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RESEARCH ARTICLE

Singular Kähler-Einstein metrics and RCD spaces

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Abstract
We study Kähler-Einstein metrics on singular projective varieties. We show that under an approximation property with constant scalar curvature metrics, the metric completion of the smooth part is a noncollapsed RCD space, and is homeomorphic to the original variety.

1. Introduction

A basic idea in complex geometry is to study complex manifolds using canonical Kähler metrics, of which perhaps the most important examples are Kähler-Einstein metrics. Yau's solution of the Calabi conjecture [61] provides Kähler-Einstein metrics on compact Kähler manifolds with negative or zero first Chern class, while Chen-Donaldson-Sun's solution of the Yau-Tian-Donaldson conjecture [17] shows that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-stable. An example of a geometric application of such metrics is Yau's proof [60] of the Miyaoka-Yau inequality.

Recently there has been increasing interest in Kähler-Einstein metrics on singular varieties. In particular Yau's theorem was extended to the singular case by Eyssidieux-Guedj-Zeriahi [29], while the singular case of the Yau-Tian-Donaldson conjecture was finally resolved by Liu-Xu-Zhuang [44] after many partial results (see, for instance, [40]). There is now a substantial literature on singular Kähler-Einstein metrics, see, for example, [4, 3, 30, 40, 33].

In order to state our main results, suppose that X is an n-dimensional normal compact Kähler space. Let us recall that a singular Kähler-Einstein metric on X can be defined to be a positive current ω_{KE} that is a smooth Kähler metric on the regular set X^{reg}, has locally bounded potentials, and satisfies the equation $\mathrm{Ric}(\omega_{KE}) = \lambda\omega_{KE}$ on X^{reg} for a constant $\lambda \in \mathbb{R}$. The metric ω_{KE} defines a length metric d_{KE} on X^{reg}, and an important problem is to understand the geometry of the metric completion $\overline{(X^{reg}, d_{KE})}$.

In recent remarkable works, Guo-Phong-Song-Sturm [32, 33] showed that this metric completion satisfies many important geometric estimates, such as bounds for their diameters, their heat kernels, as well as Sobolev inequalities, even under far more general assumptions than the Einstein condition. In particular, their results do not assume Ricci curvature bounds. It is natural to expect, however, that singular Kähler-Einstein metrics satisfy sharper results, similar to Riemannian manifolds with Ricci lower bounds. We formulate the following conjecture, which is likely folklore among experts, although we did not find it stated in the literature in this generality.

Conjecture 1. *The metric completion $\overline{(X^{reg}, d_{KE})}$, equipped with the measure ω^n_{KE}, extended trivially from X^{reg}, is a noncollapsed $RCD(\lambda, 2n)$-space, homeomorphic to X.*

The notion of noncollapsed RCD-space is due to De Philippis-Gigli [23], building on many previous works on synthetic notions of Ricci curvature lower bounds (see [53, 45, 1]). The conjecture is already known in several special cases, where in fact $\overline{(X^{reg}, d_{KE})}$ is shown to be a noncollapsed Ricci limit space – these are noncollapsed Gromov-Hausdorff limits of Riemannian manifolds with Ricci lower bounds, studied by Cheeger-Colding [12]. Settings where $\overline{(X^{reg}, d_{KE})}$ is a Ricci limit space are given, for example, by K-stable Fano manifolds with admissible singularities (see Li-Tian-Wang [40], or Song [49] for the case of crepant singularities), or smoothable K-stable Fano varieties, see Donaldson-Sun [27], Spotti [51].

Our goal in this paper is to move beyond the setting of Ricci limit spaces, and to prove the conjecture in situations where it is not clear whether the singular Kähler-Einstein space (X, ω_{KE}) can be approximated by smooth, or mildly singular, spaces with Ricci curvature bounded below. Instead, our approach is to use an approximation with constant scalar curvature Kähler metrics. The main approximation property that we require is the following.

Definition 2. We say that the singular Kähler-Einstein space (X, ω_{KE}) can be *approximated by cscK metrics*, if there is a resolution $\pi : Y \to X$, and a family of constant scalar curvature Kähler metrics ω_ϵ on Y satisfying the following:

(a) We have $\omega_\epsilon = \eta_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon$, where η_ϵ converge smoothly to $\pi^*\eta_X$ and $\eta_\epsilon \geq \pi^*\eta_X$. Here $\eta_X \in [\omega_{KE}]$ is a smooth metric on X in the sense that it is locally the restriction of smooth metrics under local embeddings into Euclidean space.
(b) We have the estimates

$$\sup_Y |u_\epsilon| < C, \quad \frac{\omega_\epsilon^n}{\eta_Y^n} > \gamma, \quad \int_Y \left(\frac{\omega_\epsilon^n}{\eta_Y^n}\right)^p \eta_Y^n < C, \tag{1}$$

for constants $C > 0, p > 1$ independent of ϵ, where η_Y is a fixed Kähler metric on Y, and γ is a non-negative continuous function on Y vanishing only along the exceptional divisor, also independent of ϵ.
(c) The metrics ω_ϵ converge locally smoothly on $\pi^{-1}(X^{reg})$ to $\pi^*\omega_{KE}$.

The cscK property of the approximating metrics ω_ϵ is used to obtain integral bounds for the Ricci and Riemannian curvatures as in Proposition 14. We expect that such an approximation is possible in all cases of interest; however, at the moment this is only known in limited settings. We have the following result.

Theorem 3. *Suppose that (X, ω_{KE}) is a singular Kähler-Einstein space with $\omega_{KE} \in c_1(L)$ for a line bundle over X, and such that X has discrete automorphism group. Assume that X admits a projective resolution $\pi : Y \to X$ for which the anticanonical bundle $-K_Y$ is relatively nef over X. Then (X, ω_{KE}) can be approximated by cscK metrics in the sense of the definition above.*

Note that recently Boucksom-Jonsson-Trusiani [6] showed the existence of cscK metrics on resolutions in this setting (and even more generally), while Pan-Tô [47] showed estimates for these approximating cscK metrics closely related to those in Definition 2, in a more general setting.

Our main result on Kähler-Einstein spaces that can be approximated by cscK metrics is the following.

Theorem 4. *Suppose that (X, ω_{KE}) can be approximated by cscK metrics, and $\omega_{KE} \in c_1(L)$ for a line bundle L on X. Then Conjecture 1 holds for $\overline{(X^{reg}, d_{KE})}$. In addition the metric singular set of $\overline{(X^{reg}, d_{KE})}$ agrees with the complex analytic singular set $X \setminus X^{reg}$, and it has Hausdorff codimension at least 4.*

It is natural to expect that Conjecture 1 can also be extended to the setting of singular Kähler-Einstein metrics ω with cone singularities along a divisor on klt pairs (X, D). In this case one can hope to approximate ω using cscK metrics with cone singularities on a log resolution of (X, D). Some results in this direction were obtained recently by Zheng [63], but we leave this extension of Theorem 4 for future work.

The RCD property implies important geometric information about the metric completion $\overline{(X^{reg}, d_{KE})}$, such as the existence of tangent cones (see De Philippis-Gigli [22]). Moreover, we expect that with only minor modifications the work of Donaldson-Sun [28] and Li-Wang-Xu [41] on the tangent cones of smoothable Kähler-Einstein spaces can be extended to the setting of Theorem 4, that is, the tangent cones of $\overline{(X^{reg}, d_{KE})}$ are unique, and are determined by the algebraic structure. Knowledge of the tangent cones can then be further leveraged to obtain more refined information about the metric, such as in Hein-Sun [35], or [18].

Using results of Honda [36], which rely on different equivalent characterizations of RCD spaces by Ambrosio-Gigli-Savaré [1], the main estimate that we need in order to prove the RCD property in Theorem 4 is that eigenfunctions of the Laplacian are Lipschitz continuous on (X^{reg}, d_{KE}). We will review Honda's result in Section 2. In order to prove a gradient estimate for eigenfunctions, we use the approximating smooth cscK spaces (Y, ω_ϵ). Note that these do not satisfy uniform gradient estimates, since they do not have uniform Ricci curvature bounds from below. Instead we will prove a weaker estimate on (Y, ω_ϵ), expressed in terms of the heat flow – roughly speaking we obtain an estimate that is valid for times $t > t_\epsilon > 0$ along the heat flow, where $t_\epsilon \to 0$ as $\epsilon \to 0$. These estimates can be passed to the limit as $\epsilon \to 0$ using the uniform estimates of Guo-Phong-Song-Sturm [32, 33] for the heat kernels, and in the limit we obtain the required gradient bound on (X^{reg}, ω_{KE}). This is discussed in Section 3.

In Section 4 we prove that $\overline{(X^{reg}, d_{KE})}$ is homeomorphic to X, and that the metric singular set has Hausdorff codimension at least 4. Some results of this type were shown by Song [49] and La Nave-Tian-Zhang [39], based on applying Hörmander's L^2-estimates, following Donaldson-Sun [27]. The main new difficulty in our setting is that a priori we do not have enough control of how large the set $\overline{(X^{reg}, d_{KE})} \setminus X^{reg}$ is in the metric sense. It was shown by Sturm [52] (see also [49]), that this set has capacity zero, which already plays an important role in the RCD property. For the approach of Donaldson-Sun [27] to apply, however, we need a slightly stronger effective bound that can be applied uniformly at all scales. In previous related results this type of estimate relied on showing that the metric regular set in $\overline{(X^{reg}, d_{KE})}$ coincides with X^{reg}, but this is not clear in our setting since our approximating Riemannian manifolds (Y, ω_ϵ) do not have lower Ricci bounds.

The new ingredient that we exploit is that the algebraic singular set of X is locally cut out by holomorphic (and therefore harmonic) functions. We show that these functions have finite order of vanishing along the singular set, and therefore we can control the size of their zero sets in any ball that is sufficiently close to a Euclidean ball, using a three annulus lemma argument, somewhat similarly to [19]. This leads to the key result that the metric and algebraic regular sets of $\overline{(X^{reg}, d_{KE})}$ coincide. After this the proof follows by now familiar lines from Donaldson-Sun [27] and other subsequent works such as [42].

In Section 5 we prove Theorem 3. The proof is based primarily on Chen-Cheng's existence theorem for cscK metrics [15] together with some extensions of their estimates by Zheng [62]. A similar result, in more general settings, was obtained recently by Boucksom-Jonsson-Trusiani [6] and Pan-Tô [47].

In Section 6, as an example application, we discuss an extension of Donaldson-Sun's partial C^0-estimate to singular Kähler-Einstein spaces with the cscK approximation property. An additional ingredient that we need is the gap result for the volume densities of (singular) Ricci flat Kähler cone metrics that arise as tangent cones, Theorem 36. This was shown very recently in the more general algebraic setting by Xu-Zhuang [59].

2. Background

2.1. Noncollapsed RCD spaces

By a metric measure space we mean a triple $(Z, d, \mathfrak{m})$, where (Z, d) is a metric space, and $\mathfrak{m}$ is a measure on Z with $\mathrm{supp}\,\mathfrak{m} = Z$. By now there are several different, but essentially equivalent, notions of synthetic lower bounds for the Ricci curvature of $(Z, d, \mathfrak{m})$, due to Sturm [53], Lott-Villani [45], and Ambrosio-Gigli-Savaré [1]. We will be particularly concerned with the notion of noncollapsed $\mathrm{RCD}(K, N)$ space introduced by De Philippis-Gigli [23]. These should be thought of as the synthetic version of noncollapsed Gromov-Hausdorff limits of N-dimensional manifolds with Ricci curvature bounded below by K.

More specifically we will be concerned with RCD spaces that are the metric completions of smooth Riemannian manifolds. In fact the spaces that we study almost fit into the setting of *almost smooth metric measure spaces*, studied by Honda [36], except we will use the standard notion of zero capacity set instead of [36, Definition 3.1, 3(b)]. The results of [36] hold with this definition too, as we will outline below. Thus we state the following slight modification of Honda's definition.

Definition 5. A compact metric measure space $(Z, d, \mathfrak{m})$ is an n-dimensional almost smooth metric measure space, if there is an open subset $\Omega \subset Z$ satisfying the following conditions.

(1) There is a smooth n-dimensional Riemannian manifold (M, g) and a homeomorphism $\phi : \Omega \to M^n$, such that ϕ defines a local isometry between (Ω, d) and (M^n, d_g).
(2) The restriction of the measure $\mathfrak{m}$ to Ω coincides with the n-dimensional Hausdorff measure.
(3) The complement $Z \setminus \Omega$ has measure zero, that is, $\mathfrak{m}(Z \setminus \Omega) = 0$, and it has zero capacity in the following sense: there is a sequence of smooth functions $\phi_i : \Omega \to [0, 1]$ with compact support in Ω such that
    (a) For any compact $A \subset \Omega$ we have $\phi_i|_A = 1$ for sufficiently large i,
    (b) We have

$$\lim_{i\to\infty} \int_\Omega |\nabla \phi_i|^2 \, d\mathcal{H}^n = 0. \tag{2}$$

As a point of comparison we remark that in [36], the condition (b) is replaced by requiring that the L^1-norm of $\Delta\phi_i$ is uniformly bounded. Note that neither of these conditions implies the other one.

In our setting we will have an n-dimensional normal projective variety X equipped with a positive current ω that is a smooth Kähler metric on X^{reg}. In addition we will assume that ω has locally bounded Kähler potentials. We use ω to define a metric structure d on the smooth locus X^{reg}:

$$d(x, y) = \inf\{\ell(\gamma) \,|\, \gamma \text{ is a smooth curve in } X^{reg} \text{ from } x \text{ to } y\}, \tag{3}$$

where $\ell(\gamma)$ denotes the length of γ with respect to ω. We define $(\hat{X}, d_{\hat{X}})$ to be the metric completion of (X^{reg}, d), and we extend the volume form ω^n to $\hat{X}$ trivially. In this way $(\hat{X}, d_{\hat{X}}, \omega^n)$ defines a metric measure space. The complement of X^{reg} has zero capacity, by the following result, due to Sturm [52] (see also Song [49, Lemma 3.7]).

Lemma 6. *There is a sequence of smooth functions $\phi_i : X^{reg} \to [0, 1]$ with compact support, such that we have: for any compact $A \subset X^{reg}$ we have $\phi_i|_A = 1$ for sufficiently large i, and*

$$\lim_{i\to\infty} \int_{X^{reg}} |\nabla \phi_i|^2 \, \omega^n = 0. \tag{4}$$

From this we have the following.

Lemma 7. *$(\hat{X}, d_{\hat{X}}, \omega^n)$ defines a $2n$-dimensional almost smooth measure metric space in the sense of Definition 5.*

Proof. The open set $\Omega \subset \hat{X}$ is the smooth locus X^{reg} viewed as a subset of its metric completion $\hat{X}$, equipped with the Kähler metric ω. The conditions (1) and (2) in Definition 5 are automatically satisfied. The fact that $\hat{X} \setminus X^{reg}$ has capacity zero follows from the existence of good cutoff functions in Lemma 6. □

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In order to show that $\hat{X}$ is an RCD space, we will use the characterization of RCD spaces in Honda [36, Corollary 3.10] (see also Ambrosio-Gigli-Savaré [1]). We state this Corollary here in our setting. Note that our notion of almost smooth metric measure space is slightly different from that in [36].

Corollary 8 (See [36]). *The metric completion $(\hat{X}, d_{\hat{X}}, \omega^n)$ is an $RCD(K, 2n)$ space, where $K \in \mathbb{R}$, if it is an almost smooth compact metric measure space associated with X^{reg} in the sense of [36, Definition 3.1], and the following conditions hold:*

1. *The Sobolev to Lipschitz property holds, that is, any $f \in W^{1,2}(\hat{X})$, with $|\nabla f|(x) \leq 1$ for ω^n-almost every x, has a 1-Lipschitz representative;*
2. *The L^2-strong compactness condition holds, that is, the inclusion $W^{1,2}(\hat{X}) \hookrightarrow L^2(\hat{X})$ is a compact operator;*
3. *Any $W^{1,2}$-eigenfunction of the Laplacian on $\hat{X}$ is Lipschitz;*
4. $\mathrm{Ric}(\omega) \geq K\omega$ *on* X^{reg}.

In these conditions the Sobolev space $W^{1,2}(\hat{X})$ is defined by taking the completion of the space of compactly supported smooth functions $C_0^\infty(X^{reg})$ on the Riemannian manifold (X^{reg}, ω) in the $W^{1,2}$-norm. By [36, Proposition 3.3] this space coincides with the $H^{1,2}(\hat{X}, d_{\hat{X}}, \omega^n)$-space defined using the Cheeger energy.

