical branching random walks, for $\rho \leq 0$, the weak limit of $\widetilde{M}_t^{-\rho}$ conditioned on survival up to time t is simpler. The limit in [14, Theorem 3] is related to the maximum of a measure-valued process (see [14, Corollary 4]).

Remark 5. It is natural to study similar problems for subcritical branching killed Lévy processes. However, in the general case, even when the spatial motion is spectrally negative, some of the main ingredients, such as Lemma 1, are much more difficult. So, to avoid technical details, we concentrate on the case of subcritical branching killed Brownian motion with drift. It also natural to study similar problems for subcritical branching killed random walks.

Organization of the paper. The rest of the paper is organized as follows. In Section 2.1, we first give some results on Brownian motion and the three-dimensional Bessel process that will be used in the proofs of our main results. Then we recall some connections between a certain evolution equation and our model in Section 2.2. The proofs of Theorems 1 and 3 are given in Section 3 and the proof of Theorem 2 is given in Section 4.

2. Preliminaries

2.1. Some useful properties of Brownian motion

Recall that $(B_t, \mathbf{P}_x)$ is a standard Brownian motion starting from x and $\mathcal{F}_t := \sigma(B_s : s \leq t)$. For any $z \in \mathbb{R}$, define $\tau_z := \inf\{t > 0 : B_t = z\}$. Note that for any $x > 0$, under $\mathbf{P}_x$, $\frac{B_t}{x}1_{\{\tau_0>t\}}$ is a positive martingale of mean 1. Define

$$\frac{\mathrm{d}\mathbf{P}_x^B}{\mathrm{d}\mathbf{P}_x}\Big|_{\mathcal{F}_t} := \frac{B_t}{x}1_{\{\tau_0>t\}} = \frac{B_t}{x}1_{\{\min_{s\leq t} B_s>0\}}. \tag{2.1}$$

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It is well known that $(B_t, \mathbf{P}_x^B)$ is a three-dimensional Bessel process with transition probability density $p_t^B(x, y)$ given by

$$p_t^B(x, y) := \frac{y}{x}\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(y-x)^2}{2t}}\left(1-\mathrm{e}^{-\frac{2xy}{t}}\right)1_{\{y>0\}}.$$

Recall that $\mathbf{P}_x^{-\rho}$ is defined in (1.11). The following result gives the asymptotic behavior of $\mathbf{P}_x^{-\rho}(\tau_0 > t, B_t > z(t, \rho))$ as $t \to \infty$ where $z(t, \rho)$ is defined in (1.10). For the case $\rho < 0$, see [5, p. 30], and for the case $\rho > 0$, see [17, (7) and Lemma 3.1]. The case $\rho = 0$ follows from the reflection principle.

Lemma 1. *Let* $x > 0$.

(i) *If* $\rho = 0$, *then for any* $z \geq 0$, *we have*

$$\lim_{t\to\infty}\sqrt{t}\mathbf{P}_x(\tau_0 > t, B_t > \sqrt{t}z) = \sqrt{\frac{2}{\pi}}x\mathrm{e}^{-\frac{z^2}{2}}.$$

(ii) *If* $\rho < 0$, *then*

$$\lim_{t\to\infty}\mathbf{P}_x^{-\rho}(\tau_0 > t) = 1-\mathrm{e}^{2\rho x}.$$

Also, for any $z \in \mathbb{R}$,

$$\lim_{t\to\infty}\mathbf{P}_x^{-\rho}\left(\tau_0 > t, B_t + \rho t > \sqrt{t}z\right) = \frac{(1-\mathrm{e}^{2\rho x})}{\sqrt{2\pi}}\int_z^\infty \mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y.$$

(iii) *If* $\rho > 0$, *then for any* $z \geq 0$,

$$\lim_{t\to\infty} t^{\frac{3}{2}}\mathrm{e}^{\frac{\rho^2}{2}t}\mathbf{P}_x^{-\rho}(\tau_0 > t, B_t > z) = \sqrt{\frac{2}{\pi}}x\mathrm{e}^{\rho x}\int_z^\infty y\mathrm{e}^{-\rho y}\mathrm{d}y.$$

In the following result, we give the asymptotic behaviors of $\mathbb{E}_x\left(\widetilde{Z}_t^{-\rho}((0, \infty))\right)$ and $\mathbb{E}_x\left(\widetilde{Z}_t^{-\rho}((z(t, \rho), \infty))\right)$ as $t \to \infty$.

Lemma 2. *Let* $x > 0$.

(i) *If* $\rho = 0$, *then for any* $z \geq 0$,

$$\lim_{t\to\infty}\sqrt{t}\mathrm{e}^{\alpha t}\mathbb{E}_x\left(\widetilde{Z}_t^{-\rho}((\sqrt{t}z, \infty))\right) = \sqrt{\frac{2}{\pi}}x\mathrm{e}^{-\frac{z^2}{2}}.$$

(ii) *If* $\rho < 0$, *we have*

$$\lim_{t\to\infty}\mathrm{e}^{\alpha t}\mathbb{E}_x\big(\widetilde{Z}_t^{-\rho}((0, \infty))\big) = 1-\mathrm{e}^{2\rho x},$$

and for any $z \in \mathbb{R}$,

$$\lim_{t\to\infty}\mathrm{e}^{\alpha t}\mathbb{E}_x\left(\widetilde{Z}_t^{-\rho}((\sqrt{t}z-\rho t, \infty))\right) = \frac{1-\mathrm{e}^{2\rho x}}{\sqrt{2\pi}}\int_z^\infty \mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y.$$

(iii) *If $\rho > 0$, then for any $z \geq 0$,*

$$\lim_{t\to\infty} t^{3/2} \mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)t} \mathbb{E}_x\left(\widetilde{Z}_t^{-\rho}((z,\infty))\right)$$
$$= \sqrt{\frac{2}{\pi}} x\mathrm{e}^{\rho x} \int_z^\infty y\mathrm{e}^{-\rho y}\mathrm{d}y = \frac{1}{\rho^2}\sqrt{\frac{2}{\pi}} x\mathrm{e}^{\rho(x-z)}(\rho z+1).$$

Proof. For any bounded measurable function F, by the many-to-one lemma (see Hardy and Harris [9, Theorem 2.8]), we have

$$\mathbb{E}_x\Big(\sum_{u\in N_t^{-\rho}} F(X_u(s),\, 0\leq s\leq t)\Big) = \mathrm{e}^{-\alpha t}\mathbf{E}_x^{-\rho}\left(F(B_s,\, 0\leq s\leq t)\right), \tag{2.2}$$

which implies that

$$\mathbb{E}_x\big(\widetilde{Z}_t^{-\rho}((0,\infty))\big) = \mathrm{e}^{-\alpha t}\mathbf{P}_x^{-\rho}(\tau_0 > t)$$

and

$$\mathbb{E}_x\big(\widetilde{Z}_t^{-\rho}((z(t,\rho),\infty))\big) = \mathrm{e}^{-\alpha t}\mathbf{P}_x^{-\rho}(B_t > z(t,\rho),\, \tau_0 > t).$$

Combining this with Lemma 1, we arrive at the desired result. □

Lemma 5 below will play an important role in the proof of Theorem 2. To prove this result, we give two elementary lemmas first. The proofs of these two lemmas are routine and we give the details for completeness.

Lemma 3. *For any $a \geq 0$, $0 < x \leq y$ and nonnegative Borel function h, we have*

$$\mathbf{E}_x\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y}h(B_s)\mathrm{d}s}\right) = \frac{x}{y}\mathbf{E}_x^B\left(\mathrm{e}^{-a\tau_y-\int_0^{\tau_y}h(B_s)\mathrm{d}s}\right).$$

Proof. Note that $\mathbf{P}_x^B(\tau_y=\infty)=0$ for any $0<x\leq y$. Since $\mathcal{F}_{\tau_y\wedge t}\subset\mathcal{F}_t$, it follows from (2.1) that

$$\begin{aligned}
\mathbf{E}_x^B\left(\mathrm{e}^{-a\tau_y-\int_0^{\tau_y}h(B_s)\mathrm{d}s}\right) &= \lim_{t\to\infty}\mathbf{E}_x^B\left(\mathrm{e}^{-a\tau_y-\int_0^{\tau_y}h(B_s)\mathrm{d}s}1_{\{\tau_y<t\}}\right)\\
&= \lim_{t\to\infty}\mathbf{E}_x\left(\frac{B_t}{x}1_{\{\tau_0>t\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y}h(B_s)\mathrm{d}s}1_{\{\tau_y<t\}}\right)\\
&= \lim_{t\to\infty}\mathbf{E}_x\left(\mathrm{e}^{-a\tau_y-\int_0^{\tau_y}h(B_s)\mathrm{d}s}1_{\{\tau_y<t\}}\mathbf{E}_x\left(\frac{B_t}{x}1_{\{\tau_0>t\}}|\mathcal{F}_{\tau_y\wedge t}\right)\right).
\end{aligned}$$

Since $\left(\frac{B_t}{x}1_{\{\tau_0>t\}}\right)_{t\geq 0}$ is a $\mathbf{P}_x$-martingale with respect to $(\mathcal{F}_t)_{t\geq0}$, by the optional stopping theorem, we have

$$\mathbf{E}_x\left(\frac{B_t}{x}1_{\{\tau_0>t\}}|\mathcal{F}_{\tau_y\wedge t}\right) = \frac{B_{\tau_y\wedge t}}{x}1_{\{\tau_0>\tau_y\wedge t\}}.$$

