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PETTY PROJECTION INEQUALITY ON THE SPHERE AND ON THE HYPERBOLIC SPACE

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Abstract. We define a spherical and hyperbolic analog to the Euclidean projection body for star bodies via the gnomonic projection from the unit sphere and stereographic projection in the hyperbolid model of hyperbolic space. We then prove a spherical and hyperbolic projection inequality for these notions by using an adaption of Steiner symmetrization for spherical, respectively hyperbolic, star bodies.

1. Introduction

In the Brunn–Minkowski theory of the Euclidean space $\mathbb{R}^n$, the two classical inequalities which connect the volume of a convex body with that of its polar projection body are the Petty and Zhang projection inequalities, see e.g. [26, 39]. The Petty projection inequality shows that among all convex bodies with the same volume, the ellipsoids have the largest volume of the polar projection body. The Zhang projection inequality shows that the simplices minimise the volume of the polar projection body. Petty projection inequalities have been extended to the L_p Petty projection inequalities and Orlicz projection inequalities, see e.g. [27, 29, 37]. Moreover, the functional versions of the Petty projection inequality—the affine Pólya–Szegö inequality and the affine Sobolev inequality have been largely studied, see e.g. [11, 20, 28, 38].

Recently some researches on isoperimetry in the Euclidean space have been extended to spherical or hyperbolic space, see e.g. [2–4, 7, 8, 12, 14, 16, 18, 31, 34, 35]. F. Besau and E. M. Werner [4, 5] introduced the spherical convex floating body for a convex body on the Euclidean unit sphere and define a new spherical area measure—the floating area. We are convinced that the floating area will become a powerful tool in the spherical convex

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geometry. F. Besau, T. Hack, P. Pivovarov and F. E. Schuster [3] introduced the spherical centroid body of a centrally symmetric convex body in the Euclidean unit sphere, studied a number of basic properties of spherical centroid bodies and proved a spherical analogue of the classical polar Busemann–Petty centroid inequality. F. Gao, D. Hug and R. Schneider [14] proved the Urysohn inequality and the Blaschke–Santaló inequality in the spherical space. Their arguments using two-point symmetrization are also applicable in the hyperbolic space which also yields the hyperbolic version of the Urysohn inequality. Later, T. Hack and P. Pivovarov [18] proved a randomized version of the spherical and hyperbolic Urysohn-type inequalities. G. Wang and C. Xia [34] solved various isoperimetric problems for the quermassintegrals and the curvature integrals in the hyperbolic space $\mathbb{H}^n$ and established quite strong Alexandrov–Fenchel type inequalities.

In comparison with the Euclidean case there are very few results and techniques available in spherical and hyperbolic space. In [14] and [18], the authors use the two-point symmetrization procedure together with rearrangement inequalities in order to prove their result. In [3], the authors used a probabilistic approach to prove the spherical centroid inequality. In [34], the authors used quermassintegral-preserving curvature flows approach to solve the isoperimetric type problems in the hyperbolic space. It is well-known that Steiner symmetrization is a fundamental tool for attacking problems regarding isoperimetry and related geometric inequalities in the Euclidean space $\mathbb{R}^n$, see e.g. [6,9,10]. In this paper, we define the spherical and hyperbolic Steiner symmetrizations on $\mathbb{S}^n$ and $\mathbb{H}^n$ which preserve the property of star bodies, the volume invariance after a Steiner symmetrization and convergence of iterative Steiner symmetrizations. Using the spherical and hyperbolic Steiner symmetrizations, we prove the spherical projection inequality and hyperbolic projection inequality, respectively.

The spherical and hyperbolic projection body operators are dependent on the center chosen and is invariant only with respect to isometries of spherical space, respectively hyperbolic space, that fix this center. In both cases this group of isometries $\overline{\mathrm{O}}(n+1)$ is isomorphic to the orthogonal group $\mathrm{O}(n)$. In contrast the Euclidean projection body operator Π does not depend on the origin and behaves affine equivalent, that is, if $K \subset \mathbb{R}^n$ is a convex body, $\phi \in \mathrm{GL}(n)$ and $x \in \mathbb{R}^n$, then

$$\Pi(\phi K + x) = |\det \phi| \phi^{-\top} \Pi K,$$

where $\phi^{-\top}$ denotes the inverse transposition of ϕ. We also show that the notion of spherical and hyperbolic projection body is continuous with respect to the Hausdorff metric on star bodies with a fixed center. Another characteristic feature of the Euclidean projection body is that it is a Minkowski valuation, see M. Ludwig [24]. However, because there is no obvious analog to Minkowksi addition in spherical and hyperbolic space, it is not clear how such a property may apply to the spherical or hyperbolic projection body.

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In this paper, first we study the spherical projection body in spherical space, which is a natural analog of Petty projection body in the Euclidean space. Using the gnomonic projection from $\mathbb{S}^n$ onto $\mathbb{R}^n$ (see [32, Section 6.1]) and the definition of classical Euclidean Petty projection body, we define the spherical Petty projection body. Using the monotonic increasing property of the measures of the polar bodies of spherical projection bodies after performing a spherical Steiner symmetrization and the continuity of spherical projection operator with respect to spherical Hausdorff distance, we proved the spherical Petty projection inequality. Let $\mathcal{S}_B(\mathbb{S}^n_+)$ denote the set of spherical star bodies with respect to a spherical cap in $\mathbb{S}^n_+$ (see Definition 5). Let $\Pi^\circ_{\mathbb{S}}(K)$ denote the spherical polar body of spherical projection body of K (see Definition 3 and Definition 10).

Theorem 1. *If $K \in \mathcal{S}_B(\mathbb{S}^n_+)$ and $K^\star$ is the spherical cap centered at e_{n+1} with the same volume as K, then*

$$\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K)\right) \le \mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K^\star)\right), \tag{1}$$

with equality if and only if $K = K^\star$.

In hyperbolic geometry, we will use the Poincaré ball model. In this model, the hyperbolic space $\mathbb{H}^n$ is identified with the open Euclidean unit ball $\mathbb{B}^n$ equipped with a certain metric. The Poincaré ball model can be obtained from the hyperboloid model of hyperbolic space via the stereographic projection, see [32, Section 4.5]. We use the stereographic projection instead of the gnomonic projection because stereographic projections can maintain hyperbolic convexity (see Section 2.3 for details). Similarly, we use the gnomonic projection instead of the stereographic projection in considering spherical projection bodies because gnomonic projections can preserve spherical convexity (see Section 2.2 for details).

In this paper, in order to construct a map from the hyperboloid model of hyperbolic space to $\mathbb{R}^n$, we introduce a transformation Φ from $\mathbb{B}^n$ to the Euclidean space $\mathbb{R}^n$. Combining the Poincaré ball model and the transformation Φ, we define the transformation Φ_p from hyperbolic space $\mathbb{H}^n$ to $\mathbb{R}^n$ (see Section 4.1 for the details). Therefore, we can use Euclidean Steiner symmetrizations, Euclidean projection bodies and Euclidean polar bodies in $\mathbb{R}^n$ to define hyperbolic Steiner symmetrizations, hyperbolic projection bodies and hyperbolic polar bodies in $\mathbb{H}^n$. Using the monotonic increasing property of the measure of the polar bodies of hyperbolic star bodies and the continuity of hyperbolic projection operator with respect to hyperbolic Hausdorff distance, we proved the hyperbolic Petty projection inequality. Let $\mathcal{S}_B(\mathbb{H}^n)$ denote the set of hyperbolic star bodies with respect to a hyperbolic ball in $\mathbb{H}^n$ (see Definition 8). Let $\Pi^\circ_{\mathbb{H}}(K)$ denote the hyperbolic polar body of hyperbolic projection body of K (see Definition 11 and Definition 13).

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Theorem 2. *If $K \in \mathcal{S}_B(\mathbb{H}^n)$ and $K^\star$ is the hyperbolic ball centered at e_{n+1} with the same volume as K, then*

$$\mathcal{H}^n\left(\Pi^\circ_{\mathbb{H}}(K)\right) \leq \mathcal{H}^n\left(\Pi^\circ_{\mathbb{H}}(K^\star)\right), \tag{2}$$

with equality if and only if $K = K^\star$.

In the proofs we rely, among other geometric observations in spherical space, respectively hyperbolic space, on tools previously developed on the projection body of (Lipschitz) star bodies, see for example the work of the first author in [21], and an adaption of Euclidean Steiner symmetrization by rescaling on the fibers, respectively radially, to preserve the spherical, respectively hyperbolic volume.

Furthermore, we focus on a characteristic feature of the polar projection body which was first observed by E. Lutwak [25] and strengthened by C. Haberl and F. Schuster [19]. That is, its volume gives an isoperimetric bound for the surface area that is better than the classical isoperimetric inequality. Indeed, together with the Petty projection inequality, we have for an Euclidean convex body $K \subset \mathbb{R}^n$, that

$$\left(\frac{\mathcal{H}^{n-1}(\partial K)}{\mathcal{H}^{n-1}(\partial B_o(1))}\right)^{\frac{1}{n-1}} \geq \left(\frac{\mathcal{H}^n(\Pi^* K)}{\mathcal{H}^n(\Pi^* B_o(1))}\right)^{-\frac{1}{n(n-1)}} \geq \left(\frac{\mathcal{H}^n(K)}{\mathcal{H}^n(B_o(1))}\right)^{\frac{1}{n}}, \tag{3}$$

where $B_o(1)$ is the Euclidean centered unit ball and $\Pi^* K$ denotes the polar body of the projection body ΠK. Note that (3) is equivalent to

$$\mathcal{H}^n(\Pi^* B_{\partial K}) \leq \mathcal{H}^n(\Pi^* K) \leq \mathcal{H}^n(\Pi^* B_K), \tag{4}$$

where $B_{\partial K}$, respectively B_K, is a Euclidean ball with the same surface area, respectively volume, as K.

In Theorem 1 and 2, we established isoperimetric inequalities for the notions of spherical and hyperbolic polar projection body that can be seen as an extension of the Petty projection inequality, that is, the second inequality in (4).

In Theorem 3 and Section 4.5, we attempt to also establish an analog of the first inequality of (4), we show that

$$\mathrm{Vol}_n^\dagger\left(\Pi^\circ_\dagger K\right) \geq \mathrm{Vol}_n^\dagger\left(\Pi^\circ_\dagger B_{\partial K}\right),$$

where $B_{\partial K}$ is a geodesic ball with the same Euclidean surface area as K, $\mathrm{Vol}_n^\dagger$ denotes the Lebesgue measure and $\Pi^\circ_\dagger$ denotes the polar projection operator in spherical space, $\dagger = \mathbb{S}$, respectively hyperbolic space, $\dagger = \mathbb{H}$. Note that this does not achieve the desired goal yet, that is, showing that the volume of the spherical, respectively hyperbolic, polar projection body strengthens the isoperimetric inequality in spherical, respectively hyperbolic space. For this, let $C^\dagger_{\partial K}$ be a geodesic ball with the same spherical, respectively hyperbolic, surface area as K, we give the following conjecture.

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Conjecture 1. *Let K be a spherical, respectively hyperbolic convex body with e_{n+1} as its interior point. Then*

$$\mathrm{Vol}_n^\dagger\left(\Pi_\dagger^\circ K\right) \geq \mathrm{Vol}_n^\dagger\left(\Pi_\dagger^\circ C_{\partial K}^\dagger\right). \tag{5}$$

2. Preliminaries

2.1. **Basic facts from Euclidean convex geometry.** We develop some notations and, for quick later references, list some basic facts about convex bodies. Good general references for the theory of convex bodies are provided by the books of Gardner [15], Gruber [17], Schneider [33] and Artstein-Avidan, Giannopoulos and Milman [1].

