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VARIATIONAL PRINCIPLES OF METRIC MEAN DIMENSION FOR RANDOM DYNAMICAL SYSTEMS

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Abstract In this paper, we establish variational principles for the metric mean dimension of random dynamical systems with infinite topological entropy. This is based on four types of measure-theoretic ϵ-entropies: Kolmogorov-Sinai ϵ-entropy, Shapira's ϵ-entropy, Katok's ϵ-entropy and Brin–Katok local ϵ-entropy. The variational principle, as a fundamental theorem, links topological dynamics and ergodic theory.

Keywords: Random dynamical system; metric mean dimension; measure-theoretic ϵ-entropy; variational principle

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1. Introduction

Mean dimension, introduced by Gromov [11], serves as a novel topological invariant for dynamical systems. It quantifies the number of parameters per second required to describe a dynamical system, analogous to how topological entropy measures the number of bits per second. It especially exhibits some applications in solving the embedding problems of dynamical systems [12, 13, 21, 24]. Inspired by the definition of Minkowski dimension, Lindenstrauss and Weiss introduced the metric mean dimension and demonstrated that it serves as an upper bound for mean dimension. See also some discussions about the applications of metric mean dimension in estimating the upper bound of mean dimension for some complex dynamical systems [32–34], some aspects of the analog compression [14] in information theory, operator algebras and L^2-invariants [7, 18, 20]. Notably, both mean dimension and metric mean dimension are zero when the topological entropy is finite.

Therefore, they are valuable tools for characterizing the topological complexity of infinite entropy systems.

Two fundamental kinds of entropies in dynamical systems are the topological entropy and measure-theoretic entropy, which are interconnected through the well-known variational principle established by Goodwyn [10] and Goodman [9]:

$$h_{top}(T) = \sup_{\mu \in M(X,T)} h_\mu(T),$$

where T denotes a homeomorphism from a compact metric space X to itself, and the supremum is taken over all T-invariant Borel probability measures on X. It is natural to expect that there are variational principles for infinite entropy systems. The absence of a role for measure-theoretic metric mean dimension is the main obstruction to obtaining such variational principles. In 2018, using the foundations of lossy data compression methods, Lindenstrauss and Tsukamoto [22] established the following variational principles for metric mean dimension in terms of rate distortion functions:

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X,T,d) = \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in M(X,T)} R_{\mu,L^\infty}(\epsilon)}{|\log \epsilon|}.$$

Additionally, if (X,d) has the tame growth of covering numbers, then for $p \in [1,\infty)$,

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X,T,d) = \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in M(X,T)} R_{\mu,p}(\epsilon)}{|\log \epsilon|},$$

where $\overline{\mathrm{mdim}}_{\mathrm{M}}(X,T,d)$ denotes upper metric mean dimension of X, $R_{\mu,p}(\epsilon)$ and $R_{\mu,L^\infty}(\epsilon)$ are referred to as the L^p and L^∞ rate distortion functions, respectively. For an extension of this result to amenable groups, see [4]. Subsequently, in 2019, Lindenstrauss and Tsukamoto [23] proved the double variational principles for mean dimension, utilizing rate-distortion dimension for systems possessing the marker property. Since then, many researchers have been devoted to obtaining the new variational relations for metric mean dimension by replacing rate-distortion functions. For instance, Velozo-Velozo [35] proved an analogous variational principle using Katok's ϵ-entropy instead of a rate distortion function, while Gutman and Spiewak [14] derived a variational principle for metric mean dimension in terms of Kolmogorov-Sinai ϵ-entropy. Additionally, Shi [30] obtained variational principles for metric mean dimension using Shapira's ϵ-entropy, Katok's ϵ-entropy, and Brin–Katok local ϵ-entropy. Inspired by the work of Feng and Huang, the authors in [36, 38] introduced the concepts of Bowen and packing metric mean dimensions for subsets and established variational principles for non-empty compact sets.

The modeling of random dynamical systems arises in some phenomena of physics, biology, climatology, economics, etc. When uncertainties or random influences, which we call noises, are taken into account, it not only compensates for the defects in some deterministic models, but also reveals some rather intrinsic phenomena. The study of the ergodic theory of random transformations can date back to 1980s, which emerged from Kifer [15], Crauel [5], Ledrappier and Young [17], Bogenschutz [2], etc. Briefly, a continuous bundle random dynamical system is a family $T = (T_\omega)_\omega$ of continuous

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transformations on the fibers of X driven by a measure-preserving system $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ and is equipped with a induced skew product transformation $\Theta : \Omega \times X \to \Omega \times X$. Bogenschutz [2] and Kifer [16] proved the variational principle of random topological entropy for random dynamical systems:

$$h^{\mathbf{r}}_{top}(T) = \sup \left\{ h^{\mathbf{r}}_{\mu}(T) : \mu \text{ is } \Theta\text{-invariant} \right\},$$

where $h^{\mathbf{r}}_{\mu}(T)$ and $h^{\mathbf{r}}_{top}(T)$ are the measure-theoretic entropy and topological entropy of random dynamical systems, respectively. Based on the previous work on $\mathbb{Z}$-actions, Ma, Yang and Chen [27] introduced the mean dimension and metric mean dimension for random dynamical systems. However, the variational principles for random metric mean dimension in the setting of random dynamical systems remain still vacant and have not been built up to now.

For a measurable subset $\mathcal{E} \subset \Omega \times X$, the fibers $\mathcal{E}_\omega = \{x \in X : (\omega, x) \in \mathcal{E}\}$ with $\omega \in \Omega$. Let $\mu \in E_{\mathbb{P}}(\mathcal{E})$ denote the set of all ergodic measures on $\mathcal{E}$ having the marginal $\mathbb{P}$ over Ω. Let $\mathcal{P}_X$ and $\mathcal{C}^o_X$ denote the set of partition and open cover of X, respectively. Our aim in this paper is to formulate some variational principles for random metric mean dimension of continuous bundle random dynamical systems. The main results of this paper are the Theorems 3.2–3.6, which can be stated as follows:

Theorem 1.1. *Let* T *be a homeomorphic bundle RDS (random dynamical system) on* $\mathcal{E}$ *over an ergodic measure-preserving system* $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$. *Then*

$$\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \limsup_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \sup_{\mu \in E_{\mathbb{P}}(\mathcal{E})} F(\mu, d, \epsilon),$$

$$\mathbb{E}\underline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \liminf_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \sup_{\mu \in E_{\mathbb{P}}(\mathcal{E})} F(\mu, d, \epsilon),$$

where

$$F(\mu, d, \epsilon) \in \left\{ \inf_{\substack{\mathrm{diam}(\alpha) \le \epsilon \\ \alpha \in \mathcal{P}_X}} h^r_\mu(T, (\Omega \times \alpha)_{\mathcal{E}}), \inf_{\substack{\mathrm{diam}(\mathcal{U}) \le \epsilon \\ \mathcal{U} \in \mathcal{C}^o_X}} h^S_\mu(T, (\Omega \times \mathcal{U})_{\mathcal{E}}), \overline{h}^K_\mu(T, \epsilon), \overline{h}^{BK}_\mu(T, \epsilon) \right\}.$$

The definitions of $\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d)$, $\mathbb{E}\underline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d)$ *and* $F(\mu, d, \epsilon)$ *see* § 2 *and* 3.

The aforementioned theorem generalizes previous variational principles of metric mean dimension in the context of $\mathbb{Z}$-actions [14, 30] whenever the space Ω is just a single point. There are still some difficulties for us to obtain above theorems from $\mathbb{Z}$-actions to random dynamical systems. It can be explained as two aspects. One is the local variational principle of Shapira's entropy is still missing for random dynamical systems. The other one is how to link different types of measure-theoretic ϵ-entropies by some proper inequalities.

This paper is structured as follows. In § 2, we revisit some fundamental definitions that are essential for our discussion. In § 3, we establish the variational principles, specifically theorems 3.2–3.6.

2. Preliminaries

2.1. The setup of random dynamical systems

In this subsection, we recall the settings and related notions of random dynamical systems investigated in [1, 5, 15].