Proof. The only place where the difference between our notion of capacity zero in Definition 5 and Honda's notion plays a role is in the proof of [36, Theorem 3.7] to deduce Equation (3.13), stating that the Hessian of f_N is in L^2 (see [36] for the meaning of f_N). We can also deduce this by using cutoff functions that satisfy our Condition (3b) in Definition 5. To simplify the notation we will write $\Omega = X^{reg}$. Let us recall Equation (3.12) from [36], which in our notation states

$$\frac{1}{2}\int_\Omega |\nabla f_N|^2 \Delta \phi_i^2 \, \omega^n \geq \int_\Omega \phi_i^2 \Big(|\mathrm{Hess}_{f_N}|^2 + \langle \nabla \Delta f_N, \nabla f_N \rangle + K |\nabla f_N|^2 \Big) \, \omega^n, \tag{5}$$

where $\mathrm{Ric}(\omega) \geq K\omega$, and we used ϕ_i^2 as the cutoff function instead of ϕ_i. Note that $0 \leq \phi_i^2 \leq 1$, and $\nabla \phi_i^2 = 2\phi_i \nabla \phi_i$, so ϕ_i^2 satisfies the same estimate as ϕ_i. In addition f_N is a Lipschitz function such that $f_N, \Delta f_N \in W^{1,2}$. We have

$$\begin{aligned} \int_\Omega |\nabla f_N|^2 \Delta \phi_i^2 \, \omega^n &= -\int_\Omega 4 |\nabla f_N| \phi_i \langle \nabla |\nabla f_N|, \nabla \phi_i \rangle \, \omega^n \\ &\leq \int_\Omega \Big(\phi_i^2 |\mathrm{Hess}_{f_N}|^2 + 4 |\nabla f_N|^2 |\nabla \phi_i|^2 \Big) \, \omega^n. \end{aligned} \tag{6}$$

It follows using this in (5) that

$$\begin{aligned} \int_\Omega \frac{1}{2} \phi_i^2 |\mathrm{Hess}_{f_N}|^2 \, \omega^n \leq \int_\Omega \Big(& 2 |\nabla f_N|^2 |\nabla \phi_i|^2 - \phi_i^2 \langle \nabla \Delta f_N, \nabla f_N \rangle \\ & - \phi_i^2 K |\nabla f_N|^2 \Big) \, \omega^n. \end{aligned} \tag{7}$$

Letting $i \to \infty$ and using that $|\nabla f_N| \in L^\infty$, we obtain that

$$\int_\Omega |\mathrm{Hess}_{f_N}|^2 \, \omega^n < \infty. \tag{8}$$

The rest of the argument is the same as in [36, Theorem 3.7]. □

Note that in our setting we have the following. In Section 3 we will show the remaining Condition (3) in the setting of Theorem 4.

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Proposition 9. *The metric measure space $(\hat{X}, d_{\hat{X}}, \omega^n)$ satisfies Conditions (1), (2), and (4) in Corollary 8, for some $K \in \mathbb{R}$.*

Proof. Condition (4) is satisfied by definition. To verify Condition (1), let $f \in W^{1,2}(\hat{X})$, such that $|\nabla f|(x) \leq 1$ for ω^n-almost every x. On X^{reg} the Sobolev to Lipschitz property holds, so we can assume that f is 1-Lipschitz on X^{reg}. By the definition of the distance d, this implies that for any $x, y \in X^{reg}$ we have $|f(x) - f(y)| \leq |x - y|$. We can then extend f uniquely to the completion $\hat{X}$ so that the same condition continues to hold. Condition (2) follows from the Sobolev inequality shown by Guo-Phong-Song-Sturm [33, Theorem 2.1]. □

Let us recall from De Philippis-Gigli [23] that an $RCD(K, N)$-space $(Z, d, \mathfrak{m})$ is called noncollapsed, if the N-dimensional Hausdorff measure on (Z, d) agrees with $\mathfrak{m}$. In particular, if an n-dimensional almost smooth metric measure space in Definition 5 satisfies the $RCD(K, n)$-property, then it is automatically noncollapsed. Noncollapsed RCD spaces satisfy many of the properties enjoyed by noncollapsed Ricci limits spaces studied by Cheeger-Colding [12]. We will now recall some results that we will use.

De Philippis-Gigli [22] showed that in a noncollapsed $RCD(K, N)$-space $(Z, d, \mathfrak{m})$, the tangent cones at every point $z \in Z$ are metric cones. In [23] they then showed that Z admits a stratification

$$S_0 \subset S_1 \subset \ldots \subset S_{N-1} \subset Z, \tag{9}$$

where S_k denotes the set of points $z \in Z$ where no tangent cone splits off an isometric factor of $\mathbb{R}^{k+1}$, and the strata satisfy the Hausdorff dimension estimate $\dim_{\mathcal{H}} S_k \leq k$. Note that in contrast with the setting of noncollapsed Ricci limit spaces, it is not necessarily the case that $S_{N-1} = S_{N-2}$, since a noncollapsed RCD-space can have boundary. In our setting, however, we have the following, which is a consequence of Bruè-Naber-Semola [8, Theorem 1.2].

Proposition 10. *Suppose that $(Z, d, \mathfrak{m})$ is a noncollapsed $RCD(K, N)$-space, and also an N-dimensional almost smooth metric measure space. Then $S_{N-1} = S_{N-2}$. Moreover any iterated tangent cone Z' of Z also satisfies $S_{N-1} = S_{N-2}$.*

Proof. Using the notation of [8] we define $\partial Z = \overline{S_{N-1} \setminus S_{N-2}}$ to be the boundary of Z. Let $\Omega \subset Z$ denote the smooth Riemannian manifold in the definition of almost smooth metric measure space. For $z \in \Omega$ the tangent cones are all $\mathbb{R}^N$, so $\partial Z \subset Z \setminus \Omega$. In particular ∂Z has capacity zero. Using [8, Theorem 1.2(i)] this implies that we must have $\partial Z = \emptyset$. If an iterated tangent cone Z' satisfied $\partial Z' \neq \emptyset$, then by [8, Theorem 1.2(i)] we would have $\partial Z \neq \emptyset$, which is a contradiction as above. □

We will be working with harmonic functions on RCD spaces, so we review some basic results. Let us suppose that $(Z, d, \mathfrak{m})$ is a noncollapsed $RCD(K, N)$-space that is also an N-dimensional almost smooth metric measure space. A function $f : U \to \mathbb{R}$ on an open set $U \subset Z$ is defined to be harmonic if $f \in W^{1,2}_{loc}(U)$, and for any Lipschitz function $\psi : U \to \mathbb{R}$ with compact support we have

$$\int_U \nabla f \cdot \nabla \psi \, d\mathfrak{m} = 0. \tag{10}$$

Note that in our setting the integration can be taken over $U \cap \Omega$, where $\Omega \subset Z$ is the dense open set in Definition 5 since $Z \setminus \Omega$ has measure zero. We will use the following result several times.

Lemma 11. *Let $u : U \to \mathbb{R}$ for an open set $U \subset Z$, such that $u \in L^\infty(U)$. Suppose that $\Delta u = 0$ on $U \cap \Omega$, using the smooth Riemannian structure on Ω. Then u is harmonic on U.*

Proof. Let ϕ_i be functions as in Condition (3) of Definition 5, and ψ a Lipschitz function with compact support in U. We have

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$$\begin{aligned}\int_U \psi^2\phi_i^2|\nabla u|^2\,d\mathfrak{m} &= -2\int_U \psi^2\phi_i u\nabla\phi_i\cdot\nabla u\,d\mathfrak{m} - 2\int_U \phi_i^2\psi u\nabla u\cdot\nabla\psi\,d\mathfrak{m}\\ &\leq \frac{1}{2}\int_U \psi^2\phi_i^2|\nabla u|^2\,d\mathfrak{m} + 4\int_U \psi^2u^2|\nabla\phi_i|^2\,d\mathfrak{m} + C_\psi\int_U u^2\,d\mathfrak{m},\end{aligned} \tag{11}$$

where C_ψ depends on $\sup_{U\cap\Omega}|\nabla\psi|$. Letting $i\to\infty$, we obtain that $u\in W^{1,2}_{loc}(U)$.

At the same time we have

$$\begin{aligned}\int_U \phi_i^2\nabla u\cdot\nabla\psi\,d\mathfrak{m} &= -2\int_U \phi_i\psi\nabla\phi_i\cdot\nabla u\,d\mathfrak{m}\\ &\leq \int_U |\nabla\phi_i|^2\,d\mathfrak{m} + \int_{\mathrm{supp}(\nabla\phi_i)}\psi^2|\nabla u|^2\,d\mathfrak{m}.\end{aligned} \tag{12}$$

Letting $i\to\infty$ we get $\int_U \nabla u\cdot\nabla\psi = 0$, so u is harmonic on U. □

We will also need the following gradient estimate, generalizing Cheng-Yau's gradient estimate.

Proposition 12 (Jiang [37], Theorem 1.1). *Let u be a harmonic function on a ball $B(p,2R)$ in an $RCD(N,K)$-space. There is a constant $C=C(R,N,K)$ such that*

$$\sup_{B(p,R)}|\nabla u|\leq C\fint_{B(p,2R)}|u|\,d\mathfrak{m}. \tag{13}$$

Note that a similar estimate holds for solutions of $\Delta u = c$ on $U\subset Z$ for a constant c, by considering $u-ct^2/2$ on the space $U\times\mathbb{R}_t$.

3. The RCD property of singular Kähler-Einstein spaces

The main result in this section will be that the completion of the Kähler-Einstein metric on X^{reg} in Theorem 4 defines a noncollapsed RCD space. We will first need some estimates for the cscK approximations of (X,ω_{KE}).

3.1. Constant scalar curvature approximations

Let (X,ω_{KE}) be a singular Kähler-Einstein space, where $\mathrm{Ric}_{\omega_{KE}}=\lambda\omega_{KE}$. Suppose that (X,ω_{KE}) can be approximated by cscK metrics as in Definition 2. In particular there is a resolution Y of X, that admits a family of cscK metrics ω_ϵ in suitable Kähler classes $[\eta_\epsilon]$, such that the η_ϵ converge to $\pi^*\eta_X$. Here η_X is a smooth metric on X in the sense that it is the restriction of a smooth metric under local embeddings into $\mathbb{C}^N$.

We will need the following, which is immediate from the work of Guo-Phong-Song-Sturm [33, Theorem 2.2].

Theorem 13. *Let $H(x,y,t)$ denote the heat kernel on (Y,ω_ϵ). There is a continuous function $\bar{H}:(0,2]\to\mathbb{R}$, depending on (X,ω_{KE}), but independent of ϵ, such that we have the upper bound*

$$H(x,y,t)\leq\bar{H}(t),\qquad \text{for } x,y\in Y \text{ and } t\in(0,2]. \tag{14}$$

Note that $\bar{H}(t)\to\infty$ as $t\to 0$.

In addition the constant scalar curvature metrics ω_ϵ satisfy the following integral bounds for their Ricci curvatures. We will use these integral bounds as a replacement for having lower bounds for the Ricci curvature, when we approximate ω_{KE} with ω_ϵ.

Proposition 14. *Let us define* $\widetilde{\mathrm{Ric}}_\omega = \mathrm{Ric}_\omega - \lambda\omega$. *We have the following estimates:*

$$\lim_{\epsilon\to 0}\int_Y |\widetilde{\mathrm{Ric}}_{\omega_\epsilon}|^2 + \frac{|\nabla\widetilde{\mathrm{Ric}}_{\omega_\epsilon}|^2}{(1+|\widetilde{\mathrm{Ric}}_{\omega_\epsilon}|^2)^{1/2}} + |\Delta(1+|\widetilde{\mathrm{Ric}}_{\omega_\epsilon}|^2)^{1/2}|\,\omega_\epsilon^n = 0, \tag{15}$$

and

$$\int_Y |\mathrm{Rm}_{\omega_\epsilon}|^2\,\omega_\epsilon^n < C, \tag{16}$$

for C independent of ϵ.

Proof. First recall the well-known result of Calabi [10] relating the L^2-norms of the scalar curvature, the Ricci and Riemannian curvature tensors of a Kähler metric. Let us denote by $R, \mathrm{Ric}, \mathrm{Rm}$ the scalar curvature, the Ricci form and the Riemannian curvature tensor. Since R_{ω_ϵ} is constant, we have

$$R_{\omega_\epsilon} = \frac{2n\pi c_1(Y)\cup[\omega_\epsilon]^{n-1}}{[\omega_\epsilon]^n}. \tag{17}$$

Note that we have

$$\lim_{\epsilon\to 0}\frac{2n\pi c_1(Y)\cup[\omega_\epsilon]^{n-1}}{[\omega_\epsilon]^n} = \frac{2n\pi c_1(X)\cup[\omega_{KE}]^{n-1}}{[\omega_{KE}]^n} = n\lambda, \tag{18}$$

since $[\omega_{KE}]^{n-1}$ vanishes when paired with the exceptional divisor of the map $Y\to X$. In addition

$$\begin{aligned}\int_Y |\mathrm{Ric}_{\omega_\epsilon}|^2\,\omega_\epsilon^n &= R^2_{\omega_\epsilon}[\omega_\epsilon]^n - 4n(n-1)\pi^2 c_1(Y)^2\cup[\omega_\epsilon]^{n-2},\\ \int_Y (|\mathrm{Ric}_{\omega_\epsilon}|^2 - |\mathrm{Rm}_{\omega_\epsilon}|^2)\,\omega_\epsilon^n &= n(n-1)\big(4\pi^2 c_1(Y)^2 - 8\pi^2 c_2(Y)\big)\cup[\omega_\epsilon]^{n-2}.\end{aligned} \tag{19}$$

Since the cohomology classes $[\omega_\epsilon] = [\eta_\epsilon]$ are uniformly bounded, and in addition $[\omega_\epsilon]^n \geq [\eta_X]^n > 0$, it follows that R_{ω_ϵ}, and the L^2 norms of $|\mathrm{Ric}_{\omega_\epsilon}|, |\mathrm{Rm}_{\omega_\epsilon}|$ are all uniformly bounded, independently of ϵ.

To see the first claim in the Proposition, note that

$$\int_Y |\mathrm{Ric}_{\omega_\epsilon} - \lambda\omega_\epsilon|^2\,\omega_\epsilon^n = (R_{\omega_\epsilon} - n\lambda)^2[\omega_\epsilon]^n - n(n-1)\big(2\pi c_1(Y) - \lambda[\omega_\epsilon]\big)^2\cup[\omega_\epsilon]^{n-2}. \tag{20}$$

As $\epsilon\to 0$, this converges to zero by (18) and the fact that $2\pi c_1(X) = \lambda[\omega_{KE}]$.