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It follows from the dominated convergence theorem that

$$\begin{aligned}\mathbf{E}_x^B\left(\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right) &= \frac{y}{x}\lim_{t\to\infty}\mathbf{E}_x\left(1_{\{\tau_y<t,\tau_0>\tau_y\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right)\\ &= \frac{y}{x}\mathbf{E}_x\left(1_{\{\tau_0>\tau_y\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right).\end{aligned}$$

This completes the proof. □

Lemma 4. *For any $a\geq 0$, $0<x\leq y$, $\rho\in\mathbb{R}$ and nonnegative Borel function h, we have*

$$\mathbf{E}_x^{-\rho}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right) = \mathrm{e}^{\rho(x-y)}\mathbf{E}_x\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-\left(a+\frac{\rho^2}{2}\right)\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right).$$

Proof. From [6, Theorem 6, p. 16], we know that $\{\tau_y<\tau_0\}\cap\{\tau_y<t\}=\{\tau_y\wedge t<\tau_0\}\cap\{\tau_y\wedge t<t\}$ is $\mathcal{F}_{\tau_y\wedge t}$-measurable. We deal with the case $a>0$ first. For $a>0$, since $\mathrm{e}^{-a\tau_y}1_{\{\tau_y=\infty\}}=0$, it follows from (1.11) that

$$\begin{aligned}&\mathbf{E}_x^{-\rho}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right) \qquad (2.3)\\ &= \lim_{t\to\infty}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}1_{\{\tau_y<t\}}\right)\\ &= \lim_{t\to\infty}\mathbf{E}_x\left(\mathrm{e}^{-\rho(B_t-x)-\frac{\rho^2}{2}t}1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}1_{\{\tau_y<t\}}\right)\\ &= \lim_{t\to\infty}\mathbf{E}_x\left(\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}1_{\{\tau_y<\tau_0,\tau_y<t\}}\mathbf{E}_x\left(\mathrm{e}^{-\rho(B_t-x)-\frac{\rho^2}{2}t}\Big|\mathcal{F}_{\tau_y\wedge t}\right)\right).\end{aligned}$$

Recall that $\left(\mathrm{e}^{-\rho(B_t-x)-\frac{\rho^2}{2}t}\right)_{t\geq 0}$ is a $\mathbf{P}_x$-martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, so by the optional stopping theorem, on $\{\tau_y<t\}$, we have

$$\mathbf{E}_x\left(\mathrm{e}^{-\rho(B_t-x)-\frac{\rho^2}{2}t}|\mathcal{F}_{\tau_y\wedge t}\right) = \mathrm{e}^{-\rho(B_{\tau_y\wedge t}-x)-\frac{\rho^2}{2}(\tau_y\wedge t)} = \mathrm{e}^{-\rho(y-x)-\frac{\rho^2}{2}\tau_y}.$$

Combining this with (2.3) and using the fact that $\mathbf{P}_x\left(\tau_y<\infty\right)=1$, we get for $a>0$,

$$\mathbf{E}_x^{-\rho}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right) = \mathrm{e}^{\rho(x-y)}\mathbf{E}_x\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-\left(a+\frac{\rho^2}{2}\right)\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right). \qquad (2.4)$$

For the case $a=0$, by the dominated convergence theorem and the display above,

$$\begin{aligned}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right) &= \lim_{\theta\to 0+}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-\theta\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right)\\ &= \lim_{\theta\to 0+}\mathrm{e}^{\rho(x-y)}\mathbf{E}_x\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-\left(\theta+\frac{\rho^2}{2}\right)\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right)\\ &= \mathrm{e}^{\rho(x-y)}\mathbf{E}_x\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-\frac{\rho^2}{2}\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right).\end{aligned}$$

This completes the proof. □

Combining Lemmas 3 and 4, we immediately get the following result.

Lemma 5. *For any $a \geq 0$, $0 < x \leq y$, $\rho \in \mathbb{R}$ and nonnegative Borel function h, we have*

$$\mathbf{E}_x^{-\rho}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-a\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right) = \frac{x}{y}\mathrm{e}^{\rho(x-y)}\mathbf{E}_x^B\left(\mathrm{e}^{-\left(a+\frac{\rho^2}{2}\right)\tau_y-\int_0^{\tau_y} h(B_s)\mathrm{d}s}\right).$$

The following result can be found in [5, p. 469].

Lemma 6. *For any $a > 0$ and $0 < x \leq y$, we have*

$$\mathbf{E}_x^B\left(\mathrm{e}^{-a\tau_y}\right) = \frac{y\sinh(x\sqrt{2a})}{x\sinh(y\sqrt{2a})}.$$

Combining Lemmas 5 and 6, we see that for any $\rho < 0$ and $x > 0$,

$$\lim_{y\to\infty}\mathbf{P}_x^{-\rho}\left(\tau_y < \tau_0\right) = \lim_{y\to\infty}\mathrm{e}^{\rho(x-y)}\frac{\sinh(-x\rho)}{\sinh(-y\rho)} = 1-\mathrm{e}^{2\rho x}. \tag{2.5}$$

2.2. An evolution equation related to branching killed Brownian motion

Recall that $(B_t, \mathbf{P}_x^{-\rho})$ is a Brownian motion with drift $-\rho$ and that τ_0 is the first time that B hits 0. Let $\xi_t := B_t 1_{\{\tau_0>t\}}$ be the Brownian motion killed upon hitting 0. Then the branching killed Brownian motion with drift is also a branching Markov process with spatial motion $(\xi_t, \mathbf{P}_x^{-\rho})$, branching rate β and offspring distribution $\{p_k : k \in \mathbb{N}\}$. Let $\widetilde{N}_t^{-\rho}$ be the set of particles alive at time t of the branching killed Brownian motion. It is well known that, for any [0,1]-valued Borel function h, the function

$$u_h(x,t) = 1-\mathbb{E}_x\left(\prod_{v\in\widetilde{N}_t^{-\rho}} h(X_v^{-\rho}(t))\right) \tag{2.6}$$

is the unique positive locally bounded solution to

$$u_h(x,t) = \mathbf{E}_x^{-\rho}(h(B_t), t<\tau_0) - \mathbf{E}_x^{-\rho}\left(\int_0^t \Phi(u_h(B_s, t-s))\mathrm{d}s, t<\tau_0\right), \tag{2.7}$$

where the function Φ is given as in (1.2) (see, for example, [16, (4.8), p. 102]). Now we check that the unique solution is given by $\mathbf{E}_x^{-\rho}\left(h(B_t)\mathrm{e}^{-\int_0^t \frac{\Phi(u_h(B_s,t-s))}{u_h(B_s,t-s)}\mathrm{d}s}, t<\tau_0\right)$. For $0 \leq s \leq t$, define

$$A_{s,t} = -\int_s^t \frac{\Phi(u_h(B_r, t-r))}{u_h(B_r, t-r)}\mathrm{d}r.$$

It is elementary to check that

$$\mathrm{e}^{A_{0,t}} = 1-\int_0^t \mathrm{e}^{A_{s,t}}\frac{\Phi(u_h(B_s, t-s))}{u_h(B_s, t-s)}\mathrm{d}s.$$

Hence we have

$$\begin{aligned}&\mathbf{E}_x^{-\rho}\left(\mathrm{e}^{A_{0,t}}h(B_t), t<\tau_0\right) \\ &= \mathbf{E}_x^{-\rho}(h(B_t), t<\tau_0) - \mathbf{E}_x^{-\rho}\left(h(B_t)\int_0^t \mathrm{e}^{A_{s,t}}\frac{\Phi(u_h(B_s, t-s))}{u_h(B_s, t-s)}\mathrm{d}s, t<\tau_0\right).\end{aligned} \tag{2.8}$$

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Now using the Markov property and the fact that

$$A_{s,t} = \int_0^{t-s} \frac{\Phi(u_h(B_{r+s}, t-s-r))}{u_h(B_{r+s}, t-s-r)} \mathrm{d}r,$$

equation (2.8) says that $\mathbf{E}_x^{-\rho}\left(h(B_t)\mathrm{e}^{-\int_0^t \frac{\Phi(u_h(B_s,t-s))}{u_h(B_s,t-s)}\mathrm{d}s}, t<\tau_0\right)$ satisfies (2.7). Thus we have

$$\begin{aligned} u_h(x,t) &= \mathbf{E}_x^{-\rho}\left(h(B_t)\mathrm{e}^{-\int_0^t \frac{\Phi(u_h(B_s,t-s))}{u_h(B_s,t-s)}\mathrm{d}s}, t<\tau_0\right) \\ &= \mathrm{e}^{-\alpha t}\mathbf{E}_x^{-\rho}\left(h(B_t)\mathrm{e}^{-\int_0^t \varphi(u_h(B_s,t-s))\mathrm{d}s}, t<\tau_0\right). \end{aligned} \tag{2.9}$$

By the Markov property, the function

$$u(x,t) := \mathbb{P}_x(\widetilde{\zeta}^{-\rho} > t) \tag{2.10}$$

satisfies $u(x,t) = 1 - \mathbb{E}_x\left(\prod_{v\in \widetilde{N}_t^{-\rho}} u(X_v^{-\rho}(s), t-s)\right)$. By taking $s=t$ in (2.8), we get that the function u defined in (2.10) has the representation (2.9) with $h(x) = 1_{\{x>0\}}$. By a similar argument, we can get that the function

$$Q_z(x,t) := \mathbb{P}_x(\widetilde{M}_t^{-\rho} > z), \qquad x, t > 0, \tag{2.11}$$

has the representation (2.9) with $h(x) = 1_{\{x>z\}}$.

Recall that φ is defined in (1.2). The next simple result will be used in the proofs of our main results.