Let $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ be a measure-preserving system, where $(\Omega, \mathscr{F}, \mathbb{P})$ is countably generated probability space and θ is an invertible measure-preserving transformation. We always assume that $\mathscr{F}$ is complete, countably generated, and separates points. Hence $(\Omega, \mathscr{F}, \mathbb{P})$ is a Lebesgue space. Let X be a compact metric space endowed with the Borel σ-algebra $\mathscr{B}_X$. This endows $\Omega \times X$ with the product σ-algebra $\mathscr{F} \otimes \mathscr{B}_X$. For a measurable subset $\mathcal{E} \subset \Omega \times X$, the fibers $\mathcal{E}_\omega = \{x \in X : (\omega, x) \in \mathcal{E}\}$ with $\omega \in \Omega$ are non-empty compact subsets of X. A *continuous (or homeomorphic) bundle random dynamical system* (RDS for short) over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is generated by mappings $T_\omega : \mathcal{E}_\omega \to \mathcal{E}_{\theta\omega}$ with iterates

$$T_\omega^n = \begin{cases} T_{\theta^{n-1}\omega} \circ \cdots \circ T_{\theta\omega} \circ T_\omega, & \text{if } n > 0 \\ id, & \text{if } n = 0 \end{cases}$$

such that $(\omega, x) \mapsto T_\omega x$ is measurable and $x \mapsto T_\omega x$ is continuous (or homeomorphic, respectively) for $\mathbb{P}$-almost all ω. The map $\Theta : \mathcal{E} \to \mathcal{E}$ defined by $\Theta(\omega, x) = (\theta\omega, T_\omega x)$ is called the *skew product transformation*.

A finite family $\mathcal{U} = \{U_i\}_{i=1}^k$ of measurable subsets of $\Omega \times X$ is said to be a *cover* if $\Omega \times X = \bigcup_{i=1}^k U_i$, and for each $i \in \{1, \cdots, k\}$ the ω-section

$$U_i(\omega) := \{x \in X : (\omega, x) \in U_i\} \subseteq X$$

is a Borel set of X. This implies that $\mathcal{U}(\omega) = \{U_i(\omega)\}_{i=1}^k$ is a Borel cover of X. A partition of $\Omega \times X$ is a cover of $\Omega \times X$ whose elements are mutually disjoint. An open cover of $\Omega \times X$ is a cover of $\Omega \times X$ whose ω-sections are open sets. Denoted by $\mathcal{P}_{\Omega\times X}$, $C_{\Omega\times X}$ and $C^0_{\Omega\times X}$ the set of all finite partitions, finite covers and finite open covers of $\Omega \times X$, respectively. Specially, by $C^{0'}_{\Omega\times X}$ we denote the set of $\mathcal{U} \in C^0_{\Omega\times X}$ formed by $\mathcal{U} = \{\Omega \times U_i\}$ with the finite open cover $\{U_i\}$ of X. The notions $\mathcal{P}_\mathcal{E}$, $C_\mathcal{E}$, $C^0_\mathcal{E}$ and $C^{0'}_\mathcal{E}$ denote the restriction of $\mathcal{P}_{\Omega\times X}$, $C_{\Omega\times X}$, $C^0_{\Omega\times X}$ and $C^{0'}_{\Omega\times X}$ on $\mathcal{E}$, respectively. Given the covers $\xi \in C_\Omega$ and $\mathcal{W} \in C_X$, we sometimes write $(\Omega \times \mathcal{W})_\mathcal{E} = \{(\Omega \times W) \cap \mathcal{E} : W \in \mathcal{W}\}$ and $(\xi \times X)_\mathcal{E} = \{(A \times X) \cap \mathcal{E} : A \in \xi\}$. Given two covers $\mathcal{U}, \mathcal{V} \in C_{\Omega\times X}$, $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (denote as $\mathcal{U} \succeq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. The join of $\mathcal{U}$ and $\mathcal{V}$ is defined by $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. For $a, b \in \mathbb{N}$ with $a \le b$ and $\mathcal{U} \in C_{\Omega\times X}$, we define $\mathcal{U}_a^b = \bigvee_{n=a}^{b} \Theta^{-n}\mathcal{U}$.

We collect some examples of continuous bundle RDSs below.

Example 1. Among interesting examples of continuous bundle RDSs are random sub-shifts, which appeared in the literature [3, 16]. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a Lebesgue space and $\theta : (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ an invertible measure-preserving transformation. Set $X = \{(x_i)_{i\in\mathbb{Z}} : x_i \in \mathbb{N} \cup \{+\infty\}, i \in \mathbb{Z}\}$, a compact metric space equipped with the metric

$$d((x_i)_{i\in\mathbb{Z}},(y_i)_{i\in\mathbb{Z}})=\sum_{i\in\mathbb{Z}}\frac{1}{2^{|i|}}|x_i^{-1}-y_i^{-1}|,$$

and let $F:X\to X$ be the translation $(x_i)_{i\in\mathbb{Z}}\to(x_{i+1})_{i\in\mathbb{Z}}$. Then, the integer group $\mathbb{Z}$ acts on $(\Omega\times X,\mathscr{F}\otimes\mathscr{B}_X)$ measurably with $(\omega,x)\to(\theta^i\omega,F^ix)$ for each $i\in\mathbb{Z}$, where $\mathscr{B}_X$ denotes the Borel σ-algebra of the space X. Now let $\mathcal{E}\in\mathscr{F}\otimes\mathscr{B}_X$ be an invariant subset of $\Omega\times X$ such that $\mathcal{E}_\omega\subset X$ is compact for $\mathbb{P}$-a.e. $\omega\in\Omega$. This defines a continuous bundle RDSs, for $\mathbb{P}$-a.e. $\omega\in\Omega$, $F_{i,\omega}$ is just the restriction of F^i over $\mathcal{E}_{\theta^i\omega}$ for $i\in\mathbb{Z}$.

A very special case is when the subset $\mathcal{E}$ is given as follows. Let k be a random $\mathbb{N}$-valued random variable satisfying

$$0<\int_\Omega\log k(\omega)d\mathbb{P}(\omega)<\infty,$$

and, for $\omega\in\mathbb{P}$, let $M(\omega)$ be a random matrix $(m_{i,j}(\omega):i=1,\cdots,k(\omega),j=1,\cdots,k(\theta(\omega)))$ with entries 0 and 1. Then the random matrix M generates a random sub-shift of finite type, where

$$\mathcal{E}=\left\{(\omega,(x_i)_{i\in\mathbb{N}}):\omega\in\Omega,1\le x_i\le k(\theta^i\omega),m_{x_i,x_{i+1}}(\theta^i\omega)=1,i\in\mathbb{Z}\right\}.$$

It is not hard to see that this is a continuous bundle RDS.

Example 2. There are many other interesting examples of random dynamical systems coming from smooth ergodic theory, see for example [19, 25]. Let M be a C^∞ compact connected Riemannian manifold without boundary and $C^r(M,M)$, $r\in\mathbb{Z}_+\cup\{+\infty\}$ the space of all C^r maps from M into itself endowed with the usual C^r topology and the Borel σ-algebra. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a Lebesgue space and $\{\phi_t:\Omega\to C^r(M,M)_{t\ge0}\}$ be a stochastic flow of $C^r(M,M)$ diffeomorphisms. It is well known that every smooth stochastic differential equation (SDE) in the finite dimensional compact manifold has a stochastic flow of diffeomorphisms as its solution flow. When the SDE is non-degenerate, it has a unique stationary measure, which is ergodic and equivalent to Lebesgue measure.

2.2. Metric mean dimension of RDSs

In this subsection, we recall the definitions of topological entropy [2, 15] and metric mean dimension introduced by Ma et al. [27] for continuous bundle random dynamical systems.

