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RESEARCH ARTICLE

Characterisation of Locally Compact Abelian Groups Having Spectral Synthesis

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Abstract
In this paper we solve a long-standing problem which goes back to Laurent Schwartz's work on mean periodic functions. Namely, we completely characterize those locally compact Abelian groups having spectral synthesis. So far a characterization theorem was available for discrete Abelian groups only. Here we use a kind of localization concept for the ideals of the Fourier algebra of the underlying group. We show that localizability of ideals is equivalent to synthesizability. Based on this equivalence we show that if spectral synthesis holds on a locally compact Abelian group, then it holds on each extensions of it by a locally compact Abelian group consisting of compact elements, and also on any extension to a direct sum with a copy of the integers. Then, using Schwartz's result and Gurevich's counterexamples, we apply the structure theory of locally compact Abelian groups to obtain our characterization theorem.

1. Introduction

The study of spectral synthesis started with the fundamental paper of L. Schwartz [1], where the following result was proved:

Theorem 1. *Every mean periodic function is the sum of a series of exponential monomials which are limits of linear combinations of translates of the function.*

Here "limit" is meant as uniform limit on compact sets. A continuous complex-valued function on the reals is called *mean periodic* if the closure – with respect to uniform convergence on compact sets – of the linear span of its translates is a proper subspace in the space of all continuous complex-valued functions. Calling this closure the variety of the function, the above result says that in the variety of each mean periodic function all exponential monomials span a dense subspace.

The basic concepts in this result can easily be generalized to more general situations. Given a commutative topological group G we denote by $\mathcal{C}(G)$ the space of all continuous complex-valued functions equipped with the topology of uniform convergence on compact sets and with the pointwise addition and pointwise multiplication with scalars. If f is in $\mathcal{C}(G)$ and y is in G, then $\tau_y f$ denotes the *translate of* f defined by

$$\tau_y f(x) = f(x + y)$$

for each x in g. A closed linear subspace V in $\mathcal{C}(G)$ is called a *variety* on G if it is *translation invariant*, that is, $\tau_y f$ is in V for each f in V and y in G. Given an f in $\mathcal{C}(G)$ the intersection of all varieties including f is denoted by $\tau(f)$, and it is called the *variety of* f.

Given a commutative topological group G continuous complex homomorphisms of G into the multiplicative group of nonzero complex numbers are called *exponentials*, and continuous complex homomorphisms of G into the additive group of complex numbers are called *additive functions*. The elements of the function algebra in $\mathcal{C}(G)$ generated by all exponentials and additive functions are called *exponential polynomials*. Functions of the form

$$f(x) = P\big(a_1(x), a_2(x), \ldots, a_k(x)\big)m(x) \tag{1}$$

are called *exponential monomials*, if $P : \mathbb{C}^k \to \mathbb{C}$ is a complex polynomial in k variables, $a_1, a_2, \ldots, a_k$ are additive functions, and m is an exponential. Every exponential polynomial is a linear combination of exponential monomials. If $m = 1$, then the above function is called a *polynomial*.

Using these concepts we say that the variety V on G is *synthesizable*, if exponential monomials span a dense subspace in it. We say that *spectral synthesis* holds on V, if every subvariety of V is synthesizable. We say that *spectral synthesis* holds on the group G, or the group G *is synthesizable*, if every variety on G is synthesizable. Hence Schwartz's theorem can be formulated by saying that spectral synthesis holds on $\mathbb{R}$. In the paper [2], M. Lefranc proved that spectral synthesis holds on $\mathbb{Z}^n$. In [4], R. J. Elliott made an attempt to prove that spectral synthesis holds on every discrete Abelian group, but his proof was incorrect. In fact, a counterexample for Elliott's statement was given in [7]. In [8], a characterization theorem was proved for discrete Abelian groups having spectral synthesis.

In the present paper we give a complete characterization of those locally compact Abelian groups on which spectral synthesis holds. Using the localization method we worked out in [9], we can show that if a locally compact Abelian group is synthesizable, then so is its extensions by a locally compact Abelian group consisting of compact elements (see [10]). Also, here we prove that if a locally compact Abelian group is synthesizable, and on its extensions to a direct sum with the group of integers (see [12]). Finally, using the results of Schwartz [1] and Gurevich [6] we apply the structure theory of locally compact Abelian groups.

2. Derivations of the Fourier algebra

In this section we recall some concepts and results concerning the Fourier algebra of locally compact Abelian groups.

Given a locally compact Abelian group G we denote by $\mathcal{M}_c(G)$ its *measure algebra*, which is the space of all compactly supported complex Borel measures on G. This space is identified with the topological dual of $\mathcal{C}(G)$ equipped with the weak*-topology. In fact, $\mathcal{M}_c(G)$ is a topological algebra with the convolution of measures defined by

$$\langle \mu * \nu, f\rangle = \int\int f(x+y)\, d\mu(x)\, d\nu(y)$$

for each μ, ν in $\mathcal{M}_c(G)$ and f in $\mathcal{C}(G)$. In addition, $\mathcal{C}(G)$ is a topological vector module over $\mathcal{M}_c(G)$. It is clear that varieties on G are exactly the closed submodules of $\mathcal{C}(G)$, and we have a one-to-one correspondence between closed ideals in $\mathcal{M}_c(G)$ and varieties in $\mathcal{C}(G)$ established by the annihilators: $V \leftrightarrow \operatorname{Ann} V$ and $I \leftrightarrow \operatorname{Ann} I$ for each variety V and closed ideal I. For the sake of simplicity, we say that *the closed ideal* I in $\mathcal{M}_c(G)$ *is synthesizable*, if the variety $\operatorname{Ann} I$ is synthesizable.