Lemma 7. *The function $\varphi(u)$ is increasing in $u \in [0,1]$. Moreover, under (1.4), for any $c>0$, we have*

$$\int_0^\infty \varphi\left(\mathrm{e}^{-ct}\right)\mathrm{d}t < \infty.$$

Proof. By the definition of φ,

$$\begin{aligned} \beta^{-1}\varphi(u) &= \frac{\sum_{k=0}^\infty p_k(1-u)^k - (1-u)}{u} - \left(1 - \sum_{k=0}^\infty kp_k\right) \\ &= \sum_{\ell=0}^\infty \left(\sum_{k=\ell+1}^\infty p_k\right) - \sum_{k=1}^\infty p_k \sum_{\ell=0}^{k-1} (1-u)^\ell = \sum_{\ell=0}^\infty \left(\sum_{k=\ell+1}^\infty p_k\right)\left(1-(1-u)^\ell\right). \end{aligned}$$

Therefore, φ is increasing in u. Combining the monotonicity of φ and (1.5), we have

$$\int_0^\infty \varphi(C_{\mathrm{sub}}\mathrm{e}^{-\alpha t})\mathrm{d}t \le \int_0^\infty \varphi(g(t))\mathrm{d}t < \infty.$$

Setting $N := -\frac{1}{\alpha}\log C_{\mathrm{sub}}$, then for any $c>0$,

$$\begin{aligned} \int_0^\infty \varphi\left(\mathrm{e}^{-ct}\right)\mathrm{d}t &= \frac{\alpha}{c}\int_0^\infty \varphi\left(\mathrm{e}^{-\alpha t}\right)\mathrm{d}t \le \frac{\alpha}{c}\int_0^N \varphi(1)\mathrm{d}t + \frac{\alpha}{c}\int_0^\infty \varphi(\mathrm{e}^{-\alpha(t-N)})\mathrm{d}t \\ &= \frac{\alpha}{c}N\varphi(1) + \frac{\alpha}{c}\int_0^\infty \varphi(C_{\mathrm{sub}}\mathrm{e}^{-\alpha t})\mathrm{d}t < \infty. \end{aligned}$$

□

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3. Proofs of Theorems 1 and 3

In this section, we prove Theorems 1 and 3 by establishing some upper and lower bounds for the functions $u(t,x)$ and $Q_z(x, t)$ defined in (2.10) and (2.11), respectively. It is easy to see that

$$Q_0(x, t) = \mathbb{P}_x\big(\widetilde{M}_t^{-\rho} > 0\big) = \mathbb{P}_x\big(\widetilde{\zeta}^{-\rho} > t\big) = u(x, t). \tag{3.1}$$

We first estimate $Q_{\sqrt{t}z-\rho t}(x, t)$ and $u(x,t)$ from below. We treat the cases $\rho = 0$ and $\rho < 0$ together since it turns out that branching and spatial motion are nearly independent in these two cases.

Lemma 8. *Suppose that $x > 0$ and $\rho \le 0$.*

(i) *If $\rho = 0$, then for any $z \ge 0$,*

$$\liminf_{t\to\infty} \sqrt{t}\mathrm{e}^{\alpha t} Q_{\sqrt{t}z}(x, t) \ge \sqrt{\frac{2}{\pi}} C_{\mathrm{sub}} x \mathrm{e}^{-\frac{z^2}{2}}.$$

(ii) *If $\rho < 0$, then*

$$\liminf_{t\to\infty} \mathrm{e}^{\alpha t} u(x, t) \ge C_{\mathrm{sub}}\big(1 - \mathrm{e}^{2\rho x}\big),$$

and for any $z \in \mathbb{R}$,

$$\liminf_{t\to\infty} \mathrm{e}^{\alpha t} Q_{\sqrt{t}z-\rho t}(x, t) \ge \frac{C_{\mathrm{sub}}(1 - \mathrm{e}^{2\rho x})}{\sqrt{2\pi}} \int_z^\infty \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y.$$

Proof. At the end of the first paragraph of Section 2.2, we have shown that $Q_z(x, t)$ defined in (2.11) admits the following expression:

$$Q_z(x, t) = \mathrm{e}^{-\alpha t} \mathbf{E}_x^{-\rho} \left(1_{\{\tau_0 > t, B_t > z\}} \mathrm{e}^{-\int_0^t \varphi(Q_z(B_s, t-s))\mathrm{d}s}\right). \tag{3.2}$$

Since $\widetilde{\zeta} \le \zeta$, we have that

$$Q_z(x, t) \le \mathbb{P}_x(\zeta > t) = g(t), \quad x, t > 0, z \ge 0. \tag{3.3}$$

Thus by Lemma 7,

$$\begin{aligned} Q_z(x, t) &\ge \mathrm{e}^{-\alpha t} \mathbf{E}_x^{-\rho} \left(1_{\{\tau_0 > t, B_t > z\}} \mathrm{e}^{-\int_0^t \varphi(g(t-s))\mathrm{d}s}\right) \\ &= \mathrm{e}^{-\int_0^t \varphi(g(s))\mathrm{d}s} \mathrm{e}^{-\alpha t} \mathbf{P}_x^{-\rho}(\tau_0 > t, B_t > z) \\ &\ge C_{\mathrm{sub}} \mathrm{e}^{-\alpha t} \mathbf{P}_x^{-\rho}(\tau_0 > t, B_t > z), \end{aligned}$$

where in the last inequality we used (1.6). Recalling (3.1) and using Lemma 1 with z replaced by 0 and $\sqrt{t}z - \rho t$, we get the desired result. □

Lemma 9. *Assume that $\rho = 0$ and $x > 0$. Then for any $z \ge 0$, we have that*

$$\limsup_{t\to\infty} \sqrt{t}\mathrm{e}^{\alpha t} Q_{\sqrt{t}z}(x, t) \le \sqrt{\frac{2}{\pi}} C_{\mathrm{sub}} x \mathrm{e}^{-\frac{z^2}{2}}.$$

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Proof. For any $y \geq x$,

$$Q_z(y, t) = \mathbb{P}_y\Big(\exists u \in N_t \quad s.t. \min_{s\leq t} X_u^{-\rho}(s) > 0,\ X_u^{-\rho}(t) > z\Big) \tag{3.4}$$
$$\geq \mathbb{P}_y\Big(\exists u \in N_t \quad s.t. \min_{s\leq t} X_u^{-\rho}(s) > y - x,\ X_u^{-\rho}(t) > z + y - x\Big)$$
$$= \mathbb{P}_x\Big(\exists u \in N_t \quad s.t. \min_{s\leq t} X_u^{-\rho}(s) > 0,\ X_u^{-\rho}(t) > z\Big) = Q_z(x, t),$$

which implies that $Q_z(x, t)$ is increasing in x. Fix an $N > 0$. For $t \geq N$, by (3.4),

$$Q_{\sqrt{t}z}(x, t) \leq \mathrm{e}^{-\alpha t}\mathbf{E}_x\left(1_{\{\tau_0>t, B_t>\sqrt{t}z\}}\mathrm{e}^{-\int_{t-N}^t \varphi(Q_{\sqrt{t}z}(B_s, t-s))\mathrm{d}s}\right) \tag{3.5}$$
$$\leq \mathrm{e}^{-\alpha t}\mathbf{E}_x\left(1_{\{\tau_0>t, B_t>\sqrt{t}z\}}\mathrm{e}^{-\int_{t-N}^t \varphi(Q_{\sqrt{t}z}(\inf_{r\in[t-N,t]} B_r, t-s))\mathrm{d}s}\right)$$
$$= \mathrm{e}^{-\alpha t}\mathbf{E}_x\left(1_{\{\tau_0>t, B_t>\sqrt{t}z\}}\mathrm{e}^{-\int_0^N \varphi(Q_{\sqrt{t}z}(\inf_{r\in[t-N,t]} B_r, s))\mathrm{d}s}\right).$$

Take a $\gamma \in (0, \frac{1}{2})$ and define

$$I_1(t) := \mathbf{E}_x\left(1_{\{\tau_0>t, B_t>\sqrt{t}z, \inf_{r\in[t-N,t]} B_r\geq\sqrt{t}z+t^\gamma\}}\mathrm{e}^{-\int_0^N \varphi(Q_{\sqrt{t}z}(\inf_{r\in[t-N,t]} B_r, s))\mathrm{d}s}\right),$$
$$I_2(t) := \mathbf{E}_x\left(1_{\{\tau_0>t, B_t>\sqrt{t}z, \inf_{r\in[t-N,t]} B_r<\sqrt{t}z+t^\gamma\}}\mathrm{e}^{-\int_0^N \varphi(Q_{\sqrt{t}z}(\inf_{r\in[t-N,t]} B_r, s))\mathrm{d}s}\right).$$

Then $Q_{\sqrt{t}z}(x, t) \leq \mathrm{e}^{-\alpha t}(I_1(t) + I_2(t))$. Since $Q_z(x, t)$ is increasing in x, we have

$$I_1(t) \leq \mathrm{e}^{-\int_0^N \varphi(Q_{\sqrt{t}z}(\sqrt{t}z+t^\gamma,\, s))\mathrm{d}s}\mathbf{P}_x(\tau_0 > t, B_t > \sqrt{t}z). \tag{3.6}$$

Set $\widetilde{M}_s := \widetilde{M}_s^0$, $M_s := M_s^0$ and $X_u^0(r) := X_u(r)$ for simplicity. For any $s \leq N$, we have

$$Q_{\sqrt{t}z}(\sqrt{t}z + t^\gamma, s) \geq \mathbb{P}_{\sqrt{t}z+t^\gamma}(\widetilde{M}_s > \sqrt{t}z, \inf_{r\leq s}\inf_{u\in N_r} X_u(r) > 0)$$
$$= \mathbb{P}_{\sqrt{t}z+t^\gamma}(M_s > \sqrt{t}z) - \mathbb{P}_{\sqrt{t}z+t^\gamma}(M_s > \sqrt{t}z, \inf_{r\leq s}\inf_{u\in N_r} X_u(r) \leq 0)$$
$$\geq \mathbb{P}_0(M_s > -t^\gamma) - \mathbb{P}_0(\inf_{r\leq s}\inf_{u\in N_r} X_u(r) \leq -(\sqrt{t}z + t^\gamma))$$
$$= \mathbb{P}_0(M_s > -t^\gamma) - \mathbb{P}_0(\max_{r\leq s} M_r \geq \sqrt{t}z + t^\gamma).$$