To estimate $\nabla\widetilde{\mathrm{Ric}}_{\omega_\epsilon}$ and $\Delta\widetilde{\mathrm{Ric}}_{\omega_\epsilon}$ note that we have the following equation satisfied by any constant scalar curvature metric:

$$\begin{aligned}\Delta|\widetilde{\mathrm{Ric}}|^2 &= \nabla_k\nabla_{\bar k}(\widetilde{\mathrm{Ric}}_{p\bar q}\widetilde{\mathrm{Ric}}_{q\bar p})\\ &= 2|\nabla_k\widetilde{\mathrm{Ric}}_{p\bar q}|^2 + \mathrm{Rm}*\widetilde{\mathrm{Ric}}*\widetilde{\mathrm{Ric}},\end{aligned} \tag{21}$$

where $*$ denotes a tensorial contraction. It follows that

$$\begin{aligned}\Delta(1+|\widetilde{\mathrm{Ric}}|^2)^{1/2} &= (1+|\widetilde{\mathrm{Ric}}|^2)^{-1/2}\Big(|\nabla\widetilde{\mathrm{Ric}}|^2 - |\nabla|\widetilde{\mathrm{Ric}}||^2 + \mathrm{Rm}*\widetilde{\mathrm{Ric}}*\widetilde{\mathrm{Ric}}\Big)\\ &\quad + \frac{|\nabla|\widetilde{\mathrm{Ric}}||^2}{(1+|\widetilde{\mathrm{Ric}}|^2)^{3/2}}.\end{aligned} \tag{22}$$

For a constant scalar curvature Kähler metric the form $\widetilde{\mathrm{Ric}}$ is harmonic, so we have the following refined Kato inequality (see Branson [7], Calderbank-Gauduchon-Herzlich [11], or Cibotaru-Zhu [20, Theorem 3.8]):

$$|\nabla|\widetilde{\mathrm{Ric}}||^2 \leq \alpha_n|\nabla\widetilde{\mathrm{Ric}}|^2 \tag{23}$$

for a dimensional constant $\alpha_n < 1$. It follows from (22) that

$$\Delta(1+|\widetilde{\mathrm{Ric}}|^2)^{1/2} \geq (1-\alpha_n)\frac{|\nabla\widetilde{\mathrm{Ric}}|^2}{(1+|\widetilde{\mathrm{Ric}}|^2)^{1/2}} - C|\mathrm{Rm}|\,|\widetilde{\mathrm{Ric}}|. \tag{24}$$

Integrating over Y, we get

$$\int_Y \frac{|\nabla\widetilde{\mathrm{Ric}}_{\omega_\epsilon}|^2}{(1+|\widetilde{\mathrm{Ric}}|^2_{\omega_\epsilon})^{1/2}}\,\omega_\epsilon^n \leq C_1\|\mathrm{Rm}_{\omega_\epsilon}\|_{L^2}\|\widetilde{\mathrm{Ric}}_{\omega_\epsilon}\|_{L^2} \to 0, \tag{25}$$

as $\epsilon \to 0$. It then follows from (22) that

$$\int_Y |\Delta(1+|\widetilde{\mathrm{Ric}}_{\omega_\epsilon}|^2)^{1/2}|\,\omega_\epsilon^n \leq \int_Y \frac{|\nabla\widetilde{\mathrm{Ric}}_{\omega_\epsilon}|^2}{(1+|\widetilde{\mathrm{Ric}}|^2_{\omega_\epsilon})^{1/2}}\,\omega_\epsilon^n + C\|\mathrm{Rm}_{\omega_\epsilon}\|_{L^2}\|\widetilde{\mathrm{Ric}}_{\omega_\epsilon}\|_{L^2} \to 0, \tag{26}$$

as $\epsilon \to 0$. □

3.2. Proof of the RCD property

In this section we assume that (X, ω_{KE}) is a singular Kähler-Einstein space, with $\mathrm{Ric}_{\omega_{KE}} = \lambda\omega_{KE}$, that can be approximated with cscK metrics as in Definition 2. Our first result is the following.

Proposition 15. *The metric completion $(\hat{X}, d, \omega_{KE}^n)$ is an $RCD(\lambda, 2n)$ space.*

Proof. From Proposition 9 it follows that it is sufficient to check condition (3) in Corollary 8, that is, to show that the eigenfunctions of the Laplacian on $\hat{X}$ are bounded. More precisely, suppose that $u \in W^{1,2}(\hat{X})$ satisfies $\Delta u = -bu$ on X^{reg} for a constant b. We will show that then $|\nabla u| \in L^\infty(X^{reg})$.

For simplicity we can assume that $\|u\|_{L^2} = 1$. Using that $u \in W^{1,2}(\hat{X})$, and also [33, Lemma 11.2], we have

$$\sup|u| + \int_{X^{reg}} |\nabla u|^2\omega_{KE}^n < C, \tag{27}$$

where C could depend on u (in particular on b).

Next we will use the approximating cscK metrics ω_ϵ on the resolution Y of X. Let us fix a large i, and let $f = \phi_i u$ for the cutoff function ϕ_i in Lemma 6. We can view f as a function on Y, supported away from the exceptional divisor, where the metrics ω_ϵ converge smoothly to ω_{KE}. Note that we have a uniform bound $\sup|f| < C$, and also

$$\int_Y |\nabla f|^2\omega_\epsilon^n \leq \int_Y 2(|u\nabla\phi_i|^2 + |\phi_i\nabla u|^2)\,\omega_\epsilon^n < 2C, \tag{28}$$

for sufficiently small ϵ.

Let us fix a point $x_0 \in X$ where $\phi_i(x_0) = 1$. We can view $x_0 \in Y$ too. We will do the following calculation on Y, using the metric ω_ϵ for sufficiently small ϵ. To simplify the notation we will omit the subscript ϵ. All geometric quantities are defined using the metric ω_ϵ. We will write $\rho_t = H(x_0, y, t)$ for the heat kernel centered at x_0 on (Y, ω_ϵ), and let f_t denote the solution of the heat equation on

(Y, ω_ϵ) with initial condition f. We will also omit the volume form ω_ϵ^n in the integrals below. We have the following.

$$\begin{aligned}\partial_s \int_Y \frac{1}{2}|\nabla f_{t-s}|^2 \rho_s &= \int_Y -\langle \nabla f_{t-s}, \nabla\Delta f_{t-s}\rangle \rho_s + \frac{1}{2}|\nabla f_{t-s}|^2 \Delta\rho_s \\ &= \int_Y \Big(|\nabla^2 f_{t-s}|^2 + \mathrm{Ric}(\nabla f_{t-s}, \nabla f_{t-s})\Big)\rho_s.\end{aligned} \tag{29}$$

In order to compensate for the Ricci term, we let $\psi^2 = (1 + |\widetilde{\mathrm{Ric}}|^2)^{1/2}$, where $\widetilde{\mathrm{Ric}} = \mathrm{Ric}_{\omega_\epsilon} - \lambda\omega_\epsilon$ as in Proposition 14. We have

$$\begin{aligned}\partial_s \int_Y \psi^2 f_{t-s}^2 \rho_s &= \int_Y -2\psi^2 f_{t-s}\Delta f_{t-s}\,\rho_s + \psi^2 f_{t-s}^2 \Delta\rho_s \\ &= \int_Y \Big(\Delta(\psi^2) f_{t-s}^2 + 2\langle \nabla\psi^2, \nabla f_{t-s}^2\rangle + 2\psi^2|\nabla f_{t-s}|^2\Big)\rho_s \\ &\geq -C\int_Y (|\Delta\psi^2| + |\nabla\psi|^2)\,\rho_s + \int_Y \psi^2|\nabla f_{t-s}|^2\rho_s,\end{aligned} \tag{30}$$

where the constant C depends on the uniform supremum bound for f_{t-s}.

Note that $\psi^2 \geq |\mathrm{Ric}| - n|\lambda|$, so if we combine (29) and (30), we get

$$\partial_s \int_Y \left(\frac{1}{2}|\nabla f_{t-s}|^2 + \psi^2 f_{t-s}^2\right)\rho_s \geq -C\int_Y (|\Delta\psi^2| + |\nabla\psi|^2)\,\rho_s - \int_Y n|\lambda||\nabla f_{t-s}|^2\rho_s. \tag{31}$$

At this point, let us fix $s_0 > 0$, and only work with $s \in [s_0, 2]$. From Proposition 14 we know that $\|\Delta\psi^2\|_{L^1}, \|\nabla\psi\|_{L^2} \to 0$ as $\epsilon \to 0$. From Theorem 13 we have a uniform upper bound for ρ_s, depending on s_0, but independent of ϵ. Therefore, if we choose ϵ sufficiently small, say $\epsilon < \epsilon_{s_0}$, then we have

$$\partial_s \int_Y \left(\frac{1}{2}|\nabla f_{t-s}|^2 + \psi^2 f_{t-s}^2\right)\rho_s \geq -1 - n|\lambda|\int_Y |\nabla f_{t-s}|^2\rho_s, \tag{32}$$

and so

$$\partial_s\, e^{2n|\lambda|s} \int_Y \left(\frac{1}{2}|\nabla f_{t-s}|^2 + \psi^2 f_{t-s}^2\right)\rho_s \geq -C. \tag{33}$$

Applying this with $t = 1 + s_0$ and integrating from $s = s_0$ to $s = 1 + s_0$, it follows that for such ϵ we have

$$e^{2n|\lambda|s_0}\int_Y \left(\frac{1}{2}|\nabla f_1|^2 + \psi^2 f_1^2\right)\rho_{s_0} \leq C + e^{2n|\lambda|(s_0+1)}\int_Y \left(\frac{1}{2}|\nabla f|^2 + \psi^2 f^2\right)\rho_{1+s_0}. \tag{34}$$

Using the uniform upper bound for ρ_{1+s_0}, together with the integral bound for $|\widetilde{\mathrm{Ric}}|^2$ from Proposition 14, we obtain that

$$\int_Y |\nabla f_1|^2 \rho_{s_0} \leq C, \tag{35}$$

where C is independent of ϵ, s_0. As $\epsilon \to 0$, the heat kernels ρ_{s_0} converge locally smoothly on X^{reg} to the heat kernel on $(\hat{X}, \omega_{KE})$, and so in the limit we obtain the estimate

$$\int_{X^{reg}} |\nabla f_1|^2 \rho_{s_0} \leq C, \tag{36}$$

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where all the quantities are computed using ω_{KE}, and recall that f_1 is simply the solution f_t of the heat flow with initial condition f at time $t = 1$. Note that the constant C does not depend on s_0, so in fact, by letting $s_0 \to 0$, we obtain the pointwise estimate

$$|\nabla f_1|^2(x_0) \leq C, \tag{37}$$

and this holds uniformly for any $x_0 \in X^{reg}$.

Recall that $f = \phi_i u$, where u is the eigenfunction that we want to estimate, and ϕ_i is a cutoff function from Lemma 6. To keep track of the dependence on i, let us now write $f^{(i)} = \phi_i u$, and write $f_1^{(i)}$ for the corresponding solutions of the heat equation at time 1. Since $f^{(i)} \to u$ in L^2, it follows that for any compact set $K \subset X^{reg}$ the solutions $f_1^{(i)}$ converge smoothly to u_1 on K. But $u_1 = e^{-b}u$, so we obtain the required pointwise bound $|\nabla u|^2(x_0) \leq e^{2b}C$ for any $x_0 \in X^{reg}$. □

Next we show that singular Kähler-Einstein metrics on projective varieties, that can be approximated by cscK metrics, define Kähler currents. This result was previously shown by Guedj-Guenancia-Zeriahi [31] for singular Kähler-Einstein metrics that are either globally smoothable, or that only have isolated smoothable singularities.

Theorem 16. *Let ω_{KE} denote a singular Kähler-Einstein metric on a normal projective variety X, which can be approximated by cscK metrics as in Definition 2. Let η_{FS} denote the pullback of the Fubini-Study metric to X under a projective embedding of X. Then there is a constant $\delta > 0$ such that $\omega_{KE} > \delta\eta_{FS}$.*

Proof. By assumption we have cscK metrics $\omega_\epsilon = \eta_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon$ on a resolution $\pi : Y \to X$, where $\eta_\epsilon \to \pi^*\eta_X$ for a smooth metric η_X on X, where $\eta_\epsilon \geq \pi^*\eta_X$. We apply the Chern-Lu inequality to the map $\pi : Y \to X$, away from the exceptional divisor E, where on Y we use the metric ω_ϵ and on X we use the pullback η_{FS} of the Fubini-Study metric under a projective embedding of X. For simplicity we write η_{FS} for $\pi^*\eta_{FS}$, and we write $g_{i\bar{j}}$ and $h_{i\bar{j}}$ for the metric components of ω_ϵ and η_{FS}, respectively. On $Y \setminus E$ we then have $|\partial\pi|^2 = \mathrm{tr}_{\omega_\epsilon}\eta_{FS}$, and (see, e.g., [46])

$$\Delta_{\omega_\epsilon} \log \mathrm{tr}_{\omega_\epsilon}\eta_{FS} \geq \frac{g^{i\bar{l}}g^{k\bar{j}}\mathrm{Ric}(\omega_\epsilon)_{i\bar{j}}h_{k\bar{l}}}{\mathrm{tr}_{\omega_\epsilon}\eta_{FS}} - A\mathrm{tr}_{\omega_\epsilon}\eta_{FS}, \tag{38}$$

where A is independent of ϵ, using that η_{FS} has bisectional curvature bounded from above. It follows that

$$\begin{aligned}\Delta_{\omega_\epsilon} \log \mathrm{tr}_{\omega_\epsilon}\eta_{FS} &\geq \frac{g^{i\bar{l}}g^{k\bar{j}}(\mathrm{Ric}(\omega_\epsilon)_{i\bar{j}} - \lambda g_{i\bar{j}})h_{k\bar{l}}}{\mathrm{tr}_{\omega_\epsilon}\eta_{FS}} + \lambda - A\mathrm{tr}_{\omega_\epsilon}\eta_{FS} \\ &\geq -|\mathrm{Ric}_{\omega_\epsilon} - \lambda\omega_\epsilon| + \lambda - A\mathrm{tr}_{\omega_\epsilon}\eta_X.\end{aligned} \tag{39}$$

We also have

$$\Delta_{\omega_\epsilon}(-u_\epsilon) = \mathrm{tr}_{\omega_\epsilon}\eta_\epsilon - n \geq \mathrm{tr}_{\omega_\epsilon}\eta_X - n \geq C_1^{-1}\mathrm{tr}_{\omega_\epsilon}\eta_{FS} - n, \tag{40}$$

for some $C_1 > 0$, using that locally both η_{FS} and η_X are given by pullbacks of smooth metrics under embeddings of X. This implies that

$$\begin{aligned}\Delta_{\omega_\epsilon}(\log \mathrm{tr}_{\omega_\epsilon}\eta_{FS} - AC_1u_\epsilon) &\geq -|\mathrm{Ric}_{\omega_\epsilon} - \lambda\omega_\epsilon| + \lambda - AC_1n \\ &\geq -|\mathrm{Ric}_{\omega_\epsilon} - \lambda\omega_\epsilon| - C_2,\end{aligned} \tag{41}$$

for some $C_2 > 0$. Let us define

$$F = \max\{0, \log \mathrm{tr}_{\omega_\epsilon}\eta_{FS} - AC_1u_\epsilon\}. \tag{42}$$

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Since ω_ϵ is a Kähler metric, F is bounded from above, and by definition F is also bounded below. In addition F satisfies the differential inequality

$$\Delta_{\omega_\epsilon} F \geq -|\mathrm{Ric}_{\omega_\epsilon} - \lambda\omega_\epsilon| - C_2 \tag{43}$$

in a distributional sense on all of Y. To see this, note first that the differential inequality is satisfied in the distributional sense on $Y \setminus E$ by the definition of F as a maximum of two functions satisfying the inequality. Then the differential inequality can be extended across E using that F is bounded, by an argument similar to Lemma 11.

Fix $x \in Y \setminus E$, and let $H(x, y, t)$ denote the heat kernel on (Y, ω_ϵ). Fix some $t_0 > 0$. For $t \in [t_0, 1]$ we have

$$\begin{aligned} \partial_t \int_Y F(y)\, H(x, y, t)\, dy &= \int_Y F(y)\, \Delta_y H(x, y, t)\, dy \\ &= \int_Y \Delta_y F(y)\, H(x, y, t)\, dy \\ &\geq \int_Y (-|\mathrm{Ric}_{\omega_\epsilon} - \lambda\omega_\epsilon|(y) - C_2) H(x, y, t)\, dy. \end{aligned} \tag{44}$$

Using the uniform upper bound for H (see Theorem 13), together with Proposition 14, we find that there exists an $\epsilon_0 = \epsilon_0(t_0)$, depending on t_0, such that if $\epsilon < \epsilon_0$, then

$$\partial_t \int_Y F(y)\, H(x, y, t)\, dy \geq -2C_2, \tag{45}$$

and so for $\epsilon < \epsilon_0$ we have

$$\int_Y F(y) H(x, y, t_0)\, dy \leq \int_Y F(y) H(x, y, 1)\, dy + 2C_2. \tag{46}$$

Note that

$$F \leq e^{-AC_1 u_\epsilon} \mathrm{tr}_{\omega_\epsilon} \eta_{FS}, \tag{47}$$

so we have (using the uniform upper bound for the heat kernel as well),

$$\begin{aligned} \int_Y F(y) H(x, y, t_0)\, dy &\leq C_3 e^{AC_1 \sup |u_\epsilon|} \int_Y \mathrm{tr}_{\omega_\epsilon} \eta_{FS}\ \omega_\epsilon^n + 2C_2 \\ &\leq C_4. \end{aligned} \tag{48}$$

Here we also used that we have a uniform bound for $\sup |u_\epsilon|$, and the cohomology classes $[\omega_\epsilon]$ are uniformly bounded. Crucially, the constant C_4 is independent of t_0.