Let G be a locally compact Abelian group and let $\mathcal{A}(G)$ denote its Fourier algebra, that is, the algebra of all Fourier transforms of compactly supported complex Borel measures on G. We recall that the Fourier transform defined by

$$\widehat{\mu}(m) = \int m(-x)\, d\mu(x)$$

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for each μ in the measure algebra is the extension of the Fourier–Laplace transform on the dual group: here m is not necessarily a unitary exponential, that is, a character of G, but it can be any complex exponential on G.

The algebra $\mathcal{A}(G)$ is topologically isomorphic to the measure algebra $\mathcal{M}_c(G)$. For the sake of simplicity, if the annihilator $\operatorname{Ann} I$ of the closed ideal I in $\mathcal{M}_c(G)$ is synthesizable, then we say that the corresponding closed ideal $\widehat{I}$ in $\mathcal{A}(G)$ is synthesizable. Given an ideal $\widehat{I}$ in $\mathcal{A}(G)$ a *root of* $\widehat{I}$ is an exponential m at which every $\widehat{\mu}$ vanishes. The set of all roots of the ideal $\widehat{I}$ is denoted by $Z(\widehat{I})$.

The continuous linear operator $D : \mathcal{A}(G) \to \mathcal{A}(G)$ is called a *derivation of order one*, if

$$D(\widehat{\mu} \cdot \widehat{\nu}) = D(\widehat{\mu}) \cdot \widehat{\nu} + \widehat{\mu} \cdot D(\widehat{\nu})$$

holds for each $\widehat{\mu}, \widehat{\nu}$ in $\mathcal{A}(G)$. For each natural number $n \geq 1$, the continuous linear operator $D : \mathcal{A}(G) \to \mathcal{A}(G)$ is called a *derivation of order* $n+1$, if the bilinear operator

$$(\widehat{\mu}, \widehat{\nu}) \to D(\widehat{\mu} \cdot \widehat{\nu}) - D(\widehat{\mu}) \cdot \widehat{\nu} - \widehat{\mu} \cdot D(\widehat{\nu})$$

is a derivation of order n in both variables. All constant multiples of the identity operator on $\mathcal{A}(G)$ are considered derivations of order 0. Finally, we call a linear operator on $\mathcal{A}(G)$ a *derivation*, if it is a derivation of order n for some natural number n. It is easy to see that all derivations on $\mathcal{A}(G)$ form a commutative algebra with unit (see [9, Theorem 4]). The elements of the subalgebra generated by derivations of order not greater than 1 are called *polynomial derivations* – in fact, they are polynomials of derivations of order at most 1.

Given a continuous linear operator F on $\mathcal{A}(G)$ and an exponential m on G the continuous function $f_{F,m} : G \to \mathbb{C}$ defined for x in G by

$$f_{F,m}(x) = F(\widehat{\delta}_x)(m)m(x)$$

is called the *generating function* of F. The following proposition shows that each continuous linear operator on $\mathcal{A}(G)$ is uniquely determined by its generating function.

Proposition 1. *Let F be a continuous linear operator on $\mathcal{A}(G)$. Then*

$$F(\widehat{\mu})(m) = \int f_{F,m}(x)\check{m}(x)\, d\mu(x) \tag{2}$$

holds for each exponential m and for every $\widehat{\mu}$ in $\mathcal{A}(G)$.

Proof. For each exponential m, the mapping $\mu \mapsto F(\widehat{\mu})(m)$ defines a continuous linear functional on the measure algebra $\mathcal{M}_c(G)$. We conclude (see e.g. [5, 3.10 Theorem]) that there exists a continuous function $\varphi_m : G \to \mathbb{C}$ such that

$$F(\widehat{\mu})(m) = \int \varphi_m(z)\, d\mu(z)$$

holds for each μ in $\mathcal{M}_c(G)$. Then we have

$$\varphi_m(x) = \int \varphi_m(z)\, d\delta_x(z) = F(\widehat{\delta}_x)(m),$$

hence $\varphi_m(x) = f_{F,m}(x)\check{m}(x)$, which yields (2). □

Clearly, the generating function of the identity operator is the identically one function, and it is easy to check that the generating function of a first order derivation is an additive function, and conversely, each additive function generates a first order derivation. It follows that the generating function of a polynomial derivation is a polynomial, and the degree of the generating polynomial is equal to the order of the corresponding polynomial derivation (see also [9]).