According to (2.2),

$$\mathbb{P}_0(M_s > -t^\gamma) \geq \mathbb{P}_0(\zeta > s, M_s > -t^\gamma) = \mathbb{P}_0(\zeta > s) - \mathbb{P}_0(\zeta > s, M_s \leq -t^\gamma) \tag{3.7}$$
$$\geq \mathbb{P}_0(\zeta > s) - \mathbb{P}_0\left(\sum_{u\in N_s} 1_{\{X_u(s)\leq t^{-\gamma}\}} \geq 1\right)$$
$$\geq \mathbb{P}_0(\zeta > s) - \mathrm{e}^{-\alpha s}\mathbf{P}_0(B_s \leq -t^\gamma).$$

Therefore, for any fixed $s \leq N$, we get

$$Q_{\sqrt{t}z}(\sqrt{t}z + t^\gamma, s) \geq g(s) - \mathrm{e}^{-\alpha s}\mathbf{P}_0(B_s \leq -t^\gamma) - \mathbb{P}_0(\max_{r\leq s} M_r \geq \sqrt{t}z + t^\gamma).$$

Plugging this into (3.6) and applying the dominated convergence theorem, we get

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{I_1(t)}{\mathbf{P}_x(\tau_0 > t,\, B_t > \sqrt{t}z)} \\
&\le \limsup_{t\to\infty} \exp\left\{-\int_0^N \varphi\left(\left(g(s) - \mathrm{e}^{-\alpha s}\mathbf{P}_0(B_s \le -t^\gamma) - \mathbb{P}_0(\max_{r\le s} M_r \ge \sqrt{t}z + t^\gamma)\right)_+\right)\mathrm{d}s\right\} \\
&= \mathrm{e}^{-\int_0^N \varphi(g(s))\mathrm{d}s}.
\end{aligned}$$

Letting $N \to \infty$, we get

$$\limsup_{N\to\infty}\limsup_{t\to\infty} \frac{I_1(t)}{\mathbf{P}_x(\tau_0 > t,\, B_t > \sqrt{t}z)} \le \mathrm{e}^{-\int_0^\infty \varphi(g(s))\mathrm{d}s} = C_{\mathrm{sub}} < \infty.$$

Therefore, applying Lemma 1(i), we obtain that

$$\limsup_{N\to\infty}\limsup_{t\to\infty} \sqrt{t}I_1(t) \le \sqrt{\frac{2}{\pi}}C_{\mathrm{sub}}x\mathrm{e}^{-z^2/2}. \tag{3.8}$$

Next, we show that $\lim_{t\to\infty} \sqrt{t}I_2(t) = 0$. For $\delta > 0$, it holds that

$$\begin{aligned}
I_2(t) &\le \mathbf{P}_x(\tau_0 > t,\, B_t > \sqrt{t}z,\, \inf_{r\in[t-N,t]} B_r < \sqrt{t}z + t^\gamma) \qquad (3.9)\\
&\le \mathbf{P}_x\big(\tau_0 > t,\, \sqrt{t}z < B_t < \sqrt{t}(z+\delta)\big) \\
&\quad + \mathbf{P}_x\Big(B_t \ge \sqrt{t}(z+\delta),\, \inf_{r\in[t-N,t]} B_r < \sqrt{t}z + t^\gamma\Big).
\end{aligned}$$

Note that $\mathrm{e}^{-u}(1-\mathrm{e}^{-x}) \le x$ for all $u, x > 0$. Thus by (2.1), we get

$$\begin{aligned}
&\mathbf{P}_x\left(\tau_0 > t,\, \sqrt{t}z < B_t < \sqrt{t}(z+\delta)\right) = \mathbf{E}_x^B\left(\frac{x}{B_t}1_{\{\sqrt{t}z < B_t < \sqrt{t}(z+\delta)\}}\right) \qquad (3.10)\\
&= \int_{\sqrt{t}z}^{\sqrt{t}(z+\delta)} \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(y-x)^2}{2t}}\left(1 - \mathrm{e}^{-\frac{2xy}{t}}\right)\mathrm{d}y \le \frac{\delta}{\sqrt{2\pi}}\frac{2x(z+\delta)}{\sqrt{t}}.
\end{aligned}$$

For any $t \ge N$, by the reflection principle, we have

$$\begin{aligned}
&\mathbf{P}_x\Big(B_t \ge \sqrt{t}(z+\delta),\, \inf_{r\in[t-N,t]} B_r < \sqrt{t}z + t^\gamma\Big) \qquad (3.11)\\
&\le \mathbf{P}_0\Big(\inf_{r\in[0,N]} B_r < -\delta\sqrt{t} + t^\gamma\Big) = \mathbf{P}_0\left(|B_N| > \delta\sqrt{t} - t^\gamma\right).
\end{aligned}$$

Combining (3.9), (3.10) and (3.11), letting $t \to \infty$ first and then $\delta \to 0$, we get

$$\lim_{t\to\infty} \sqrt{t}I_2(t) = 0.$$

Combining this with (3.5) and (3.8), we get the desired assertion. □

Lemma 10. *Assume that* $x > 0$ *and* $\rho < 0$.

(i) *It holds that*

$$\limsup_{t\to\infty} \mathrm{e}^{\alpha t}u(x, t) \le C_{\mathrm{sub}}(1 - \mathrm{e}^{2\rho x}).$$

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(ii) *For any $z \in \mathbb{R}$, we have*

$$\limsup_{t\to\infty} e^{\alpha t} Q_{\sqrt{t}z-\rho t}(x,t) \le \frac{C_{\mathrm{sub}}(1-e^{2\rho x})}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{y^2}{2}}\,dy.$$

Proof. We will prove (i) and (ii) in one stroke. For (i) we put $z_t = 0$ and for (ii) we put $z_t = \sqrt{t}z - \rho t$. Then taking $z = z_t$ in (3.2), we get

$$\begin{aligned} Q_{z_t}(x,t) &\le e^{-\alpha t}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t, B_t>z_t\}} e^{-\int_{t-N}^t \varphi(Q_{z_t}(B_s, t-s))ds}\right) \\ &\le e^{-\alpha t}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t, B_t>z_t\}} e^{-\int_0^N \varphi(Q_{z_t}(\inf_{r\in[t-N,t]} B_r, s))ds}\right). \end{aligned} \tag{3.12}$$

Take a $\gamma \in (0, \frac{1}{2})$ and define

$$\begin{aligned} C_1(t) &:= \mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t, B_t>z_t, \inf_{r\in[t-N,t]} B_r \ge z_t + t^\gamma\}} e^{-\int_0^N \varphi(Q_{z_t}(\inf_{r\in[t-N,t]} B_r, s))ds}\right), \\ C_2(t) &:= \mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t, B_t>z_t, \inf_{r\in[t-N,t]} B_r < z_t + t^\gamma\}} e^{-\int_0^N \varphi(Q_{z_t}(\inf_{r\in[t-N,t]} B_r, s))ds}\right). \end{aligned}$$

Then $Q_{z_t}(x,t) \le e^{-\alpha t}(C_1(t) + C_2(t))$. Using (3.4), we have

$$C_1(t) \le e^{-\int_0^N \varphi(Q_{z_t}(z_t + t^\gamma, s))ds}\mathbf{P}_x^{-\rho}(\tau_0 > t, B_t > z_t). \tag{3.13}$$

For any $s \le N$, similarly to (3.7), for t large enough such that $z_t \ge 0$, we have

$$\begin{aligned} Q_{z_t}(z_t + t^\gamma, s) &\ge \mathbb{P}_{z_t+t^\gamma}(\widetilde{M}_s^{-\rho} > z_t, \inf_{r\le s}\inf_{u\in N_r} X_u^{-\rho}(r) > 0) \\ &= \mathbb{P}_{z_t+t^\gamma}(M_s^{-\rho} > z_t) - \mathbb{P}_{z_t+t^\gamma}(M_s^{-\rho} > z_t, \inf_{r\le s}\inf_{u\in N_r} X_u^{-\rho}(r) \le 0) \\ &\ge \mathbb{P}_0(M_s^{-\rho} > -t^\gamma) - \mathbb{P}_0(\inf_{r\le s}\inf_{u\in N_r} X_u^{-\rho}(r) \le -(z_t + t^\gamma)) \\ &\ge \mathbb{P}_0(M_s > -t^\gamma) - \mathbb{P}_0(\max_{r\le s} M_r^\rho \ge t^\gamma), \end{aligned}$$

where the last inequality follows from $M_s^{-\rho} \ge M_s$ and $z_t \ge 0$. Combining this with (3.7), we get

$$Q_{z_t}(z_t + t^\gamma, s) \ge g(s) - e^{-\alpha s}\mathbf{P}_0(B_s \le -t^\gamma) - \mathbb{P}_0(\max_{r\le s} M_r^\rho \ge t^\gamma).$$

Letting $N \to \infty$ in (3.13) and combining the resulting conclusion with the above, we get

$$\limsup_{N\to\infty}\limsup_{t\to\infty} \frac{C_1(t)}{\mathbf{P}_x^{-\rho}(\tau_0 > t, B_t > z_t)} \le e^{-\int_0^\infty \varphi(g(s))ds} = C_{\mathrm{sub}}.$$