Note that as $\epsilon \to 0$, the heat kernels $H(x, y, t)$ for (Y, ω_ϵ) converge locally smoothly on $Y \setminus E$ to the heat kernel for (X, ω_{KE}). At the same time, the function $F(y)$ converges locally uniformly on $Y \setminus E$ to

$$\max\{0, \log \mathrm{tr}_{\omega_{KE}} \eta_{FS} - AC_1 u_{KE}\}. \tag{49}$$

It follows that in the limit, for any $t > 0$, we have

$$\int_{X^{reg}} (\log \mathrm{tr}_{\omega_{KE}} \eta_{FS} - AC_1 u_{KE})(y)\, H_{\omega_{KE}}(x, y, t)\, \omega_{KE}^n(y) \leq C_4. \tag{50}$$

Letting $t \to 0$ we obtain a pointwise bound $\mathrm{tr}_{\omega_{KE}} \eta_{FS} < C_5$, as required. $\square$

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4. Homeomorphism with the underlying variety

In this section our goal is to show that the metric completion $\hat{X}$ of the smooth locus of a singular Kähler-Einstein metric (X, ω_{KE}) is homeomorphic to X, under suitable assumptions. These assumptions hold in the setting of Theorem 4, where (X, ω_{KE}) can be approximated with cscK metrics.

We assume that X is a normal projective variety of dimension n, and we have a Kähler current ω on X with bounded local potentials, such that $\omega \in c_1(L)$ for a line bundle L on X. We will write η_{FS} for the pullback of the Fubini-Study metric to X under a projective embedding. We make the following assumptions:

(1) The Ricci form of ω, as a current, satisfies $\mathrm{Ric}(\omega) = \lambda\omega$ for a constant $\lambda \in \mathbb{R}$ on the regular part X^{reg} of X.
(2) ω is a Kähler current, that is, $\omega \geq c\eta_{FS}$ on X for some $c > 0$.
(3) The metric completion $(\hat{X}, d_{\hat{X}})$ of (X^{reg}, ω) is a noncollapsed $RCD(2n, \lambda)$ space, where the measure on $\hat{X}$ is the pushforward of ω^n from X^{reg}.
(4) We have $\omega^n = F\eta_{FS}^n$, where $F \in L^p(X, \eta_{FS}^n)$ for some $p > 1$.

We have seen that Conditions (1)–(3) are satisfied for singular Kähler-Einstein metrics (X, ω_{KE}), with $\omega_{KE} \in c_1(L)$, that can be approximated with cscK metrics in the sense of Definition 2. For Condition (4), see Eyssidieux-Guedj-Zeriahi [29, Section 7].

The main result of this section is the following, and the proof will be completed after Proposition 27 below.

Theorem 17. *Let (X, ω) satisfy the conditions (1)–(4) above. Then the metric completion $\hat{X}$ is homeomorphic to X.*

Rescaling the metric ω we can assume that L is a very ample line bundle on X. The sections of L define a holomorphic embedding $\Phi_X : X \to \mathbb{CP}^N$, and we can identify the image of this embedding with X. By the assumption that ω is a Kähler current, we have that the map

$$\Phi_X : (X^{reg}, \omega) \to (X, \eta_{FS}) \subset \mathbb{CP}^N \tag{51}$$

is Lipschitz continuous, where we use the length metric as defined in (3). In particular Φ_X extends to a Lipschitz continuous map

$$\hat{\Phi}_X : \hat{X} \to (X, \eta_{FS}). \tag{52}$$

Note that $\hat{\Phi}_X$ is surjective, since the image of X^{reg} is dense in X, so our task is to prove that $\hat{\Phi}_X$ is injective, that is, to show that the sections of L separate points of $\hat{X}$. In fact we will work with L^k for large k, however since L is very ample, the map defined by section of L^k is obtained by composing the map defined by sections of L with an embedding of $\mathbb{CP}^N$ into a larger projective space.

The general strategy for showing that sections of L^k separate points of $\hat{X}$ is similar to the work of Donaldson-Sun [27]. We will apply the following form of Hörmander's estimate (see, e.g., [25, Theorem 6.1]):

Theorem 18. *Let (P, h_P) be a Hermitian holomorphic line bundle on a Kähler manifold (M, ω_M), which admits some complete Kähler metric. Suppose that the curvature form of h_P satisfies $\sqrt{-1}F_{h_P} \geq c\omega_M$ for some constant $c > 0$. Let $\alpha \in \Omega^{n,1}(P)$ be such that $\bar{\partial}\alpha = 0$. Then there exists $u \in \Omega^{n,0}(P)$ such that $\bar{\partial}u = \alpha$, and*

$$\|u\|_{L^2}^2 \leq \frac{1}{c}\|\alpha\|_{L^2}^2, \tag{53}$$

provided the right hand side is finite.

We will apply this result to $M = X^{reg}$, with the metric $\omega_M = k\omega$. Note that it follows from Demailly [24, Theorem 0.2], that X^{reg} admits a complete Kähler metric. For the line bundle P we will take

$P = L^k \otimes K_M^{-1}$, so that an $(n,0)$-form valued in P is simply a section of L^k. For the metric on P we take the metric induced by the metric h^k on L^k whose curvature is $k\omega$, together with the metric given by ω^n on K_M. The curvature of h_P then satisfies

$$\sqrt{-1}F_{h_P} = k\omega + \mathrm{Ric}_\omega = (k+\lambda)\omega > \frac{1}{2}\omega_M, \tag{54}$$

for large enough k.

We will need the following L^∞ and gradient estimates for holomorphic sections of L^k.

Proposition 19. *Let f be a holomorphic section of L^k over $M = X^{reg}$. We then have the following estimates*

$$\sup_M |f|_{h^k} + |\nabla f|_{h^k,\omega_M} \leq K_1 \|f\|_{L^2(M,h^k,\omega_M)}, \tag{55}$$

where we emphasize that we are using the metrics h^k and $\omega_M = k\omega$ to measure the various norms, and K_1 does not depend on k.

Proof. Note first that f extends to a holomorphic section of L^k over X, using that X is normal. Using that ω has locally bounded potentials, we have that $\sup_X |f|_{h^k} < \infty$.

Next we show that $|\nabla f|_{h^k,\omega_M} < \infty$. For any $\hat{x} \in \hat{X}$, let $x = \hat{\Phi}_X(\hat{x}) \in X$. We can find a section $s \in H^0(X,L)$ and some $r > 0$ such that $s(y) \neq 0$ for $y \in B_{\eta_{FS}}(x,r)$. The assumption that ω is a Kähler current implies that we have constants $r' > 0$ and $C > 0$ (depending on $\hat{x}$) such that if we write $|s|_h^2 = e^{-u}$, then $|u| < C$ on $X^{reg} \cap B_{\omega_M}(\hat{x},r')$. We have $\Delta_{\omega_M} u = n$ on $X^{reg} \cap B_{\omega_M}(\hat{x},r'/2)$, and since u is bounded, this equation extends to $B_{\omega_M}(\hat{x},r'/2)$ by Lemma 11 and Lemma 6. The gradient estimate in Proposition 12 then implies that $|\nabla u| < C_1$ on $B_{\omega_M}(\hat{x},r'/2)$. This implies that $|\nabla s| < C_2$ on $B_{\omega_M}(\hat{x},r'/2)$. If f is any holomorphic section of L^k, then on $B_{\omega_M}(\hat{x},r'/2)$ the ratio f/s^k is a bounded harmonic function, so using the gradient estimate again, together with the bounds for s, we find that $|\nabla f| < C_3$ on $B_{\omega_M}(\hat{x},r'/4)$. We can cover $\hat{X}$ with finitely many balls of this type, showing that $|\nabla f|_{h^k,\omega_M} < \infty$ globally.

We can obtain the effective estimates claimed in the proposition as follows. Since the curvature of h^k is ω_M, on M we have

$$\Delta_{\omega_M}|f|^2_{h^k} = |\nabla f|^2_{h^k,\omega_M} - n|f|^2_{h^k}. \tag{56}$$

Let ϕ_i denote cutoff functions as in Lemma 6. We have, omitting the subscripts,

$$\begin{aligned}\int_M \phi_i^2|\nabla f|^2\omega_M^n &= \int_M \phi_i^2(\Delta|f|^2 + n|f|^2)\,\omega_M^n \\ &= \int_M (-4\phi_i|f|\nabla\phi_i\cdot\nabla|f| + \phi_i^2 n|f|^2)\,\omega_M^n \\ &\leq \int_M \left(\frac{1}{2}\phi_i^2|\nabla f|^2 + 8|\nabla\phi_i|^2|f|^2 + \phi_i^2 n|f|^2\right)\omega_M^n.\end{aligned} \tag{57}$$

Letting $i \to \infty$, and using that $|f| \in L^\infty$, we get

$$\int_M |\nabla f|^2\,\omega_M^n \leq 2n\int_M |f|^2\,\omega_M^n. \tag{58}$$

We also have the following Bochner-type formula on M (see, e.g., La Nave-Tian-Zhang [39, Lemma 3.1]):

$$\Delta|\nabla f|^2 \geq \mathrm{Ric}_{\omega_M}(\nabla f,\nabla f) - (n+2)|\nabla f|^2 \geq -(n+2+|\lambda|)|\nabla f|^2, \tag{59}$$

where we are using the metrics h^k, ω_M as above.

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Both (56) and (59) are of the form

$$\Delta v \geq -Av, \tag{60}$$

where v is a smooth L^∞ function on M. We can argue using the cutoff functions ϕ_i, as in the proof of Lemma 11, to show that v satisfies this differential inequality on all of $\hat{X}$ in a weak sense, that is, for any Lipschitz test function $\rho \geq 0$ we have

$$\int_M (-\nabla\rho \cdot \nabla v + A\rho v)\, \omega_M^n \geq 0. \tag{61}$$

Using this, together with estimates for the heat kernel on $\hat{X}$, we can obtain the required L^∞ bound for $v = |f|^2$ and $v = |\nabla f|^2$. More precisely, using [38, Theorem 1.2], together with the RCD property in Proposition 15, we obtain an L^2-bound for the heat kernel $H(x, y, 1)$ on M, independently of k. Using (60), for any $x \in M$ we have

$$\frac{d}{dt}\int_M v(y)H(x,y,t)\,\omega_M^n(y) = \int_M v(y)\Delta_y H(x,y,t)\,\omega_M^n(y) \geq -A\int_M v(y)H(x,y,t)\,\omega_M^n(y), \tag{62}$$

so

$$v(x) \leq e^A \int_M v(y)H(x,y,1)\,\omega_M^n(y) \leq e^A C\|v\|_{L^2}, \tag{63}$$

as required. □

In order to show that sections of L^k separate points of $\hat{X}$ for large k (and therefore also for $k = 1$), we follow the approach of Donaldson-Sun [27], constructing suitable sections of L^k using Hörmander's L^2-estimate. For this the basic ingredient in [27] is to consider a tangent cone Z of $\hat{X}$ at x, and use that the regular part of Z is a Kähler cone, while at the same time the singular set can be excised by a suitable cutoff function. The main new difficulty in our setting is that along the pointed convergence of a sequence of rescalings

$$(\hat{X}, \lambda_i d_{\hat{X}}, x) \to (Z, d_Z, o), \tag{64}$$

with $\lambda_i \to \infty$, we do not know that compact subsets $K \subset Z^{reg}$ of the (metric) regular set in Z are obtained as smooth limits of subsets of the (complex analytic) regular set X^{reg}. For example, a priori it may happen that along the convergence in (64), even if $Z = \mathbb{R}^{2n}$, the singular set $X \setminus X^{reg}$ converges to a dense subset of Z. This is similar to the issue dealt with in Chen-Donaldson-Sun [17], but in that work it is used crucially that the singular spaces considered are limits of smooth manifolds with lower Ricci bounds.

To deal with this issue in our setting, we exploit the fact that $X \setminus X^{reg}$ is locally contained in the zero set of holomorphic functions, which also define harmonic functions on the RCD space $\hat{X}$. Crucially, these functions have a bound on their order of vanishing (Lemma 20), which can be used to control the size of the zero set at different scales, at least on balls that are sufficiently close to a Euclidean ball. This can be used to show that balls in $\hat{X}$ that are almost Euclidean are contained in X^{reg} (Proposition 24). This is the main new ingredient in our argument. Given this, we can closely follow the arguments in Donaldson-Sun [27] or [42] to construct holomorphic sections of L^k.

Let us write $\Gamma = X \setminus X^{reg}$ for the algebraic singular set. Observe that Γ can locally be cut out by holomorphic functions. Therefore, we can cover X with open sets U'_k and we have nonzero holomorphic functions s_k on U'_k such that $\Gamma \cap U'_k \subset s_k^{-1}(0)$. We can assume that the s_k are bounded, and that we have relatively compact open sets $U_k \subset\subset U'_k$ that still cover X. We let $\hat{U}_k, \hat{U}'_k$ be the corresponding open sets pulled back to $\hat{X}$. Using Lemma 6, we can extend the s_k to complex valued harmonic functions on $\hat{X}$, which vanish along Γ. Our first task will be to show that we have a bound for the order of vanishing of

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the s_k at each point. Note first that by the assumption that ω is a Kähler current, there exists an $r_0 > 0$ such that if $p \in \hat{U}_k$, then $B(p, r_0) \subset \hat{U}'_k$. Here, and below, a ball $B(p, r)$ always denotes the metric ball using the metric $d_{\hat{X}}$ on $\hat{X}$ induced by d_ω on X^{reg}.

Lemma 20. *There are constant $c_1, N > 0$, depending on (X, ω), such that for any $\hat{x} \in \hat{U}_k$ and $r \in (0, r_0)$, we have*

$$\int_{B(\hat{x},r)} |s_k|^2 \, \omega^n \geq c_1 r^N, \tag{65}$$

for all $r < r_0$.

Proof. First note that since $\hat{X}$ is a noncollapsed RCD space, we have a constant $\nu > 0$ such that $\mathrm{vol}\, B(\hat{x}, r) > \nu r^{2n}$ for all $r < 1$. At the same time we can bound the volume of sublevel sets $U'_k \cap \{|s_k| < t\}$ from above, using the assumptions on ω. Indeed, on U'_k we have $\omega^n = F\eta^n_{FS}$, and $F \in L^p(X, \eta^n_{FS})$ for some $p > 1$. It follows that for any $t > 0$ we have

$$\begin{aligned}
\mathrm{vol}(U'_k \cap \{|s_k| < t\}, \omega^n) &= \int_{U'_k \cap \{|s_k|<t\}} \omega^n \\
&= \int_{U'_k \cap \{|s_k|<t\}} F\, \eta^n_{FS} \\
&\leq C_1 \mathrm{vol}(U'_k \cap \{|s_k| < t\}, \eta^n_{FS})^{1/p'} \left(\int_{U'_k} F^p \, \eta^n_{FS} \right)^{1/p} \\
&\leq C_2 \mathrm{vol}(U'_k \cap \{|s_k| < t\}, \eta^n_{FS})^{1/p'},
\end{aligned} \tag{66}$$

for suitable constants C_1, C_2 independent of t, and p' is the conjugate exponent of p. Since $|s_k|^{-\epsilon} \, \eta^n_{FS}$ is integrable for some $\epsilon > 0$, it follows that we have a bound

$$\mathrm{vol}(U'_k \cap \{|s_k| < t\}, \eta^n_{FS}) \leq C_3 t^\epsilon, \tag{67}$$

and so in sum we have

$$\mathrm{vol}(U'_k \cap \{|s_k| < t\}, \omega^n) \leq C_4 t^\alpha, \tag{68}$$

for some $C_4, \alpha > 0$ independent of t. Given a small $r > 0$ such that $B(\hat{x}, r) \subset \hat{U}'_k$, choose t_r such that

$$C_4 t_r^\alpha = \frac{1}{2} \nu r^{2n}, \tag{69}$$

that is,

$$t_r = \left(\frac{\nu}{2C_4} \right)^{1/\alpha} r^{2n/\alpha} = c_5 r^{2n\alpha^{-1}}, \tag{70}$$

for suitable $c_5 > 0$. By our estimates for the volumes, we then have

$$\mathrm{vol}(B(\hat{x}, r) \cap \{|\hat{s}_k| \geq t_r\}) \geq \frac{1}{2} \nu r^{2n}, \tag{71}$$

and so

$$\int_{B(\hat{x},r)} |\hat{s}_k|^2 \, \omega^n \geq \frac{c_5^2 r^{4n\alpha^{-1}}}{2} \nu r^{2n} = c_1 r^N, \tag{72}$$

for some $c_1, N > 0$, independent of r, as required. □

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Next we need a version of the three annulus lemma for almost Euclidean balls, similar to [26, Theorem 0.7].