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In general, there may exist nonpolynomial derivations on the Fourier algebra. However, the generating function φ of any derivation is a so-called *generalized polynomial*, which, by definition, satisfies the higher order difference equation

$$\Delta_{y_1,y_2,\dots,y_{n+1}}\varphi(x) = 0. \tag{3}$$

Here $\Delta_y = \tau_y - \tau_0$, and $\Delta_{y_1,y_2,\dots,y_{n+1}}$ is the product of the linear operators $\tau_{y_i} - \tau_0$ for $i = 1, 2, \dots, n+1$ (see [9]). Polynomials are generalized polynomials, but the converse is not true. Still all generalized polynomials generate derivations, which are not polynomial derivations. We shall see that the existence of nonpolynomial derivations is closely related to the failure of spectral synthesis.

Given a derivation D and an exponential m we denote by $\widehat{I}_{D,m}$ the set of all functions $\widehat{\mu}$ in $\mathcal{A}(G)$ which are annihilated at m by all derivations of the form

$$\widehat{\mu} \mapsto \int \varphi(x)\check{m}(x)\,d\mu(x),$$

where φ belongs to the translation invariant linear space in $\mathcal{C}(G)$ generated by $f_{D,m}$. In other words, $\widehat{I}_{D,m}$ is the set of those functions $\widehat{\mu}$ in $\mathcal{A}(G)$ which satisfy $\widehat{\mu}(m) = D\widehat{\mu}(m) = 0$, and

$$\int [\Delta_{y_1,y_2,\dots,y_k} f_{D,m}](x)\cdot \check{m}(x)\,d\mu(x) = 0$$

for each positive integer k and $y_1, y_2, \dots, y_k$ in G. It is easy to see that for every derivation D on $\mathcal{A}(G)$ and for each exponential m, we have the equation $I_{D,m} = \operatorname{Ann}\tau(\check{f}_{D,m}m)$ (see [9]). As a by-product we obtain that $I_{D,m}$, as well as $\widehat{I}_{D,m}$ is a closed ideal, hence so is the intersection $\widehat{I}_{\mathcal{D},m} = \bigcap_{D\in\mathcal{D}} \widehat{I}_{D,m}$ for any family $\mathcal{D}$ of derivations.

We note that for a polynomial derivation $P(D_1, D_2, \dots, D_k)$ the set $\widehat{I}_{D,m}$ consists of those Fourier transforms $\widehat{\mu}$ in $\mathcal{A}(G)$ that satisfy

$$(\partial^{\alpha_1}\partial^{\alpha_2}\cdots\partial^{\alpha_k}P)(D_1, D_2, \dots, D_k)(\widehat{\mu})(m) = 0$$

for every choice of the nonnegative integers α_i.

The dual concept is the following: given a closed ideal $\widehat{I}$ in $\mathcal{A}(G)$ and an exponential m, the set of all derivations annihilating $\widehat{I}$ at m is denoted by $\mathcal{D}_{\widehat{I},m}$. The subset of $\mathcal{D}_{\widehat{I},m}$ consisting of all polynomial derivations is denoted by $\mathcal{P}_{\widehat{I},m}$. Clearly, we have the inclusion

$$\widehat{I} \subseteq \bigcap_m \widehat{I}_{\mathcal{D}_{\widehat{I},m},m} \subseteq \bigcap_m \widehat{I}_{\mathcal{P}_{\widehat{I},m},m}. \tag{4}$$

We note that if m is not a root of $\widehat{I}$, then $\mathcal{D}_{\widehat{I},m} = \mathcal{P}_{\widehat{I},m} = \{0\}$, consequently $\widehat{I}_{\mathcal{D}_{\widehat{I},m},m} = \widehat{I}_{\mathcal{P}_{\widehat{I},m},m} = \mathcal{A}(G)$, hence those terms have no effect on the intersection.

Proposition 2. *Let $\mathcal{D}$ be a family of derivations on $\mathcal{A}(G)$. The ideal $\widehat{I}$ in $\mathcal{A}(G)$ has the property*

$$\widehat{I} \supseteq \bigcap_m \widehat{I}_{\mathcal{D},m} \tag{5}$$

if and only if the functions $\check{f}_{D,m}m$ with D in $\mathcal{D}$ span a dense subspace in $\operatorname{Ann} I$.

Proof. Let $\widehat{J} = \bigcap_m \widehat{I}_{\mathcal{D},m}$, and assume that $\widehat{J} \subseteq \widehat{I}$. If the subspace spanned by all functions of the form $\check{f}_{D,m}m$ with D in $\mathcal{D}$ is not dense in $\operatorname{Ann} I$, then there exists a μ_0 not in $\operatorname{Ann}\operatorname{Ann} I = I$ such that μ_0 annihilates all functions of the form $\check{f}_{D,m}m$ with D in $\mathcal{D}$. In other words, for each x in G we have

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$$0 = (\mu_0 * \check{f}_{D,m})m(x) = \int \check{f}_{D,m}(x-y)m(x-y)\,d\mu_0(y)$$
$$= \int f_{D,m}(y-x)\check{m}(y-x)\,d\mu_0(y) = m(x)\int f_{D,m}(y-x)\check{m}(y)\,d\mu_0(y).$$

In particular, for $x = 0$

$$0 = \mu_0 * \check{f}_{D,m}m(0) = \int f_{D,m}(y)\check{m}(y)\,d\mu_0(y) = D(\mu_0)(m)$$

holds for each D in $\mathcal{D}$ and for every m. Consequently, $\widehat{\mu}_0$ is in $\widehat{I}_{\mathcal{D},m}$ for each m, hence it is in the set $\widehat{J}$, but not in $\widehat{I}$ – a contradiction.