Applying Lemma 1(ii), we get that for $z_t = 0$,

$$\limsup_{N\to\infty}\limsup_{t\to\infty} C_1(t) \le C_{\mathrm{sub}}(1 - e^{2\rho x}), \tag{3.14}$$

and for $z_t = \sqrt{t}z - \rho t$,

$$\limsup_{N\to\infty}\limsup_{t\to\infty} C_1(t) \le \frac{C_{\mathrm{sub}}(1-e^{2\rho x})}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{y^2}{2}}\,dy. \tag{3.15}$$

Next, we show that $\lim_{t\to\infty} C_2(t)=0$. For $\delta>0$, we have

$$\begin{aligned}C_2(t) &\le \mathbf{P}_x^{-\rho}(\tau_0>t,\, B_t>z_t,\, \inf_{r\in[t-N,t]} B_r<z_t+t^\gamma)\\ &\le \mathbf{P}_x^{-\rho}(z_t<B_t<z_t+\sqrt{t}\delta)\\ &\quad+\mathbf{P}_x^{-\rho}(B_t\ge z_t+\sqrt{t}\delta,\, \inf_{r\in[t-N,t]} B_r<z_t+t^\gamma).\end{aligned}$$

Since the density of B_t under $\mathbf{P}_x^{-\rho}$ is equal to $\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(y-x+\rho t)^2}{2t}}\le\frac{1}{\sqrt{2\pi t}}$, we have

$$\mathbf{P}_x^{-\rho}(z_t<B_t<z_t+\sqrt{t}\delta)\le\int_{z_t}^{z_t+\sqrt{t}\delta}\frac{1}{\sqrt{2\pi t}}\mathrm{d}y=\frac{\delta}{\sqrt{2\pi}}.$$

Moreover, for any fixed $N>0$, similarly to (3.11), we have, for $t\ge N$,

$$\mathbf{P}_x^{-\rho}(B_t\ge z_t+\sqrt{t}\delta,\, \inf_{r\in[t-N,t]} B_r<z_t+t^\gamma)\le \mathbf{P}_0\big(|B_N|>\delta\sqrt{t}-t^\gamma-N\rho\big).$$

Letting $t\to\infty$ first and then $\delta\to 0$, we get that, for any $\rho<0$, $\lim_{t\to\infty} C_2(t)=0$. Combining this with (3.12), (3.14) and (3.15), we get the desired assertion. □

Now we consider the asymptotic behavior of $Q_z(x,t)$ as $t\to\infty$ for $\rho>0$. Fix an $N>0$ and define

$$f_N^z(y):=\mathbf{E}_y^{-\rho}\Big(1_{\{\tau_0>N,B_N>z\}}\mathrm{e}^{-\int_0^N\varphi(Q_z(B_s,N-s))\mathrm{d}s}\Big),\qquad y>0,\, z\ge 0.$$

Obviously, f_N^z is a bounded function on $(0,\infty)$. Combining with (3.2), we easily see that

$$f_N^z(y)=\mathrm{e}^{\alpha N}\mathbb{P}_y\big(\widetilde{M}_N^{-\rho}>z\big), \tag{3.16}$$

which implies that f_N^z is increasing with respect to y.

Lemma 11. *Assume that $\rho>0$, $x>0$ and $z\ge 0$. It holds that*

$$\begin{aligned}&\lim_{t\to\infty} t^{3/2}\mathrm{e}^{\frac{\rho^2}{2}t}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t,B_t>z\}}\mathrm{e}^{-\int_{t-N}^t\varphi(Q_z(B_s,t-s))\mathrm{d}s}\right)\\ &=\sqrt{\frac{2}{\pi}}x\mathrm{e}^{\rho x}\mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)N}\int_0^\infty\mathbb{P}_y\left(\widetilde{M}_N^{-\rho}>z\right)y\mathrm{e}^{-\rho y}\mathrm{d}y.\end{aligned}$$

Proof. By the Markov property,

$$\begin{aligned}&\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t,B_t>z\}}\mathrm{e}^{-\int_{t-N}^t\varphi(Q_z(B_s,t-s))\mathrm{d}s}\right)\\ &=\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t-N\}}\mathbf{E}_{B_{t-N}}^{-\rho}\left(1_{\{\tau_0>N,B_N>z\}}\mathrm{e}^{-\int_0^N\varphi(Q_z(B_s,N-s))\mathrm{d}s}\right)\right)\\ &=\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t-N\}}f_N^z(B_{t-N})\right)=\mathbf{E}_x^{-\rho}\left(f_N^z(B_{t-N})|\tau_0>t-N\right)\mathbf{P}_x^{-\rho}(\tau_0>t-N).\end{aligned}$$

Since f_N^z is increasing and bounded, it is almost everywhere continuous. Then applying Lemma 1(iii), we get that

$$\begin{aligned}&\lim_{t\to\infty} t^{3/2}\mathrm{e}^{\frac{\rho^2}{2}(t-N)}\mathbf{E}_x^{-\rho}\left(f_N^z(B_{t-N})|\tau_0>t-N\right)\mathbf{P}_x^{-\rho}(\tau_0>t-N)\\ &=\rho^2\int_0^\infty f_N^z(y)y\mathrm{e}^{-\rho y}\mathrm{d}y\times\sqrt{\frac{2}{\pi}}x\rho^{-2}\mathrm{e}^{\rho x}=\sqrt{\frac{2}{\pi}}x\mathrm{e}^{\rho x}\int_0^\infty f_N^z(y)y\mathrm{e}^{-\rho y}\mathrm{d}y,\end{aligned}$$

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which implies the desired result together with (3.16). □

Proofs of Theorem 1 and Theorem 3. Parts (i) and (ii) of both Theorem 1 and Theorem 3 follow directly from Lemmas 8, 9, 10 and 2. So we only need to prove part (iii) of both theorems. By (3.1), it suffices to prove (iii) of Theorem 3. Fix $\rho > 0$, $N > 0$ and $z \geq 0$. By (3.2), we have, for $t \geq N$,

$$\begin{aligned} Q_z(x, t) &= \mathrm{e}^{-\alpha t}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t, B_t>z\}}\mathrm{e}^{-\int_0^t \varphi(Q_z(B_s,t-s))\mathrm{d}s}\right) \\ &\leq e^{-\alpha t}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t, B_t>z\}}\mathrm{e}^{-\int_{t-N}^t \varphi(Q_z(B_s,t-s))\mathrm{d}s}\right). \end{aligned} \tag{3.17}$$

Applying Lemma 11, we get

$$\begin{aligned} &\limsup_{t\to\infty} t^{3/2}\mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)t}Q_z(x, t) \\ &\leq \sqrt{\frac{2}{\pi}}xe^{\rho x}\liminf_{N\to\infty}\mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)N}\int_0^\infty \mathbb{P}_y\left(\widetilde{M}_N^{-\rho} > z\right)y\mathrm{e}^{-\rho y}\mathrm{d}y. \end{aligned} \tag{3.18}$$

It follows from (3.3) that

$$Q_z(x, t) \geq \mathrm{e}^{-\alpha t}\mathbf{E}_x^{-\rho}\left(1_{\{\tau_0>t, B_t>z\}}\mathrm{e}^{-\int_{t-N}^t \varphi(Q_z(B_s,t-s))\mathrm{d}s}\right)e^{-\int_0^{t-N}\varphi(g(t-s))\mathrm{d}s}. \tag{3.19}$$

Recall that the moment condition (1.4) is equivalent to (1.7), which implies that

$$1 \geq \mathrm{e}^{-\int_0^{t-N}\varphi(g(t-s))\mathrm{d}s} = \mathrm{e}^{-\int_N^t \varphi(g(s))\mathrm{d}s} \geq \mathrm{e}^{-\int_N^\infty \varphi(g(s))\mathrm{d}s} \overset{N\to\infty}{\longrightarrow} 1.$$

Using Lemma 11 again, we get

$$\begin{aligned} &\liminf_{t\to\infty} t^{3/2}e^{\left(\alpha+\frac{\rho^2}{2}\right)t}Q_z(x, t) \\ &\geq \limsup_{N\to\infty}\mathrm{e}^{-\int_N^\infty \varphi(g(s))\mathrm{d}s}\sqrt{\frac{2}{\pi}}xe^{\rho x}e^{\left(\alpha+\frac{\rho^2}{2}\right)N}\int_0^\infty \mathbb{P}_y\left(\widetilde{M}_N^{-\rho} > z\right)y\mathrm{e}^{-\rho y}\mathrm{d}y \\ &= \sqrt{\frac{2}{\pi}}xe^{\rho x}\limsup_{N\to\infty}\mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)N}\int_0^\infty \mathbb{P}_y\left(\widetilde{M}_N^{-\rho} > z\right)ye^{-\rho y}\mathrm{d}y. \end{aligned} \tag{3.20}$$

Combining (3.18) and (3.20), we get

$$\begin{aligned} &\lim_{t\to\infty} t^{3/2}\mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)t}Q_z(x, t) \\ &= \sqrt{\frac{2}{\pi}}xe^{\rho x}\lim_{N\to\infty} e^{\left(\alpha+\frac{\rho^2}{2}\right)N}\int_0^\infty \mathbb{P}_y\left(\widetilde{M}_N^{-\rho} > z\right)y\mathrm{e}^{-\rho y}\mathrm{d}y = \sqrt{\frac{2}{\pi}}xe^{\rho x}C_z(\rho), \end{aligned} \tag{3.21}$$

where $C_z(\rho) := \lim_{N\to\infty} e^{\left(\alpha+\frac{\rho^2}{2}\right)N}\int_0^\infty \mathbb{P}_y\left(\widetilde{M}_N^{-\rho} > z\right)y\mathrm{e}^{-\rho y}\mathrm{d}y$. Now we show that $C_z(\rho) \in (0, \infty)$. First, applying (3.17), we get

$$Q_z(x, t) \leq \mathrm{e}^{-\alpha t}\mathbf{P}_x^{-\rho}(\tau_0 > t, B_t > z).$$