Let $\omega\in\Omega$, $n\in\mathbb{N}$ and $\epsilon>0$. For each $x,y\in\mathcal{E}_\omega$, the *n-th Bowen metric* d_n^ω on $\mathcal{E}_\omega$ is defined by

$$d_n^\omega(x,y)=\max\{d(T_\omega^ix,T_\omega^iy):0\le i<n\}.$$

Then the (n,ϵ,ω)-Bowen ball around x with radius ϵ in the metric d_n^ω is given by

$$B_{d_n^\omega}(x,\epsilon)=\{y\in\mathcal{E}_\omega:d_n^\omega(x,y)<\epsilon\}.$$

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Fix $\omega \in \Omega$ and let $\alpha = \{A_i : 1 \leq i \leq m\}$ be a finite open cover of $\mathcal{E}_\omega$. We define the mesh of α with respect to the metric d_n^ω as follows

$$\mathrm{diam}(\alpha, d_n^\omega) = \max_{1 \leq i \leq m} \mathrm{diam}(A_i, d_n^\omega),$$

where the diameter of a set A_i with respect to the metric d_n^ω is given by

$$\mathrm{diam}(A_i, d_n^\omega) = \sup \{d_n^\omega(x, y) : x, y \in A_i\}.$$

Let $\#(\mathcal{E}, \omega, \epsilon, n) = \inf \left\{|\alpha| : \alpha \in C^0_{\mathcal{E}_\omega}, \mathrm{diam}(\alpha, d_n^\omega) < \epsilon\right\}$. A set $E \subset \mathcal{E}_\omega$ is said to be an (ω, ϵ, n)-*separated set* if $x, y \in E$, $x \neq y$ implies that $d_n^\omega(x, y) > \epsilon$. The maximum cardinality of (ω, ϵ, n)-separated sets is denoted by $\mathrm{sep}(\mathcal{E}, \omega, \epsilon, n)$. A subset F of $\mathcal{E}_\omega$ is said to be an (ω, ϵ, n)-*spanning set* if for any $x \in \mathcal{E}_\omega$, there exists $y \in F$ such that $d_n^\omega(x, y) \leq \epsilon$. The smallest cardinality of (ω, n, ϵ)-spanning sets is denoted by $\mathrm{span}(\mathcal{E}, \omega, n, \epsilon)$. Let

$$S'(\mathcal{E}, \omega, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \#(\mathcal{E}, \omega, \epsilon, n).$$

Set

$$S'(\mathcal{E}, \epsilon) = \int S'(\mathcal{E}, \omega, \epsilon) d\mathbb{P}(\omega). \tag{2.2}$$

The quantity (2.2) is non-decreasing as $\epsilon \to 0$. One can define a quantity to measure how fast $S'(\mathcal{E}, \epsilon)$ increases as follows:

$$\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \limsup_{\epsilon \to 0} \frac{S'(\mathcal{E}, \epsilon)}{|\log \epsilon|}, \tag{2.3}$$

$$\mathbb{E}\underline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \liminf_{\epsilon \to 0} \frac{S'(\mathcal{E}, \epsilon)}{|\log \epsilon|}.$$

We call (2.3) the *upper and lower metric mean dimension* of $\mathcal{E}$ for RDSs, respectively.

It is easy to show that

$$\#(\mathcal{E}, \omega, 2\epsilon, n) \leq \mathrm{sep}(\mathcal{E}, \omega, \epsilon, n) \leq \#(\mathcal{E}, \omega, \epsilon, n). \tag{2.4}$$

Notice that $\mathrm{sep}(\mathcal{E}, \omega, \epsilon, n)$ is measurable in ω [16, lemma 2.1]. Then metric mean dimension can also defined by separated sets. Set

$$S(\mathcal{E}, \omega, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(\mathcal{E}, \omega, \epsilon, n)$$

and

$$h^{\mathbf{r}}_{top}(T, \mathcal{E}, d, \epsilon) = \int S(\mathcal{E}, \omega, \epsilon) d\mathbb{P}(\omega).$$

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By (2.4) and the fact that $\frac{|\log \epsilon|}{|\log 2\epsilon|} = 1$, we have

$$\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \limsup_{\epsilon \to 0} \frac{h^{\mathbf{r}}_{top}(T, \mathcal{E}, d, \epsilon)}{|\log \epsilon|},$$
$$\mathbb{E}\underline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \liminf_{\epsilon \to 0} \frac{h^{\mathbf{r}}_{top}(T, \mathcal{E}, d, \epsilon)}{|\log \epsilon|}.$$

Clearly, the metric mean dimension depends on the metrics on X and hence is not topologically invariant. Furthermore, Ma et al. [26] proved that any finite entropy systems have zero metric mean dimension in the setting of random dynamical systems. So metric mean dimension is a useful quantity to describe the topological complexity of infinite random entropy systems.

3. Variational principles for metric mean dimension

In this section, we establish four types of variational principles for metric mean dimension. The main results are Theorems 3.2–3.6.

3.1. Variational principle I: Kolmogorov-Sinai ϵ-entropy

In this subsection, we first introduce the local variational principle for the topological entropy of a fixed finite open covers in terms of measure-theoretic entropy of a fixed finite open covers given in [6, 26]. Subsequently, we prove the first main result Theorem 3.2 by using the local variational principle of RDSs.

By $\mathcal{P}_{\mathbb{P}}(\Omega \times X)$ we denote the space of probability measures on $\Omega \times X$ with the marginal $\mathbb{P}$ on Ω. Let $\mathcal{P}_{\mathbb{P}}(\mathcal{E}) = \{\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X) : \mu(\mathcal{E}) = 1\}$. It is well-known [5, proposition 3.6] that $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ on $\mathcal{E}$ can be disintegrated as $d\mu(\omega, x) = d\mu_\omega(x)d\mathbb{P}(\omega)$, where μ_ω is the regular conditional probabilities with respect to the σ-algebra $\mathscr{F}_{\mathcal{E}}$ formed by all sets $(A \times X) \cap \mathcal{E}$ with $A \in \mathscr{F}$. The set of Θ-invariant measures μ of $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ is denoted by $M_{\mathbb{P}}(\mathcal{E})$. By Bogenschutz [2], the measure $\mu \in M_{\mathbb{P}}(\mathcal{E})$ if and only if $T_\omega \mu_\omega = \mu_{\theta\omega}$ for $\mathbb{P}$-a.e. ω. And the set of ergodic elements in $M_{\mathbb{P}}(\mathcal{E})$ is denoted by $E_{\mathbb{P}}(\mathcal{E})$. This means that μ_ω is a probability measure on $\mathcal{E}_\omega$ for $\mathbb{P}$-a.e.ω and for any measurable set $R \subset \mathcal{E}$, $\mu_\omega(R(\omega)) = \mu(R|\mathscr{F}_{\mathcal{E}})$, where $\mu(R|\mathscr{F}_{\mathcal{E}})$ is the conditional expectation of the characterization function 1_R of R with respect to $\mathscr{F}_{\mathcal{E}}$, $R_\omega = \{x \in \mathcal{E}_\omega : (\omega, x) \in R_i\}$ and so $\mu(R) = \int \mu_\omega(R(\omega))d\mathbb{P}(\omega)$. Let $\mathcal{R} = \{R_i\}$ be a finite measurable partition of $\mathcal{E}$ and $R_i(\omega) = \{x \in \mathcal{E}_\omega : (\omega, x) \in R_i\}$. Then $\mathcal{R}(\omega) = \{R_i(\omega)\}$ is a finite partition of $\mathcal{E}_\omega$. Set $\mathscr{F}_{\mathcal{E}} = \{(A \times X) \cap \mathcal{E} : A \in \mathscr{F}\}$.

The *conditional entropy* of $\mathcal{R}$ for the given σ-algebra $\mathscr{F}_{\mathcal{E}}$ is defined by

$$H_\mu(\mathcal{R}|\mathscr{F}_{\mathcal{E}}) = -\int \sum_i \mu(R_i|\mathscr{F}_{\mathcal{E}}) \log \mu(R_i|\mathscr{F}_{\mathcal{E}}) d\mathbb{P}(\omega) = \int H_{\mu_\omega}(\mathcal{R}(\omega))d\mathbb{P}(\omega),$$

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where $H_{\mu_\omega}(P)$ denotes the usual partition entropy of P. Let $\mu \in M_{\mathbb{P}}(\mathcal{E})$, $\xi \in \mathcal{P}_{\mathcal{E}}$ and define

$$h_\mu^{\mathbf{r}}(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} (\Theta^i)^{-1}\xi | \mathscr{F}_{\mathcal{E}}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \int H_{\mu_\omega}\left(\bigvee_{i=0}^{n-1} (T_\omega^i)^{-1}\xi(\theta^i\omega)\right) d\mathbb{P}(\omega),$$

where the limit exists due to the subadditivity of conditional entropy[15]. If $\mathbb{P}$ is ergodic, then $h_\mu^{\mathbf{r}}(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H_{\mu_\omega}\left(\bigvee_{i=0}^{n-1} (T_\omega^i)^{-1}\xi(\theta^i\omega)\right)$ for $\mathbb{P}$-a.e. ω.

Let $\mathcal{U} \in C_{\mathcal{E}}^0$ and $\mu \in M_{\mathbb{P}}(\mathcal{E})$. We define the *measure-theoretic entropy of open cover $\mathcal{U}$ w.r.t. μ* as

$$h_\mu^{\mathbf{r}}(T,\mathcal{U}) = \inf_{\alpha \succeq \mathcal{U}, \alpha\in\mathcal{P}_{\mathcal{E}}} h_\mu^{\mathbf{r}}(T,\alpha).$$

For each $\mathcal{U} \in C_{\mathcal{E}}^{0\prime}$, it is not difficult to verify (see [2, 6, 15]) that infimum above can only take over the partitions Q of $\mathcal{E}$ into sets Q_i of the form $Q_i = (\Omega \times P_i)\cap\mathcal{E}$, where $\mathcal{P} = \{P_i\}$ is a finite partition of X.