Lemma 21. *For any $\mu > 0$, $\mu \notin \mathbb{Z}$, there is an $\epsilon > 0$ depending on μ, n with the following property. Suppose that $B(p,1)$ is a unit ball in a noncollapsed $RCD(-1,2n)$-space such that*

$$d_{GH}(B(p,1), B(0_{\mathbb{R}^{2n}}, 1)) < \epsilon, \tag{73}$$

where $0_{\mathbb{R}^{2n}}$ denotes the origin in Euclidean space. Let $u : B(p,1) \to \mathbb{C}$ be a harmonic function such that

$$\left(⨍_{B(p,1/2)} |u|^2 \right)^{1/2} \geq 2^{\mu} \left(⨍_{B(p,1/4)} |u|^2 \right)^{1/2}. \tag{74}$$

Then

$$\left(⨍_{B(p,1)} |u|^2 \right)^{1/2} \geq 2^{\mu} \left(⨍_{B(p,1/2)} |u|^2 \right)^{1/2}. \tag{75}$$

Proof. The proof is by contradiction, similarly to [26], based on the fact that on the Euclidean space $\mathbb{R}^{2n}$ every homogeneous harmonic function has integer degree. □

Combining the previous two results, we have the following, controlling the decay rate of the defining functions $\hat{s}_k$ around almost regular points.

Lemma 22. *There exists an $\epsilon_0, r_0 > 0$, depending on (X, ω), such that if $\hat{x} \in \hat{U}_k$ and for some $r_1 \in (0, r_0)$ we have*

$$d_{GH}(B(\hat{x}, r_1), B(0_{\mathbb{R}^{2n}}, r_1)) < r_1\epsilon_0, \tag{76}$$

then

$$\limsup_{r\to 0} \frac{⨍_{B(\hat{x},r)} |\hat{s}_k|^2\,\omega^n}{⨍_{B(\hat{x},r/2)} |\hat{s}_k|^2\,\omega^n} \leq 2^{2N}, \tag{77}$$

for the N in Lemma 20.

Proof. Fix $\mu \in (N/2, N)$ such that $\mu \notin \mathbb{Z}$. If ϵ_0 and r_0 are sufficiently small (depending on μ), then the inequality (76) implies that for any $r \leq r_1$ we have

$$d_{GH}(B(\hat{x}, r), B(0_{\mathbb{R}^{2n}}, r)) < r\epsilon, \tag{78}$$

for the ϵ in Lemma 21, and so the conclusion of that Lemma holds. It follows that if

$$\left(⨍_{B(\hat{x},r)} |\hat{s}_k|^2\,\omega^n \right)^{1/2} \geq 2^{\mu} \left(⨍_{B(\hat{x},r/2)} |\hat{s}_k|^2\,\omega^n \right)^{1/2}, \tag{79}$$

for some $r \leq r_1$, then applying Lemma 21 inductively, we have

$$\left(⨍_{B(\hat{x},2^j r)} |\hat{s}_k|^2\,\omega^n \right)^{1/2} \geq 2^{j\mu} \left(⨍_{B(\hat{x},r/2)} |\hat{s}_k|^2\,\omega^n \right)^{1/2}, \tag{80}$$

as long as $2^j r \leq r_1$. Given any $r \leq r_1$, if we let $\bar{j}$ denote the largest j such that $2^j r \leq r_1$, then we obtain

$$\left(⨍_{B(\hat{x},r/2)} |\hat{s}_k|^2\,\omega^n \right)^{1/2} \leq 2^{-\bar{j}\mu} C, \tag{81}$$

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where C is independent of r, but depends on the L^2-norm of $\hat{s}_k$ on $B(\hat{x}, r_1)$. Applying Lemma 20 we then have

$$2^{-\bar{j}\mu} C \geq c_1^{1/2} (r/2)^{N/2}. \tag{82}$$

Since $2^{\bar{j}+1} r > r_1$, it follows that $2^{-\bar{j}\mu} < (2r/r_1)^{\mu}$, so

$$c_1^{1/2} (r/2)^{N/2} < (2r/r_1)^{\mu} C. \tag{83}$$

Since $\mu > N/2$, this inequality implies a lower bound for r satisfying (79). The required conclusion (77) follows. □

Using this result, we will show that almost Euclidean balls are contained in the complex analytically regular set $X^{reg} \subset \hat{X}$. Note that the assumption (85) will hold on sufficiently small balls around a given point, by the previous lemma.

Proposition 23. *There exists an $\epsilon_2 > 0$, depending on (X, ω), with the following property. Suppose that $\hat{x} \in \hat{U}_j$ and $k > 0$ is a large integer such that $\epsilon_2^{-1} k^{-2} < \epsilon_2$. Suppose in addition that*

$$d_{GH}\Big(B(\hat{x}, \epsilon_2^{-1} k^{-2}), B_{\mathbb{R}^{2n}}(0, \epsilon_2^{-1} k^{-2})\Big) < \epsilon_2 k^{-2}, \tag{84}$$

and that

$$\fint_{B(\hat{x}, \epsilon_2^{-1} k^{-2})} |\hat{s}_j|^2 \,\omega^n \leq 2^{2N} \fint_{B(\hat{x}, \frac{1}{2}\epsilon_2^{-1} k^{-2})} |\hat{s}_j|^2 \,\omega^n, \tag{85}$$

for the N in Lemma 20. Then $\hat{x} \in X^{reg}$, where X^{reg} is the complex analytically regular set of X, viewed as a subset of $\hat{X}$.

Proof. We will argue by contradiction, similarly to [42, Proposition 3.1] which in turn is based on Donaldson-Sun [27]. Suppose that no suitable ϵ_2 exists. Then we have a sequence of points $\hat{x}_i$, and integers $k_i > i$ such that the hypotheses are satisfied (with $\epsilon_2 = 1/i$). We will show that for sufficiently large i we have $\hat{x}_i \in X^{reg}$ by constructing holomorphic coordinates in a neighborhood of $\hat{x}_i$.

By a slight abuse of notation we will write $\hat{U}_i, \hat{s}_i$ instead of $\hat{U}_{j_i}$ and $\hat{s}_{j_i}$ to simplify the notation. The assumptions imply that the rescaled balls

$$k_i^{1/2} B(\hat{x}_i, i k_i^{-1/2}) \to \mathbb{R}^{2n}, \tag{86}$$

in the pointed Gromov-Hausdorff sense. Using Lemma 21 together with the condition (85), we can extract a nontrivial limit of the normalized functions

$$\tilde{s}_i = \frac{\hat{s}_i}{\left(\fint_{B(\hat{x}_i, k_i^{-1/2})} |\hat{s}_i|^2 \,\omega^n\right)^{1/2}}. \tag{87}$$

Indeed, we have

$$\fint_{B(\hat{x}_i, k_i^{-1/2})} |\tilde{s}_i|^2 \,\omega^n = 1, \tag{88}$$

and using Lemma 21 with some $\mu \in (N, 2N)$, together with (85), implies that for sufficiently large i we have

$$\fint_{B(\hat{x}_i, 2^{-j} \epsilon_2^{-1} k_i^{-1/2})} |\tilde{s}_i|^2 \,\omega^n \leq 2^{4N} \fint_{B(\hat{x}_i, 2^{-j-1} \epsilon_2^{-1} k_i^{-1/2})} |\tilde{s}_i|^2 \,\omega^n, \tag{89}$$

for all $j \geq 0$. In particular, viewed as functions on the rescaled balls $k_i^{1/2} B(\hat{x}_i, i k_i^{-1/2})$, the L^2 norms of the $\tilde{s}_i$ are bounded independently of i on any R-ball. Using the gradient estimate, Proposition 12, it

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follows that up to choosing a subsequence, the functions $\tilde{s}_i$ converge locally uniformly to a harmonic function $\tilde{s}_\infty : \mathbb{R}^{2n} \to \mathbb{C}$. As a consequence, $\tilde{s}_\infty$ is smooth, and because of the normalization (88), $\tilde{s}_\infty$ is nonzero.

Note that if we take a sequence of rescalings of $\mathbb{R}^{2n}$ with factors going to infinity, and consider the corresponding pullbacks of $\tilde{s}_\infty$, normalized to have unit L^2-norm on the unit balls, then this new sequence of harmonic functions will converge to the leading order homogeneous piece of the Taylor expansion of $\tilde{s}_\infty$ at the origin (up to a constant factor). This means that in the procedure above, up to replacing the integers k_i by suitable larger integers, we can assume that the limit $\tilde{s}_\infty$ is in fact homogeneous.

Let us write $\Sigma = \tilde{s}_\infty^{-1}(0)$. Our next goal is to show that under the convergence in (86), the set $\mathbb{R}^{2n} \setminus \Sigma$ is the locally smooth limit of subsets of X^{reg}, and that $\tilde{s}_\infty$ is actually a holomorphic function under an identification $\mathbb{R}^{2n} = \mathbb{C}^n$. Then we will be able to follow the argument in the proof of [42, Proposition 3.1] with the cone $V = \mathbb{C}^n$, but treating Σ as the singular set.

Note that since $\tilde{s}_\infty$ is a nonzero harmonic function, the set $\mathbb{R}^{2n} \setminus \Sigma$ is open and dense in $\mathbb{R}^{2n}$. Suppose that $V \subset \mathbb{R}^{2n}$ is an open subset such that $\bar{V}$ is compact and $|\tilde{s}_\infty| > 0$ on $\bar{V}$. Then, because of the local uniform convergence of $\tilde{s}_i$ to $\tilde{s}_\infty$, and the fact that the sets $\tilde{s}_i \neq 0$ are contained in X^{reg}, it follows that we have open subsets $V_i \subset\subset k_i^{-2} B(\hat{x}_i, ik_i^{-1/2}) \cap X^{reg}$, which converge in the Gromov-Hausdorff sense to V. The metrics on the V_i are smooth noncollapsed Kähler-Einstein metrics, so using Anderson's ϵ-regularity result [2], up to choosing a subsequence, the complex structures on V_i converge to a complex structure on V with respect to which the Euclidean metric is Kähler. Note that we do not yet know that $\mathbb{R}^{2n} \setminus \Sigma$ is connected, and in principle we may get different complex structures on different connected components. Our next goal is to show that the Hausdorff dimension of Σ is at most $2n-2$, which will show that the complement of Σ is connected.

We can assume that the holomorphic functions $\tilde{s}_i$ on V_i converge to a holomorphic function $\tilde{s}_\infty$ on V. Writing $\tilde{s}_\infty = u_\infty + \sqrt{-1}v_\infty$, we therefore have $\langle \nabla u_\infty, \nabla v_\infty \rangle = 0$ and $|\nabla v_\infty| = |\nabla u_\infty|$ on $\mathbb{R}^{2n} \setminus \Sigma$, and by density these relations extend to all of $\mathbb{R}^{2n}$. We can assume that u_∞ is nonconstant. Let $\alpha > 2n-2$, and suppose that the Hausdorff measure $\mathcal{H}^\alpha(\Sigma) > 0$. By Caffarelli-Friedman [9] (see also Han-Lin [34]) we know that $\mathcal{H}^\alpha(\Sigma \cap |\nabla u_\infty|^{-1}(0)) = 0$, and so we can find an α-dimensional point of density q of $\Sigma \setminus |\nabla u_\infty|^{-1}(0)$. Since $\nabla u_\infty(q) \neq 0$, it follows that $\nabla v_\infty(q) \neq 0$ and $\langle \nabla v_\infty(q), \nabla u_\infty(q) \rangle = 0$. Therefore in a neighborhood of q the set Σ is a smooth $2n-2$-dimensional submanifold, contradicting that q is an α-dimensional point of density. In conclusion $\dim_{\mathcal{H}} \Sigma \leq 2n-2$, and so $\mathbb{R}^{2n-2} \setminus \Sigma$ is connected.

We can therefore assume that in the argument above the complex structure that we obtain on $\mathbb{R}^{2n} \setminus \Sigma$ agrees with the standard structure on $\mathbb{C}^n$, and $\tilde{s}_\infty$ is a holomorphic function on $\mathbb{C}^n \setminus \Sigma$, but since it is smooth, it is actually holomorphic on $\mathbb{C}^n$. In particular $\tilde{s}_\infty^{-1}(0)$ is a complex hypersurface defined by a homogeneous holomorphic function.

At this point we can closely follow the proof of [42, Proposition 3.1]), treating the zero set $\tilde{s}_\infty^{-1}(0)$ as the singular set Σ in [42]. The properties of the set Σ that are used are that the tubular ρ-neighborhood Σ_ρ satisfies the volume bounds $\mathrm{vol}(\Sigma_\rho \cap B(0,R)) \leq C_R \rho^2$, where the constant C_R in our setting could depend on $R, \tilde{s}_\infty$. In addition if $B(p, 2r) \in \mathbb{R}^{2n} \setminus \Sigma$, then $B(p,r)$ is the Gromov-Hausdorff limit of balls $B(p_i, r) \subset (M, k_i\omega)$ in Kähler-Einstein manifolds, and so by Anderson's result [2] we have good holomorphic charts on the $B(p_i, r)$ for sufficiently large i, analogous to those in [42, Theorem 1.4]. The rest of the proof is then identical to the argument in the proof of [42, Proposition 3.1] (see also Donaldson-Sun [27]) to show that for sufficiently large i we can construct holomorphic sections $s_0, \dots, s_n$ of $L^{k_i'}$ for suitable powers k_i', such that $\frac{s_1}{s_0}, \dots, \frac{s_n}{s_0}$ define a generically one-to-one map from a neighborhood of $x_i = \hat{\Phi}_X(\hat{x}_i)$ in X to a subset of $\mathbb{C}^n$. Since X is normal, it follows that the map is one-to-one, and so $x_i \in X^{reg}$. Therefore $\hat{x}_i \in X^{reg}$ as claimed. □

For any $\epsilon > 0$, let us define the ϵ-regular set $\mathcal{R}_\epsilon(Y)$ in a noncollapsed RCD space Y to be the set of points p that satisfy

$$\lim_{r \to 0} r^{-2n} \mathrm{vol}(B(p,r)) > \omega_{2n} - \epsilon, \tag{90}$$

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where ω_{2n} is the volume of the $2n$-dimensional Euclidean unit ball. Then $\mathcal{R}_\epsilon(Y)$ is an open set, and from the previous result we obtain the following.

Proposition 24. *There exists an $\epsilon_3 > 0$, depending on (X, ω), such that the ϵ_3-regular set $\mathcal{R}_{\epsilon_3}(\hat{X}) \subset \hat{X}$ coincides with the complex analytically regular set X^{reg}.*

Proof. It is clear that $X^{reg} \subset \mathcal{R}_{\epsilon_3}(\hat{X})$. To see the reverse inclusion, note that by Cheeger-Colding [12], and De Philippis-Gigli [23] in the setting of noncollapsed RCD spaces, given the $\epsilon_2 > 0$ in Proposition 23, there exists an $\epsilon_3 > 0$ such that if $\hat{x} \in \mathcal{R}_{\epsilon_3}$, then for all sufficiently large k (depending on $\hat{x}$), we have

$$d_{GH}\Big(B(\hat{x}, \epsilon_2^{-1}k^{-2}), B_{\mathbb{R}^{2n}}(0, \epsilon_2^{-1}k^{-2})\Big) < \epsilon_2 k^{-2}. \tag{91}$$

Using also Lemma 22 (and choosing ϵ_3 smaller if necessary), we have the growth estimate (85). Proposition 23 then implies that $\hat{x} \in X^{reg}$. □

This has the following immediate corollary.

Corollary 25. *There is an $\epsilon > 0$, depending on (X, ω), such that the ϵ-regular set $\mathcal{R}_\epsilon(\hat{X})$ coincides with the metric regular set of $\hat{X}$, that is, the points $\hat{x} \in \hat{X}$ where the tangent cone is $\mathbb{R}^{2n}$.*

Given these preliminaries, we have the following result, analogous to [42, Proposition 3.1] in our setting.

Proposition 26. *Let (V, o) be a metric cone, such that for any $\epsilon > 0$ the singular set $V \setminus \mathcal{R}_\epsilon(V)$ has zero capacity (in the sense of (3) in Definition 5). Let $\zeta > 0$. There are $K, \epsilon, C > 0$, depending on $\zeta, (X, \omega), V$ satisfying the following property. Suppose that k is a large integer such that $\epsilon^{-1}k^{-1/2} < \epsilon$ and for some $\hat{x} \in \hat{X}$*

$$d_{GH}\Big(B(\hat{x}, \epsilon^{-1}k^{-1/2}), B(o, \epsilon^{-1}k^{-1/2})\Big) < \epsilon k^{-1/2}. \tag{92}$$

Then for some $m < K$ the line bundle L^{mk} admits a holomorphic section s over $M = X^{reg} \setminus D$ such that $\|s\|_{L^2(h^{mk}, mk\omega)} < C$ and

$$\Big| |s(z)| - e^{-mkd(z,\hat{x})^2/2} \Big| < \zeta \tag{93}$$

for $z \in M$.