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Combining this with Lemma 1, we get that

$$\limsup_{t\to\infty} t^{\frac{3}{2}} \mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)t} Q_z(x, t) \le \sqrt{\frac{2}{\pi}} x e^{\rho x} \int_z^\infty y e^{-\rho y} \mathrm{d}y. \tag{3.22}$$

Therefore, $C_z(\rho) \le \int_z^\infty y e^{-\rho y} \mathrm{d}y < \infty$. Next, by (3.3), we have

$$\begin{aligned} Q_z(x, t) &\ge \mathrm{e}^{-\int_0^t \varphi(g(s))\mathrm{d}s} \mathrm{e}^{-\alpha t} \mathbf{P}_x^{-\rho} (\tau_0 > t, B_t > z) \\ &\ge C_{\mathrm{sub}} \mathrm{e}^{-\alpha t} \mathbf{P}_x^{-\rho} (\tau_0 > t, B_t > z), \end{aligned} \tag{3.23}$$

where the last inequality follows from (1.6). Using Lemma 1 (iii) again, we get

$$\liminf_{t\to\infty} t^{\frac{3}{2}} \mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)t} Q_z(x, t) \ge C_{\mathrm{sub}} \sqrt{\frac{2}{\pi}} x e^{\rho x} \int_z^\infty y e^{-\rho y} \mathrm{d}y. \tag{3.24}$$

Therefore, we see that

$$C_z(\rho) \ge C_{\mathrm{sub}} \int_z^\infty y e^{-\rho y} \mathrm{d}y > 0.$$

Combining (3.21) and Lemma 2, we get (1.13).

Proof of (1.12). First using Theorem 1(iii), and then Lemma 11 and (3.19) with $N = 1$ and $z = 0$, we see that

$$\begin{aligned} C_0(\rho) &= \sqrt{\frac{\pi}{2}} \frac{1}{x e^{\rho x}} \lim_{t\to\infty} t^{\frac{3}{2}} \mathrm{e}^{\left(\alpha+\frac{\rho^2}{2}\right)t} Q_0(x, t) \\ &\ge \sqrt{\frac{\pi}{2}} \frac{1}{x e^{\rho x}} \lim_{t\to\infty} t^{\frac{3}{2}} \mathrm{e}^{\frac{\rho^2}{2}t} \mathbf{E}_x^{-\rho} \left(1_{\{\tau_0 > t\}} \mathrm{e}^{-\int_{t-1}^t \varphi(Q_z(B_s, t-s))\mathrm{d}s}\right) \mathrm{e}^{-\int_1^t \varphi(g(s))\mathrm{d}s} \\ &= \mathrm{e}^{-\int_1^\infty \varphi(g(s))\mathrm{d}s} \mathrm{e}^{\alpha+\frac{\rho^2}{2}} \int_0^\infty \mathbb{P}_y \left(\widetilde{M}_1^{-\rho} > 0\right) y e^{-\rho y} \mathrm{d}y \\ &= \mathrm{e}^{-\int_1^\infty \varphi(g(s))\mathrm{d}s} \mathrm{e}^{\frac{\rho^2}{2}} \int_0^\infty \mathbf{E}_y^{-\rho} \left(1_{\{\tau_0 > 1\}} \mathrm{e}^{-\int_0^1 \varphi(Q_0(B_s, 1-s))\mathrm{d}s}\right) y e^{-\rho y} \mathrm{d}y, \end{aligned}$$

where in the last equality we used (3.2). Since $Q_0(B_s, 1-s) \le g(1-s)$ by the definition of Q, applying the inequality $\mathrm{e}^x \ge 1 + x$, $x \ge 0$, we conclude from the display above that

$$\begin{aligned} C_0(\rho) &\ge \mathrm{e}^{-\int_0^\infty \varphi(g(s))\mathrm{d}s} \mathrm{e}^{\frac{\rho^2}{2}} \int_0^\infty \mathbf{E}_y^{-\rho} \left(1_{\{\tau_0 > 1\}} \mathrm{e}^{\int_0^1 \varphi(g(1-s)) - \varphi(Q_0(B_s, 1-s))\mathrm{d}s}\right) y e^{-\rho y} \mathrm{d}y \\ &\ge C_{\mathrm{sub}} \mathrm{e}^{\frac{\rho^2}{2}} \int_0^\infty \mathbf{E}_y^{-\rho} \left(1 + \int_0^1 \varphi(g(1-s)) - \varphi(Q_0(B_s, 1-s))\mathrm{d}s, \tau_0 > 1\right) y e^{-\rho y} \mathrm{d}y \\ &> C_{\mathrm{sub}} \mathrm{e}^{\frac{\rho^2}{2}} \int_0^\infty \mathbf{P}_y^{-\rho} (\tau_0 > 1)\, y e^{-\rho y} \mathrm{d}y. \end{aligned} \tag{3.25}$$

Now combining (1.11), (2.1) and (3.25), we obtain that

$$\begin{aligned}C_0(\rho) &> C_{\text{sub}}\int_0^\infty \mathbf{E}_y\left(\mathrm{e}^{-\rho B_1}1_{\{\tau_0>1\}}\right)y\mathrm{d}y = C_{\text{sub}}\int_0^\infty \mathbf{E}_y^B\left(\frac{\mathrm{e}^{-\rho B_1}}{B_1}\right)y^2\mathrm{d}y\\ &= C_{\text{sub}}\int_0^\infty\int_0^\infty \frac{\mathrm{e}^{-\rho a}}{\sqrt{2\pi}}y\mathrm{e}^{-\frac{(a-y)^2}{2}}\left(1-\mathrm{e}^{-2ay}\right)\mathrm{d}y\mathrm{d}a\\ &= C_{\text{sub}}\int_0^\infty \mathrm{e}^{-\rho a}\mathrm{d}a\int_0^\infty \frac{y}{\sqrt{2\pi}}\left(\mathrm{e}^{-\frac{(a-y)^2}{2}}-\mathrm{e}^{-\frac{(a+y)^2}{2}}\right)\mathrm{d}y.\end{aligned}$$

Noticing that

$$\int_0^\infty \frac{y}{\sqrt{2\pi}}\left(\mathrm{e}^{-\frac{(a-y)^2}{2}}-\mathrm{e}^{-\frac{(a+y)^2}{2}}\right)\mathrm{d}y = \int_{-\infty}^\infty \frac{y}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(a-y)^2}{2}}\mathrm{d}y = a,$$

we obtain that

$$C_0(\rho) > C_{\text{sub}}\int_0^\infty a\mathrm{e}^{-\rho a}\mathrm{d}a. \tag{3.26}$$

Therefore, combining (1.3), Theorem 1(iii) and Lemma 1(iii), we get (1.12).

Proof of Corollary 1. We only give the proof of (iii). Taking $N=0$ in (3.17), by (3.23) with $z=0$, we have

$$\begin{aligned}\mathbb{P}_x\left(\widetilde{M}_t^{-\rho}>z\big|\widetilde{\zeta}^{-\rho}>t\right) &= \frac{Q_z(x,t)}{u(x,t)}\\ &\le \frac{\mathbf{P}_x^{-\rho}\left(\tau_0>t, B_t>z\right)}{C_{\text{sub}}\mathbf{P}_x^{-\rho}\left(\tau_0>t\right)} = \frac{1}{C_{\text{sub}}}\mathbf{P}_x^{-\rho}\left(B_t>z\big|\tau_0>t\right).\end{aligned}$$

By Lemma 1(iii), the tightness of $\widetilde{M}_t^{-\rho}$ follows from the tightness of B_t under $\mathbf{P}_x^{-\rho}\left(\cdot\big|\tau_0>t\right)$. Therefore, the weak convergence of $\widetilde{M}_t^{-\rho}$ is a consequence of the existence of $C_z(\rho)$ in Theorem 3(iii), which implies the desired result.

4. Proof of Theorem 2

Proof of Theorem 2. For $x, y>0$, define $v(x,y):=\mathbb{P}_x(\widetilde{M}^{-\rho}>y)$. We divide the proof into three steps. In Step 1, we use the Feynman–Kac formula and the strong Markov property to rewrite $v(x,y)$ as the product of two factors $A_1(x,y)$ and $A_2(y)$; see (4.3) below. In Steps 2 and 3, we study the asymptotic behavior of $A_1(x,y)$ and $A_2(y)$ respectively as $y\to\infty$. Combining these results, we arrive at the assertion of the theorem.