Let $\mathcal{U} \in C_{\mathcal{E}}^0$, $n \in \mathbb{N}$ and $\omega \in \Omega$. Put

$$N(T,\omega,\mathcal{U},n) = \min\left\{\#F : F \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} (T_\omega^i)^{-1}\mathcal{U}(\theta^i\omega) \text{ over } \mathcal{E}_\omega\right\},$$

By the proof of [16, proposition 1.6], the quantity $N(T,\omega,\mathcal{U},n)$ is measurable with respect to ω. The Kingman's subadditive ergodic theorem then gives us the following:

$$h_{top}^{\mathbf{r}}(T,\mathcal{U}) := \int \lim_{n\to\infty} \frac{1}{n} \log N(T,\omega,\mathcal{U},n) d\mathbb{P}(\omega) \tag{3.1}$$

$$= \lim_{n\to\infty} \frac{1}{n} \int \log N(T,\omega,\mathcal{U},n) d\mathbb{P}(\omega),$$

and (3.1) remains valid for $\mathbb{P}$-a.e ω without taking the integral on the right-hand side if $\mathbb{P}$ is ergodic.

The proof of the variational principle I as stated in Theorem 3.2 is based on the random version of the local variational principle for entropy of a fixed open cover. Local entropy theory for deterministic dynamical systems has been studied by Romagnoli [28] and proved by Glasner and Weiss [8]. In the case of random dynamical systems, authors [6, 26] established the following local variational principle.

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Theorem 3.1. *Let* T *be a homeomorphic bundle RDS on* $\mathcal{E}$ *over a measure-preserving system* $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$*. If* $\mathcal{U} \in C_{\mathcal{E}}^{0'}$*, then*

$$h_{top}^{r}(T, \mathcal{U}) = \max_{\mu \in M_{\mathbb{P}}(\mathcal{E})} h_{\mu}^{r}(T, \mathcal{U}).$$

Additionally, if $\mathbb{P}$ *is ergodic, then*

$$h_{top}^{r}(T, \mathcal{U}) = \sup_{\mu \in E_{\mathbb{P}}(\mathcal{E})} h_{\mu}^{r}(T, \mathcal{U}).$$

Given a finite open cover $\mathcal{U}$ of X, We define $\mathrm{diam}(\mathcal{U})$ as the *diameter* of $\mathcal{U}$, i.e., the maximal diameter of the elements of $\mathcal{U}$. The Lebesgue number of $\mathcal{U}$, denoted by $\mathrm{Leb}(\mathcal{U})$, is the largest positive number δ with the property that every open ball of X with radius δ is contained in an element of $\mathcal{U}$.

Lemma 3.1 ([14, lemma 3.4]). *For any compact metric space* (X, d) *and* $\epsilon > 0$*, there exists a finite open cover* $\mathcal{U}$ *of* X *such that* $\mathrm{diam}(\mathcal{U}) \leq \epsilon$ *and* $\mathrm{Leb}(\mathcal{U}) \geq \frac{\epsilon}{4}$*.*

Lemma 3.2. *Let* $\sigma = \{A_i\}$ *be a finite open cover of* X*. Let* $\mathcal{U} = (\Omega \times \sigma)_{\mathcal{E}} = \{(\Omega \times A_i) \cap \mathcal{E} : A_i \in \sigma\}$ *be a finite open cover of* $\mathcal{E}$*. Then for each fixed* ω*,*

$$S(\mathcal{E}, \omega, \mathrm{diam}(\sigma)) \leq \lim_{n\to\infty} \frac{1}{n} \log N(T, \omega, \mathcal{U}, n) \leq S(\mathcal{E}, \omega, \mathrm{Leb}(\sigma)). \tag{3.2}$$

Proof. One can obtain the desired result by using

$$\mathrm{sep}(\mathcal{E}, \omega, \mathrm{diam}(\sigma), n) \leq N(T, \omega, \mathcal{U}, n) \leq \mathrm{sep}(\mathcal{E}, \omega, \mathrm{Leb}(\sigma), n).$$

□

Theorem 3.2. *Let* T *be a homeomorphic bundle RDS on* $\mathcal{E}$ *over a measure-preserving system* $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$*. Then*

$$\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \limsup_{\epsilon\to 0} \frac{1}{|\log \epsilon|} \sup_{\mu \in M_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\mathrm{diam}(\alpha) \leq \epsilon, \\ \alpha \in \mathcal{P}_X}} h_{\mu}^{r}(T, (\Omega \times \alpha)_{\mathcal{E}}),$$

$$\mathbb{E}\underline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \liminf_{\epsilon\to 0} \frac{1}{|\log \epsilon|} \sup_{\mu \in M_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\mathrm{diam}(\alpha) \leq \epsilon, \\ \alpha \in \mathcal{P}_X}} h_{\mu}^{r}(T, (\Omega \times \alpha)_{\mathcal{E}}).$$

Additionally, if $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ *is ergodic, then the results are also valid by changing the supremum into* $\sup_{\mu \in E_{\mathbb{P}}(\mathcal{E})}$*.*

Proof. It suffices to show the variational principles hold for $\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d)$. Let $\epsilon > 0$. From Lemma 3.1, there exists a finite open cover $\mathcal{U}$ of X such that $\mathrm{diam}(\mathcal{U}) \leq \epsilon$ and $\mathrm{Leb}(\mathcal{U}) \geq \frac{\epsilon}{4}$.

Note that, for any finite Borel partition α of X satisfying $\alpha \succeq \mathcal{U}$, we have $\operatorname{diam}(\alpha) \leq \epsilon$. By Theorem 3.1, we obtain

$$\sup_{\mu \in M_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\operatorname{diam}(\alpha) \leq \epsilon, \\ \alpha \in \mathcal{P}_X}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}) \leq \sup_{\mu \in M_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\alpha \succeq \mathcal{U}, \\ \alpha \in \mathcal{P}_X}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}) = h_{top}^{\mathbf{r}}(T, (\Omega \times \mathcal{U})_{\mathcal{E}}). \tag{3.3}$$

Using Lemma 3.2,

$$h_{top}^{\mathbf{r}}(T, (\Omega \times \mathcal{U})_{\mathcal{E}}) \leq \int S(\mathcal{E}, \omega, \operatorname{Leb}(\mathcal{U})) d\mathbb{P}(\omega) \leq \int S(\mathcal{E}, \omega, \frac{\epsilon}{4}) d\mathbb{P}(\omega). \tag{3.4}$$

It follows from inequalities (3.3) and (3.4) that

$$\sup_{\mu \in M_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\operatorname{diam}(\alpha) \leq \epsilon, \\ \alpha \in \mathcal{P}_X}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}) \leq \int S(\mathcal{E}, \omega, \frac{\epsilon}{4}) d\mathbb{P}(\omega).$$

So we get

$$\limsup_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \sup_{\mu \in M_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\operatorname{diam}(\alpha) \leq \epsilon, \\ \alpha \in \mathcal{P}_X}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}) \leq \mathbb{E}\overline{\operatorname{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d).$$

On the other hand, for every finite Borel partition α of X such that $\operatorname{diam}(\alpha) \leq \frac{\epsilon}{8}$, we have $\alpha \succ \mathcal{U}$. Then Theorem 3.1 and Lemma 3.2 give us

$$\begin{aligned}
\sup_{\mu \in M_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\operatorname{diam}(\alpha) \leq \frac{\epsilon}{8}, \\ \alpha \in \mathcal{P}_X}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}) &\geq \sup_{\mu \in M_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\alpha \succeq \mathcal{U}, \\ a \in \mathcal{P}_X}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}) \\
&= h_{top}^{\mathbf{r}}(T, (\Omega \times \mathcal{U})_{\mathcal{E}}) \\
&\geq \int S(\mathcal{E}, \omega, \operatorname{diam}(\mathcal{U})) d\mathbb{P}(\omega) \geq \int S(\mathcal{E}, \omega, \epsilon) d\mathbb{P}(\omega),
\end{aligned}$$

which yields the desired results. If $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ is ergodic, one can get the variational principles by the similar arguments. □

3.2. Variational principle II: Shapira's ϵ-entropy

In this subsection, we introduce the notion of Shapira's entropy in the setting of random dynamical systems and prove Theorem 3.3, which reflects the relationship between Shapira's entropy and the measure-theoretic entropy of a fixed finite open cover $\mathcal{U}$ for random dynamical systems. Using this result, we can establish the variational principle of Shapira's ϵ-entropy.