Given the results above, the argument is essentially the same as that in [42] (see also Donaldson-Sun [27]). One main difference is that in the setting of noncollapsed RCD spaces the sharp estimates of Cheeger-Jiang-Naber [14] do not yet seem to be available in the literature. However, the proof of [42, Proposition 3.1] applies under the assumption that for any $\epsilon > 0$ the singular set $\Sigma = V \setminus \mathcal{R}_\epsilon(V)$ has zero capacity.

We can rule out nonflat (iterated) tangent cones that split off a Euclidean factor of $\mathbb{R}^{2n-2}$, following the approach of Chen-Donaldson-Sun [16, Proposition 12] (see also [42, Proposition 3.2]).

Proposition 27. *Suppose that $\hat{x}_j \in \hat{X}$ and for a sequence of integers $k_j \to \infty$ the rescaled pointed sequence $(\hat{X}, k_j^2 d_{\hat{X}}, \hat{x}_j)$ converges to $\mathbb{R}^{2n-2} \times C(S^1_\gamma)$ in the pointed Gromov-Hausdorff sense. Here $C(S^1_\gamma)$ is the cone over a circle of length γ. Then $\gamma = 2\pi$, that is, $C(S^1_\gamma) = \mathbb{R}^2$.*

Proof. If $V = \mathbb{R}^{2n-2} \times C(S^1_\gamma)$, then the singular set of V has capacity zero, and so Proposition 26 can be applied. Then, as in [16, Proposition 12], it follows that for sufficiently large j, we can find a biholomorphism F_j from a neighborhood Ω_j of $\hat{x}_j$ to the unit ball $B(0,1) \subset \mathbb{C}^n$. In particular $B(\hat{x}_j, \frac{1}{2}k_j^{-2}) \subset X^{reg}$, and then the limit $\mathbb{R}^{2n-2} \times C(S^1_\gamma)$ of $(\hat{X}, k_j^2 d_{\hat{X}}, \hat{x}_j)$ must be smooth at the origin. Therefore $\gamma = 2\pi$. □

As a consequence of this result we can prove Theorem 17.

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Proof of Theorem 17. Using Propositions 10 and 27, and De Philippis-Gigli's dimension estimate [23] for the singular set (extending Cheeger-Colding [12]), it follows that the singular set of any iterated tangent cone of $\hat{X}$ has Hausdorff codimension at least 3. Using Proposition 24 we know that the singular set is closed, and so as in Donaldson-Sun [27, Proposition 3.5] we see that the singular set of any iterated tangent cone has capacity zero. In particular Proposition 26 can be applied to any (V, o) that arises as a rescaled limit of $\hat{X}$.

Suppose that $p \neq q$ are points in $\hat{X}$. Applying Proposition 26 to tangent cones at p, q, we can find sections s_p and s_q of some powers L^{m_p}, L^{m_q}, such that $|s_p(p)| > |s_p(q)|$, and $|s_q(q)| > |s_q(p)|$. Taking powers we find that the sections $s_p^{m_q}$ and $s_q^{m_p}$ of $L^{m_p m_q}$ separate the points p, q, and so the map $\hat{\Phi}_X$ is injective as required. □

To complete the proofs of Theorem 4, it remains to show the codimension bounds for the singular set of $\hat{X}$. By the dimension estimate of [23], it suffices to show the following. Note that this result would follow from a version of Cheeger-Colding-Tian [13, Theorem 9.1] for RCD spaces, but in our setting we can give a more direct proof.

Proposition 28. *In the setting of Theorem 4, suppose that a tangent cone $\hat{X}_p$ at $p \in \hat{X}$ splits off an isometric factor of $\mathbb{R}^{2n-3}$. Then $\hat{X}_p = \mathbb{R}^{2n}$. In particular in the stratification of the singular set of $\hat{X}$ we have $S_{2n-1} = S_{2n-4}$, and so $\dim_{\mathcal{H}} S \leq 2n-4$.*

Proof. Suppose that $\hat{X}$ has a tangent cone of the form $\hat{X}_p = C(Z) \times \mathbb{R}^{2n-3}$, where Z is two-dimensional. If Z had a singular point, necessarily with tangent cone $C(S^1_\gamma)$ for some $\gamma < 2\pi$, then $\hat{X}$ would have an iterated tangent cone of the form $\mathbb{R}^{2n-2} \times C(S^1_\gamma)$. This is ruled out by Proposition 27. Therefore Z is actually a smooth two-dimensional Einstein manifold with metric satisfying $\mathrm{Ric}(h) = h$. This implies that Z is the unit 2-sphere, and it follows that $\hat{X}_p = \mathbb{R}^{2n}$ so that p is a regular point. Therefore the singular set of $\hat{X}$ coincides with S_{2n-4}, as required. □

5. CscK approximations

In this section we will prove Theorem 3. Thus, let (X, ω_{KE}) be an n-dimensional singular Kähler-Einstein space, such that the automorphism group of X is discrete and $\omega_{KE} \in c_1(L)$ for an ample $\mathbb{Q}$-line bundle on X. On the regular part we have $\mathrm{Ric}(\omega_{KE}) = \lambda\omega_{KE}$ for a constant $\lambda \in \mathbb{R}$. We will assume that $\lambda \in \{0, -1, 1\}$. In the latter two cases we have $L = \pm K_X$. We first recall the properness of the Mabuchi K-energy in this singular setting. This has been well studied in the Fano setting (see Darvas [21] for example), but we were not able to find the corresponding much easier result in the literature for singular varieties in the case when $\lambda \leq 0$.

First recall the definitions of certain functionals (see Darvas [21] or Boucksom-Eyssidieux-Guedj-Zeriahi [5] for instance). We choose a smooth representative $\omega \in c_1(L)$. This means that $m\omega$ is the pullback of the Fubini-Study metric under an embedding using sections of L^m for large m. In general we define a function $f : U \to \mathbb{R}$ on an open set $U \subset X$ to be smooth, and write $f \in C^\infty(U)$, if it is the restriction of a smooth function under an embedding $U \subset \mathbb{C}^N$. We let

$$\begin{aligned} \mathcal{H}_\omega(X) &= \{u \in C^\infty(X) \,:\, \omega_u := \omega + \sqrt{-1}\partial\bar{\partial}u > 0\}, \\ PSH_\omega(X) &= \{u \in L^1(X) \,:\, \omega_u := \omega + \sqrt{-1}\partial\bar{\partial}u \geq 0\}. \end{aligned} \tag{94}$$

We define the $\mathcal{J}_\omega$ functional on $PSH_\omega(X) \cap L^\infty$ by setting $\mathcal{J}_\omega(0) = 0$ and the variation

$$\delta\mathcal{J}_\omega(u) = n \int_{X^{reg}} \delta u(\omega - \omega_u) \wedge \omega_u^{n-1}. \tag{95}$$

Let us choose a smooth metric h on K_X, that is, if σ is a local nonvanishing section of K_X^r, then the norm $|\sigma|^2_{h^r}$ is a smooth function. The adapted measure μ is defined using such local trivializing sections to be (see [29, Section 6.2])

$$\mu = (i^{rn^2}\sigma \wedge \bar{\sigma})^{1/r}|\sigma|_{h^r}^{-2/r} \text{ on } X^{reg}, \tag{96}$$

extended trivially to X. Recall that if X has klt singularities, then μ has finite total mass. Moreover, if $\pi : Y \to X$ is a resolution, and Ω is a smooth volume form on Y, then we have

$$\pi^*\mu = F\Omega \text{ on } \pi^{-1}(X^{reg}), \tag{97}$$

where $F \in L^p(\Omega)$ for some $p > 1$ (see [29, Lemma 6.4]). In our three cases $\lambda \in \{0, -1, 1\}$ we can choose the metric h in such a way that the curvature of h is given by $-\lambda\omega$ for the smooth metric ω.

We define the Mabuchi K-energy, for $u \in PSH_\omega(X) \cap L^\infty$, by

$$M_\omega(u) = \int_{X^{reg}} \log\left(\frac{\omega_u^n}{\mu}\right)\omega_u^n - \lambda\mathcal{J}_\omega(u). \tag{98}$$

The first term (the entropy) is defined to be ∞, unless $\omega_u^n = f\mu$ and $f\log f$ is integrable with respect to μ. We have the following result.

Proposition 29. *The functional M_ω is proper in the sense that there are constants $\delta, B > 0$ such that for all $u \in PSH_\omega(X) \cap L^\infty$ we have*

$$M_\omega(u) > \delta\mathcal{J}_\omega(u) - B. \tag{99}$$

Proof. The case when $\lambda = 1$ is well known, going back to Tian [56] in the smooth setting, who proved a weaker version of properness. The properness in the form (99) was shown by Phong-Song-Sturm-Weinkove [48]. In the singular setting the result was shown in Darvas [21, Theorem 2.2]. Note that we are assuming that X has discrete automorphism group and admits a Kähler-Einstein metric.

The cases $\lambda = 0, -1$ are much easier (see Tian [57] or Song-Weinkove [50, Theorem 1.2] for a similar result). For this, note that $\mathcal{J}_\omega \geq 0$, and so when $\lambda \leq 0$, we have

$$M_\omega(u) \geq \frac{1}{V}\int_{X^{reg}} \log\left(\frac{\omega_u^n}{\mu}\right)\omega_u^n. \tag{100}$$

At the same time, using Tian [54], we know that there are $\alpha, C_1 > 0$ such that for all $u \in PSH_\omega(X)$ with $\sup_X u = 0$ we have

$$\int_Y e^{-\alpha\pi^*u}\,\Omega < C_1, \tag{101}$$

and so with $p^{-1} + q^{-1} = 1$ (such that F in (97) is in L^p) we have

$$\begin{aligned}
\int_{X^{reg}} e^{-\alpha q^{-1}u}\,d\mu &= \int_{\pi^{-1}(X^{reg})} e^{-\alpha q^{-1}\pi^*u}\,\pi^*\mu \\
&= \int_{\pi^{-1}(X^{reg})} e^{-\alpha q^{-1}\pi^*u}\,F\Omega \\
&\leq \left(\int_{\pi^{-1}(X^{reg})} e^{-\alpha\pi^*u}\,\Omega\right)^{1/q}\left(\int_{\pi^{-1}(X^{reg})} F^p\Omega\right)^{1/p} \\
&\leq C_2.
\end{aligned} \tag{102}$$

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Using the convexity of the exponential function we then have, as in [50, Lemma 4.1],

$$\int_{X^{reg}} \log\left(\frac{\omega_u^n}{\mu}\right) \omega_u^n \geq \alpha q^{-1} \int_{X^{reg}} (-u)\, \omega_u^n - C_3, \tag{103}$$

for all $u \in PSH_\omega(X)$ with $\sup_X u = 0$. As the same time, if $\sup_X u = 0$ and $u \in L^\infty$, then we have

$$\int_{X^{reg}} (-u)\, \omega_u^n \geq \mathcal{J}_\omega(u). \tag{104}$$

To see this, note that

$$\begin{aligned}
\int_{X^{reg}} (-u)\, \omega_u^n &= \int_0^1 \frac{d}{dt} \int_{X^{reg}} (-tu)\, \omega_{tu}^n \, dt \\
&= \int_0^1 \int_{X^{reg}} (-u)\, \omega_{tu}^n - ntu\sqrt{-1}\partial\bar{\partial}u \wedge \omega_{tu}^{n-1}\, dt \\
&\geq \int_0^1 n \int_{X^{reg}} u(\omega - \omega_{tu}) \wedge \omega_{tu}^{n-1}\, dt \\
&= \int_0^1 \frac{d}{dt} \mathcal{J}_\omega(tu)\, dt = \mathcal{J}_\omega(u).
\end{aligned} \tag{105}$$

So combining the estimates above we obtain (99). □

Suppose that $\pi : Y \to X$ is a projective resolution such that the anticanonical bundle $-K_Y$ is relatively nef. Let us write E for the exceptional divisor. The relatively nef assumption implies (see Boucksom-Jonsson-Trusiani [6]), that we have a smooth volume form Ω on Y, whose Ricci form $\mathrm{Ric}(\Omega)$ satisfies

$$\mathrm{Ric}(\Omega) \geq -C\pi^*\omega \tag{106}$$

for suitable $C > 0$. Let us fix a smooth Kähler metric η_Y on Y, with volume form Ω, and we let $\eta_\epsilon = \pi^*\omega + \epsilon\eta_Y$, which is a smooth Kähler metric on Y. For any closed $(1,1)$-form α on Y, we define the functional $\mathcal{J}_{\eta_\epsilon,\alpha}$ on $PSH_{\eta_\epsilon}(Y) \cap L^\infty$ by letting $\mathcal{J}_{\eta_\epsilon,\alpha}(0) = 0$ and its variation

$$\delta\mathcal{J}_{\eta_\epsilon,\alpha}(u) = n\int_Y \delta u\big(\alpha - c_\alpha \eta_{\epsilon,u}\big) \wedge \eta_{\epsilon,u}^{n-1}. \tag{107}$$

Here c_α is the constant determined by

$$\int_Y \big(\alpha - c_\alpha\eta_{\epsilon,u}\big) \wedge \eta_{\epsilon,u}^{n-1} = 0, \tag{108}$$

and $\eta_{\epsilon,u} = \eta_\epsilon + \sqrt{-1}\partial\bar{\partial}u$.

We write $\mathcal{J}_{\eta_\epsilon} = \mathcal{J}_{\eta_\epsilon,\eta_\epsilon}$, which is consistent with the earlier definition. The twisted Mabuchi K-energy in the class $[\eta_\epsilon]$ is defined, for $u \in PSH_{\eta_\epsilon}(Y) \cap L^\infty$ by

$$M_{\eta_\epsilon,s}(u) = \int_Y \log\left(\frac{\eta_{\epsilon,u}^n}{\Omega}\right) \eta_{\epsilon,u}^n + \mathcal{J}_{\eta_\epsilon, s\eta_\epsilon - \mathrm{Ric}(\Omega)}. \tag{109}$$

Note that

$$M_{\eta_\epsilon,s}(u) \geq M_{\eta_\epsilon} := M_{\eta_\epsilon,0} \tag{110}$$

for $s \geq 0$. The critical points of this functional are the twisted cscK metrics $\eta_{\epsilon,u} \in [\eta_\epsilon]$, satisfying

$$R(\eta_{\epsilon,u}) - s\,\mathrm{tr}_{\eta_{\epsilon,u}}\eta_\epsilon = \text{const.} \tag{111}$$

The following result uses our assumption that $-K_Y$ is relatively nef.

Lemma 30. *Assuming that $-K_Y$ is relatively nef, there is a constant $C_2 > 0$ such that $\mathcal{J}_{\eta_\epsilon,-\mathrm{Ric}(\Omega)} \geq -C_2\mathcal{J}_{\eta_\epsilon}$ on $PSH_{\eta_\epsilon}(Y) \cap L^\infty$. In particular there are constants $s_0, \epsilon_0 > 0$ (depending on (X, ω_{KE})) such that for $s \geq s_0$ and $\epsilon < \epsilon_0$ the twisted K-energy is proper:*

$$M_{\eta_\epsilon,s}(u) \geq \mathcal{J}_{\eta_\epsilon}(u), \tag{112}$$

for all $u \in PSH_{\eta_\epsilon}(Y) \cap L^\infty$.