Conversely, assume that the subspace spanned by all functions of the form $\check{f}_{D,m}m$ with D in $\mathcal{D}$, is dense in $\operatorname{Ann} I$. It follows that any μ in $\mathcal{M}_c(G)$, which satisfies

$$\int \check{f}_{D,m}(x-y)m(x-y)\,d\mu(y) = 0 \tag{6}$$

for all D in $\mathcal{D}$ and x in G, belongs to $I = \operatorname{Ann}\operatorname{Ann} I$. Now let $\widehat{\mu}$ be in $\widehat{I}_{\mathcal{D},m}$ for some m, and suppose that D is in $\mathcal{D}$. Then for each x in G, the function $\widehat{\mu}\cdot\widehat{\delta}_{-x}$ is in $\widehat{I}_{\mathcal{D},m}$, hence

$$0 = D(\widehat{\mu}\cdot\widehat{\delta}_{-x})(m) = \int \check{f}_{D,m}(x-y)m(x-y)\,d\mu(y),$$

that is, $\widehat{\mu}$ satisfies (6) for each D in $\mathcal{D}$. This implies that μ is in I, and the theorem is proved. □

Corollary 1. *Let $\widehat{I}$ be a closed ideal in $\mathcal{A}(G)$. Then $\widehat{I} = \bigcap_{m\in Z(\widehat{I})} \widehat{I}_{\mathcal{P}_{\widehat{I},m},m}$ holds if and only if all functions of the form $\check{f}_{D,m}m$ with m in $Z(\widehat{I})$ and D in $\mathcal{P}_{\widehat{I},m}$ span a dense subspace in the variety* $\operatorname{Ann} I$.

3. Localization

The ideal $\widehat{I}$ is called *localizable*, if we have equalities in (4). Roughly speaking, localizability of an ideal means that the ideal is completely determined by the values of "derivatives" of the functions belonging to this ideal. Nonlocalizability of the ideal $\widehat{I}$ means that there is a $\widehat{\nu}$ not in $\widehat{I}$, which is annihilated by all polynomial derivations which annihilate $\widehat{I}$ at its zeros.

Theorem 2. *Let G be a locally compact Abelian group. The ideal $\widehat{I}$ in $\mathcal{A}(G)$ is localizable if and only if it is synthesizable.*

Proof. Assume that $\operatorname{Ann} I$ is not synthesizable. Then the linear span of the exponential monomials in $\operatorname{Ann} I$ is not dense. In other words, there is a $\widehat{\nu}$ not in $\widehat{I}$ such that $\nu * pm = 0$ for every polynomial p such that pm is in $\operatorname{Ann} I$. For each such pm we consider the polynomial derivation

$$D(\widehat{\mu})(m) = \int \check{p}(x)\check{m}(x)\,d\mu(x)$$

whenever $\widehat{\mu}$ is in $\mathcal{A}(G)$. As pm is in $\operatorname{Ann} I$, hence D is in $\mathcal{P}_{\widehat{I},m}$. On the other hand, every derivation in $\mathcal{P}_{\widehat{I},m}$ has this form with some pm in $\operatorname{Ann} I$. As $\nu * pm(0) = 0$ for all these functions, we have

$$D(\widehat{\nu})(m) = \int \check{p}(x)\check{m}(x)\,d\nu(x) = \int p(0-x)m(0-x)\,d\nu(x) = \nu * pm(0) = 0,$$

holds for each D in $\mathcal{P}_{\widehat{I},m}$. This means that $\widehat{\nu}$ is annihilated by all derivations in $\mathcal{P}_{\widehat{I},m}$, but $\widehat{\nu}$ is not in $\widehat{I}$, which contradicts the localizability.

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Now we assume that $\operatorname{Ann} I$ is synthesizable. This means that all functions of the form $\check{f}_{D,m}m$ with m in $Z(\widehat{I})$ and D in $\mathcal{P}_{\widehat{I},m}$ span a dense subspace in the variety $\operatorname{Ann} I$. By Corollary 1,

$$\widehat{I} = \bigcap_{m \in Z(\widehat{I})} \widehat{I}_{\mathcal{P}_{\widehat{I},m},m}.$$

We show that this ideal is localizable. Assuming the contrary, there is an exponential m in $Z(\widehat{I})$ and there is a $\widehat{\nu}$ not in $\widehat{I}_{\mathcal{P}_{\widehat{I},m},m}$ such that $D(\widehat{\nu})(m) = 0$ for each derivation D in $\mathcal{P}_{\widehat{I},m}$. In other words, $\widehat{\nu}$ is annihilated at m by all derivations in $\mathcal{P}_{\widehat{I},m}$, and still $\widehat{\nu}$ is not in $\widehat{I}_{\mathcal{P}_{\widehat{I},m},m}$ – a contradiction. □

4. Compact elements

In this section we show that if spectral synthesis holds on a locally compact Abelian group, then it also holds on every extension by a locally compact Abelian group consisting of compact elements.