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space. Let o denote the origin of $\mathbb{R}^n$. Let $e_1, \cdots, e_n$ denote the standard orthonormal basis of $\mathbb{R}^n$. Let $x \cdot y$ denote the Euclidean scalar product for $x, y \in \mathbb{R}^n$. Let $\mathbb{R}^{n-1}$ denote the subspace of $\mathbb{R}^n$, where the n-th component is 0, i.e., $\mathbb{R}^{n-1} := \{x \in \mathbb{R}^n,\ x \cdot e_n = 0\}$. Let $u^\perp$ denote the orthogonal complementary space of the unit vector $u \in S^{n-1}$. Let $\mathbb{S}^{n-1}$ denote the set of unit vectors of $\mathbb{R}^n$. Let $B_o(r)$ denote the closed ball centered at the origin o with radius r in $\mathbb{R}^n$. Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure. Let ω_n denote the volume of $B_o(1)$, i.e., $\omega_n := \mathcal{H}^n(B_o(1))$. A convex body K is a compact convex subset of $\mathbb{R}^n$. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}(\mathbb{R}^n)$. We denote by $\mathcal{K}_o(\mathbb{R}^n)$ the set of convex bodies that contain the origin in their interiors. Let $\|\cdot\|$ denote the Euclidean norm. For $K \in \mathcal{K}(\mathbb{R}^n)$, K is uniquely determined by its *support function* $h(K, \cdot)$ defined by

$$h(K, x) := \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$

The support function is homogeneous of degree 1, i.e.,

$$h(K, rx) = rh(K, x), \quad \text{for } r > 0. \tag{6}$$

For $K \in \mathcal{K}_o(\mathbb{R}^n)$, its *radial function* is defined by

$$\rho(K, x) := \max\{r > 0 :\ rx \in K\}, \quad x \in \mathbb{R}^n \backslash \{o\}. \tag{7}$$

The radial function is homogeneous of degree -1, i.e.,

$$\rho(K, rx) = \frac{1}{r}\rho(K, x), \quad \text{for } r > 0. \tag{8}$$

A compact set $K \subset \mathbb{R}^n$ is a *star-shaped set* with respect to $z \in K$ if the intersection of every straight line through z with K is either a line segment or a single point set $\{z\}$. Let $K \subset \mathbb{R}^n$ be a compact star shaped set with respect to $z \in K$, the radial function $\rho_z(K, \cdot)$: $\mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ is defined by

$$\rho_z(K, x) := \max\{r \geq 0 :\ z + rx \in K\}. \tag{9}$$

If $\rho_z(K, \cdot)$ is strictly positive and continuous, then we call K a *star body* with respect to z, denotes the class of star bodies in $\mathbb{R}^n$ by $\mathcal{S}_z(\mathbb{R}^n)$. If $K \subset \mathbb{R}^n$ is a star body with respect

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to each point of ball $B_o(r)$, then we say K is a *star body with respect to a ball*. The class of star bodies with respect to ball $B_o(r)$ will be denoted by $\mathcal{S}_B(\mathbb{R}^n)$. It is clear that $\mathcal{K}_o(\mathbb{R}^n) \subset \mathcal{S}_B(\mathbb{R}^n)$, i.e., any convex body with the origin as its interior is a star body with respect to a ball. For $K \in \mathcal{S}_B(\mathbb{R}^n)$, we have the following volume formula

$$\mathcal{H}^n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho^n(K,u)du \tag{10}$$

and the surface area formula

$$\mathcal{H}^{n-1}(\partial K) = \int_{\mathbb{S}^{n-1}} \rho^{n-1}(K,u)\left[u \cdot \nu^K(\rho(K,u)u)\right]^{-1} du, \tag{11}$$

where $\nu^K(\rho(K,u)u)$ denotes the outer unit normal vector of K at the boundary point $\rho(K,u)u$.

For $K \in \mathcal{S}_o(\mathbb{R}^n)$, its *polar body* is defined by

$$K^* := \{y \in \mathbb{R}^n : \; y \cdot x \leq 1 \;\text{ for any } x \in K\}.$$

It is well-known that if $K \in \mathcal{K}_o(\mathbb{R}^n)$,

$$(K^*)^* = K. \tag{12}$$

The support function and radial function of $K \in \mathcal{K}_o(\mathbb{R}^n)$ have the following relationship:

$$h(K,x)\rho(K^*,x) = 1, \qquad x \in \mathbb{R}^n \backslash \{o\}. \tag{13}$$

The *radial distance* between $K, L \in \mathcal{S}_o(\mathbb{R}^n)$ is defined by

$$d_R(K,L) := \max_{u\in\mathbb{S}^{n-1}} |\rho(K,u) - \rho(L,u)|. \tag{14}$$

The *Hausdorff distance* between the compact sets $K, L \subset \mathbb{R}^n$ is defined by

$$d_H(K,L) := \min\{r \geq 0 : \; K \subset L + B_o(r), \; L \subset K + B_o(r)\}. \tag{15}$$

For $K \in \mathcal{S}_B(\mathbb{R}^n)$, its *Petty projection body* is defined with its support function:

$$h(\Pi K, z) := \frac{1}{2}\int_{\partial K} \left|\nu^K(x) \cdot z\right| d\mathcal{H}^{n-1}(x), \tag{16}$$

where ∂K denotes the boundary of K, $\nu^K(x)$ denotes the unit outer normal vector of K at the boundary point $x \in \partial K$. The polar body of ΠK will be denoted by $\Pi^* K$ rather than $(\Pi K)^*$.

The following lemma shows that the Petty projection operator $\Pi : \; \mathcal{S}_B(\mathbb{R}^n) \to \mathcal{K}_o(\mathbb{R}^n)$ is continuous when K_i converges to K_∞ in the Hausdorff distance.

Lemma 1. *[21, Proposition 4.1] Let $K_\infty, K_i \in \mathcal{S}_B(\mathbb{R}^n)$, $i \in \mathbb{N}$. If $K_i \to K_\infty$ in the Hausdorff distance and $\mathcal{H}^{n-1}(\partial K_i) \to \mathcal{H}^{n-1}(\partial K_\infty)$, then $\Pi K_i \to \Pi K_\infty$ in the Hausdorff distance.*

For $K \in \mathcal{S}_o(\mathbb{R}^n)$, its *Steiner symmetrization* along the direction $u \in \mathbb{S}^{n-1}$ is defined by

$$S_u K := \bigcup_{x' \in K|_{u^\perp}} \left\{ x' + tu : \ t \in \left[-\frac{1}{2}\mathcal{H}^1(K_{u,x'}), \frac{1}{2}\mathcal{H}^1(K_{u,x'}) \right] \right\}, \tag{17}$$

where $u^\perp$ denotes the orthogonal complementary space of u, i.e., $u^\perp := \{z \in \mathbb{R}^n : \ z \cdot u = 0\}$; $K|_{u^\perp}$ denotes the orthogonal projection of K onto $u^\perp$, i.e.,

$$K|_{u^\perp} := \{x' \in u^\perp : \ \text{there exits some } r \in \mathbb{R} \text{ such that } x' + ru \in K\};$$

$K_{u,x'}$ denotes the intersection of K and the straight line parallel to u and passing through point x', i.e.,

$$K_{u,x'} := \{x \in K : \ x = x' + ru, \ r \in \mathbb{R}\}.$$

For simplicity of notation, we write SK, K' and $K_{x'}$ instead of $S_{e_n}K$, $K|_{e_n^\perp}$ and $K_{e_n,x'}$, respectively.

Let $K \in \mathcal{S}_o(\mathbb{R}^n)$. Its *symmetric rearrangement* $K^\star$ is the closed centered ball whose volume agrees with K,

$$K^\star := \{x \in \mathbb{R}^n : \ \omega_n \|x\|^n \leq \mathcal{H}^n(K)\}.$$

The following lemma provides some properties on the Steiner symmetrizations of star bodies.

Lemma 2. *[22, Lemma 5.1] If $K \in \mathcal{S}_B(\mathbb{R}^n)$, then $S_u K \in \mathcal{S}_B(\mathbb{R}^n)$ for every $u \in \mathbb{S}^{n-1}$.*

Lemma 3. *[22, Theorem 2.2] Let $K, K_i \in \mathcal{S}_B(\mathbb{R}^n)$, $i \in \mathbb{N}$. Then, the fact that K_i converges to K in Hausdorff distance is equivalent to the fact that K_i converges to K in radial distance.*

Lemma 4. *[22, Theroem 2.3] If $K \in \mathcal{S}_B(\mathbb{R}^n)$ and T is a dense subset of $\mathbb{S}^{n-1}$, then there is a sequence $\{u_i\} \subset T$ such that $K_i := S_{u_i} \cdots S_{u_1} K$ converges to $K^\star$ in radial distance.*

The following lemma characterizes the structures of the boundaries of star bodies with respect to a ball.

Lemma 5. *[23, Theorem 3.1] Let $K \in \mathcal{S}_B(\mathbb{R}^n)$. Then, for almost all $u \in \mathbb{S}^{n-1}$, there is a sequence of disjoint open subsets $G_m \subset K'$, and two sequences of graph functions $f_{m,j}, g_{m,j} : G_m \to \mathbb{R}$, $1 \leq j \leq m$, satisfying*
(i) $\bigcup_{m=1}^\infty G_m$ is open dense in K', and $f_{m,1} < g_{m,1} < \cdots < f_{m,j} < g_{m,j}$;
(ii) K has the representation (if we neglect an $\mathcal{H}^n$-null set)

$$K = \bigcup_{m=1}^\infty \bigcup_{\substack{x' \in G_m \\ 1 \leq j \leq m}} \left\{ (x', x_n) : f_{m,j}(x') \leq x_n \leq g_{m,j}(x') \right\}$$

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and ∂K has the representation (if we neglect an $\mathcal{H}^{n-1}$-null set)

$$\partial K = \bigcup_{m=1}^{\infty}\bigcup_{j=1}^{m}\left\{\left(x', f_{m,j}(x')\right) : x' \in G_m\right\} \cup \left\{\left(x', g_{m,j}(x')\right) : x' \in G_m\right\};$$

(iii) $f_{m,j}$, $g_{m,j}$ are differentiable at each $x' \in G_m$, and

$$\nu^K(x', f_{m,j}(x')) = \frac{(\nabla f_{m,j}(x'), -1)}{\sqrt{1 + \left|\nabla f_{m,j}(x')\right|^2}}, \quad \nu^K(x', g_{m,j}(x')) = \frac{(-\nabla g_{m,j}(x'), 1)}{\sqrt{1 + \left|\nabla g_{m,j}(x')\right|^2}}.$$

The following well-known fact, which is provided by Lutwak, Yang and Zhang [29], establishes the relationship between Steiner symmetrizations and polar bodies.

Lemma 6. *[29, Lemma 1.1.] For two convex bodies $K, L \in \mathcal{K}_o(\mathbb{R}^n)$,*

$$S_{e_n}L^* \subset K^*$$

if and only if

$$h(L,(z',t)) = h(L,(z',-s)) = 1, \text{ with } t \neq -s \implies h\left(K,\left(z', \frac{s+t}{2}\right)\right) \leq 1.$$

In addition, if $S_{e_n}L^* = K^*$, then $h(K,(z',\frac{s+t}{2})) = 1$ for any $(z',t),(z',-s) \in \mathbb{R}^{n-1}\times\mathbb{R}$ satisfying $t \neq -s$ and $h(L,(z',t)) = h(L,(z',-s)) = 1$.