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Let $\mathcal{U}=\{U_i\}_{i=1}^k$ be a finite open cover of $\mathcal{E}$ and $\mu\in E_{\mathbb{P}}(\mathcal{E})$. Given $\omega\in\Omega$ and $0<\delta<1$, we define

$$N_{\mu_\omega}(\mathcal{U},\delta)=\min\left\{\#I:\mu_\omega\left(\bigcup_{i\in I}U_i(\omega)\right)>1-\delta\right\}.$$

In order to define Shapira's entropy of random dynamical systems, we need to prove the measurability of $N_{\mu_\omega}(\mathcal{U},\delta)$.

Proposition 3.1. *Let* T *be a continuous bundle RDS over a measure-preserving system* $(\Omega,\mathscr{F},\mathbb{P},\theta)$*. Let* $\mathcal{U}\in C^0_{\mathcal{E}}$*. Then the function* $\omega\mapsto N_{\mu_\omega}(\mathcal{U},\delta)$ *is measurable.*

Proof. For every $q>0$, we have

$$\begin{aligned}&\Omega_q:=\{\omega:N_{\mu_\omega}(\mathcal{U},\delta)=q\}\\ &=\bigcup_{\substack{\#I=q,\\ I\subset\{1,\cdots,\#\mathcal{U}\}}}\left\{\omega:\mu_\omega\left(\bigcup_{i\in I}U_i(\omega)\right)>1-\delta\right\}\bigcap\left(\bigcap_{\substack{\#J<q,\\ J\subset\{1,\cdots,\#\mathcal{U}\}}}\left\{\omega:\mu_\omega\left(\bigcup_{i\in J}U_i(\omega)\right)\le 1-\delta\right\}\right).\end{aligned}$$

For each $I\subset\{1,\cdots,\#\mathcal{U}\}$, the graph$(A_I)$= $\{(\omega,x):x\in\bigcup_{i\in I}U_i(\omega)\}=\cup_{i\in I}U_i\cap\mathcal{E}$ is a measurable set of $\Omega\times X$. By [5, corollary 3.4], the map $\omega\to\mu_\omega(U_i(\omega))$ is measurable. Then Ω_q is a measurable set of Ω. This implies that $\omega\mapsto N_{\mu_\omega}(\mathcal{U},\delta)$ is measurable since the map only takes finite many values. □

Using proposition 3.1, we can define *Shapira's entropy of* $\mathcal{U}\in C^0_{\mathcal{E}}$ *with respect to* μ as

$$\overline{h}^S_\mu(T,\mathcal{U}):=\lim_{\delta\to0}\limsup_{n\to\infty}\frac{1}{n}\int\log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\delta)d\mathbb{P}(\omega).$$
$$\underline{h}^S_\mu(T,\mathcal{U}):=\lim_{\delta\to0}\liminf_{n\to\infty}\frac{1}{n}\int\log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\delta)d\mathbb{P}(\omega).$$

If the limit supremum and infimum agree, we denote the common value by $h^S_\mu(T,\mathcal{U})$. By the above definition, the essence of Shapira's entropy is the alternative of Katok's entropy defined by open covers. The Lemma 3.3 states the well-known Shannon–McMillan–Breiman Theorem for RDSs [39]. The result of topological dynamical systems can be seen in [31]. For $\xi\in\mathcal{P}_{\mathcal{E}}$ and $n\in\mathbb{N}$, denote by $A^n_{\xi,\omega}(x)$ be the atom of $\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\xi(\theta^i\omega)$ containing the point $x\in\mathcal{E}_\omega$.

Lemma 3.3 (Shannon-McMillan-Breiman Theorem). *Let* T *be a continuous bundle RDS on* $\mathcal{E}$ *over a measure-preserving system* $(\Omega,\mathscr{F},\mathbb{P},\theta)$ *and* $\mu\in M_{\mathbb{P}}(\mathcal{E})$*. Then for any* $\xi\in\mathcal{P}_{\mathcal{E}}$*, we have*

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(1)

$$\lim_{n\to\infty}\frac{1}{n}I_{\mu_\omega}(\bigvee_{i=0}^{n-1}(T_\omega^i)^{-1}\xi(\theta^i\omega))(x)$$
$$= E_\mu(r|\tau)(\omega,x)\quad \mu-a.e. and\ in\ L^1(\mu),$$

where $I_{\mu_\omega}(\bigvee_{i=0}^{n-1}(T_\omega^i)^{-1}\xi(\theta^i\omega)) = -\log\mu_\omega(A^n_{\xi,\omega}(x))$ *is the information function,* $r(\omega,x) = I_{\mu_\omega}(\xi_\omega|\bigvee_{i=1}^{\infty}(T_\omega^i)^{-1}\xi(\theta^i\omega)(x)$ *and* τ *is the* σ*-algebra of* Θ*-invariant sets;*

(2) $h_\mu^r(T,\xi) = \int H_{\mu_\omega}(\xi(\omega)|\bigvee_{i=1}^{\infty}(T_\omega^i)^{-1}\xi(\theta^i\omega))d\mathbb{P}(\omega)$*;*

(3) if μ *is ergodic, then*

$$\lim_{n\to\infty}-\frac{1}{n}\log\mu_\omega(A^n_{\xi,\omega}(x)) = h_\mu^r(T,\xi)\quad \mu-a.e. and\ in\ L^1(\mu).$$

Adapting the ideas from [29] and [37, lemma 6.1], the following theorem establishes the bridge between Shapira's entropy and measure-theoretic entropy of a fixed finite open cover $\mathcal{U}$ for random dynamical systems.

Theorem 3.3. *Let* T *be a homeomorphic bundle RDS on* $\mathcal{E}$ *over a measure-preserving system* $(\Omega,\mathscr{F},\mathbb{P},\theta)$*. Let* $\mathcal{U}\in C^0_{\mathcal{E}}$ *and* $\mu\in E_{\mathbb{P}}(\mathcal{E})$*. Then*

$$\overline{h}_\mu^S(T,\mathcal{U}) = \underline{h}_\mu^S(T,\mathcal{U}) = h_\mu^S(T,\mathcal{U}) = h_\mu^r(T,\mathcal{U}).$$

Proof. Step 1: We prove $h_\mu^{\mathbf{r}}(T,\mathcal{U}) \geq \overline{h}_\mu^S(T,\mathcal{U})$.

Take any finite measurable partition ξ of $\mathcal{E}$ such that $\xi\succeq\mathcal{U}$. According to Lemma 3.3, there exists $F\subset\mathcal{E}$ such that $\mu(F)=1$ and for each $(\omega,x)\in F$,

$$\lim_{n\to\infty}-\frac{1}{n}\log\mu_\omega(A^n_{\xi,\omega}(x)) = h_\mu^{\mathbf{r}}(T,\xi).$$

Fix $\omega\in\pi_\Omega(F)$ and let $a>0$. Set

$$L_{\omega,n} = \left\{x\in\mathcal{E}_\omega : -\frac{1}{m}\log\mu_\omega(A^m_{\xi,\omega}(x)) \leq h_\mu^{\mathbf{r}}(T,\xi)+a, \forall m\geq n\right\}.$$

By Lemma 3.3, $\mu_\omega(L_{\omega,n})>1-\delta$ for n sufficiently large. Fix n and choose a finite subset $G_{\omega,n}=\{x_1,\cdots,x_{s_{\omega,n}}\}$ of $L_{\omega,n}$ such that $L_{\omega,n}\subset\bigcup_{i=1}^{s_{\omega,n}}A^n_{\xi,\omega}(x_i)$. Since the sets $A^n_{\xi,\omega}(x_i)$ are distinct and μ_ω measure of each member of them is not less than $\exp(-n(h_\mu^{\mathbf{r}}(T,\xi)+a))$, then

$$\#G_{\omega,n} = s_{\omega,n} \leq \exp(n(h_\mu^{\mathbf{r}}(T,\xi)+a)).$$

Note that $\mu_\omega(L_{\omega,n})>1-\delta$, we have

$$N_{\mu_\omega}(\mathcal{U}_0^{n-1},\delta) \leq N_{\mu_\omega}(\xi^n,\delta) \leq \exp(n(h_\mu^{\mathbf{r}}(T,\xi)+a)). \tag{3.5}$$