Proof. For $u \in PSH_{\eta_\epsilon}(Y) \cap L^\infty$ with $\sup_Y u = 0$, we have

$$\begin{aligned} -\mathcal{J}_{\eta_\epsilon,\mathrm{Ric}(\Omega)}(u) &= n\int_0^1\int_Y (-u)(\mathrm{Ric}(\Omega) - c\eta_{\epsilon,tu}) \wedge \eta_{\epsilon,tu}^{n-1} \\ &\geq -n\int_0^1\int_Y (-u)(C\pi^*\omega + c\eta_{\epsilon,tu}) \wedge \eta_{\epsilon,tu}^{n-1} \\ &\geq -C_1 n\int_0^1\int_Y (-u)(\eta_\epsilon + \eta_{\epsilon,tu}) \wedge \eta_{\epsilon,tu}^{n-1} \\ &\geq -C_2 J_{\eta_\epsilon}(u). \end{aligned} \tag{113}$$

Note that since the entropy term is nonnegative, we have $M_{\eta_\epsilon,s} \geq \mathcal{J}_{\eta_\epsilon,s\eta_\epsilon-\mathrm{Ric}(\Omega)}$ and also

$$\mathcal{J}_{\eta_\epsilon,s\eta_\epsilon-\mathrm{Ric}(\Omega)} = s\mathcal{J}_{\eta_\epsilon,\eta_\epsilon} - \mathcal{J}_{\eta_\epsilon,\mathrm{Ric}(\Omega)}. \tag{114}$$

It follows that for $s > C_2 + 1$,

$$M_{\eta_\epsilon,s}(u) \geq J_{\eta_\epsilon}(u). \tag{115}$$

□

It follows from this result, using the work of Chen-Cheng [15], that if $\epsilon < \epsilon_0$ and $s > s_0$, then there exists a twisted cscK metric $\eta_{\epsilon,u} \in [\eta_\epsilon]$ satisfying

$$R(\eta_{\epsilon,u}) - s\,\mathrm{tr}_{\eta_{\epsilon,u}}\eta_\epsilon = \text{const.} \tag{116}$$

We will use a continuity method to construct twisted cscK metrics in $[\eta_\epsilon]$ for sufficiently small ϵ, that satisfy (116) for $s \in [0, s_0]$, and so in particular we obtain a cscK metric in $[\eta_\epsilon]$. For this we will need a refinement of Chen-Cheng's estimates, which are uniform in the degenerating cohomology classes $[\eta_\epsilon]$ as $\epsilon \to 0$. Such a refinement was shown by Zheng [62] who worked in the more complicated setting of cscK metrics with cone singularities. See also Pan-Tô [47].

Note that in Zheng's work the cscK metrics are expressed relative to metrics with a fixed volume form, rather than metrics of the form η_ϵ. Let us write $\tilde{\eta_\epsilon} \in [\eta_\epsilon]$ for the metrics with $\tilde{\eta_\epsilon}^n = c_\epsilon\Omega$ provided by Yau [61], where the c_ϵ are bounded above and below uniformly. Note that we have $\tilde{\eta_\epsilon} = \eta_\epsilon + \sqrt{-1}\partial\bar{\partial}v_\epsilon$ with a uniform bound on $\sup|v_\epsilon|$, independent of ϵ, so it does not matter whether we obtain L^∞ bounds for potentials relative to η_ϵ or relative to $\tilde{\eta_\epsilon}$.

In order to state the estimates in a form that we will use, we make the following definition.

Definition 31. Fix an exhaustion $K_1 \subset K_2 \subset \ldots \subset \pi^{-1}(X^{reg})$ of $\pi^{-1}(X^{reg})$ by compact sets. Let $a_0, a_1, \ldots$ be a sequence of positive numbers, and $p > 1$. We say that a potential $u \in PSH_{\eta_\epsilon}(Y)$ is $\{p, a_j\}_{j\geq 0}$-bounded, if we have

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$$\left\|\frac{\eta_{\epsilon,u}^n}{\Omega}\right\|_{L^p(\Omega)} + \sup_Y |u| \leq a_0, \qquad \sup_{K_j}\left|\log\frac{\eta_{\epsilon,u}^n}{\Omega}\right| + \|u\|_{C^4(K_j,\eta_Y)} \leq a_j. \tag{117}$$

In other words such a potential is uniformly bounded globally, has volume form in L^p, is locally bounded in C^4, and its volume form is locally bounded above and below away from the exceptional divisor E.

We then have the following.

Proposition 32. *Suppose that $\epsilon \in (0, \epsilon_0)$, $s \in (0, s_0]$, and $\eta_{\epsilon,u} := \eta_\epsilon + \sqrt{-1}\partial\bar{\partial}u$ satisfies the twisted cscK equation*

$$R(\eta_{\epsilon,u}) - s\,\mathrm{tr}_{\eta_{\epsilon,u}}\eta_\epsilon = c_{s,\epsilon}, \tag{118}$$

where $c_{s,\epsilon}$ is a constant determined by s, ϵ through cohomological data. Assume that $\sup u = 0$. Let $\phi = \log|s_E|^2$, where s_E is a section of $\mathcal{O}(E)$ vanishing along E, and we are using a smooth metric on $\mathcal{O}(E)$ to compute the norm. There are constants $C, a > 0$, $p > 1$, depending on Y, η_Y, η_0, s_0, as well as on the entropy $\int_Y \log\left(\frac{\eta_{\epsilon,u}^n}{\Omega}\right)\eta_{\epsilon,u}^n$, but not on ϵ, s, such that we have the following estimates:

1.

$$\sup_Y\left(\log\frac{\eta_{\epsilon,u}^n}{\Omega} + a\phi\right) + \left\|\frac{\eta_{\epsilon,u}^n}{\Omega}\right\|_{L^p(\eta_Y)} + \sup_Y |u| < C, \tag{119}$$

2.

$$\inf_Y\left(\log\frac{\eta_{\epsilon,u}^n}{\Omega} - a\phi\right) > C, \tag{120}$$

3.

$$\|e^{a\phi}\mathrm{tr}_{\eta_Y}\eta_{\epsilon,u}\|_{L^q(\eta_Y)} < C, \text{ for any } q > 1. \tag{121}$$

In particular there exist $p > 1$ and $a_j > 0$ such that u is $\{p, a_j\}_{j\geq 0}$-bounded.

Proof. The estimates (1) are shown in [62, Proposition 5.12], the estimate (2) is in [62, Proposition 5.15], and the estimate (3) is [62, Proposition 5.18]. Since the notation in [62] is quite different, and they consider a more general situation including conical singularities along a divisor, we recall their setup. In [62, Section 5], the author considers the equations

$$\begin{aligned} F &= \log\frac{\eta_{\epsilon,u}^n}{\Omega}, \\ \Delta_{\eta_{\epsilon,u}}F &= \mathrm{tr}_{\eta_{\epsilon,u}}\Theta - c_{s,\epsilon}. \end{aligned} \tag{122}$$

Here Ω is a smooth volume form on Y as above, with Ricci curvature $\mathrm{Ric}(\Omega) = \theta$, and we define $\Theta = \theta - s\eta_\epsilon$. The coupled equations then imply

$$-R(\eta_{\epsilon,u}) + \mathrm{tr}_{\eta_{\epsilon,u}}\theta = \mathrm{tr}_{\eta_{\epsilon,u}}(\theta - s\eta_{\epsilon,u}) - c_{s,\epsilon}, \tag{123}$$

which is (118). Note that in [62] the resolution is called X and the singular variety is Y, which is the opposite of our notation. There is also an additional function f which we take to be zero. Zheng considers a semipositive form ω_{sr} on Y, which we can take to be $\pi^*\omega$, and a Kähler form ω_K on Y, which we take to be η_Y, so $\eta_\epsilon = \pi^*\omega + \epsilon\eta_Y$ is what Zheng calls ω_ϵ.

In order to deal with the degeneracy of η_ϵ as $\epsilon \to 0$, Zheng uses the technique of Tsuji [58], relying on the fact that if we choose a suitable smooth metric on the line bundle $\mathcal{O}(E)$ for a divisor supported

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on the exceptional set of π, then for any $a > 0$ the current $\pi^*\omega + a\sqrt{-1}\partial\bar\partial \log|s_E|^2$ dominates a Kähler form on Y, where s_E vanishes along E. We can assume that on $Y \setminus E$ we have

$$\eta_Y = \pi^*\omega + a\sqrt{-1}\partial\bar\partial \log|s_E|^2. \tag{124}$$

We apply [62, Proposition 5.12], to deduce the estimates (1). For this we need to check the condition that $e^{-\phi_l} \in L^{p_0}$ for some $p_0 > 1$, where ϕ_l in Zheng's notation is defined in his Lemma 5.6. Since in that Lemma ϕ_{θ_ϵ} is uniformly bounded, it is enough to check the integrability of $e^{-p_0\tilde\phi_l}$ where we define

$$\tilde\phi_l = (\inf_{(X,\eta_Y)} \Theta)(-a\log|s_E|^2). \tag{125}$$

We claim that the infimum $\inf_{(X,\eta_Y)}\Theta$ is bounded below, independently of the choice of small $a > 0$ (note that the choice of a affects the definition of η_Y and so also Θ). To see this, we use the condition $\theta = \mathrm{Ric}(\Omega) \geq -C\pi^*\omega$, so that we have

$$\begin{aligned}\Theta = \theta - s\eta_\epsilon &\geq -(C+s)\pi^*\omega - s\epsilon\eta_Y \\ &= -C'\eta_Y,\end{aligned} \tag{126}$$

for C' depending on C and s_0. Here we also used that if $a > 0$ is sufficiently small, then $\eta_Y > \frac{1}{2}\pi^*\omega_X$. It follows from this that

$$-\tilde\phi_l \leq -C'a\log|s_E|^2, \tag{127}$$

and so if $a > 0$ is sufficiently small, then $e^{-\tilde\phi_l} \in L^{p_0}$ for $p_0 > 1$, as required in Zheng's Proposition 5.12. The conclusion is the estimates (1). Note that the quantities that the estimate in Proposition 5.12 depends on are all uniformly bounded in s, ϵ in our setting. Similarly, Propositions 5.15 and 5.18 imply the estimates (2) and (3).

Note that the L^p-bound on the trace of $\eta_{\epsilon,u}$ implies higher order estimates for u on compact sets away from E, using Chen-Cheng's local estimate [15, Proposition 6.1]. This leads to the $\{p, a_j\}$-boundedness of u. See also [47, Theorem C] for similar estimates. □

Next we show that by Proposition 29, the Mabuchi energy M_{η_ϵ} is proper on $\{p, a_j\}$-bounded classes of potentials, when ϵ is sufficiently small.

Proposition 33. *Given $p > 1$ and a sequence $\{a_j\}_{j\geq 0}$, let $V \subset PSH_{\eta_\epsilon}(Y)$ denote the $\{p, a_j\}_{j\geq 0}$-bounded potentials. Then for sufficiently small ϵ, depending on the p, a_j, the K-energy M_{η_ϵ} is proper on V in the sense that*

$$M_{\eta_\epsilon}(u) > \delta\mathcal{J}_{\eta_\epsilon}(u) - B_2, \textit{ for all } u \in V. \tag{128}$$

Here δ is the same constant as in Proposition 29, while B_2 is a constant depending on (X, ω) and Ω, but not on the p, a_j.

Proof. We argue by contradiction. Suppose that we have a sequence $\epsilon_i \to 0$, and $u_i \in PSH_{\eta_{\epsilon_i}}(Y)$ that are $\{p, a_j\}_{j\geq 0}$-bounded, such that

$$M_{\eta_{\epsilon_i}}(u_i) \leq \delta\mathcal{J}_{\eta_\epsilon}(u) - B_2, \tag{129}$$

for B_2 to be determined below. Up to choosing a subsequence we can assume that $u_i \to u_\infty$ in L^1 and also in $C^{3,\alpha}$ on compact sets away from the exceptional divisor E. We have $u_\infty \in PSH_{\pi^*\omega}(Y)$, and we have an identification $PSH_{\pi^*\omega}(Y) = PSH_\omega(X)$. We will next show that in terms of F in (97) we have

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$$
\begin{aligned}
&M_{\eta_{\epsilon_i}}(u_i) \to M_\omega(u_\infty) + \int_Y \log F\, \eta_0^n, \\
&\mathcal{J}_{\eta_{\epsilon_i}}(u_i) \to \mathcal{J}_\omega(u_\infty).
\end{aligned} \tag{130}
$$

Let us first consider the relevant entropy terms. Note that

$$
\int_Y \log\left(\frac{\eta^n_{\epsilon_i,u_i}}{\Omega}\right)\eta^n_{\epsilon_i,u_i} = \int_Y \log\left(\frac{\eta^n_{\epsilon_i,u_i}}{\Omega}\right)\frac{\eta^n_{\epsilon_i,u_i}}{\Omega}\,\Omega. \tag{131}
$$

Our assumptions mean that the integrand has a uniform $L^p(\Omega)$-bound for some $p > 1$. Using this, and the $C^{3,\alpha}$-convergence $u_i \to u_\infty$ on compact sets away from E, it follows that

$$
\int_Y \log\left(\frac{\eta^n_{\epsilon_i,u_i}}{\Omega}\right)\eta^n_{\epsilon_i,u_i} \to \int_Y \log\left(\frac{\eta^n_{0,u_\infty}}{\Omega}\right)\eta^n_{0,u_\infty}. \tag{132}
$$

Using (97) we have

$$
\int_Y \log\left(\frac{\eta^n_{0,u_\infty}}{\Omega}\right)\eta^n_{0,u_\infty} = \int_{X^{reg}} \log\left(\frac{\omega^n_{u_\infty}}{\mu}\right)\omega^n_{u_\infty} + \int_Y \log F\,\eta^n_{0,u_\infty}. \tag{133}
$$

The last term can be computed by writing

$$
\begin{aligned}
\int_Y \log F\,\eta^n_{0,u_\infty} &= \int_Y \log F\,\eta_0^n + \int_0^1 \frac{d}{dt}\int_Y \log F\,\eta^n_{0,tu_\infty}\,dt \\
&= \int_Y \log F\,\eta_0^n + \int_0^1 n\int_Y u_\infty\sqrt{-1}\partial\bar\partial \log F\ \wedge \eta^{n-1}_{0,tu_\infty}\,dt \\
&= \int_Y \log F\,\eta_0^n + \int_0^1 n\int_Y u_\infty(\mathrm{Ric}(\Omega) - \mathrm{Ric}(\pi^*\mu)) \wedge \eta^{n-1}_{0,u_\infty}\,dt \\
&= \int_Y \log F\,\eta_0^n + \mathcal{J}_{\eta_0,\mathrm{Ric}(\Omega)}(u_\infty) - \lambda\mathcal{J}_\omega(u_\infty).
\end{aligned} \tag{134}
$$

For the last step note that η_0 vanishes along E, so although $\mathrm{Ric}(\pi^*\mu)$ has current contributions along E, the only part that survives in the integral is $\mathrm{Ric}(\mu) = \lambda\omega$ on X. In conclusion we have that

$$
\begin{aligned}
\int_Y \log\left(\frac{\eta^n_{\epsilon_i,u_i}}{\Omega}\right)\eta^n_{\epsilon_i,u_i} \to &\int_{X^{reg}} \log\left(\frac{\omega^n_{u_\infty}}{\mu}\right)\omega^n_{u_\infty} + \int_Y \log F\,\eta_0^n \\
&+ \mathcal{J}_{\eta_0,\mathrm{Ric}(\Omega)}(u_\infty) - \lambda\mathcal{J}_\omega(u_\infty).
\end{aligned} \tag{135}
$$

Next we consider the $\mathcal{J}$-functional terms. Consider a general smooth, closed (1,1)-form α on Y. We claim that we have $\mathcal{J}_{\eta_{\epsilon_i},\alpha}(u_i) \to \mathcal{J}_{\eta_0,\alpha}(u_\infty)$. Using the variational definition of $\mathcal{J}$, the local $C^{3,\alpha}$-convergence, and the uniform L^∞-bound for the u_i, it is enough to show that for every $\kappa > 0$ there is a compact set $K \subset Y \setminus E$, such that

$$
\int_{Y\setminus K} \eta_1 \wedge \eta^{n-1}_{\epsilon_i,u_i} + \int_{Y\setminus K}\eta^n_{\epsilon_i,u_i} < \kappa,\ \text{for all } i. \tag{136}
$$

To see this, let $h = -\log|s_E|^2$, where s_E is a section of the line bundle $\mathcal{O}(E)$ over Y vanishing along the exceptional divisor E, and we use a smooth metric on $\mathcal{O}(E)$. We have

$$
\sqrt{-1}\partial\bar\partial h = \chi - [E], \tag{137}
$$

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where χ is a smooth form on Y. We can assume that $h \geq 0$, and note that $h \to \infty$ along E. We show by induction that for each $k = 0, \dots, n$ there is a constant $C_k > 0$, independent of i, such that

$$\int_Y h\eta_1^{n-k} \wedge \eta_{\epsilon_i,u_i}^k \leq C_k. \tag{138}$$

For $k = 0$ this is clear since h has logarithmic singularities. Suppose that the bound has been established for a value of k. Then

$$\begin{aligned}\int_Y h\eta_1^{n-k-1} \wedge \eta_{\epsilon_i,u_i}^{k+1} &= \int_Y h\eta_1^{n-k-1} \wedge (\eta_{\epsilon_i} + \sqrt{-1}\partial\bar\partial u_i) \wedge \eta_{\epsilon_i,u_i}^k \\ &= \int_Y h\eta_1^{n-k-1} \wedge \eta_{\epsilon_i} \wedge \eta_{\epsilon_i,u_i}^k + \int u_i\sqrt{-1}\partial\bar\partial h \wedge \eta_1^{n-k-1} \wedge \eta_{\epsilon_i,u_i}^k \\ &\leq \int_Y h\eta_1^{n-k} \wedge \eta_{\epsilon_i,u_i}^k + \int_Y u_i\chi \wedge \eta_1^{n-k-1} \wedge \eta_{\epsilon_i,u_i}^k - \int_E u_i\eta_1^{n-k-1} \wedge \eta_{\epsilon_i,u_i}^k \\ &\leq C_k(1+C) - \int_E u_i\eta_1^{n-k-1} \wedge \eta_{\epsilon_i,u_i}^k \\ &\leq C_k(1+C) + C',\end{aligned} \tag{139}$$

where C, C' depend on χ and the uniform L^∞ bound for u_i.