Let $f : \mathbb{R}^n \to [0,+\infty)$ be a nonnegative measurable function that *vanishes at infinity*, in the sense that all its positive level sets have finite measure,

$$\mathcal{H}^n(\{x : f(x) > t\}) < \infty, \quad \text{for all } t > 0.$$

We define the *symmetric decreasing rearrangement* $f^\star$ of f by symmetrizing its level sets,

$$f^\star(x) = \int_0^\infty \chi_{\{f>t\}^\star}(x)dt, \tag{18}$$

where χ_E denote the characteristic function of $E \subset \mathbb{R}^n$, i.e.,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Lemma 7. *(Hardy-Littlewood inequality) If $f, g : \mathbb{R}^n \to [0,\infty)$ are nonnegative measurable functions that vanish at infinity, then*

$$\int_{\mathbb{R}^n} f(x)g(x)dx \leq \int_{\mathbb{R}^n} f^\star(x)g^\star(x)dx \tag{20}$$

in the sense that the left hand side is finite whenever the right hand side is finite.

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2.2. **Basic facts from Spherical convex geometry.** Let us recall some facts about spherical convex geometry; see e.g. [4, 30, 32]. Let $\mathbb{R}^{n+1}$ denote $(n+1)$-dimensional Euclidean space. Let $\mathbb{R}^n := \left\{x \in \mathbb{R}^{n+1} : x \cdot e_{n+1} = 0\right\}$. Let o denote the origin of $\mathbb{R}^{n+1}$. Let $e_1, \cdots, e_n, e_{n+1}$ denote the standard orthonormal basis of $\mathbb{R}^{n+1}$. We denote the Euclidean unit sphere in $\mathbb{R}^{n+1}$ by $\mathbb{S}^n, n \geq 2$. For $u \in \mathbb{S}^n$, let $\mathbb{S}_u$ denote the set $\{v \in \mathbb{S}^n : \ v \cdot u = 0\}$. Let $\mathbb{S}_u^+$ denote the set $\{v \in \mathbb{S}^n : \ v \cdot u > 0\}$ and $\mathbb{S}_u^-$ denote the set $\{v \in \mathbb{S}^n : \ v \cdot u < 0\}$. To simplify notation, we let $\mathbb{S}^{n-1}$ denote $\mathbb{S}_{e_{n+1}}$ and $\mathbb{S}^n_\pm$ denote $\mathbb{S}^\pm_{e_{n+1}}$. A set $A \subseteq \mathbb{S}^n$ is called *(spherical) convex* if its radial extension

$$\operatorname{rad} A = \left\{rv \in \mathbb{R}^{n+1} : r \geq 0 \text{ and } v \in A\right\}$$

is convex in $\mathbb{R}^{n+1}$. If $A \subseteq \mathbb{S}^n$ is convex, then $\operatorname{rad} A$ is a convex cone with o as its vertex in $\mathbb{R}^{n+1}$. A closed convex subset of $\mathbb{S}^n$ is called a *(spherical) convex body*. The set of convex bodies is denoted by $\mathcal{K}(\mathbb{S}^n)$. Furthermore, the set of convex bodies contained in $\mathbb{S}^n_+$ with e_{n+1} as its interior point is denoted by $\mathcal{K}_o(\mathbb{S}^n_+)$. And the set of convex bodies contained in $\mathbb{S}^n_-$ with $-e_{n+1}$ as its interior point is denoted by $\mathcal{K}_o(\mathbb{S}^n_-)$.

The natural spherical distance $\hat{d}_s$ is given by $\hat{d}_s(u, v) = \arccos(u \cdot v)$ for $u, v \in \mathbb{S}^n$. For spherical compact sets $K, L \subset \mathbb{S}^n$, the *spherical Hausdorff distance* of K and L is defined by

$$\hat{d}_s(K, L) := \inf\{r > 0 : \ K \subseteq L_r \ \text{ and } \ L \subseteq K_r\}, \tag{21}$$

where L_r denotes the spherical parallel set of L, which is defined by

$$L_r := \{w \in \mathbb{S}^n : \text{ there exists } v \in L \text{ such that } \hat{d}_s(w, v) \leq r\}.$$

Let $B_s(\alpha)$ denote the spherical cap of radius $\alpha \in (0, \pi/2)$ and center e_{n+1} in $\mathbb{S}^n$, i.e.,

$$B_s(\alpha) := \{v \in \mathbb{S}^n : \ \hat{d}_s(v, e_{n+1}) \leq \alpha\}.$$

The *convex hull* $\operatorname{conv} A$ of $A \subseteq \mathbb{S}^n$ is the intersection of all convex bodies in $\mathbb{S}^n$ that contain A. The convex hull of two spherical convex bodies K, L is denoted by $\operatorname{conv}(K, L)$, i.e.,

$$\operatorname{conv}(K, L) := \operatorname{conv}(K \cup L).$$

The segment spanned by two points $u, v \in \mathbb{S}^n, u \neq -v$, is given by $\operatorname{conv}(u, v) = \operatorname{conv}(\{u\}, \{v\})$. A k-sphere S, $k \in \{0, \ldots, n\}$, is the intersection of a $(k+1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$. Let S be a k-sphere and let $K \in \mathcal{K}(\mathbb{S}^n)$. Then the *spherical projection* $K \mid S$ is defined by

$$K \mid S := \operatorname{conv}(S^\circ, K) \cap S,$$

where $S^\circ := \{w \in \mathbb{S}^n : \ w \cdot u = 0 \text{ for all } u \in S\}$. The spherical projection of a point is given by $x \mid S := \{x\} \mid S$.

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Definition 1. *For $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, the spherical support function $h(K,\cdot) : \mathbb{S}^{n-1} \to \left(0, \frac{\pi}{2}\right)$ of K is defined by*

$$h_s(K,v) = \max\left\{\operatorname{sgn}(v\cdot w)\hat{d}_s\left(e_{n+1}, w \mid \mathbb{S}^1_{e_{n+1},v}\right) : w \in K\right\}, \quad v \in \mathbb{S}^{n-1}, \tag{22}$$

where $\mathbb{S}^1_{e_{n+1},v}$ denotes the 1-sphere spanned by e_{n+1} and v.

The intuitive interpretation of the spherical support function is as follows: If $v \in \mathbb{S}^{n-1}$ then the projection $K \mid \mathbb{S}^1_{e_{n+1},v}$ is a spherical segment and the spherical support function measures the width along the direction v with respect to e_{n+1}. We have

$$K \mid \mathbb{S}^1_{e_{n+1},v} = \{\cos(\alpha)e_{n+1} + \sin(\alpha)v : \alpha \in [-h(K,-v), h(K,v)]\}.$$

Definition 2. *For $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, its spherical radial function is defined by*

$$\rho_s(K,v) := \max\left\{\operatorname{sgn}(v\cdot w)\hat{d}_s(e_{n+1}, w) : w \in K \cap \mathbb{S}^1_{e_{n+1},v}\right\}, \quad v \in \mathbb{S}^{n-1}. \tag{23}$$

Definition 3. *For $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, its spherical polar body K° is defined by*

$$K^\circ = \{v \in \mathbb{S}^n : \ v\cdot w \le 0 \ \text{ for all } w \in K\}. \tag{24}$$

By the above definition, if $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, then $K^\circ \in \mathcal{K}_o(\mathbb{S}^n_-)$. Moreover, the spherical polar body K° is the intersection of $\mathbb{S}^n$ and $(\operatorname{rad} A)^*$, here

$$(\operatorname{rad} A)^* := \left\{z \in \mathbb{R}^{n+1} : \ z\cdot x \le 0, \ x \in \operatorname{rad} A\right\}$$

denotes the polar of the corresponding convex cone $\operatorname{rad} A$.

Definition 4. *The gnomonic projection $g : \mathbb{S}^n_+ \to \mathbb{R}^n$ is defined by*

$$g(v) := \frac{v}{e_{n+1}\cdot v} - e_{n+1}.$$

The inverse gnomonic projection $g^{-1} : \mathbb{R}^n \to \mathbb{S}^n_+$ is defined by

$$g^{-1}(x) := \frac{x + e_{n+1}}{\|x + e_{n+1}\|}.$$

By the definition of gnomonic projection of $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, the following equalities show the relations between spherical support function (spherical radial function) of K and the Euclidean support function (Euclidean radial function) of $g(K)$:

$$h(g(K),v) = \tan h_s(K,v), \quad v \in \mathbb{S}^{n-1}, \tag{25}$$

$$\rho(g(K),v) = \tan \rho_s(K,v), \quad v \in \mathbb{S}^{n-1}. \tag{26}$$

Moreover, it is easy to prove that for $K \in \mathcal{K}_o(\mathbb{S}^n_+)$,

$$g(K)^* = -g(-K^\circ). \tag{27}$$

By (12) and (27), for $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, we have

$$(K^\circ)^\circ = K. \tag{28}$$

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By (25), (26) and (27),

$$\tan h_s(K,v) = h(g(K),v) = \frac{1}{\rho(g(K)^*,v)} = \frac{1}{\rho(g(-K^\circ),-v)} = \frac{1}{\tan\rho_s(-K^\circ,-v)}.$$

Therefore, for $K \in \mathcal{K}_o(\mathbb{S}^n_+)$ and $v \in \mathbb{S}^{n-1}$,

$$h_s(K,v) + \rho_s(-K^\circ,-v) = \frac{\pi}{2}. \tag{29}$$

By Definition 4 and the definitions of star bodies in $\mathbb{R}^n$, we define spherical star bodies as follows.

Definition 5. *For a spherical compact set $K \subset \mathbb{S}^n_+$, if its gnomonic projection $g(K)$ is a star body with respect to o in $\mathbb{R}^n$, then K is called a spherical star body with respect to e_{n+1}. If $g(K)$ is a star body with respect to a ball $B_o(\tan\alpha)$ in $\mathbb{R}^n$, then K is called a spherical star body with respect to a spherical cap $B_s(\alpha)$.*

The set of spherical star bodies with respect to e_{n+1} is denoted by $\mathcal{S}_o(\mathbb{S}^n_+)$. The set of spherical star bodies with respect to $B_s(\alpha)$ is denoted by $\mathcal{S}_B(\mathbb{S}^n_+)$. Similarly, $\mathcal{K}_o(\mathbb{S}^n_+) \subset \mathcal{S}_B(\mathbb{S}^n_+)$. For $K \in \mathcal{S}_o(\mathbb{S}^n_+)$, its spherical radial function ρ_K can be defined as in (23).

Definition 6. *For $\bar{K} \in \mathcal{K}_o(\mathbb{R}^n)$, its spherical measure is defined by*

$$\mu_{s,n}(\bar{K}) := \int_{\bar{K}} \left(1 + \|x\|^2\right)^{-\frac{n+1}{2}} dx. \tag{30}$$

By [3, Lemma 2.3], the spherical measure of $\bar{K} \in \mathcal{K}_o(\mathbb{R}^n)$ equals the n-Hausdorff measure of its inverse gnomonic projection $\mathcal{H}^n\left(g^{-1}(\bar{K})\right)$. Thus, for $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, we have

$$\mathcal{H}^n(K) = \mu_{s,n}(g(K)). \tag{31}$$

Lemma 8. *([36, Lemma 6.5.1]). Let S be a k-sphere, $0 \le k \le n-1$, and let $f : \mathbb{S}^n \to \mathbb{R}$ be a non-negative measurable function. Then*

$$\int_{\mathbb{S}^n} f(w)dw = \int_S \int_{\mathrm{conv}(S^\circ,v)} \sin\left(d_s\left(S^\circ,u\right)\right)^k f(u)dudv. \tag{32}$$

2.3. **Basic facts from Hyperbolic convex geometry.** Let us recall some facts about hyperbolic geometry; see e.g. [32]. Recall that $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, where

$$\mathbb{R}^n := \left\{x \in \mathbb{R}^{n+1} : x \cdot e_{n+1} = 0\right\}.$$

In $\mathbb{R}^{n+1}$, let

$$\mathbb{H}^n := \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} :\ \|x\|^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\right\}$$

denote the upper sheet of a two-sheet hyperboloid.