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Thus for any $a > 0$

$$\limsup_{n\to\infty} \frac{1}{n} \int \log N_{\mu_\omega}(\mathcal{U}_0^{n-1}, \delta) d\mathbb{P}(\omega) \le h_\mu^{\mathbf{r}}(T, \xi) + a.$$

Letting $a \to 0$, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \int \log N_{\mu_\omega}(\mathcal{U}_0^{n-1}, \delta) d\mathbb{P}(\omega) \le h_\mu^{\mathbf{r}}(T, \xi).$$

Taking infimum over $\xi \succeq \mathcal{U}$ and $\delta \to 0$, we have

$$\overline{h}_\mu^S(T, \mathcal{U}) \le h_\mu^{\mathbf{r}}(T, \mathcal{U}).$$

Applying the approach from [37, lemma 6.1], we have

Claim 1. *For any $\mathcal{V} \in C_{\mathcal{E}}^0$ and $0 < \delta < 1$, there exists $\beta \in \mathcal{P}_{\mathcal{E}}$ such that $\beta \succeq \mathcal{V}$ and $N_{\mu_\omega}(\beta, \delta) \le N_{\mu_\omega}(\mathcal{V}, \delta)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. Let $\pi_\Omega : (\mathcal{E}, \mathscr{F}_{\mathcal{E}}, \mu, \Theta) \to (\Omega, \mathscr{F}, \mathbb{F}, \theta)$ be a factor map. Let $\mathcal{V} = \{V_1, \cdots, V_m\} \in C_{\mathcal{E}}^0$. For $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $I_\omega \subset \{1, \cdots, m\}$ with cardinality $N_{\mu_\omega}(\mathcal{V}, \delta)$ such that $\mu_\omega(\bigcup_{i\in I_\omega} V_j(\omega)) \ge 1 - \delta$. Hence we can find $\omega_1, \cdots, \omega_s \in \Omega$ such that for $\mathbb{P}$-a.e.$\omega \in \Omega$, $I_\omega = I_{\omega_i}$ for some $i \in \{1, \cdots, s\}$. For $i = 1, \cdots, s$, define

$$\Omega_i = \left\{ \omega \in \Omega : \mu_\omega(\bigcup_{j\in I_{\omega_i}} V_j(\omega)) \ge 1 - \delta \right\}.$$

Let $C_1 = \Omega_1$, $C_i = \Omega_i \backslash \bigcup_{j=1}^{i-1} \Omega_j$, $i = 2, \cdots, s$. Fix $i \in \{1, \cdots, s\}$. Assume that $I_{\omega_i} = \{k_1, \cdots, k_{t_i}\}$, where $t_i = N_{\mu_{\omega_i}}(\mathcal{V}, \delta)$. Take $\left\{W_1^{\omega_i}, \cdots, W_{t_i}^{\omega_i}\right\}$ such that

$$W_1^{\omega_i} = V_{k_1}, W_2^{\omega_i} = V_{k_2} \setminus V_{k_1}, \cdots, W_{t_i}^{\omega_i} = V_{k_{t_i}} \setminus \cup_{j=1}^{t_i-1} V_{k_j}.$$

Define $A := \mathcal{E} \setminus \left(\cup_{i=1}^s (\pi_\Omega^{-1} C_i \cap \cup_{j=1}^{t_i} W_j^{\omega_j}) \right)$. Set $A_1 = A \cap V_1$, $A_l := A \cap (V_l \setminus \cup_{j=1}^{l-1} V_j)$, $l = 2, \cdots, m$. Finally, take

$$\beta = \left\{ \pi_\Omega^{-1} C_1 \cap W_1^{\omega_1}, \cdots, \pi_\Omega^{-1} C_1 \cap W_{t_1}^{\omega_1}, \cdots, \pi_\Omega^{-1} C_s \cap W_1^{\omega_s}, \cdots, \pi_\Omega^{-1} C_s \cap W_{t_s}^{\omega_s}, A_1, \cdots, A_m \right\}.$$

Then $\beta \succeq \mathcal{V}$ and $N_{\mu_\omega}(\beta, \delta) \le N_{\mu_\omega}(\mathcal{V}, \delta)$ for $\mathbb{P}$-a.e. ω. □

Definition 3.1. *A measure-preserving system $(X, \mathcal{B}, \mu, T)$ is said to be aperiodic, if for every $n \in \mathbb{N}$, $\mu(\{x | T^n x = x\}) = 0$.*

Lemma 3.4 (Lemma 1.5.4 in [30]). *If $\delta < \frac{1}{2}$, then $\sum_{j\le\delta K} \binom{K}{j} \le 2^{H(\delta)}$, where $H(\delta) = -\delta \log \delta - (1 - \delta) \log(1 - \delta)$.*

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The following lemma is the strong Rohlin Lemma [29, lemma 2.5].

Lemma 3.5. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic, aperiodic system and let $\alpha \in \mathcal{P}_X$. Then for any $\delta > 0$ and $n \in \mathbb{N}$, one can find a set $B \in \mathcal{B}$ such that* B, TB, $\cdots$, $T^{n-1}B$ *are mutually disjoint,* $\mu\left(\bigcup_{i=0}^{n-1} T^i B\right) > 1 - \delta$ *and the distribution of α is the same as the distribution of the partition $\alpha|_B$ that α induces on* B.

The data (n, δ, B, α) will be called a strong Rohlin tower of height n and error δ with respect to α and with B as a base.
Step 2: Our aim is to prove

$$h_\mu^{\mathbf{r}}(T, \mathcal{U}) \le \underline{h}_\mu^S(T, \mathcal{U})$$

where μ is an ergodic measure and $\mathcal{U} \in C_{\mathcal{E}}^0$.

Case 1: If the system $(\mathcal{E}, \mathscr{F}_{\mathcal{E}}, \mu, \Theta)$ is periodic, then μ is supported on a periodic point of Θ. In this case, it is straightforward to see that

$$h_\mu^{\mathbf{r}}(T, \mathcal{U}) = \underline{h}_\mu^S(T, \mathcal{U}) = 0.$$

Now, we can assume that the system is aperiodic. Let $\mathcal{U} = \{U_1, \cdots, U_M\}$ be a open cover of $\mathcal{E}$. For fixed $n \in \mathbb{N}$, by claim 1, we can find a partition $\beta \in \mathcal{P}_{\mathcal{E}}$ such that $\beta \succeq \mathcal{U}_0^{n-1}$. There exists a subset A of $\mathcal{E}$ such that $\mu(A) < \rho$ and for any $(\omega, x) \in A$, we have $N_{\mu_\omega}(\beta, \rho) \le N_{\mu_\omega}(\mathcal{U}_0^{n-1}, \rho)$. Choose $\delta > 0$ with $0 < \rho + \delta < 1/4$. By Lemma 3.5, we can construct a strong Rohlin tower with respect to β with height n and error less than δ. Let $\tilde{B}$ denote the base of tower and $B = \tilde{B} \setminus A$. That is

- the sets $\{\tilde{B}, \Theta\tilde{B}, \cdots, \Theta^{n-1}\tilde{B}\}$ are disjoint and $\mu(\cup_{i=0}^{n-1}\Theta^i\tilde{B}) > 1 - \delta$;
- $\mu(A) = \mu_{\tilde{B}}(A)$ for any $A \in \beta$.

Note that β is constructed according to Claim 1, we get that A is the union of the atoms of β. Since the distribution of β is equal to the distribution of $\beta_{\tilde{B}}$, we have $\mu(B) > (1 - \rho)\mu(\tilde{B})$. Define $E = \cup_{i=0}^{n-1}\Theta^i B$, then $\mu(\Theta^i B) \ge (1 - \rho)\mu(\Theta^i \tilde{B})$ and hence $\mu(E) > (1 - \delta)(1 - \rho) = 1 - (\delta + \rho) + \delta \cdot \rho > 1 - (\delta + \rho)$.

Since $\beta_{\tilde{B}} \succeq \mathcal{U}_0^{n-1}$, there exist sequences $i_0, \cdots, i_{n-1}$ and $B_{i_0, \cdots, i_{n-1}} \in \beta|_{\tilde{B}}$, such that $\Theta^j B_{i_0, \cdots, i_{n-1}} \subset U_{i_j}$ for every $0 \le j \le n-1$. Let $\hat{\alpha} = \left\{\hat{A}_1, \cdots, \hat{A}_M\right\}$ be a partition of E defined by

$$\hat{A}_m := \bigcup \left\{\Theta^j B_{i_0, \cdots, i_{n-1}} : 0 \le j \le n-1, i_j = m\right\}.$$

Note that $\hat{A}_m \subset U_m$ for every $1 \le m \le M$. Extend $\hat{\alpha}$ to a partition α of $\mathcal{E}$ in some way such that $\alpha \succeq \mathcal{U}$ and $\#\alpha = 2M$.