Since $h \to \infty$ along E, it follows from (138) that for any $\kappa > 0$ we can find a compact set $K \subset Y \setminus E$ such that (136) holds. It follows that

$$\mathcal{J}_{\eta_{\epsilon_i,-\mathrm{Ric}(\Omega)}}(u_i) \to \mathcal{J}_{\eta_{0,-\mathrm{Ric}(\Omega)}}(u_\infty), \tag{140}$$

and also

$$\mathcal{J}_{\eta_{\epsilon_i}}(u_i) \to \mathcal{J}_\omega(u_\infty). \tag{141}$$

From this, together with (135), we have

$$\begin{aligned}M_{\eta_{\epsilon_i}}(u_i) &\to \int_{X^{reg}} \log\left(\frac{\omega_{u_\infty}^n}{\mu}\right)\omega_{u_\infty}^n - \lambda\mathcal{J}_\omega(u_\infty) + \int_Y \log F\,\eta_0^n \\ &= M_\omega(u_\infty) + \int_Y \log F\,\eta_0^n.\end{aligned} \tag{142}$$

From (129) we therefore get

$$M_\omega(u_\infty) + \int_Y \log F\,\eta_0^n \leq \delta\mathcal{J}_\omega(u_\infty) - B_2. \tag{143}$$

Choosing $B_2 = B - \int_Y \log F\,\eta_0^n$ for the B in Proposition 29, we get a contradiction. □

We are now ready to combine the different ingredients to prove the main result of this section.

Proof of Theorem 3. We will choose suitable $p > 0, a_j > 0$ shortly. By Proposition 33, for a given p, a_j we have some $\epsilon_1 > 0$ such that once $\epsilon < \epsilon_1$ and for any $s \geq 0$, we have

$$M_{\eta_\epsilon,s}(u) \geq M_{\eta_\epsilon}(u) > \delta\mathcal{J}_{\eta_\epsilon}(u) - B_2, \tag{144}$$

for $\{p.a_j\}$-bounded potentials u. Recall that δ, B_2 do not depend on $\{p, a_j\}$. For small $\kappa > 0$ we have

$$\begin{aligned} M_{\eta_\epsilon,s}(u) &\geq \kappa \int_Y \left(\frac{\eta^n_{\epsilon,u}}{\Omega}\right) \eta^n_{\epsilon,u} + \kappa \mathcal{J}_{\eta_\epsilon, s\eta_\epsilon - \mathrm{Ric}(\Omega)}(u) + (1-\kappa)\delta \mathcal{J}_{\eta_\epsilon}(u) - (1-\kappa)B_2 \\ &= \kappa \int_Y \left(\frac{\eta^n_{\epsilon,u}}{\Omega}\right) \eta^n_{\epsilon,u} + (\kappa s + (1-\kappa)\delta)\mathcal{J}_{\eta_\epsilon}(u) + \kappa \mathcal{J}_{\eta_\epsilon, -\mathrm{Ric}(\Omega)}(u) - (1-\kappa)B_2. \end{aligned} \tag{145}$$

If κ is chosen sufficiently small (depending on δ), then by Lemma 30 we find that

$$M_{\eta_\epsilon,s}(u) \geq \kappa \int_Y \log\left(\frac{\eta^n_{\epsilon,u}}{\Omega}\right) \eta^n_{\epsilon,u} - B_2. \tag{146}$$

We also have

$$M_{\eta_\epsilon,s}(0) = \int_Y \log\left(\frac{\eta^n_\epsilon}{\Omega}\right) \eta^n_\epsilon < C_3, \tag{147}$$

for a constant $C_3 > 0$ independent of ϵ. Since twisted cscK metrics minimize the twisted Mabuchi K-energy, it follows that if $\eta_{\epsilon,u} \in [\eta_\epsilon]$ is a twisted cscK metric, then we have $M_{\eta_\epsilon,s}(u) < C_3$. From (146) we get

$$\int_Y \log\left(\frac{\eta^n_{\epsilon,u}}{\Omega}\right) \eta^n_{\epsilon,u} \leq \kappa^{-1}(C_3 + B_2), \tag{148}$$

and in particular the entropy of $\eta_{\epsilon,u}$ is bounded independently of ϵ. We apply Proposition 32. As long as $s \leq s_0$, for the s_0 determined by Lemma 30, we find that if $\eta_{\epsilon,u} = \eta_\epsilon + \sqrt{-1}\partial\bar{\partial}u$ is a solution of the twisted cscK equation

$$R(\eta_{\epsilon,u}) - s\,\mathrm{tr}_{\eta_{\epsilon,u}}\eta_\epsilon = \text{const.}, \tag{149}$$

then u is $\{p, a_j\}$-bounded, for suitable p, a_j, determined by s_0 and the entropy bound (148). From now we fix this choice of p, a_j.

We can now use a continuity method to show that if $\epsilon < \epsilon_1$, for the ϵ_1 determined by $\{p, a_j\}$, for all $s \in [0, s_0]$ we can solve the twisted cscK equation (149). To see this, let us fix $\epsilon < \epsilon_1$, and set

$$S = \{s \in [0, s_0] \ : \ \text{the equation (149) has a solution}\}. \tag{150}$$

We have $s_0 \in S$, and it follows from the implicit function theorem that S is open. To see that it is closed, note that the twisted cscK metrics for $s \in S$ automatically satisfy the entropy bound (148). Using the main estimates of Chen-Cheng [15], we find that the potentials of the corresponding twisted cscK metrics satisfy a priori C^k-estimates, and the metrics are bounded below uniformly (these estimates depend on ϵ, but now ϵ is fixed). It follows that S is closed.

It follows that for sufficiently small $\epsilon > 0$ the classes $[\eta_\epsilon]$ on Y admit cscK metrics. The estimates required by Definition 2 follow from Proposition 32. □

Remark 34. To conclude this section we give an example where the assumption that $-K_Y$ is relatively nef is satisfied. Let M be a smooth Fano manifold, and suppose that P is a line bunde over M such that $P^r = -K_M$ for some $r > 0$. We let V denote the total space of P^{-1}, with the zero section blown down to a point o. Suppose that X has one isolated singularity p, and a neighborhood of p is isomorphic to the neighborhood of $o \in V$. In this case we can consider a resolution $\pi : Y \to X$, obtained by blowing up the singular point. Then

$$K_Y = \pi^* K_X + rE, \tag{151}$$

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where the exceptional divisor E isomorphic to M, and is in particular irreducible. It follows that in this case $-K_Y$ is relatively nef (in fact relatively ample). Note that this family of examples does not fit into the framework of admissible singularities studied by Li-Tian-Wang [40].

6. Partial C^0-estimate

An important result of Donaldson-Sun [27] is the partial C^0-estimate for smooth Kähler-Einstein manifolds, conjectured by Tian [55]. More precisely, suppose that (X, ω_{KE}) is a smooth Kähler-Einstein manifold, with $\omega_{KE} \in c_1(L)$ for an ample line bundle, and such that for some constant $D > 0$ we have

1. noncollapsing: $\mathrm{vol}\, B_{\omega_{KE}}(p, 1) > D^{-1}$ for a basepoint $p \in X$,
2. bounded volume: $\mathrm{vol}(X, \omega_{KE}) < D$,
3. bounded Ricci curvature: $\mathrm{Ric}(\omega_{KE}) = \lambda\omega_{KE}$ for $|\lambda| < D$.

For any integer $k > 0$ the density of states function $\rho_{k,\omega_{KE}}$ is defined by

$$\rho_{k,\omega_{KE}}(x) = \sum_j |s_j|^2(x), \tag{152}$$

where the s_j form an L^2-orthonormal basis of $H^0(X, L^k)$ in terms of the metric induced by $k\omega_{KE}$. Then, by Donaldson-Sun [27], there is a power $k_0 = k_0(n, D)$, and $b = b(n, D) > 0$, depending on the dimension and the constant D, such that $\rho_{k_0,\omega_{KE}} > b$. In this section we show the following extension of this result to singular Kähler-Einstein spaces that admit good cscK approximations.

Theorem 35. *Given $n, D > 0$ there are constants $k_0(n, D), b(n, D) > 0$ with the following property. Suppose that (X, ω_{KE}) is a singular Kähler-Einstein variety of dimension n, such that $\omega_{KE} \in c_1(L)$ for a line bundle L. Assume that (X, ω_{KE}) can be approximated by cscK metrics, and in addition the conditions (1), (2), (3) above hold. Then the corresponding density of states function satisfies $\rho_{k,\omega_{KE}} > b$.*

The proof of the result follows the same strategy as Donaldson-Sun [27], arguing by contradiction. We suppose that the sequence $(X_i, \omega_{KE,i})$ satisfies the bounds (1)–(3), but no fixed power L_i^k of the corresponding line bundles is very ample. The corresponding metric completions $\hat{X}_i$ are noncollapsed RCD spaces by Proposition 15, and we can pass to the Gromov-Hausdorff limit $\hat{X}_\infty$ along a subsequence. We would then like to use the structure of the tangent cones of $\hat{X}_\infty$ to construct suitable holomorphic sections of a suitable power L_i^k for large i, leading to a contradiction.

The difficulty in executing this strategy is that we do not have good control of the convergence of $\hat{X}_i$ to $\hat{X}_\infty$ on the regular set of $\hat{X}_\infty$, because in Corollary 25 the constant ϵ depends on the singular Kähler-Einstein space X that we are considering. As such it is a priori possible that the singular set of $\hat{X}_\infty$, consisting of points where the tangent cone is not given by $\mathbb{R}^{2n}$, is dense. In order to rule this out, we prove the following. Note that recently this result was shown in the more general algebraic setting by Xu-Zhuang [59] (see also Liu-Xu [43] for the three-dimensional case).

Theorem 36. *There is an $\epsilon > 0$, depending only on the dimension n, with the following property. Suppose that $\hat{X}$ is the metric completion of a singular Kähler-Einstein space as in Theorem 17, that is, one that can be approximated by cscK metrics. Let $(\hat{X}_p, o)$ be a tangent cone of $\hat{X}$, such that $\hat{X}_p \neq \mathbb{R}^{2n}$. Then*

$$\mathrm{vol} B(o, 1) < \omega_{2n} - \epsilon, \tag{153}$$

where ω_{2n} is the volume of the Euclidean unit ball in $\mathbb{R}^{2n}$.

Proof. We will argue by contradiction. If the stated result is not true, then we can find a sequence $\hat{X}_i$, and a sequence of singular points $p_i \in \hat{X}_i$ with tangent cones V_{p_i} such that $V_{p_i} \to \mathbb{R}^{2n}$ in the pointed Gromov-Hausdorff sense.

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We will prove a more general statement about almost smooth metric measure spaces in the sense of Definition 5, of any dimension, which satisfy the following conditions.

Definition 37. We say that an almost smooth metric measure space V satisfies Condition $(*)$ if the following conditions hold:

1. For some $\epsilon > 0$ (possibly depending on V), the ϵ-regular set $\mathcal{R}_\epsilon \subset V$, defined by (90), can be chosen to be the set Ω in Definition 5.
2. The Riemannian metric on Ω is Ricci flat.
3. If a tangent cone V' of V is of the form $C(S^1_\gamma) \times \mathbb{R}^{2n-2}$, then $V' = \mathbb{R}^{2n}$.

Note that by Propositions 24 and 27, the (iterated) tangent cones of the spaces $\hat{X}_i$ satisfy Condition $(*)$. Moreover, if a space $V = W \times \mathbb{R}^j$ satisfies Condition $(*)$, then so does W, and so do the tangent cones of V.

We argue by induction on the dimension to show that if a sequence of k-dimensional cones V_j satisfies Condition $(*)$, and $V_j \to \mathbb{R}^k$ in the pointed Gromov-Hausdorff sense, then $V_j = \mathbb{R}^k$ for sufficiently large i. For $k = 2$ this follows directly from Condition $(*)$.

Assuming $k > 2$, suppose first that for all sufficiently large j the cones V_j have smooth link (i.e., the singular set consists of only the vertex). In this case $V_j = C(Y_j)$, where the (Y_j, h_j) are $(k-1)$-dimensional smooth Einstein manifolds satisfying $\mathrm{Ric}(h_j) = (k-2)h_j$. Moreover the (Y_j, h_j) converge in the Gromov-Hausdorff sense to the unit $(k-1)$-sphere. As long as $k - 1 > 1$, it follows that for sufficiently large j we have $\mathrm{vol}(Y_j, h_j) = \mathrm{vol}(S^{k-1}, g_{S^{k-1}})$, using that Einstein metrics are critical points of the Einstein-Hilbert action. The Bishop-Gromov comparison theorem then implies that in fact (Y_j, h_j) is isometric to the unit $(k-1)$-sphere for sufficiently large j, so that $V_j = \mathbb{R}^k$. If $k - 1 = 1$, then V_j is a cone over a circle, so by Condition $(*)$ we have $V_j = \mathbb{R}^2$. Either way, we have a contradiction.

We can therefore assume, up to choosing a subsequence, that the V_j all have singularities q_j away from the vertex. By taking tangent cones at the q_j, we obtain a new sequence of cones, V'_j, which still satisfy the Condition $(*)$, they converge to $\mathbb{R}^k$, and they all split off an isometric factor of $\mathbb{R}$, that is, $V'_j = W_j \times \mathbb{R}$. The cones W_j are then $k-1$ dimensional, they also satisfy Condition $(*)$, and $W_j \to \mathbb{R}^{k-1}$. We can then apply the inductive hypothesis. It follows that $W_j = \mathbb{R}^{k-1}$ for large j, so $V'_j = \mathbb{R}^k$, contradicting that the q_j are singular points. ◻

Given this result, we can follow the argument of Donaldson-Sun [27] to prove Theorem 35.

Proof of Theorem 35. We argue by contradiction. Suppose that there are singular Kähler-Einstein spaces $(X_i, \omega_{KE,i})$, that can be approximated by cscK metrics, with $\omega_{KE,i} \in c_1(L_i)$, satisfying the conditions (1)–(3) before the statement of Theorem 35, but such that there is no fixed power L_i^k of the line bundles L_i whose density of states functions are bounded away from zero uniformly. Up to choosing a subsequence, we can assume that the corresponding RCD spaces $\hat{X}_i$ converge to $\hat{X}_\infty$ in the Gromov-Hausdorff sense. Theorem 36 implies that for some $\epsilon > 0$, the ϵ-regular subset of $\hat{X}_\infty$ coincides with the regular set $\mathcal{R} \subset \hat{X}_\infty$ (given by the points with tangent cone $\mathbb{R}^{2n}$). Therefore the set $\mathcal{R}$ is open, and by Theorem 36 together with Proposition 24, it follows that the convergence $\hat{X}_i \to \hat{X}_\infty$ is locally smooth on $\mathcal{R}$. In addition, using the argument in Proposition 27, we know that no iterated tangent cone of $\hat{X}_\infty$ is given by $C(S^1_\gamma) \times \mathbb{R}^{2n-2}$ with $\gamma < 2\pi$. This means that we are in essentially the same setting as Donaldson-Sun [27], and can closely follow their arguments to show that there is a $k_0 > 0$, such that the density of states functions of the sections of $L_i^{k_0}$ are bounded away from zero for all sufficiently large i. ◻

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