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Definition 7. *(Poincaré ball model) Let* $o' := -e_{n+1}$. *For any* $\bar{x} := (x, x_{n+1}) \in \mathbb{H}^n$, *the Poincaré ball model projection point of* $\bar{x}$, *denoted by* $P(\bar{x})$, *is the intersection of the half-line* $o'\bar{x}$ *and* $e_{n+1}^{\perp}$. *In the Poincaré ball model,* $\mathbb{H}^n$ *is identified with the following open unit ball equipped with a certain metric*

$$\mathbb{B}^n := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : \ \|x\| < 1, \ x_{n+1} = 0\}. \tag{33}$$

We call the projection $P : \ \mathbb{H}^n \to \mathbb{B}^n$ as *Poincaré ball model projection*. In the Poincaré ball model, the corresponding metric is

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{\left(1 - \left(x_1^2 + \cdots + x_n^2\right)\right)^2}. \tag{34}$$

In this metric, geodesic segments are arcs of the circles orthogonal to the boundary of the ball $\mathbb{B}^n$. If a segment passes through the origin, then the circle becomes a straight line. We say that a body $\bar{K} \subset \mathbb{B}^n$ is *hyperbolic convex* if it is convex with respect to the metric (34). The *hyperbolic convexity* means that for any two points x and y in $\bar{K}$ the geodesic segment connecting these two points is also in $\bar{K}$.

For two compact sets $K, L \subset \mathbb{H}^n$, their *hyperbolic Hausdorff distance* is defined by

$$d_h(K, L) := \inf\{r > 0 : \ K \subset L_r, \ \ L \subset K_r\}, \tag{35}$$

where

$$K_r := \{\bar{y} \in \mathbb{H}^n : \ \text{there exits } \bar{x} \in K \text{ such that } ds^2(P(\bar{x}), P(\bar{y})) \le r^2\}.$$

Let $B_h(\alpha) \subset \mathbb{H}^n$ denote the *hyperbolic ball* centered at e_{n+1} with radius α, i.e.,

$$B_h(\alpha) := \left\{\bar{x} \in \mathbb{H}^n : \ ds^2\left(P(\bar{x}), o\right) \le \alpha^2\right\}.$$

The volume element of the metric (34) equals

$$d\mu_{h,n} = 2^n \frac{dx_1 \cdots dx_n}{\left(1 - (x_1^2 + \cdots + x_n^2)\right)^n} = 2^n \frac{dx}{(1 - \|x\|^2)^n}. \tag{36}$$

Therefore, for $\bar{K} \subset \mathbb{B}^n$, the hyperbolic volume is then given by

$$\mathrm{hvol}_n(\bar{K}) = \int_{\bar{K}} d\mu_{h,n} = 2^n \int_{\bar{K}} \frac{dx}{(1 - \|x\|^2)^n}. \tag{37}$$

If $\bar{K}$ is a star body in $\mathbb{B}^n$, we can write its hyperbolic volume in polar coordinates,

$$\mathrm{hvol}_n(\bar{K}) = 2^n \int_{\mathbb{S}^{n-1}} \int_0^{\rho_{\bar{K}}(u)} \frac{r^{n-1}}{(1 - r^2)^n} dr du, \tag{38}$$

where $\rho_{\bar{K}}$ denotes the radial function of $\bar{K}$ in $\mathbb{R}^n$ given in (7).

Definition 8. *For* $K \subset \mathbb{H}^n$, *if* $P(K)$ *is a star body with respect to the origin in* $\mathbb{R}^n$, *then we call* K *is a hyperbolic star body with respect to* e_{n+1}. *If* $P(K)$ *is a star body with respect to a ball* $B_o(r)$ *in* $\mathbb{R}^n$, *then* K *is called as a hyperbolic star body with respect to some hyperbolic ball.*

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We denote the class of hyperbolic star bodies with respect to e_{n+1} in $\mathbb{H}^n$ by $\mathcal{S}_o(\mathbb{H}^n)$ and denote the class of hyperbolic star bodies with respect to a hyperbolic ball by $\mathcal{S}_B(\mathbb{H}^n)$.

3. Spherical projection body and Spherical projection inequality

3.1. **Spherical Steiner symmetrization.** In this section, we define the spherical Steiner symmetrization for spherical star bodies and study some of its fundamental properties. Without loss of generality, we only consider the Steiner symmetrization along the direction e_n. If $K \in \mathcal{S}_B(\mathbb{S}^n_+)$, then $g(K) \in \mathcal{S}_B(\mathbb{R}^n)$. By Lemma 5, there is a sequence of disjoint open subsets $G_m \subset g(K)'$, and two sequences of graph functions

$$f_{m,j}, g_{m,j} : G_m \to \mathbb{R},\ 1 \le j \le m,$$

satisfying (i), (ii) and (iii) in Lemma 5. In particular, $g(K)$ has the representation (if we neglect an $\mathcal{H}^n$-null set)

$$g(K) = \bigcup_{m=1}^{\infty} \bigcup_{\substack{x' \in G_m \\ 1 \le j \le m}} \left\{(x', x_n) : f_{m,j}(x') \le x_n \le g_{m,j}(x')\right\}. \tag{39}$$

By (17), the Euclidean Steiner symmetrization $S\,g(K)$ of $g(K)$ along the direction e_n (if we neglect an $\mathcal{H}^n$-null set)

$$S\,g(K) = \left\{(x', x_n) \in \mathbb{R}^n : \quad x' \in \bigcup_{m=1}^{\infty} G_m,\ \ \underline{\varrho}(x') \le x_n \le \overline{\varrho}(x')\right\}, \tag{40}$$

where for $x' \in G_m$,

$$\overline{\varrho}(x') = \sum_{j=1}^{m} \frac{g_{m,j}(x') - f_{m,j}(x')}{2} = -\underline{\varrho}(x'). \tag{41}$$

The following lemma shows that the spherical measure of $S\,g(K)$ is not less than the spherical measure of $g(K)$.

Lemma 9. *For convex body $K \in \mathcal{K}_o(\mathbb{S}^n)$, we have*

$$\mu_{s,n}\left(S\,g(K)\right) \ge \mu_{s,n}\left(g(K)\right), \tag{42}$$

with the equality if and only if $S\,g(K) = g(K)$.

Proof. By Fubini's theorem and the definition of spherical measure (30), we only need to prove that for any $x' \in g(K)|_{\mathbb{R}^{n-1}}$,

$$\int_{(S\,g(K))_{x'}} \left(1 + \|x'\|^2 + |x_n|^2\right)^{-\frac{n+1}{2}} dx_n \ge \int_{g(K)_{x'}} \left(1 + \|x'\|^2 + |x_n|^2\right)^{-\frac{n+1}{2}} dx_n. \tag{43}$$

Let

$$f_1(x_n) := \left(1 + \|x'\|^2 + |x_n|^2\right)^{-\frac{n+1}{2}}, \quad f_2(x_n) := \chi_{g(K)_{x'}}(x_n).$$

Then

$$f_1^{\bigstar} = f_1, \quad \text{and} \quad f_2^{\bigstar} = \chi_{S\,g(K)_{x'}},$$

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where $f^\star$ denotes the symmetric decreasing rearrangement of f (see (18) for specific definition).

By Hardy-Littlewood inequality (see Lemma 7), we have

$$\begin{aligned} &\int_{g(K)_{x'}} \left(1+\|x'\|^2+|x_n|^2\right)^{-\frac{n+1}{2}} dx_n \\ =& \int_{\mathbb{R}} f_1(x_n)f_2(x_n)dx_n \\ \le& \int_{\mathbb{R}} f_1^\star(x_n)f_2^\star(x_n)dx_n \\ =& \int_{(S\,g(K))_{x'}} \left(1+\|x'\|^2+|x_n|^2\right)^{-\frac{n+1}{2}} dx_n. \end{aligned} \tag{44}$$

Moreover, since f_1 is an even nonnegative unimodal integrable function, the equality in (44) holds if and only if $g(K)_{x'} = S\,g(K)_{x'}$. Thus, the equality in (42) holds if and only if $S\,g(K) = g(K)$. □

By Lemma 9, $\mu_{s,n}(S\,g(K)) \ge \mu_{s,n}(g(K))$. Thus, there exists some real number $r_K \in (0,1]$ such that

$$\bar{S}\,g(K) := \left\{(x', x_n) \in \mathbb{R}^n : \quad x' \in g(K)|_{\mathbb{R}^{n-1}}, \quad r_K\underline{\varrho}(x') \le x_n \le r_K\overline{\varrho}(x')\right\} \tag{45}$$

satisfies

$$\mu_{s,n}\left(\bar{S}\,g(K)\right) = \mu_{s,n}\left(g(K)\right). \tag{46}$$

Definition 9. *For $K \in \mathcal{K}_o(\mathbb{S}^n)$, its spherical Steiner symmetrization $\hat{S}_{e_n}K$ along the direction $e_n \in \mathbb{S}^{n-1}$ is defined by*

$$\hat{S}_{e_n}(K) := g^{-1}\left(\bar{S}\,g(K)\right). \tag{47}$$

For simplicity of notation, we write $\hat{S}K$ instead of $\hat{S}_{e_n}(K)$. In Definition 9, g and g^{-1} denote the gnomonic projection and the inverse gnomonic projection (see Definition 4). If $K \in \mathcal{K}_o(\mathbb{S}^n)$, then $g(K) \in \mathcal{K}_o(\mathbb{R}^n)$. Thus $S\,g(K) \in \mathcal{K}_o(\mathbb{R}^n)$. By (45), $\bar{S}\,g(K) \in \mathcal{K}_o(\mathbb{R}^n)$. Thus by (47),

$$K \in \mathcal{K}_o(\mathbb{S}^n_+) \Longrightarrow \hat{S}K \in \mathcal{K}_o(\mathbb{S}^n_+). \tag{48}$$

By Lemma 2, (45), (47) and the definition of spherical star bodies with respect to a spherical cap (see Definition 5),

$$K \in \mathcal{S}_B(\mathbb{S}^n_+) \Longrightarrow \hat{S}K \in \mathcal{S}_B(\mathbb{S}^n_+). \tag{49}$$

By (31) and (46), the spherical Steiner symmetrization maintains the invariance of n-Hausdorff measure, i.e.,

$$\mathcal{H}^n(\hat{S}K) = \mathcal{H}^n(K). \tag{50}$$

Similarly, for a compact set $K \subset \mathbb{S}^n$, we define the *spherical symmetric rearrangement* $K^\bigstar$ as follows

$$K^\bigstar := \left\{ v \in \mathbb{S}^n : \ \hat{d}_s(v, e_{n+1}) \le \alpha, \ \ \mathcal{H}^n(K) = \mathcal{H}^n(B_s(\alpha)) \right\}. \tag{51}$$

Lemma 10. *For $K \in \mathcal{S}_B(\mathbb{S}^n_+)$, there exists a sequence of directions $\{u_i\}_{i=1}^\infty \subset \mathbb{S}^{n-1}$ such that the sequence of successive spherical Steiner symmetrizations of K converges to $K^\bigstar$ in spherical Hausdorff distance, i.e.,*

$$\lim_{i\to\infty} d_s(\hat{S}_{u_i} \cdots \hat{S}_{u_1}(K), K^\bigstar) = 0. \tag{52}$$

Proof. By Lemma 4, there exists a sequence of directions $\{u_i\}_{i=1}^\infty \subset \mathbb{S}^{n-1}$ such that

$$\lim_{i\to\infty} d_R(S_{u_i} \cdots S_{u_1}(g(K)), g(K)^\bigstar) = 0. \tag{53}$$