Step 1: For $0<x<y$, comparing the first branching time with τ_y, we have

$$\begin{aligned}v(x,y) &= \int_0^\infty \beta\mathrm{e}^{-\beta s}\mathbf{P}_x^{-\rho}(\tau_y<\tau_0, \tau_y\le s)\mathrm{d}s\\ &\quad+\int_0^\infty \beta\mathrm{e}^{-\beta s}\mathbf{E}_x^{-\rho}\left(\left(1-\sum_{k=0}^\infty p_k\left(1-v(B_s,y)\right)^k\right)1_{\{\tau_y\wedge\tau_0>s\}}\right)\\ &= \mathbf{E}_x^{-\rho}\left(\mathrm{e}^{-\beta\tau_y}1_{\{\tau_y<\tau_0\}}\right)+\int_0^\infty \beta\mathrm{e}^{-\beta s}\mathbf{E}_x^{-\rho}\left(\left(1-\sum_{k=0}^\infty p_k\left(1-v(B_s,y)\right)^k\right)1_{\{\tau_y\wedge\tau_0>s\}}\right)\mathrm{d}s.\end{aligned}$$

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By [8, Lemma 4.1], the above equation is equivalent to

$$v(x,y)+\beta\int_0^\infty \mathbf{E}_x^{-\rho}\left(v(B_s,y)1_{\{\tau_y\wedge\tau_0>s\}}\right)\mathrm{d}s$$
$$=\mathbf{P}_x^{-\rho}\left(\tau_y<\tau_0\right)+\beta\int_0^\infty \mathbf{E}_x^{-\rho}\left(\left(1-\sum_{k=0}^\infty p_k\left(1-v(B_s,y)\right)^k\right)1_{\{\tau_y\wedge\tau_0>s\}}\right)\mathrm{d}s,$$

which is also equivalent to

$$v(x,y)=\mathbf{P}_x^{-\rho}\left(\tau_y<\tau_0\right)-\mathbf{E}_x^{-\rho}\left(\int_0^{\tau_y\wedge\tau_0}\Phi(v(B_s,y))\mathrm{d}s\right),$$

where Φ is defined in (1.2). By repeating the argument leading to (2.9), we get that

$$\begin{aligned}v(x,y)&=\mathbf{E}_x^{-\rho}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-\alpha\tau_y-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right)\\&=\frac{x}{y}\mathrm{e}^{\rho(x-y)}\mathbf{E}_x^B\left(\mathrm{e}^{-\left(\alpha+\frac{\rho^2}{2}\right)\tau_y-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right),\end{aligned}\tag{4.1}$$

where the last equality follows from Lemma 5. Combining the first equality in (4.1) and (2.4) (with $h=0$), we have that

$$v(x,y)\le\mathbf{E}_x^{-\rho}\left(\mathrm{e}^{-\alpha\tau_y}\right)=\mathrm{e}^{\rho(x-y)}\mathbf{E}_x\left(\mathrm{e}^{-(\alpha+\frac{\rho^2}{2})\tau_y}\right)=\mathrm{e}^{\left(\rho+\sqrt{2\alpha+\rho^2}\right)(x-y)}.\tag{4.2}$$

Fix a $\gamma\in(0,1)$. By the strong Markov property of three-dimensional Bessel processes, we have

$$\begin{aligned}v(x,y)&=\frac{x}{y}\mathrm{e}^{\rho(x-y)}\mathbf{E}_x^B\left(\mathrm{e}^{-\left(\alpha+\frac{\rho^2}{2}\right)\tau_{(y-y^\gamma)}-\int_0^{\tau_{(y-y^\gamma)}}\varphi(v(B_s,y))\mathrm{d}s}\right)\\&\quad\times\mathbf{E}_{y-y^\gamma}^B\left(\mathrm{e}^{-\left(\alpha+\frac{\rho^2}{2}\right)\tau_y-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right)\\&=:\frac{x}{y}\mathrm{e}^{\rho(x-y)}A_1(x,y)A_2(y),\end{aligned}\tag{4.3}$$

where

$$A_1(x,y):=\mathbf{E}_x^B\left(\mathrm{e}^{-\left(\alpha+\frac{\rho^2}{2}\right)\tau_{(y-y^\gamma)}-\int_0^{\tau_{(y-y^\gamma)}}\varphi(v(B_s,y))\mathrm{d}s}\right)$$

and

$$A_2(y):=\mathbf{E}_{y-y^\gamma}^B\left(\mathrm{e}^{-\left(\alpha+\frac{\rho^2}{2}\right)\tau_y-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right).$$

Step 2: In this step, we study the asymptotic behavior of $A_1(x, y)$ as $y \to \infty$. By Lemma 5 with $a = 0$, ρ replaced by $-\sqrt{2\alpha + \rho^2}$, y replaced by $y - y^\gamma$, and $h = \varphi \circ v(\cdot, y)$, we get

$$\begin{aligned}A_1(x, y) &= \frac{y - y^\gamma}{x} \mathrm{e}^{-\sqrt{2\alpha+\rho^2}(y-y^\gamma-x)} \mathbf{E}_x^{\sqrt{2\alpha+\rho^2}} \left(1_{\{\tau_{(y-y^\gamma)}<\tau_0\}} \mathrm{e}^{-\int_0^{\tau_{(y-y^\gamma)}} \varphi(v(B_s,y))\mathrm{d}s}\right)\\ &=: \frac{y - y^\gamma}{x} \mathrm{e}^{-\sqrt{2\alpha+\rho^2}(y-y^\gamma-x)} \hat{A}_1(x, y).\end{aligned}$$

By the inequality $1 - \mathrm{e}^{-|x|} \le |x|$, we obtain that

$$\begin{aligned}0 &\le \mathbf{P}_x^{\sqrt{2\alpha+\rho^2}} \left(\tau_{(y-y^\gamma)} < \tau_0\right) - \hat{A}_1(x, y) \\ &= \mathbf{E}_x^{\sqrt{2\alpha+\rho^2}} \left(1_{\{\tau_{(y-y^\gamma)}<\tau_0\}} \left(1 - \mathrm{e}^{-\int_0^{\tau_{(y-y^\gamma)}} \varphi(v(B_s,y))\mathrm{d}s}\right)\right) \\ &\le \mathbf{E}_x^{\sqrt{2\alpha+\rho^2}} \left(\int_0^{\tau_{(y-y^\gamma)}} \varphi(v(B_s, y))\mathrm{d}s\right).\end{aligned} \tag{4.4}$$

Now set $y_*(x) := \inf\{w \ge y - y^\gamma : w - x \in \mathbb{N}\}$ to be the smallest number w greater than or equal to $y - y^\gamma$ such that $w - x$ is a positive integer and $c_* := \rho + \sqrt{2\alpha + \rho^2} > 0$. By (4.2),

$$\begin{aligned}\mathbf{E}_x^{\sqrt{2\alpha+\rho^2}} \left(\int_0^{\tau_{(y-y^\gamma)}} \varphi(v(B_s, y))\mathrm{d}s\right) &\le \mathbf{E}_x^{\sqrt{2\alpha+\rho^2}} \left(\int_0^{\tau_{y_*(x)}} \varphi(\mathrm{e}^{c_*(B_s-y)})\mathrm{d}s\right) \\ &= \sum_{k=0}^{y_*(x)-x-1} \mathbf{E}_x^{\sqrt{2\alpha+\rho^2}} \left(\int_{\tau_{x+k}}^{\tau_{x+k+1}} \varphi(\mathrm{e}^{c_*(B_s-y)})\mathrm{d}s\right) \\ &\le \sum_{k=0}^{y_*(x)-x-1} \mathbf{E}_x^{\sqrt{2\alpha+\rho^2}} \left(\tau_{x+k+1} - \tau_{x+k}\right) \varphi(\mathrm{e}^{c_*(x+k+1-y)}) \\ &= \mathbf{E}_0^{\sqrt{2\alpha+\rho^2}} (\tau_1) \sum_{k=1}^{y_*(x)-x} \varphi\left(\mathrm{e}^{-c_*(y-1-y_*(x)+k)}\right).\end{aligned}$$

According to the definition of $y_*(x)$, for y large enough,

$$y - 1 - y_*(x) \ge y - 1 - (y - y^\gamma + 1) = y^\gamma - 2.$$

Therefore, when y is large enough so that $y^\gamma - 2 \ge y^{\gamma/2}$, by Lemma 7, we have

$$\begin{aligned}\mathbf{E}_x^{\sqrt{2\alpha+\rho^2}} \left(\int_0^{\tau_{(y-y^\gamma)}} \varphi(v(B_s, y))\mathrm{d}s\right) &\le \mathbf{E}_0^{\sqrt{2\alpha+\rho^2}} (\tau_1) \sum_{k=1}^{\infty} \varphi\left(\mathrm{e}^{-c_*(y^{\gamma/2}+k)}\right) \\ &\le \mathbf{E}_0^{\sqrt{2\alpha+\rho^2}} (\tau_1) \int_0^\infty \varphi\left(\mathrm{e}^{-c_*(y^{\gamma/2}+z)}\right)\mathrm{d}z \\ &= \mathbf{E}_0^{\sqrt{2\alpha+\rho^2}} (\tau_1) \int_{y^{\gamma/2}}^\infty \varphi\left(\mathrm{e}^{-c_* z}\right)\mathrm{d}z \overset{y\to\infty}{\longrightarrow} 0.\end{aligned}$$

Combining the above limit with (4.4), it holds that

$$\lim_{y\to\infty} \left(\mathbf{P}_x^{\sqrt{2\alpha+\rho^2}} \left(\tau_{(y-y^\gamma)} < \tau_0\right) - \hat{A}_1(x, y)\right) = 0.$$

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Combining (2.5) and the definition of $\hat{A}_1$, we conclude that

$$\lim_{y\to\infty}\frac{A_1(x,y)}{y}\mathrm{e}^{\sqrt{2\alpha+\rho^2}(y-y^\gamma)}=\frac{\mathrm{e}^{\sqrt{2\alpha+\rho^2}x}}{x}\lim_{y\to\infty}\mathbf{P}_x^{\sqrt{2\alpha+\rho^2}}\left(\tau_{(y-y^\gamma)}<\tau_0\right)\qquad(4.5)$$
$$=\frac{2}{x}\sinh\left(x\sqrt{2\alpha+\rho^2}\right).$$