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Set $\eta^4 = \rho + \delta$. For large enough $k > n$ large enough, define

$$f_k(\omega, x) = \frac{1}{k}\sum_{i=0}^{k-1} 1_E(\Theta^i(\omega, x))$$

and $L_k := \{(\omega, x) \in \mathcal{E} : f_k(\omega, x) > 1 - \eta^2\}$. By Birkhoff ergodic theorem, we have $\int f_k d\mu > 1 - \eta^4$ and

$$\eta^2 \mu(L_k^c) \leq \int_{L_k^c} 1 - f_k d\mu \leq \int_{\mathcal{E}} 1 - f_k d\mu \leq \eta^4.$$

It follows that $\mu(L_k) \geq 1 - \eta^2$. Since E is measurable, L_k is measurable with respect to $(\omega, x) \in \mathcal{E}$. For all $j \geq k$, take

$$J_k = \{(\omega, x) \in \mathcal{E} : \mu_\omega(A^j_{\alpha,\omega}(x)) < \exp(-(h^{\mathbf{r}}_\mu(T, \alpha) - \eta)j)\} \bigcap$$
$$\left\{(\omega, x) \in \mathcal{E} : \left|\frac{1}{j}\sum_{i=0}^{j-1} \log N_{\mu_{\theta^i\omega}}(\mathcal{U}_0^{n-1}, \rho) 1_B(\Theta^i(\omega, x)) - \int \log N_{\mu_\omega}(\mathcal{U}_0^{n-1}, \rho) 1_B(\omega, x) d\mu\right| \leq \eta\right\}.$$

Applying Lemma 3.3 and the Birkhoff ergodic theorem, we conclude that $\mu(J_k) > 1 - \eta^2$ for k sufficiently large k. Then by [5, corollary 3.4], the set

$$\{(\omega, x) \in \mathcal{E} : \mu_\omega(A^j_{\alpha,\omega}(x)) < \exp(-(h^{\mathbf{r}}_\mu(T, \alpha) - \eta)j), \forall j \geq k\},$$

is measurable. Since $N_{\mu_\omega}(\mathcal{U}_0^{n-1}, \rho)$ is measurable with respect to $\omega \in \Omega$, then the function $N_{\mu_\omega}(\mathcal{U}_0^{n-1}, \rho)$ is measurable with respect to $(\omega, x) \in \mathcal{E}$ by [5, lemma 1.1]. And $1_B(\Theta^i(\omega, x)) = 1_{\Theta^{-i}B}(\omega, x)$ is a measurable function. Then J_k is measurable. Set $G_k = L_k \cap J_k$, then G_k is measurable and $\mu(G_k) > 1 - 2\eta^2$. Define $\tilde{G}_k = \{(\omega, x) \in G_k : \mu_\omega(G_k(\omega)) \geq 1 - 4\eta\}$, we have

$$\tilde{G}_k^c \cap \mathcal{E} = \{(\omega, x) \in G_k : \mu_\omega(G_k^c(\omega)) > 4\eta\} \cup (G_k^c \cap \mathcal{E}).$$

Therefore,

$$\mu(\tilde{G}_k^c \cap \mathcal{E}) \cdot 4\eta \leq \int \mu_\omega(G_k^c(\omega)) d\mathbb{P} + \mu(G_k^c) = 2\mu(G_k^c) \leq 4\eta^2,$$

i.e., $\mu(\tilde{G}_k^c \cap \mathcal{E}) \leq \eta$.

Fix $\omega \in \pi_\Omega(\tilde{G}_k)$ and choose a sufficiently large $k > n$. Let $0 \leq i_1 \leq \cdots \leq i_m \leq k - n$, and set

$$C_\omega = \{x \in \tilde{G}_k(\omega) : T_\omega^{i_1} x \in B(\theta^{i_1}\omega), \cdots, T_\omega^{i_m} x \in B(\theta^{i_m}\omega)\}.$$

Because each element of this partition corresponds to a collection of subintervals of $[0, k-1]$ of length n, which covers all but at most $\eta^2 k + 2n$ elements of $[0, k-1]$ in a

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one-to-one manner, we have the number of elements in the partition of $\tilde{G}_k(\omega)$ is bounded above by

$$\sum_{j<\eta^2 k+2n} \binom{k}{j}.$$

In the sequel, we will want to estimate the number of $\alpha_0^{k-1}(\omega)$-elements needed to cover it. If $0 \le i_1 \le \cdots \le i_m \le k-n$ are the times elements of C_ω visit $B(\theta^{i_1}\omega), \cdots, B(\theta^{i_m}\omega)$, then we need at most $N_{\mu_{\theta^{i_j}\omega}}(\mathcal{U}_0^{n-1}, \rho)\ \alpha_{i_j}^{i_j+n-1}(\omega)$-elements to cover C_ω. Because the size of $[0, k-1] \setminus \cup_j [i_j, i_j+n-1]$ is at most $\eta^2 k + 2n$, we need at most $\Pi_{j=1}^m N_{\mu_{\theta^j\omega}}(\mathcal{U}_0^{n-1}, \rho) \cdot (2M)^{\eta^2 k+2n}\ \alpha_0^{k-1}(\omega)$-elements to cover C_ω. Since $\tilde{G}_k \subset J_k$, we know that $\tilde{G}_k(\omega)$ can be covered by no more than

$$\sum_{j<\eta^2 k+2n} \binom{k}{j} \cdot (2M)^{\eta^2 k+2n} \cdot e^{k(\int_B \log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\rho)d\mu+\eta)} \tag{3.6}$$

$\alpha_0^{k-1}(\omega)$ elements. By Lemma 3.4, the above equation (3.6) less than

$$e^{kH(\eta^2+2n/k)} \cdot (2M)^{\eta^2 k+2n} \cdot e^{k(\int_B \log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\rho)d\mu+\eta)}.$$

Note that $\omega \in \pi_\Omega(\tilde{G}_k)$, then we have

$$\begin{aligned} &1-4\eta \le \mu_\omega(\tilde{G}_k(\omega)) \le \\ &\exp(-(h_\mu^{\mathbf{r}}(T,\alpha)-\eta)k) \cdot \exp(kH(\eta^2+2n/k)) \cdot \\ &(2M)^{\eta^2 k+2n} \cdot \exp\left(k(\int \log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\rho) 1_B(\omega,x)d\mu+\eta)\right). \end{aligned} \tag{3.7}$$

Combining with (3.7), we obtain

$$\begin{aligned} h_\mu^{\mathbf{r}}(T,\alpha) &\le \eta + H(\eta^2) + \eta^2\log(2M) + \int \log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\rho)1_B(\omega,x)d\mu + \eta \\ &\le 2\eta + H(\eta^2) + \eta^2\log(2M) + \frac{1}{n}\int \log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\rho)1_B(\omega,x)d\mu \\ &\le 2\eta + H(\eta^2) + \eta^2\log(2M) + \frac{1}{n}\int \log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\rho)1_{\Omega\times X}(\omega,x)d\mu_\omega d\mathbb{P}(\omega) \\ &\le 2\eta + H(\eta^2) + \eta^2\log(2M) + \frac{1}{n}\int \log N_{\mu_\omega}(\mathcal{U}_0^{n-1},\rho)d\mathbb{P}(\omega). \end{aligned}$$

Letting $n \to \infty$ and then $\rho \to 0$, we get

$$h_\mu^{\mathbf{r}}(T,\mathcal{U}) \le h_\mu^{\mathbf{r}}(T,\alpha) \le \underline{h}_\mu^S(T,\mathcal{U}).$$

□

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Theorem 3.4. *Let* T *be a homeomorphic bundle RDS on* $\mathcal{E}$ *over an ergodic measure-preserving system* $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$*. Then*

$$\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \limsup_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \sup_{\mu \in E_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\mathrm{diam}(\mathcal{U}) \leq \epsilon, \\ \mathcal{U} \in \mathcal{C}_X^o}} h_\mu^S(T, (\Omega \times \mathcal{U})_{\mathcal{E}}).$$

$$\mathbb{E}\underline{\mathrm{mdim}}_{\mathrm{M}}(T, \mathcal{E}, d) = \liminf_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \sup_{\mu \in E_{\mathbb{P}}(\mathcal{E})} \inf_{\substack{\mathrm{diam}(\mathcal{U}) \leq \epsilon, \\ \mathcal{U} \in \mathcal{C}_X^o}} h_\mu^S(T, (\Omega \times \mathcal{U})_{\mathcal{E}}).$$