Let $r_1 \in (0, 1]$ satisfy

$$\mu_{s,n}\left(\bar{S}_{u_1,r_1}(g(K))\right) = \mathcal{H}^n(K),$$

where $\bar{S}_{u_1,r_1}(g(K))$ denotes the star body with the overgraph function on the direction u_1

$$\overline{\varrho}_{u_1}\left(\bar{S}_{u_1,r_1}(g(K)), \cdot\right) = r_1 \overline{\varrho}_{u_1}\left(S_{u_1}(g(K)), \cdot\right)$$

and the undergraph function on the direction u_1

$$\underline{\varrho}_{u_1}\left(\bar{S}_{u_1,r_1}(g(K)), \cdot\right) = r_1 \underline{\varrho}_{u_1}\left(S_{u_1}(g(K)), \cdot\right).$$

Let $r' \in (0, 1]$ satisfy

$$\mu_{s,n}\left(\bar{S}_{u_1,r_1}(g(K))\right) = \mu_{s,n}\left(r' g(K)^\bigstar\right). \tag{54}$$

Let $r_2 \in (0, 1]$ satisfy

$$\mu_{s,n}\left(\bar{S}_{u_2,r_2}\left(g(\hat{S}_{u_1}(K))\right)\right) = \mathcal{H}^n(\hat{S}_{u_1}(K)).$$

Repeating the previous process, we can get a sequence of real numbers $\{r_i\}_{i=1}^\infty \subset (0, 1]$ such that

$$\mu_{s,n}\left(\bar{S}_{u_i,r_i}\left(g\left(\hat{S}_{u_{i-1}} \cdots \hat{S}_{u_1}(K)\right)\right)\right) = \mathcal{H}^n\left(\hat{S}_{u_{i-1}} \cdots \hat{S}_{u_1}(K)\right). \tag{55}$$

By (53), (54) and (55), we have

$$\lim_{i\to\infty} d_R\left(\bar{S}_{u_i,r_i} \cdots \bar{S}_{u_1,r_1}(g(K)), r' g(K)^\bigstar\right) = 0. \tag{56}$$

By the definition of spherical Steiner symmetrizations, we have

$$\begin{aligned}
&\lim_{i\to\infty} d_s\left(\hat{S}_{u_i} \cdots \hat{S}_{u_1}(K), g^{-1}\left(r' g(K)^\bigstar\right)\right) \\
= &\lim_{i\to\infty} d_s\left(g^{-1}\left(\bar{S}_{u_i,r_i}\left(g\left(\hat{S}_{u_{i-1}} \cdots \hat{S}_{u_1}(K)\right)\right)\right), g^{-1}\left(r' g(K)^\bigstar\right)\right) \\
&\vdots \\
= &\lim_{i\to\infty} d_s\left(g^{-1}\left(\bar{S}_{u_i,r_i} \cdots \bar{S}_{u_1,r_1}(g(K))\right), g^{-1}\left(r' g(K)^\bigstar\right)\right).
\end{aligned} \tag{57}$$

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By (56), (57) and the continuity of inverse gnomonic projection, we have

$$\lim_{i\to\infty} d_s\left(\hat{S}_{u_i}\cdots\hat{S}_{u_1}(K), g^{-1}\left(r' g(K)^{\star}\right)\right) = 0. \tag{58}$$

Let $K^{\star} = g^{-1}\left(r' g(K)^{\star}\right)$. The desired conclusion now follows from (58). ◻

3.2. **Spherical projection bodies.** In this section, using the Euclidean projection bodies, we introduce the notion of spherical projection bodies and study some elementary properties.

Definition 10. *For $K \in \mathcal{S}_B(\mathbb{S}^n_+)$, its spherical projection body $\Pi_{\mathbb{S}}(K)$ is defined by*

$$\Pi_{\mathbb{S}} K := g^{-1}\left(\Pi g(K)\right). \tag{59}$$

By the definition of spherical projection body (59), (25) and (16), for $u \in \mathbb{S}^{n-1}$,

$$\tan h_s\left(\Pi_{\mathbb{S}} K, u\right) = \frac{1}{2}\int_{\partial g(K)} \left|u \cdot \nu^{g(K)}(y)\right| d\mathcal{H}^{n-1}(y). \tag{60}$$

The following lemma shows that the spherical projection operator $\Pi_{\mathbb{S}} : \mathcal{S}_B(\mathbb{S}^n_+) \to \mathcal{K}_o(\mathbb{S}^n)$ is continuous.

Lemma 11. *For a sequence of spherical star bodies $\{K_i\}_{i=0}^{\infty} \subset \mathcal{S}_B(\mathbb{S}^n_+)$, if*

$$\lim_{i\to\infty} d_s(K_i, K_\infty) = 0, \tag{61}$$

then

$$\lim_{i\to\infty} d_s(\Pi_{\mathbb{S}} K_i, \Pi_{\mathbb{S}} K_\infty) = 0. \tag{62}$$

Proof. By the continuity of gnomonic projection and (61), we have

$$\lim_{i\to\infty} d_H(g(K_i), g(K_\infty)) = 0. \tag{63}$$

Since $\{K_i\}_{i=0}^{\infty} \subset \mathcal{S}_B(\mathbb{S}^n_+)$, $g(K_i) \in \mathcal{S}_B(\mathbb{R}^n)$. By Lemma 3 and (63), $g(K_i)$ converges to $g(K_\infty)$ in radial distance. By (11) and Lemma 1, we have

$$\lim_{i\to\infty} d_H\left(\Pi(g(K_i)), \Pi(g(K_\infty))\right) = 0.$$

Thus by the definition of spherical projection body and the continuity of gnomonic projection, we have

$$\begin{aligned}
\lim_{i\to\infty} d_s\left(\Pi_{\mathbb{S}} K_i, \Pi_{\mathbb{S}} K_\infty\right) &= \lim_{i\to\infty} d_s\left(g^{-1}\left(\Pi(g(K_i))\right), g^{-1}\left(\Pi(g(K_\infty))\right)\right) \\
&= \lim_{i\to\infty} d_H\left(\Pi(g(K_i)), \Pi(g(K_\infty))\right) \\
&= 0.
\end{aligned}$$

This is the desired conclusion. ◻

Let $\overline{\mathrm{O}}(n+1)$ denote the set of rotation transformations around the x_{n+1}-axis in $\mathbb{R}^{n+1}$. The following lemma demonstrates the rotation covariance of the spherical projection operator.

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Lemma 12. *Let $\phi \in \overline{\mathrm{O}}(n+1)$ be a rotation transformation on $\mathbb{R}^{n+1}$ and $K \in \mathcal{S}_B(\mathbb{S}^n)$. Then*

$$\Pi_{\mathbb{S}}(\phi K) = \phi \Pi_{\mathbb{S}} K. \tag{64}$$

Proof. For $\phi \in \overline{\mathrm{O}}(n+1)$, there exists a rotation transformation $\bar{\phi} \in \mathrm{O}(n)$ on $\mathbb{R}^n$ such that

$$g(\phi K) = \bar{\phi}\left(g(K)\right). \tag{65}$$

By [23, Lemma 6.4], we have

$$\Pi\left(\bar{\phi}\left(g(K)\right)\right) = \bar{\phi}\Pi\left(g(K)\right). \tag{66}$$

Therefore,

$$\Pi_{\mathbb{S}}(\phi K) = g^{-1}\left(\Pi(g(\phi K))\right) = g^{-1}\left(\Pi(\bar{\phi} g(K))\right) = g^{-1}\left(\bar{\phi}\Pi(g(K))\right) = \phi g^{-1}\left(\Pi(g(K))\right) = \phi \Pi_{\mathbb{S}} K,$$

where first equality is due to (59), the second is due to (65), the third is due to (66), the fourth is due to (65) and the last equality is due to the definition of spherical projection body (59). □

3.3. **Spherical projection inequality.**

Lemma 13. *Let $K \in \mathcal{S}_B(\mathbb{S}^n_+)$. Then*

$$\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}\left(\hat{S} K\right)\right) \geq \mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}} K\right), \tag{67}$$

with equality if and only if $\hat{S} K = K$.

Proof. If $K \in \mathcal{S}_B(\mathbb{S}^n_+)$, then $g(K) \in \mathcal{S}_B(\mathbb{R}^n)$. For $g(K)$, by Lemma 5, there is a sequence of disjoint open subsets $G_m \subset g(K)'$, and two sequences of graph functions

$$f_{m,j}, g_{m,j} : G_m \to \mathbb{R},\ 1 \leq j \leq m,$$

satisfying (i), (ii) and (iii) in Lemma 5. In particular, $\partial g(K)$ has the representation (if we neglect an $\mathcal{H}^{n-1}$-null set)

$$\partial g(K) = \bigcup_{m=1}^{\infty}\bigcup_{j=1}^{m}\left\{\left(x', f_{m,j}\left(x'\right)\right) : x' \in G_m\right\} \cup \left\{\left(x', g_{m,j}\left(x'\right)\right) : x' \in G_m\right\}.$$

Let $(z', t), (z', -s) \in \partial\Pi^*\left(g(K)\right)$ and $t \neq -s$, s.t.,

$$h\left(\Pi\left(g(K)\right), (z', t)\right) = 1 = h\left(\Pi\left(g(K)\right), (z', -s)\right). \tag{68}$$

By the definition of projection body (68), (16) and (iii) in Lemma 5, we have

$$\begin{aligned} 1 &= h\left(\Pi\left(g(K)\right), (z', t)\right) \\ &= \frac{1}{2}\int_{\partial g(K)} \left|(z', t) \cdot \nu^{g(K)}(x)\right| d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{\{(x', f_{m,j}(x')) :\, x' \in G_m\}} \frac{\left|(z', t) \cdot \left(\nabla f_{m,j}(x'), -1\right)\right|}{\sqrt{1 + \left|\nabla f_{m,j}(x')\right|^2}} d\mathcal{H}^{n-1}(x) \end{aligned} \tag{69}$$

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$$
\begin{aligned}
&+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{\{(x',g_{m,j}(x')):\ x'\in G_m\}}\frac{\left|(z',t)\cdot\left(-\nabla g_{m,j}(x'),1\right)\right|}{\sqrt{1+\left|\nabla g_{m,j}(x')\right|^2}}d\mathcal{H}^{n-1}(x)\\
=\;&\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m}\left|(z',t)\cdot\left(\nabla f_{m,j}(x'),-1\right)\right|dx'+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m}\left|(z',t)\cdot\left(-\nabla g_{m,j}(x'),1\right)\right|dx'.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
1 =\;& h\left(\Pi\left(g(K)\right),(z',-s)\right) \qquad (70)\\
=\;&\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m}\left|(z',-s)\cdot\left(\nabla f_{m,j}(x'),-1\right)\right|dx'+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m}\left|(z',-s)\cdot\left(-\nabla g_{m,j}(x'),1\right)\right|dx'.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& h\left(\Pi\left(\bar{S}g(K)\right),\left(z',\frac{t+s}{2}\right)\right) \qquad (71)\\
=\;&\frac{1}{2}\int_{K'}\left|\left(z',\frac{t+s}{2}\right)\cdot(-r_K\nabla\overline{\varrho}(x'),1)\right|dx'+\frac{1}{2}\int_{K'}\left|\left(z',\frac{t+s}{2}\right)\cdot\left(r_K\nabla\underline{\varrho}(x'),-1\right)\right|dx'\\
\le\;&\frac{1}{2}\int_{K'}\left|\left(z',\frac{t+s}{2}\right)\cdot(-\nabla\overline{\varrho}(x'),1)\right|dx'+\frac{1}{2}\int_{K'}\left|\left(z',\frac{t+s}{2}\right)\cdot\left(\nabla\underline{\varrho}(x'),-1\right)\right|dx'\\
=\;&\frac{1}{2}\int_{K'}\left|\left(z',\frac{t+s}{2}\right)\cdot\left(-\sum_{j=1}^{m}\frac{\nabla g_{m,j}(x')-\nabla f_{m,j}(x')}{2},1\right)\right|dx'\\
&+\frac{1}{2}\int_{K'}\left|\left(z',\frac{t+s}{2}\right)\cdot\left(-\sum_{j=1}^{m}\frac{\nabla g_{m,j}(x')-\nabla f_{m,j}(x')}{2},-1\right)\right|dx'\\
\le\;&\frac{1}{4}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m}\left|(z',t)\cdot\left(-\nabla g_{m,j}(x'),1\right)\right|dx'+\frac{1}{4}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m}\left|(z',-s)\cdot\left(\nabla f_{m,j}(x'),-1\right)\right|dx'\\
&+\frac{1}{4}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m}\left|(z',t)\cdot\left(\nabla f_{m,j}(x'),-1\right)\right|dx'+\frac{1}{4}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\int_{G_m}\left|(z',-s)\cdot\left(-\nabla g_{m,j}(x'),1\right)\right|dx'\\
=\;&\frac{1}{2}h\left(\Pi(g(K)),(z',t)\right)+\frac{1}{2}h\left(\Pi(g(K)),(z',-s)\right)=1,
\end{aligned}
$$

where the first equality is due to (45) and the same reasoning process as (69), the first inequality is due to $r_K \le 1$ and the monotonically increasing property of $|a+bt|+|a-bt|$ on $t>0$ for any $a,b\in\mathbb{R}$, the second equality is due to (41), the second inequality is due to the triangle inequalities for absolute value functions, the last two equalities are due to (69) and (70).