Theorem 3. *Let G be a locally compact Abelian group and let B denote the closed subgroup of G consisting of all compact elements. Then spectral synthesis holds on G if and only if it holds on G/B.*

Proof. If spectral synthesis holds on G, then it obviously holds on every continuous homomorphic image of G (see [11, Theorem 3.1]), in particular, it holds on G/B.

Conversely, we assume that spectral synthesis holds on G/B. This means that every closed ideal in the Fourier algebra of G/B is localizable, and we need to show the same for all closed ideals of the Fourier algebra of G.

First we remark that the polynomial rings over G and over G/B can be identified. Indeed, polynomials on G are built up from additive functions on G, which clearly vanish on compact elements, as the additive topological group of complex numbers has no nontrivial compact subgroups. Consequently, if a is an additive function and x, y are in the same coset of B, then $x - y$ is in B, and $a(x - y) = 0$, which means $a(x) = a(y)$. So, the additive functions on G arise from the additive functions of G/B, hence the two polynomial rings can be identified.

Now we define a projection of the Fourier algebra of G into the Fourier algebra of G/B as follows. Let $\Phi : G \to G/B$ denote the natural mapping. For each measure μ in $\mathcal{M}_c(G)$ we define μ_B as the linear functional

$$\langle \mu_B, \varphi \rangle = \langle \mu, \varphi \circ \Phi \rangle$$

whenever $\varphi : G/B \to \mathbb{C}$ is a continuous function. It is straightforward that the mapping $\widehat{\mu} \mapsto \widehat{\mu}_B$ is a continuous algebra homomorphism of the Fourier algebra of G into the Fourier algebra of G/B. As Φ is an open mapping, closed ideals are mapped onto closed ideals.

For a given closed ideal $\widehat{I}$ in $\mathcal{A}(G)$, we denote by $\widehat{I}_B$ the closed ideal in $\mathcal{A}(G/B)$ which corresponds to $\widehat{I}$ under the above homomorphism. If m is a root of the ideal $\widehat{I}_B$, then $\widehat{\mu}_B(m) = 0$ for each $\widehat{\mu}$ in $\widehat{I}$. In other words,

$$\langle \widehat{\mu}, \check{m} \circ \Phi \rangle = \langle \widehat{\mu}_B, \check{m} \rangle = 0,$$

hence $m \circ \Phi$, which is clearly an exponential on G, is a root of $\widehat{I}$. Suppose that D is a derivation in $\mathcal{P}_{\widehat{I},m\circ\Phi}$, then it has the form

$$D\widehat{\mu}(m \circ \Phi) = \int p \cdot (\check{m} \circ \Phi)\, d\mu$$

with some polynomial p on G. According to our remark above, the polynomial p can uniquely be written as $p_B \circ \Phi$, where p_B is a polynomial on G/B. In other words,

$$D\widehat{\mu}(m \circ \Phi) = \langle \widehat{\mu}, (p_B \circ \Phi)(\check{m} \circ \Phi) \rangle = \langle \widehat{\mu}_B, p_B \check{m} \rangle = D_B(\widehat{\mu}_B)(m),$$

which defines a derivation D_B on $\mathcal{A}(G/B)$ with generating function $f_{D_B,m} = p_B$.

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It follows that every derivation in $\mathcal{P}_{\widehat{I},m\circ\Phi}$ arises from a derivation in $\mathcal{P}_{\widehat{I}_B,m}$. On the other hand, if d is a derivation in $\mathcal{P}_{\widehat{I}_B,m}$, then we have

$$d\widehat{\mu}_B(m) = \int p\check{m}\,d\mu_B = \langle \mu_B, p\check{m}\rangle = \langle \mu, (p\circ\Phi)(\check{m}\circ\Phi)\rangle$$
$$= \int p\big(\Phi(x)\big)(\check{m}\circ\Phi)(x)\,d\mu(x),$$

which defines a derivation D in $\mathcal{P}_{\widehat{I},m\circ\Phi}$.

We summarize our assertions. Let $\widehat{I}$ be a proper closed ideal in $\mathcal{A}(G)$ and assume that $\widehat{I}$ is nonlocalizable. It follows that there is a function $\widehat{\nu}$ not in $\widehat{I}$ which is annihilated at M by all polynomial derivations in $\mathcal{P}_{\widehat{I},M}$, for each exponential M on G. In particular, $\widehat{\nu}$ is annihilated at $m\circ\Phi$ by all polynomial derivations in $\mathcal{P}_{\widehat{I},m\circ\Phi}$, for each exponential m on G/B. We have seen above that this implies that $\widehat{\nu}_B$ is annihilated at m by all polynomial derivations in $\mathcal{P}_{\widehat{I}_B,m}$ and for each exponential m on G/B. As spectral synthesis holds on G/B, the ideal $\widehat{I}_B$ is localizable, hence $\widehat{\nu}_B$ is in $\widehat{I}_B$, but this contradicts the assumption that $\widehat{\nu}$ is not in $\widehat{I}$. The proof is complete. □