Step 3: In this step, we study the limit behavior for A_2. We define $v(x,y)=0$ for $x\le 0$. By Lemma 5, we have

$$A_2(y)=\frac{y\mathrm{e}^{-\sqrt{2\alpha+\rho^2}y^\gamma}}{y-y^\gamma}\mathbf{E}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}\left(1_{\{\tau_y<\tau_0\}}\mathrm{e}^{-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right)\qquad(4.6)$$
$$=\frac{y\mathrm{e}^{-\sqrt{2\alpha+\rho^2}y^\gamma}}{y-y^\gamma}\left(\mathbf{E}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}\left(\mathrm{e}^{-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right)-\mathbf{E}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}\left(1_{\{\tau_y\ge\tau_0\}}\mathrm{e}^{-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right)\right),$$

where, under $\mathbf{P}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}$, B is a Brownian motion with drift $\sqrt{2\alpha+\rho^2}$ starting from $y-y^\gamma$. We claim that

$$\lim_{y\to\infty}\mathbf{E}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}\left(\mathrm{e}^{-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right)=C_*(\rho)>0,\qquad(4.7)$$

$$\lim_{y\to\infty}\mathbf{E}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}\left(1_{\{\tau_y\ge\tau_0\}}\mathrm{e}^{-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right)=0.\qquad(4.8)$$

We prove (4.8) first. In fact, by Lemma 5 and 6, we have

$$\mathbf{E}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}\left(1_{\{\tau_y\ge\tau_0\}}\mathrm{e}^{-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right)\le\mathbf{P}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}(\tau_y\ge\tau_0)=1-\mathbf{P}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}(\tau_y<\tau_0)$$
$$=1-\frac{y-y^\gamma}{y}\mathrm{e}^{-\sqrt{2\alpha+\rho^2}y^\gamma}\mathbf{E}_{y-y^\gamma}^{B}\left(\mathrm{e}^{-\frac{2\alpha+\rho^2}{2}\tau_y}\right)$$
$$=1-\mathrm{e}^{-\sqrt{2\alpha+\rho^2}y^\gamma}\frac{\sinh\left((y-y^\gamma)\sqrt{2\alpha+\rho^2}\right)}{\sinh\left(y\sqrt{2\alpha+\rho^2}\right)}\xrightarrow{y\to\infty}0,$$

which gives (4.8). To prove (4.7), for any $y>0$, define

$$G(y):=\mathbf{E}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}}\left(\mathrm{e}^{-\int_0^{\tau_y}\varphi(v(B_s,y))\mathrm{d}s}\right).$$

For $z>y$, by the strong Markov property, we have

$$G(z)=\mathbf{E}_0^{\sqrt{2\alpha+\rho^2}}\left(\mathrm{e}^{-\int_0^{\tau_{z^\gamma}}\varphi(v(B_s+z-z^\gamma,z))\mathrm{d}s}\right)$$
$$=\mathbf{E}_0^{\sqrt{2\alpha+\rho^2}}\left(\mathrm{e}^{-\int_0^{\tau_{(z^\gamma-y^\gamma)}}\varphi(v(B_s+z-z^\gamma,z))\mathrm{d}s}\right)\mathbf{E}_{z^\gamma-y^\gamma}^{\sqrt{2\alpha+\rho^2}}\left(\mathrm{e}^{-\int_0^{\tau_{z^\gamma}}\varphi(v(B_s+z-z^\gamma,z))\mathrm{d}s}\right)$$

The first term of the above display is dominated by 1 from above, and the second is equal to $\mathbf{E}_0^{\sqrt{2\alpha+\rho^2}}\left(\mathrm{e}^{-\int_0^{\tau_{y^\gamma}}\varphi(v(B_s+z-y^\gamma,z))\mathrm{d}s}\right)$. Hence, $G(z)$ is bounded from above by

$$G(z)\le\mathbf{E}_0^{\sqrt{2\alpha+\rho^2}}\left(\mathrm{e}^{-\int_0^{\tau_{y^\gamma}}\varphi(v(B_s+y-y^\gamma+z-y,y+z-y))\mathrm{d}s}\right).\qquad(4.9)$$

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Note that, for $w > 0$, when $x > 0$, we have that

$$
\begin{aligned}
v(x+w, y+w) &= \mathbb{P}_{x+w}(\exists\, t > 0,\ u \in N_t^{-\rho}\ s.t. \min_{s\le t} X_u(s) > 0,\ X_u(t) > y+w) \\
&\ge \mathbb{P}_{x+w}(\exists\, t > 0,\ u \in N_t^{-\rho}\ s.t. \min_{s\le t} X_u(s) > w,\ X_u(t) > y+w) = v(x, y).
\end{aligned}
$$

When $x \le 0$, the above inequality holds trivially since $v(x, y) = 0$. Combining this with (4.9), we get that

$$
G(z) \le \mathbf{E}_0^{\sqrt{2\alpha+\rho^2}} \left(\mathrm{e}^{-\int_0^{\tau_{y^\gamma}} \varphi(v(B_s+y-y^\gamma, y))\mathrm{d}s} \right) = G(y), \quad z > y.
$$

Thus the limit $C_*(\rho) := \lim_{y\to\infty} G(y)$ exists. Combining (4.6), (4.7) and (4.8), we get

$$
\lim_{y\to\infty} A_2(y)\mathrm{e}^{\sqrt{2\alpha+\rho^2}y^\gamma} = C_*(\rho). \tag{4.10}
$$

It is obvious that $C_*(\rho) \in [0, 1]$. Now we show that $C_*(\rho)$ is positive. Combining the definition of G, (4.2) and Jensen's inequality, we get that

$$
\begin{aligned}
G(y) &\ge \mathbf{E}_{y-y^\gamma}^{\sqrt{2\alpha+\rho^2}} \left(\exp\left\{ -\int_0^{\tau_y} \varphi\left(\mathrm{e}^{\left(\rho+\sqrt{2\alpha+\rho^2}\right)(B_s-y)} \right) \mathrm{d}s \right\} \right) \\
&= \mathbf{E}_{y^\gamma}^{-\sqrt{2\alpha+\rho^2}} \left(\exp\left\{ -\int_0^{\tau_0} \varphi\left(\mathrm{e}^{-\left(\rho+\sqrt{2\alpha+\rho^2}\right)B_s} \right) \mathrm{d}s \right\} \right) \\
&\ge \exp\left\{ -\mathbf{E}_{y^\gamma}^{-\sqrt{2\alpha+\rho^2}} \left(\int_0^{\tau_0} \varphi\left(\mathrm{e}^{-\left(\rho+\sqrt{2\alpha+\rho^2}\right)B_s} \right) \mathrm{d}s \right) \right\}.
\end{aligned} \tag{4.11}
$$

Combining Lemma 1(ii) and [4, Proposition 17(i) and (ii), p. 172], we know that the renewal functions of the ladder height process of B under $\mathbf{P}_0^{-\sqrt{2\alpha+\rho^2}}$ and $\mathbf{P}_0^{\sqrt{2\alpha+\rho^2}}$ are equal to $K_1(1-\mathrm{e}^{-2\sqrt{2\alpha+\rho^2}x})$ and K_2x for some constants $K_1, K_2 > 0$, respectively. Therefore, combining (4.11) and [4, Proposition 20, p. 176], there exists a constant $K > 0$ such that

$$
\begin{aligned}
C_*(\rho) = \lim_{y\to\infty} G(y) &\ge \lim_{y\to\infty} \exp\left\{ -K \int_0^\infty \mathrm{e}^{-2\sqrt{2\alpha+\rho^2}z}\mathrm{d}z \int_0^{y^\gamma} \varphi\left(\mathrm{e}^{-(\rho+\sqrt{2\alpha+\rho^2})(y^\gamma+z-x)} \right) \mathrm{d}x \right\} \\
&= \lim_{y\to\infty} \exp\left\{ -K \int_0^\infty \mathrm{e}^{-2\sqrt{2\alpha+\rho^2}z}\mathrm{d}z \int_z^{y^\gamma+z} \varphi\left(\mathrm{e}^{-(\rho+\sqrt{2\alpha+\rho^2})x} \right) \mathrm{d}x \right\} \\
&\ge \exp\left\{ -K \int_0^\infty \mathrm{e}^{-2\sqrt{2\alpha+\rho^2}z}\mathrm{d}z \int_0^\infty \varphi\left(\mathrm{e}^{-(\rho+\sqrt{2\alpha+\rho^2})x} \right) \mathrm{d}x \right\} > 0,
\end{aligned}
$$

where in the last inequality we used Lemma 7. This implies $C_*(\rho) > 0$. Combining (4.3), (4.5) and (4.10), we conclude that

$$
\lim_{y\to\infty} \mathrm{e}^{(\sqrt{2\alpha+\rho^2}+\rho)y} v(x, y) = 2C_*(\rho)\mathrm{e}^{\rho x} \sinh(x\sqrt{2\alpha+\rho^2}),
$$

which completes the proof of the theorem.

https://doi.org/10.1017/apr.2025.17 Published online by Cambridge University Press

Acknowledgements

We thank the referees for the very helpful comments. Part of the research for this paper was done while the third-named author was visiting Jiangsu Normal University, where he was partially supported by a grant from the National Natural Science Foundation of China (11931004, Yingchao Xie). Yaping Zhu is the corresponding author.

Funding information

The research of this project is supported by the National Key R&D Program of China (No. 2020YFA0712900). The research of Haojie Hou is supported by the China Postdoctoral Science Foundation (No. 2024M764112). The research of Yan-Xia Ren is supported by NSFC (Grants numbers 12071011 and 12231002) and the Fundamental Research Funds for the Central Universities, Peking University LMEQF. Research supported in part by a grant from the Simons Foundation (#960480, Renming Song). The research of Yaping Zhu is supported by the China Postdoctoral Science Foundation (No. 2024M760056).

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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