By (71) and Lemma 6,

$$S\,\Pi^*\left(g(K)\right)\subseteq\Pi^*\left(\bar{S}g(K)\right). \qquad (72)$$

By Lemma 9 and the above containment relationship, we have

$$\mu_{s,n}\left(\Pi^*\left(g(K)\right)\right)\le\mu_{s,n}\left(S\,\Pi^*\left(g(K)\right)\right)\le\mu_{s,n}\left(\Pi^*\left(\bar{S}g(K)\right)\right).$$

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By the above inequality, the definition of spherical projection body (59), the definition of spherical Steiner symmetrization (47) and (27), we have

$$\mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(K)\right) \leq \mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(\hat{S}K)\right).$$

If $\mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(K)\right) = \mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(\hat{S}K)\right)$, then the equality in the first inequality of (71) is established. Thus by the arbitrariness of z', $r_K = 1$. By Lemma 9, $Sg(K) = g(K)$. Therefore, $\hat{S}K = K$. □

Proof of Theorem 1. By Lemma 10, there exists a sequence of directions $\{u_i\}_{i=1}^{\infty}$ such that $\hat{S}_{u_i}\cdots\hat{S}_{u_1}K$ converges to $K^{\bigstar}$ in spherical Hausdorff distance. By the continuity of spherical projection operator (see Lemma 11), we have

$$\lim_{i\to\infty} d_s\left(\Pi_{\mathbb{S}}\left(\hat{S}_{u_i}\cdots\hat{S}_{u_1}K\right), \Pi_{\mathbb{S}}(K^{\bigstar})\right) = 0. \tag{73}$$

By Lemma 13, $\mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}\left(\hat{S}_{u_i}\cdots\hat{S}_{u_1}K\right)\right)$ is increasing with respect to i. Thus, by (73), we have

$$\mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(K)\right) \leq \mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(K^{\bigstar})\right). \tag{74}$$

If $K \neq K^{\bigstar}$, then there exists some $u_o \in \mathbb{S}^{n-1}$ such that $\hat{S}_{u_o}(K) \neq K$. By Lemma 13, we have

$$\mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(K)\right) < \mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}\left(\hat{S}_{u_o}(K)\right)\right). \tag{75}$$

By (74) and (75), we have

$$\mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(K)\right) < \mathcal{H}^n\left(\Pi_{\mathbb{S}}^{\circ}(K^{\bigstar})\right).$$

Therefore, the equality in (74) holds if and only if $K = K^{\bigstar}$. □

3.4. **Spherical projection inequality and an inequality on surface areas.** In this section, we shall prove that spherical projection inequality is stronger than an inequality on surface areas.

Let

$$F_1(t) := \frac{\pi}{2} - \arctan t, \quad t \in (0, \infty)$$

and

$$F_2(s) := \int_0^s (\sin r)^{n-1} dr, \quad s \in (0, \frac{\pi}{2}).$$

Let $F := F_2 \circ F_1$ be the composition function of F_1 and F_2. It is easily to check that F_1 and F_2 are strictly convex functions, F_2 is strictly increasing and F_1 is strictly decreasing. Thus F is strictly convex and strictly decreasing. Let F^{-1} be the inverse function of F, then F^{-1} is also strictly convex and strictly decreasing.

Theorem 3. *Let $K \in \mathcal{S}_B(\mathbb{S}^n_+)$ and $c_o = \omega_{n-1}/(n\omega_n)$. Then*

$$c_o\mathcal{H}^{n-1}\left(\partial g(K^\bigstar)\right) = F^{-1}\left(\frac{1}{n\omega_n}\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K^\bigstar)\right)\right) \tag{76}$$
$$\leq F^{-1}\left(\frac{1}{n\omega_n}\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K)\right)\right) \leq c_o\mathcal{H}^{n-1}\left(\partial g(K)\right).$$

Moreover, $\mathcal{H}^{n-1}\left(\partial g(K^\bigstar)\right) = \mathcal{H}^{n-1}\left(\partial g(K)\right)$ if and only if $K^\bigstar = K$.

Proof. For $K \in \mathcal{S}_B(\mathbb{S}^n_+)$, by Lemma 8, (29) and (60),

$$\begin{aligned}
\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K)\right) &= \mathcal{H}^n\left(-\Pi^\circ_{\mathbb{S}}(K)\right) \\
&= \int_{\mathbb{S}^{n-1}}\int_{\mathrm{conv}(e_{n+1},v)} \sin\left(\hat{d}_s\left(e_{n+1},u\right)\right)^{n-1}\chi_{-\Pi^\circ_{\mathbb{S}}(K)}(u)\,du\,dv \qquad (77)\\
&= \int_{\mathbb{S}^{n-1}}\int_0^{\rho_s(-\Pi^\circ_{\mathbb{S}}(K),v)} (\sin t)^{n-1}\,dt\,dv \\
&= \int_{\mathbb{S}^{n-1}}\int_0^{\frac{\pi}{2}-h_s(\Pi_{\mathbb{S}}(K),v)} (\sin t)^{n-1}\,dt\,dv \\
&= \int_{\mathbb{S}^{n-1}}\int_0^{\frac{\pi}{2}-\arctan\left(\frac{1}{2}\int_{\partial g(K)}\left|v\cdot\nu^{g(K)}(y)\right|d\mathcal{H}^{n-1}(y)\right)} (\sin t)^{n-1}\,dt\,dv.
\end{aligned}$$

By (77), we have

$$\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K)\right) = \int_{\mathbb{S}^{n-1}} F\left(\frac{1}{2}\int_{\partial g(K)}\left|v\cdot\nu^{g(K)}(y)\right|d\mathcal{H}^{n-1}(y)\right)dv. \tag{78}$$

By Theorem 1 and the strict decreasing and strictly convex properties of F^{-1},

$$F^{-1}\left(\frac{1}{n\omega_n}\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K^\bigstar)\right)\right) \leq F^{-1}\left(\frac{1}{n\omega_n}\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K)\right)\right). \tag{79}$$

By Jensen's inequality, (78) and Fubini's theorem, we have

$$\begin{aligned}
F^{-1}\left(\frac{1}{n\omega_n}\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K)\right)\right) &\leq \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}\left(\frac{1}{2}\int_{\partial g(K)}\left|v\cdot\nu^{g(K)}(y)\right|d\mathcal{H}^{n-1}(y)\right)dv \\
&= \frac{1}{2n\omega_n}\int_{\partial g(K)}\left(\int_{\mathbb{S}^{n-1}}\left|v\cdot\nu^{g(K)}(y)\right|dv\right)d\mathcal{H}^{n-1}(y) \\
&= \frac{\omega_{n-1}}{n\omega_n}\mathcal{H}^{n-1}\left(\partial g(K)\right). \qquad (80)
\end{aligned}$$

Similarly, by the equality case of Jensen's inequality, (78) and Fubini's theorem, we have

$$\begin{aligned}
F^{-1}\left(\frac{1}{n\omega_n}\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K^\bigstar)\right)\right) &= \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}\left(\frac{1}{2}\int_{\partial g(K^\bigstar)}\left|v\cdot\nu^{g(K^\bigstar)}(y)\right|d\mathcal{H}^{n-1}(y)\right)dv \\
&= \frac{1}{2n\omega_n}\int_{\partial g(K^\bigstar)}\left(\int_{\mathbb{S}^{n-1}}\left|v\cdot\nu^{g(K^\bigstar)}(y)\right|dv\right)d\mathcal{H}^{n-1}(y) \\
&= \frac{\omega_{n-1}}{n\omega_n}\mathcal{H}^{n-1}\left(\partial g(K^\bigstar)\right). \qquad (81)
\end{aligned}$$

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Note that the first equality of (81) is due to the fact that the following integral is a constant for any $v \in \mathbb{S}^{n-1}$,

$$\int_{\partial g(K^\star)} \left| v \cdot v^{g(K^\star)}(y) \right| d\mathcal{H}^{n-1}(y).$$

Then, the desired inequality follows from (79), (80) and (81).

If $\mathcal{H}^{n-1}\left(\partial g(K^\star)\right) = \mathcal{H}^{n-1}\left(\partial g(K)\right)$, then by strict monotonicity of F^{-1} and (76), we have $\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K^\star)\right) = \mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K)\right)$. Thus, the equality case of Theorem 1 gives $K^\star = K$. □

4. Hyperbolic projection body and Hyperbolic projection inequality

4.1. **Transformation Φ_p of Hyperbolic space.** First, we introduce a transformation from the Poincaré ball model $\mathbb{B}^n$ onto $\mathbb{R}^n$. Let $\Phi : \mathbb{B}^n \to \mathbb{R}^n$ be a transformation given by

$$y = \Phi(x) := \tan\left(2 \arctan \|x\|\right) \frac{x}{\|x\|} = \frac{2x}{1 - \|x\|^2}. \tag{82}$$

Then its inverse transformation $\Phi^{-1} : \mathbb{R}^n \to \mathbb{B}^n$ is

$$x = \Phi^{-1}(y) = \frac{y}{1 + \sqrt{1 + \|y\|^2}}. \tag{83}$$

Let $\Phi_p := \Phi \circ P$ denote the composite of transformation Φ defined in (82) and the Poincaré ball model projection defined in Definition 7.