From this result it follows immediately that if every element of a locally compact Abelian group is compact, then spectral synthesis holds on this group. In particular, spectral synthesis holds on every compact Abelian group. Also, we can provide the following simple proof for the characterization theorem of discrete synthesizable Abelian groups (see [8]):

Corollary 2. *Spectral synthesis holds on a discrete Abelian group if and only if its torsion free rank is finite.*

Proof. If the torsion free rank of G is infinite, then there is a generalized polynomial on G, which is not a polynomial (see [7]), hence there is a nonpolynomial derivation on the Fourier algebra. Consequently, we have the chain of inclusions

$$\widehat{I} \subseteq \widehat{I}_{\mathcal{D}_{\widehat{I},m},m} \subsetneq \widehat{I}_{\mathcal{P}_{\widehat{I},m},m},$$

which implies that $\widehat{I} \neq \widehat{I}_{\mathcal{P}_{\widehat{I},m},m}$, hence $\widehat{I}$ is not synthesizable.

Conversely, let G have finite torsion free rank. The subgroup B of compact elements coincides with the set T of all elements of finite order, and G/T is a (continuous) homomorphic image of $\mathbb{Z}^n$ with some nonnegative integer n. As spectral synthesis holds on $\mathbb{Z}^n$ (see [2]), it holds on its homomorphic images. □

5. Extension by the integers

In this section we show that if spectral synthesis holds on a locally compact Abelian group, then it also holds on the group obtained by adding $\mathbb{Z}$ to it as a direct summand.

It is known that every exponential $e:\mathbb{Z}\to\mathbb{C}$ has the form

$$e(k) = \lambda^k$$

for k in $\mathbb{Z}$, where λ is a nonzero complex number, which is uniquely determined by e. For this exponential we use the notation e_λ. It follows that for every commutative topological group G, the exponentials on $G\times\mathbb{Z}$ have the form $m\otimes e_\lambda : (g,k)\mapsto m(g)e_\lambda(k)$, where m is an exponential on G, and λ is a nonzero complex number. Hence the Fourier–Laplace transforms in $\mathcal{A}(G\times\mathbb{Z})$ can be thought as two variable functions defined on the pairs (m,λ), where m is an exponential on G, and λ is a nonzero complex number.

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Let G be a locally compact Abelian group. For each measure μ in $\mathcal{M}_c(G\times\mathbb{Z})$ and for every k in $\mathbb{Z}$ we let

$$S_k(\mu)=\{g:\ g\in G\ \text{ and }\ (g,k)\in\operatorname{supp}\mu\}.$$

As μ is compactly supported, there are only finitely many k's in $\mathbb{Z}$ such that $S_k(\mu)$ is nonempty. We have

$$\operatorname{supp}\mu=\bigcup_{k\in\mathbb{Z}}(S_k(\mu)\times\{k\}),$$

and

$$S_k(\mu)\times\{k\}=(G\times\{k\})\cap\operatorname{supp}\mu.$$

It follows that the sets $S_k(\mu)\times\{k\}$ are pairwise disjoint compact sets in $G\times\mathbb{Z}$, and they are nonempty for finitely many k's only. The restriction of μ to $S_k(\mu)\times\{k\}$ is denoted by μ_k. Then, by definition

$$\langle\mu_k,f\rangle=\int f\cdot\chi_k\,d\mu$$

for each f in $\mathcal{C}(G\times\mathbb{Z})$, where χ_k denotes the characteristic function of the set $S_k(\mu)\times\{k\}$. In other words,

$$\int f\,d\mu_k=\int f(g,k)\,d\mu(g,l)$$

holds for each k in $\mathbb{Z}$ and for every f in $\mathcal{C}(G\times\mathbb{Z})$. Clearly, $\mu=\sum_{k\in\mathbb{Z}}\mu_k$, and this sum is finite.

Lemma 1. *Let μ be in $\mathcal{M}_c(G\times\mathbb{Z})$. Then, for each k in $\mathbb{Z}$, we have*

$$\mu_k=\mu_0*\delta_{(0,k)}.$$

Here $\delta_{(0,k)}$ denotes the Dirac measure at the point $(0,k)$ in $G\times\mathbb{Z}$.