By (83), we have

$$dx_i = \frac{dy_i}{1 + \sqrt{1 + \|y\|^2}} - \frac{y_i (y \cdot dy)}{\left(1 + \sqrt{1 + \|y\|^2}\right)^2 \sqrt{1 + \|y\|^2}}. \tag{84}$$

Combining (34), (83) and (84), let $|dx|^2 := dx_1^2 + \cdots + dx_n^2$, the metric in the Poincaré ball model

$$\begin{aligned} ds^2 = 4\frac{|dx|^2}{(1 - \|x\|^2)^2} &= \sum_{i=1}^n \left(dy_i - \frac{y_i\,(y \cdot dy)}{\left(1 + \sqrt{1 + \|y\|^2}\right)\sqrt{1 + \|y\|^2}} \right)^2 \\ &= |dy|^2 - \frac{2\sqrt{1 + \|y\|^2} + 2 + \|y\|^2}{\left(1 + \sqrt{1 + \|y\|^2}\right)^2 (1 + \|y\|^2)} (y \cdot dy)^2 \\ &= |dy|^2 - \frac{(y \cdot dy)^2}{1 + \|y\|^2}. \end{aligned} \tag{85}$$

By the above equality and the Cauchy-Schwarz inequality, we have

$$|dy|^2 \geq ds^2 \geq |dy|^2 - \frac{\|y\|^2 |dy|^2}{1 + \|y\|^2} = \frac{|dy|^2}{1 + \|y\|^2}. \tag{86}$$

By (83) and the area formula (see [13, Theorem 3.8]), we have the volume element

$$dx := \frac{1}{\sqrt{1 + \|y\|^2}\left(1 + \sqrt{1 + \|y\|^2}\right)^n} dy. \tag{87}$$

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By (82) and (83), we have

$$\frac{1-\|x\|^2}{2} = \frac{1}{\sqrt{1+\|y\|^2}+1}. \tag{88}$$

Combining (36), (88) and (87) we obtain

$$d\mu_{h,n} = \frac{1}{\sqrt{1+\|y\|^2}}dy. \tag{89}$$

Therefore, for $K \in \mathcal{S}_o(\mathbb{H}^n)$,

$$\mu_{h,n}(K) = \int_{\Phi_p(K)} \frac{1}{\sqrt{1+\|y\|^2}}dy. \tag{90}$$

Lemma 14. *Let $K \in \mathcal{S}_o(\mathbb{H}^n)$. Then*

$$\int_{S\Phi_p(K)} \frac{1}{\sqrt{1+\|y\|^2}}dy \geq \int_{\Phi_p(K)} \frac{1}{\sqrt{1+\|y\|^2}}dy, \tag{91}$$

with equality if and only if $S\Phi_p(K) = \Phi_p(K)$.

Proof. By Fubini's theorem, we only need to prove that for any $y' \in \Phi_p(K)|_{\mathbb{R}^{n-1}}$,

$$\int_{(S\Phi_p(K))_{y'}} \left(1+\|y'\|^2+|y_n|^2\right)^{-\frac{1}{2}} dy_n \geq \int_{(\Phi_p(K))_{y'}} \left(1+\|y'\|^2+|y_n|^2\right)^{-\frac{1}{2}} dy_n. \tag{92}$$

Let

$$f_1(y_n) := \left(1+\|y'\|^2+|y_n|^2\right)^{-\frac{1}{2}}, \quad f_2(y_n) := \chi_{(\Phi_p(K))_{y'}}(y_n).$$

Then

$$f_1^\star = f_1, \quad \text{and} \quad f_2^\star = \chi_{(S\Phi_p(K))_{y'}},$$

where $f^\star$ denotes the symmetric decreasing rearrangement of f (see (18) for specific definition).

By Hardy-Littlewood inequality (see Lemma 7), we have

$$\begin{aligned} & \int_{(\Phi_p(K))_{y'}} \left(1+\|y'\|^2+|y_n|^2\right)^{-\frac{1}{2}} dy_n \\ = & \int_{\mathbb{R}} f_1(y_n) f_2(y_n) dy_n \\ \leq & \int_{\mathbb{R}} f_1^\star(y_n) f_2^\star(y_n) dy_n \\ = & \int_{(S\Phi_p(K))_{y'}} \left(1+\|y'\|^2+|y_n|^2\right)^{-\frac{1}{2}} dy_n. \end{aligned} \tag{93}$$

Moreover, since f_1 is an even nonnegative unimodal integrable function, the equality in (93) holds if and only if $(\Phi_p(K))_{y'} = (S\Phi_p(K))_{y'}$. Thus, the equality in (91) holds if and only if $S\Phi_p(K) = \Phi_p(K)$. □

Lemma 15. *For $K \subset \mathbb{H}^n$, $K \in \mathcal{S}_o(\mathbb{H}^n)$ is equivalent to $\Phi_p(K) \in \mathcal{S}_o(\mathbb{R}^n)$. Moreover, $K \in \mathcal{S}_B(\mathbb{H}^n)$ is equivalent to $\Phi_p(K) \in \mathcal{S}_B(\mathbb{R}^n)$.*

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Proof. By (82) and (83), $P(K)$ is a star-shaped set with respect to origin if and only if $\Phi_p(K)$ is a star-shaped set with respect to origin in $\mathbb{R}^n$. Moreover, by (82) and (83), we have

$$\rho(\Phi_p(K), x) = \frac{2\rho(P(K), x)}{1 - \|x\|^2\rho^2(P(K), x)}, \quad x \in \mathbb{R}^n\backslash\{o\} \tag{94}$$

and

$$\rho(P(K), x) = \frac{\rho(\Phi_p(K), x)}{1 + \sqrt{1 + \|x\|^2\rho^2(\Phi_p(K), x)}}, \quad x \in \mathbb{R}^n\backslash\{o\}. \tag{95}$$

Therefore, $\rho(P(K), \cdot)$ is strictly positive and continuous in $\mathbb{R}^n\backslash\{o\}$ if and only if $\rho(\Phi_p(K), \cdot)$ is strictly positive and continuous in $\mathbb{R}^n\backslash\{o\}$. Moreover, by (94) and (95), $\rho(P(K), \cdot)$ is locally Lipschitz continuous in $\mathbb{R}^n\backslash\{o\}$ if and only if $\rho(\Phi_p(K), \cdot)$ is locally Lipschitz continuous in $\mathbb{R}^n\backslash\{o\}$. By [22, Theorem 2.1], a Lipschitz star body (its radial function is locally Lipschitz continuous in $\mathbb{R}^n\backslash\{o\}$) is a star body with respect to a ball and vice versa. This shows the desired conclusion. □

Definition 11. *For $K \in \mathcal{S}_o(\mathbb{H}^n)$, its hyperbolic polar body K° is defined by*

$$K^\circ := \Phi_p^{-1}\left(\left(\Phi_p(K)\right)^*\right). \tag{96}$$

By the above definition and Lemma 15, if $K \in \mathcal{S}_o(\mathbb{H}^n)$, then $K^\circ \in \mathcal{S}_o(\mathbb{H}^n)$.

4.2. **Hyperbolic Steiner symmetrization.**

Definition 12. *For $K \in \mathcal{S}_B(\mathbb{H}^n)$, its hyperbolic Steiner symmetrization $\check{S}(K)$ is defined by*

$$\check{S}(K) := \Phi_p^{-1}\left(r_K S\,\Phi_p(K)\right), \tag{97}$$

where $r_K \in (0, 1]$ satisfies $\mu_{h,n}\left(\check{S}(K)\right) = \mu_{h,n}(K)$.

By [22, Lemma 5.1], if $\tilde{K} \in \mathcal{S}_B(\mathbb{R}^n)$, then $S\tilde{K} \in \mathcal{S}_B(\mathbb{R}^n)$. Moreover, it is clear that $rS\tilde{K} \in \mathcal{S}_B(\mathbb{R}^n)$ for $r \in (0, 1]$. Thus, the hyperbolic Steiner symmetrization maintains the property of star bodies, i.e., if $K \in \mathcal{S}_B(\mathbb{H}^n)$, then $\check{S}(K) \in \mathcal{S}_B(\mathbb{H}^n)$. Moreover, by Definition 12, the hyperbolic Steiner symmetrization maintains the invariance of $\mu_{h,n}$ measure.

Similarly, for compact set $K \in \mathbb{H}^n$, we define the *hyperbolic symmetric rearrangement* $K^\star$ as following

$$K^\star := \left\{v \in \mathbb{H}^n : \; ds^2(P(v), o) \le \alpha^2, \;\; \mu_{h,n}(K) = \mu_{h,n}(B_h(\alpha))\right\}. \tag{98}$$

Next, we prove the convergence of hyperbolic Steiner symmetrizations.

Lemma 16. *For any $K \in \mathcal{S}_B(\mathbb{H}^n)$, there exists a sequence of directions $\{u_i\}_{i=1}^{\infty} \subset \mathbb{S}^{n-1}$ such that*

$$\lim_{i\to\infty} d_h\left(\check{S}_{u_i}\cdots\check{S}_{u_1}(K), K^\star\right) = 0. \tag{99}$$

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Proof. Since $K \in \mathcal{S}_B(\mathbb{H}^n)$, $\Phi_p(K) \in \mathcal{S}_B(\mathbb{R}^n)$. By [21, Theorem 3.1], for the compact set $\Phi_p(K)$, there exists a sequence of directions $\{u_i\}_{i=1}^{\infty} \subset \mathbb{S}^{n-1}$ and a ball $B(r_o) \subset \mathbb{R}^n$ with the same volume as $\Phi_p(K)$ such that

$$\lim_{i\to\infty} d_H\left(S_{u_i}\cdots S_{u_1}\Phi_p(K), B(r_o)\right) = 0. \tag{100}$$

Let $r_1 \in (0,1]$ satisfy

$$\Phi_p^{-1}\left(r_1 S_{u_1}\Phi_p(K)\right) = \check{S}_{u_1}(K).$$

Let $r_2 \in (0,1]$ satisfy

$$\Phi_p^{-1}\left(r_2 S_{u_2}\Phi_p(\check{S}_{u_1}(K))\right) = \check{S}_{u_2}\check{S}_{u_1}(K).$$

Repeating the previous argument, we get a sequence of positive real numbers $\{r_i\}_{i=1}^{\infty}$ satisfying $r_i \in (0,1]$ and

$$\Phi_p^{-1}\left(r_i S_{u_i}\Phi_p(\check{S}_{u_{i-1}}\cdots\check{S}_{u_1}(K))\right) = \check{S}_{u_i}\check{S}_{u_{i-1}}\cdots\check{S}_{u_1}(K). \tag{101}$$

Let $\bar{r} := \lim_{i\to\infty}(r_i r_{i-1}\cdots r_1)$, by (101), (86) and (100),

$$\begin{aligned}
&\lim_{i\to\infty} d_h\left(\check{S}_{u_i}\cdots\check{S}_{u_1}(K), \Phi_p^{-1}(r_i\cdots r_1 B(r_o))\right)\\
=\;&\lim_{i\to\infty} d_h\left(\Phi_p^{-1}\left(r_i\cdots r_1 S_{u_i}\cdots S_{u_1}\Phi_p(K)\right), \Phi_p^{-1}(r_i\cdots r_1 B(r_o))\right)\\
\le\;&\lim_{i\to\infty} d_H\left(r_i\cdots r_1 S_{u_i}\cdots S_{u_1}\Phi_p(K), r_i\cdots r_1 B(r_o)\right)\\
=\;&\bar{r}\lim_{i\to\infty} d_H\left(S_{u_i}\cdots S_{u_1}\Phi_p(K), B(r_o)\right)\\
=\;&0.
\end{aligned}$$

Since $\mu_{h,n}\left(\check{S}_{u_i}\cdots\check{S}_{u_1}(K)\right) = \mu_{h,n}(K)$ for any $i \in \mathbb{N}$, $K^{\star} = \Phi_p^{-1}(\bar{r}B(r_o))$. This completes the proof. □

4.3. **Hyperbolic projection body.**

Definition 13. *For $K \in \mathcal{S}_B(\mathbb{H}^n)$, its hyperbolic projection body $\Pi_{\mathbb{H}}(K)$ is defined by*

$$\Pi_{\mathbb{H}}(K) := \Phi_p^{-1}\left(\Pi\left(\Phi_p(K)\right)\right). \tag{102}$$

By (90),

$$\mu_{h,n}(\Pi_{\mathbb{H}}(K)) = \int_{\Pi(\Phi_p(K))}\frac{1}{\sqrt{1+\|y\|^2}}dy = \int_{\mathbb{S}^{n-1}}\int_o^{\rho_{\Pi(\Phi_p(K))}(u)}\frac{r^{n-1}}{\sqrt{1+r^2}}drdu. \tag{103}$$

Next, we show that the hyperbolic projection operator $\mathcal{S}_B(\mathbb{H}^n) \to \mathcal{S}_B(\mathbb{H}^n)$ is continuous.