Proof. We have for each f in $\mathcal{C}(G\times\mathbb{Z})$:

$$\begin{aligned}\langle\mu_0*\delta_{(0,k)},f\rangle&=\int\int f(g+h,l+n)\,d\mu_0(g,l)\,d\delta_{(0,k)}(h,n)\\&=\int f(g,l+k)\,d\mu_0(g,l)=\int f(g,k)\,d\mu(g,l)=\langle\mu_k,f\rangle.\end{aligned}$$

□

For each μ in $\mathcal{M}_c(G\times\mathbb{Z})$, we define the measure μ_G in $\mathcal{M}_c(G)$ by

$$\langle\mu_G,\varphi\rangle=\int\varphi(g)\,d\mu(g,l),$$

whenever φ is in $\mathcal{C}(G)$. Clearly, every φ in $\mathcal{C}(G)$ can be considered as a function in $\mathcal{C}(G\times\mathbb{Z})$, hence this definition makes sense, further we have

$$\langle\mu_G,\varphi\rangle=\int\varphi(g)\,d\mu_0(g,l).$$

Lemma 2. *If I is a closed ideal in $\mathcal{M}_c(G\times\mathbb{Z})$, then the set I_G of all measures μ_G with μ in I, is a closed ideal in $\mathcal{M}_c(G)$.*

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Proof. Clearly $\mu_G + \nu_G = (\mu + \nu)_G$ and $\lambda \cdot \mu_G = (\lambda \cdot \mu)_G$. Let μ_G be in I and ξ in $\mathcal{M}_c(G)$. Then we have for each φ in $\mathcal{C}(G)$:

$$\langle \xi * \mu_G, \varphi \rangle = \int \int \varphi(g+h)\, d\xi(g)\, d\mu_G(h) = \int \int \varphi(g+h)\, d\xi(g)\, d\mu(h,l).$$

On the other hand, we extend ξ from $\mathcal{M}_c(G)$ to $\mathcal{M}_c(G \times \mathbb{Z})$ by the definition

$$\langle \tilde{\xi}, f \rangle = \int f(g,0)\, d\xi(g)$$

whenever f is in $\mathcal{C}(G \times \mathbb{Z})$. Then

$$\langle \tilde{\xi}_G, \varphi \rangle = \int \varphi(g)\, d\tilde{\xi}_0(g,l) = \int \varphi(g)\, d\xi(g) = \langle \xi, \varphi \rangle,$$

that is $\tilde{\xi}_G = \xi$. Finally, a simple calculation shows that

$$\langle \xi * \mu_G, \varphi \rangle = \langle (\tilde{\xi} * \mu)_G, \varphi \rangle,$$

hence $\xi * \mu_G = (\tilde{\xi} * \mu)_G$ is in I_G, as $\tilde{\xi} * \mu$ is in I.

Now we show that the ideal I_G is closed. Assume that (μ_α) is a generalized sequence in I such that the generalized sequence $(\mu_{\alpha,G})$ converges to ξ in $\mathcal{M}_c(G)$. This means that

$$\lim_\alpha \int \varphi(g)\, d\mu_{\alpha,G}(g) = \int \varphi(g)\, d\xi(g)$$

holds for each φ in $\mathcal{C}(G)$. In particular, for each exponential m on G we have

$$\lim_\alpha \int \check{m}(g)\, d\mu_{\alpha,0}(g,l) = \lim_\alpha \int \check{m}(g)\, d\mu_{\alpha,G}(g) = \int \check{m}(g)\, d\xi(g) = \int \check{m}(g)\, d\tilde{\xi}_0(g,l).$$

In other words,

$$\lim_\alpha \widehat{\mu}_{\alpha,0} = \widehat{\tilde{\xi}}_0$$

holds. It follows

$$\lim_\alpha \mu_{i,0} = \tilde{\xi}_0,$$

consequently

$$\tilde{\xi}_k = \tilde{\xi}_0 * \delta_{(0,k)} = \lim_\alpha \mu_{\alpha,0} * \delta_{(0,k)} = \lim_\alpha \mu_{\alpha,k}.$$

Then we infer

$$\tilde{\xi} = \sum_k \tilde{\xi}_k = \sum_k \lim_\alpha \mu_{\alpha,k} = \lim_\alpha \sum_k \mu_{\alpha,k} = \lim_\alpha \mu_\alpha,$$

where we can interchange the sum and the limit using the fact that in each sum the number of nonzero terms is finite. As I is closed, $\tilde{\xi}$ is in I, which proves that $\xi = \tilde{\xi}_G$ is in I_G, that is, I_G is closed. □

Now we can derive the following theorem.

Theorem 4. *Let G be a locally compact Abelian group. Then spectral synthesis holds on G if and only if it holds on $G \times \mathbb{Z}$.*

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Proof. If spectral synthesis holds on $G \times \mathbb{Z}$, then it obviously holds on its continuous homomorphic images, in particular, it holds on G, which is the projection of $G \times \mathbb{Z}$ onto the first component.

Conversely, we assume that spectral synthesis holds on G. This means that every closed ideal in the Fourier algebra of G is localizable, and we need to show the same for all closed ideals of the Fourier algebra of $G \times \mathbb{Z}$.

We consider the closed ideal $\widehat{I}$ in the Fourier algebra $\mathcal{A}(G \times \mathbb{Z})$, and we assume that $\widehat{I}$ is nonlocalizable, that is, there is a measure ν in $\mathcal{M}_c(G \times \mathbb{Z})$ such that $\widehat{\nu}$ is annihilated by $\mathcal{P}_{\widehat{I},m,\lambda}$ for each m and λ, but $\widehat{\nu}$ is not in $\widehat{I}$. We show that $\widehat{\nu}_G$ is in $\widehat{I}_G$; then it will follow that $\widehat{\nu}$ is in $\widehat{I}$, a contradiction.

Suppose that a polynomial derivation d annihilates $\widehat{I}_G$ at m. Then we have

$$d\widehat{\mu}_G(m) = \int p_{d,m}(g)\check{m}(g)\, d\mu_G(g) = \int p_{d,m}(g)\check{m}(g)\, d\mu(g,l) = 0$$

for each $\widehat{\mu}$ in $\widehat{I}_G$ and for every exponential m on G, where $p_{d,m} : G \to \mathbb{C}$ is the generating polynomial of d at m. Then we define the polynomial derivation D on the Fourier algebra $\mathcal{A}(G \times \mathbb{Z})$ by

$$D\widehat{\mu}(m,\lambda) = \int p_{d,m}(g)\check{m}(g)\lambda^{-l}\, d\mu(g,l).$$

If $\widehat{\mu}$ is in $\widehat{I}$, then we have

$$D\widehat{\mu}_k(m,\lambda) = \int p_{d,m}(g)\check{m}(g)\lambda^{-l}\, d\mu_k(g,l) = \lambda^{-k} \cdot \int p_{d,m}(g)\check{m}(g)\, d\mu(g,l) = 0$$

for each k in $\mathbb{Z}$. As $\widehat{\mu} = \sum_{k\in\mathbb{Z}} \widehat{\mu}_k$, it follows that $D\widehat{\mu}(m,\lambda) = 0$ for each $\widehat{\mu}$ in $\widehat{I}$. In other words, D is in $\mathcal{P}_{\widehat{I},m,\lambda}$ for each exponential m and nonzero complex number λ. In particular, $\widehat{\nu}$ is annihilated by D:

$$D\widehat{\nu}(m,\lambda) = \int p_{d,m}\check{m}(g)\lambda^{-l}\, d\nu(g,l) = 0.$$

It follows

$$d\widehat{\nu}_G(m) = D\widehat{\nu}_0(m,\lambda) = \int p_{d,m}(g)\check{m}(g)\, d\nu(g,l) = 0.$$

As d is an arbitrary polynomial derivation which annihilates $\widehat{I}_G$ at m, we have that $\widehat{\nu}_G$ is annihilated by $\mathcal{P}_{\widehat{I}_G,m}$ for each m. As spectral synthesis holds on G, the ideal $\widehat{I}_G$ is localizable, consequently $\widehat{\nu}_G$ is in $\widehat{I}_G$, which implies that $\widehat{\nu}$ is in $\widehat{I}$, and our theorem is proved. □

6. Characterization theorems

Corollary 3. *Let G be a compactly generated locally compact Abelian group. Then spectral synthesis holds on G if and only if G is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times F$, where $a \le 1$ and b are nonnegative integers, and F is an arbitrary compact Abelian group.*

Proof. By the Structure Theorem of compactly generated locally compact Abelian groups (see [3, (9.8) Theorem]) G is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times F$, where a, b are nonnegative integers, and F is a compact Abelian group. If spectral synthesis holds on G, then it holds on its projection $\mathbb{R}^a$. By the results in [1, 6], spectral synthesis holds on $\mathbb{R}^a$ if and only if $a \le 1$, hence G is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times F$ where $a \le 1$ and b are nonnegative integers, and F is a compact Abelian group.

Conversely, let $G = \mathbb{R} \times \mathbb{Z}^b \times F$ with b a nonnegative integer, and F a compact Abelian group. By [1], spectral synthesis holds on $\mathbb{R}$. By repeated application of Theorem 4, we have that spectral synthesis holds on $\mathbb{R} \times \mathbb{Z}^b$ with any nonnegative integer b. Finally, by Theorem 3, spectral synthesis holds on $\mathbb{R} \times \mathbb{Z}^b \times F$. Our proof is complete. □

Corollary 4. *Let G be a locally compact Abelian group. Let B denote the closed subgroup of all compact elements in G. Then spectral synthesis holds on G if and only if G/B is topologically isomorphic to $\mathbb{R}^n \times F$, where $n \leq 1$ is a nonnegative integer, and F is a discrete torsion free Abelian group of finite rank.*

Proof. First we prove the necessity. If spectral synthesis holds on G, then it holds on G/B. By [3, (24.34) Theorem], G/B has sufficiently enough real characters. By [3, (24.35) Corollary], G/B is topologically isomorphic to $\mathbb{R}^n \times F$, where n is a nonnegative integer, and F is a discrete torsion-free Abelian group. As spectral synthesis holds on $\mathbb{R}^n \times F$, it holds on the continuous projections $\mathbb{R}^n$ and F. Then we have $n \leq 1$, and the torsion-free rank of F is finite, by [8].

For the sufficiency, if F is a torsion-free discrete Abelian group with finite rank, then it is the (continuous) homomorphic image of $\mathbb{Z}^k$ with some nonnegative integer k. By repeated application of Theorem 4, we have that spectral synthesis holds on $\mathbb{R} \times \mathbb{Z}^k$, and then it holds on its continuous homomorphic image $\mathbb{R} \times F$. Finally, by Theorem 3, we have that spectral synthesis holds on G. □

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