Lemma 17. *For $K_\infty, K_i \in \mathcal{S}_B(\mathbb{H}^n)$, $i = 0,1,2,\ldots$, if*

$$\lim_{i\to\infty} d_h(K_i, K_\infty) = 0, \tag{104}$$

then

$$\lim_{i\to\infty} d_h(\Pi_{\mathbb{H}}(K_i), \Pi_{\mathbb{H}}(K_\infty)) = 0. \tag{105}$$

Proof. By the relation between ds^2 and $|dy|^2$ (see (86)), if (104) holds, then

$$\lim_{i\to\infty} d_H\left(\Phi_p(K_i), \Phi_p(K_\infty)\right) = 0.$$

By [22, Theorem 2.2] and the above equality, $\Phi_p(K_i)$ converges to $\Phi_p(K_\infty)$ in radial distance. Thus by (11), their surface areas satisfy

$$\lim_{i\to\infty} \mathcal{H}^n\left(\partial\Phi_p(K_i)\right) = \mathcal{H}^n\left(\partial\Phi_p(K_\infty)\right).$$

By the above equality and the continuity of projection operator (see [21, Proposition 4.1]), we have

$$\lim_{i\to\infty} d_H\left(\Pi\left(\Phi_p(K_i)\right), \Pi\left(\Phi_p(K_\infty)\right)\right) = 0. \tag{106}$$

By (102), (86) and (106),

$$\begin{aligned}\lim_{i\to\infty} d_h\left(\Pi_{\mathbb{H}}(K_i), \Pi_{\mathbb{H}}(K_\infty)\right) &= \lim_{i\to\infty} d_h\left(\Phi_p^{-1}\left(\Pi\left(\Phi_p(K_i)\right)\right), \Phi_p^{-1}\left(\Pi\left(\Phi_p(K_\infty)\right)\right)\right) \\ &\le \lim_{i\to\infty} d_H\left(\Pi\left(\Phi_p(K_i)\right), \Pi\left(\Phi_p(K_\infty)\right)\right) = 0.\end{aligned}$$

This completes the proof. □

The following lemma shows that the rotation invariance of the hyperbolic projection operator.

Lemma 18. *If $K \in \mathcal{S}_B(\mathbb{H}^n)$ and $\phi \in \overline{\mathrm{O}}(n+1)$, then*

$$\Pi_{\mathbb{H}}(\phi K) = \phi\Pi_{\mathbb{H}}(K). \tag{107}$$

Proof. For $\phi \in \overline{\mathrm{O}}(n+1)$, there exists a rotation transformation $\bar{\phi} \in \mathrm{O}(n)$ in $\mathbb{R}^n$ such that

$$\Phi_p(\phi(K)) = \bar{\phi}\left(\Phi_p(K)\right).$$

By the definition of hyperbolic projection body (102) and the affine invariance of Euclidean projection body on Lipschitz star bodies (see [23, Lemma 6.4]), we have

$$\begin{aligned}\Pi_{\mathbb{H}}(\phi K) &= \Phi_p^{-1}\left(\Pi\left(\Phi_p(\phi K)\right)\right) = \Phi_p^{-1}\left(\Pi\left(\bar{\phi}\Phi_p(K)\right)\right) \\ &= \Phi_p^{-1}\left(\bar{\phi}\Pi\left(\Phi_p(K)\right)\right) = \phi\Phi_p^{-1}\left(\Pi\left(\Phi_p(K)\right)\right) = \phi\Pi_{\mathbb{H}}(K).\end{aligned}$$

This completes the proof. □

4.4. **Hyperbolic projection inequality.**

Lemma 19. *For $K \in \mathcal{S}_B(\mathbb{H}^n)$, we have*

$$\mu_{h,n}\left(\Pi_{\mathbb{H}}^\circ(K)\right) \le \mu_{h,n}\left(\Pi_{\mathbb{H}}^\circ(\check{S}K)\right), \tag{108}$$

with equality if and only if $K = \check{S}K$.

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Proof. By Lemma 15, if $K \in \mathcal{S}_B(\mathbb{H}^n)$, then $\Phi_p(K) \in \mathcal{S}_B(\mathbb{R}^n)$. By [23, Theorem 7.1],

$$S\Pi^*\left(\Phi_p(K)\right) \subset \Pi^*\left(S\Phi_p(K)\right). \tag{109}$$

Therefore,

$$\begin{aligned}\mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(\check{S}K)\right) &= \int_{\Phi_p(\Pi^\circ_{\mathbb{H}}(\check{S}K))}\frac{1}{\sqrt{1+\|y\|^2}}dy = \int_{\Pi^*(r_K S\Phi_p(K))}\frac{1}{\sqrt{1+\|y\|^2}}dy\\ &\geq \int_{\Pi^*(S\Phi_p(K))}\frac{1}{\sqrt{1+\|y\|^2}}dy \geq \int_{S\Pi^*(\Phi_p(K))}\frac{1}{\sqrt{1+\|y\|^2}}dy\\ &\geq \int_{\Pi^*(\Phi_p(K))}\frac{1}{\sqrt{1+\|y\|^2}}dy = \mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(K)\right),\end{aligned} \tag{110}$$

where the first equality is from (89), the second equality from (96), (102) and (97), the first inequality from $r_K \in (0,1]$, the second inequality from (109), the third inequality from (91) and the last equality is from (89) and (102).

If the equality in (108) holds, then the equality in the first inequality of (110) holds. This implies $r_K = 1$. By the sufficient and necessary conditions of the equality holds in (91), $S\Phi_p(K) = \Phi_p(K)$. Thus, $K = \check{S}K$. □

Proof of Theorem 2. By the convergence of hyperbolic Steiner symmetrizations (see Lemma 16), for $K \in \mathcal{S}_B(\mathbb{H}^n)$, there exists a sequence of directions $\{u_i\}_{i=1}^\infty \subset \mathbb{S}^{n-1}$ such that the iterative hyperbolic Steiner symmetriztions $\check{S}_{u_i}\cdots\check{S}_{u_1}K$ converge to $K^\star$ in hyperbolic Hausdorff distance. Then by the continuity of hyperbolic projection operator (see Lemma 17), the sequence of the hyperbolic projection bodies $\Pi_{\mathbb{H}}\left(\check{S}_{u_i}\cdots\check{S}_{u_1}K\right)$ converge to $\Pi_{\mathbb{H}}\left(K^\star\right)$ in hyperbolic Hausdorff distance. By the monotonically increasing property of $\mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(K)\right)$ with respect to hyperbolic Steiner symmetrization (see Lemma 19), we have

$$\mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(K)\right) \leq \mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(K^\star)\right). \tag{111}$$

If there exists $u \in \mathbb{S}^{n-1}$ such that $K \neq \check{S}_uK$, then by Lemma 19, $\mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(K)\right) < \mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(\check{S}_uK)\right)$. By (111) and the above inequality, $\mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(K)\right) < \mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(K^\star)\right)$. Therefore, if the equality in (111) holds, then for any direction $u \in \mathbb{S}^{n-1}$, $K = \check{S}_uK$. By the arbitrariness of $u \in \mathbb{S}^{n-1}$, $K = K^\star$. □

4.5. **Hyperbolic projection inequality and isoperimetric inequality on $\mathbb{H}^n$.** In this section, we show that the hyperbolic projection inequality is stronger than the hyperbolic isoperimetric inequality with respect to the transformation Φ_p.

On the one hand,

$$\begin{aligned}\mu_{h,n}\left(\Pi^\circ_{\mathbb{H}}(K)\right) &= \mu_{h,n}\left(\Phi_p^{-1}\left(\left(\Phi_p\left(\Pi_{\mathbb{H}}(K)\right)\right)^*\right)\right) = \mu_{h,n}\left(\Phi_p^{-1}\left(\left(\Pi\Phi_p(K)\right)^*\right)\right)\\ &= \int_{\Pi^*(\Phi_p(K))}\frac{1}{\sqrt{1+\|y\|^2}}dy = \int_{\mathbb{S}^{n-1}}\int_0^{1/h(\Pi(\Phi_p(K)),u)}\frac{r^{n-1}}{\sqrt{1+r^2}}drdu,\end{aligned} \tag{112}$$

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where the first equality is from the definition of hyperbolic polar body (96), the second equality from the definition of hyperbolic projection body (102), the third from (89) and the last equality from (103) and (13). Let $G_1(t) := \frac{1}{t}, \quad t > 0$, then G_1 is a strictly decreasing convex function. Let

$$G_2(s) := \int_0^s \frac{r^{n-1}}{\sqrt{1+r^2}} dr, \quad s > 0,$$

then G_2 is a strictly increasing convex function. Thus, the composite function $G := G_2 \circ G_1$ is a strictly decreasing convex function. Then its inverse function G^{-1} is also strictly decreasing convex function. Therefore, by (112), Jensen's inequality, (16) and Fubini's theorem,

$$\begin{aligned} G^{-1}\left(\frac{1}{n\omega_n}\mu_{h,n}\left(\Pi_{\mathbb{H}}^{\circ}(K)\right)\right) &= G^{-1}\left(\frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}\int_0^{h^{-1}(\Pi(\Phi_p(K)),u)} \frac{r^{n-1}}{\sqrt{1+r^2}} dr du\right) \qquad (113) \\ &= G^{-1}\left(\frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} G\left(h\left(\Pi\left(\Phi_p(K)\right),u\right)\right)du\right) \\ &\leq \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} h\left(\Pi\left(\Phi_p(K)\right),u\right)du \\ &= \frac{1}{2n\omega_n}\int_{\mathbb{S}^{n-1}}\left(\int_{\partial\Phi_p(K)} \left|u\cdot \nu^{\Phi_p(K)}(x)\right| d\mathcal{H}^{n-1}(x)\right)du \\ &= \frac{1}{2n\omega_n}\int_{\partial\Phi_p(K)}\left(\int_{\mathbb{S}^{n-1}} \left|u\cdot \nu^{\Phi_p(K)}(x)\right| du\right)d\mathcal{H}^{n-1}(x) \\ &= \frac{\omega_{n-1}}{n\omega_n}\mathcal{H}^{n-1}\left(\partial\Phi_p(K)\right). \end{aligned}$$

On the other hand, the same reasoning applies to $K^\bigstar$, the only difference being the equality in Jensen's inequality, we have

$$G^{-1}\left(\frac{1}{n\omega_n}\mu_{h,n}\left(\Pi_{\mathbb{H}}^{\circ}(K^\bigstar)\right)\right) = \frac{\omega_{n-1}}{n\omega_n}\mathcal{H}^{n-1}\left(\partial\Phi_p(K^\bigstar)\right). \qquad (114)$$

Let $c_o := \omega_{n-1}/(n\omega_n)$. By (2), (113), (114) and the monotonicity of G^{-1},

$$c_o\mathcal{H}^{n-1}\left(\partial\Phi_p(K^\bigstar)\right) = G^{-1}\left(\frac{1}{n\omega_n}\mu_{h,n}\left(\Pi_{\mathbb{H}}^{\circ}(K^\bigstar)\right)\right) \leq G^{-1}\left(\frac{1}{n\omega_n}\mu_{h,n}\left(\Pi_{\mathbb{H}}^{\circ}(K)\right)\right) \leq c_o\mathcal{H}^{n-1}\left(\partial\Phi_p(K)\right).$$

Therefore, hyperbolic projection inequality is stronger than the hyperbolic isoperimetric inequality on transformation Φ_h.

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