Proof. Fix $\epsilon > 0$ and $\mu \in E_{\mathbb{P}}(\mathcal{E})$. Then by Theorem 3.3, we have

$$\begin{aligned} \inf_{\substack{\mathrm{diam}(\mathcal{U}) \leq \epsilon, \\ \mathcal{U} \in C_X^o}} h_\mu^S(T, (\Omega \times \mathcal{U})_{\mathcal{E}}) &= \inf_{\substack{\mathrm{diam}(\mathcal{U}) \leq \epsilon, \\ \mathcal{U} \in C_X^o}} h_\mu^{\mathbf{r}}(T, (\Omega \times \mathcal{U})_{\mathcal{E}}), \\ &= \inf_{\mathrm{diam}(\mathcal{U}) \leq \epsilon, \alpha \succeq \mathcal{U}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}) \\ &\geq \inf_{\substack{\mathrm{diam}(\alpha) \leq \epsilon, \\ \alpha \in P_X}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}). \end{aligned} \tag{3.8}$$

By Lemma 3.1, we can choose a finite open cover $\mathcal{U}'$ of X with $\mathrm{diam}(\mathcal{U}') \leq \epsilon$ and $\mathrm{Leb}(\mathcal{U}') \geq \frac{\epsilon}{4}$. Then

$$\begin{aligned} \inf_{\substack{\mathrm{diam}(\mathcal{U}) \leq \epsilon \\ \mathcal{U} \in C_X^o}} h_\mu^S(T, (\Omega \times \mathcal{U})_{\mathcal{E}}) &\leq h_\mu^S(T, (\Omega \times \mathcal{U}')_{\mathcal{E}}) \\ &= h_\mu^{\mathbf{r}}(T, (\Omega \times \mathcal{U}')_{\mathcal{E}}), \text{ by Theorem 3.3} \\ &= \inf_{\alpha \succeq \mathcal{U}', \alpha \in P_X} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}) \\ &\leq \inf_{\substack{\mathrm{diam}(\alpha) \leq \frac{\epsilon}{8}, \\ \alpha \in P_X}} h_\mu^{\mathbf{r}}(T, (\Omega \times \alpha)_{\mathcal{E}}). \end{aligned} \tag{3.9}$$

We finally get the desired results by the inequalities (3.8), (3.9) and Theorem 3.2. □

3.3. Variational principle III: Katok's ϵ-entropy

In this subsection, we prove the third main result by replacing Shapira's ϵ-entropy with Katok local ϵ-entropy. Given $\mu \in M_{\mathbb{P}}(\mathcal{E})$, let

$$N_{\mu_\omega}^\delta(n, \epsilon) = \min \left\{ j : \mu_\omega \left(\bigcup_{i=1}^{j} B_{d_n^\omega}(x_i, \epsilon) \right) > 1 - \delta \right\}.$$

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Based on proposition 3.1, we similarly obtain the measurability of $N^{\delta}_{\mu_\omega}(n,\epsilon)$ and define the *upper and lower Katok's ϵ-entropies of μ* as follows

$$\overline{h}^K_\mu(T,\epsilon)=\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\int \log N^{\delta}_{\mu_\omega}(n,\epsilon)d\mathbb{P}(\omega),$$
$$\underline{h}^K_\mu(T,\epsilon)=\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\int \log N^{\delta}_{\mu_\omega}(n,\epsilon)d\mathbb{P}(\omega).$$

Theorem 3.5. *Let* T *be a homeomorphic bundle RDS on $\mathcal{E}$ over an ergodic measure-preserving system $(\Omega,\mathscr{F},\mathbb{P},\theta)$. Then*

$$\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T,\mathcal{E},d)=\limsup_{\epsilon\to 0}\frac{1}{|\log\epsilon|}\sup_{\mu\in E_{\mathbb{P}}(\mathcal{E})}\overline{h}^K_\mu(T,\epsilon),$$
$$\mathbb{E}\underline{\mathrm{mdim}}_{\mathrm{M}}(T,\mathcal{E},d)=\liminf_{\epsilon\to 0}\frac{1}{|\log\epsilon|}\sup_{\mu\in E_{\mathbb{P}}(\mathcal{E})}\overline{h}^K_\mu(T,\epsilon).$$

The results are valid if we change $\overline{h}^K_\mu(T,\epsilon)$ into $\underline{h}^K_\mu(T,\epsilon)$.

Proof. It suffices to show the results hold for $\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T,\mathcal{E},d)$ since the second one follows similarly. Fix $\epsilon>0$. Let $0<\delta<1$ and $\mu\in E_{\mathbb{P}}(\mathcal{E})$. Let $\mathcal{U}=\{U_1,\cdots,U_l\}$ be a finite open cover of X with $\mathrm{diam}(\mathcal{U})<\epsilon$. Then the family $\mathcal{U}(\omega)$ formed by the sets $U\cap\mathcal{E}_\omega$ with $U\in\mathcal{U}$ is an open cover of $\mathcal{E}_\omega$. This implies that each element of $\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\mathcal{U}(\theta^i\omega)$ can be contained in an (n,ϵ,ω)-Bowen ball. So

$$N^{\delta}_{\mu_\omega}(n,\epsilon)\le N_{\mu_\omega}\left(\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\mathcal{U}(\theta^i\omega),\delta\right).$$

This shows

$$\overline{h}^K_\mu(T,\epsilon)\le\inf_{\substack{\mathrm{diam}(\mathcal{U})\le\epsilon,\\ \mathcal{U}\in\mathcal{C}^o_X}}\overline{h}^S_\mu(T,(\Omega\times\mathcal{U})_{\mathcal{E}}). \tag{3.10}$$

By Lemma 3.1, we can choose a finite cover $\mathcal{U}$ of X such that $\mathrm{diam}(\mathcal{U})\le\epsilon$ and $\mathrm{Leb}(\mathcal{U})\ge\frac{\epsilon}{4}$. Since each $(n,\frac{\epsilon}{4},\omega)$-Bowen ball is contained in some element of $\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\mathcal{U}(\theta^i\omega)$, then $N_{\mu_\omega}(\bigvee_{i=0}^{n-1}(T^i_\omega)^{-1}\mathcal{U}(\theta^i\omega),\delta)\le N^{\delta}_{\mu_\omega}(n,\frac{\epsilon}{4})$. This shows

$$\inf_{\substack{\mathrm{diam}(\mathcal{U})\le\epsilon,\\ \mathcal{U}\in\mathcal{C}^o_X}}\overline{h}^S_\mu(T,(\Omega\times\mathcal{U})_{\mathcal{E}})\le\overline{h}^K_\mu(T,\frac{\epsilon}{4}). \tag{3.11}$$

Since $\mu\in E_{\mathbb{P}}(\mathcal{E})$, we have $\overline{h}^S_\mu(T,(\Omega\times\mathcal{U})_{\mathcal{E}})=h^S_\mu(T,(\Omega\times\mathcal{U})_{\mathcal{E}})$ by Theorem 3.3. Therefore, by inequalities (3.10), (3.11) and Theorem 3.4, we get the desired results. □

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3.4. Variational principle IV: Brin–Katok local ϵ-entropy

In this subsection, we borrow the Shannon-McMillan-Breiman theorem of random dynamical systems and Theorem 3.5 to establish the fourth variational principle for metric mean dimensions in terms of Brin–Katok local ϵ-entropy.

Let $\mu \in M_{\mathbb{P}}(\mathcal{E})$, $\omega \in \Omega$ and $x \in \mathcal{E}_\omega$. Put

$$\overline{h}^{BK}_{\mu_\omega}(T, x, \epsilon) = \limsup_{n\to\infty} -\frac{1}{n}\log \mu_\omega(B_{d_n^\omega}(x, \epsilon)),$$
$$\underline{h}^{BK}_{\mu_\omega}(T, x, \epsilon) = \liminf_{n\to\infty} -\frac{1}{n}\log \mu_\omega(B_{d_n^\omega}(x, \epsilon)).$$

We define the *upper and lower Brin–Katok local ϵ-entropies of μ at x* as

$$\overline{h}^{BK}_{\mu}(T, \epsilon) = \int \overline{h}^{BK}_{\mu_\omega}(T, x, \epsilon)d\mu(\omega, x),$$
$$\underline{h}^{BK}_{\mu}(T, \epsilon) = \int \underline{h}^{BK}_{\mu_\omega}(T, x, \epsilon)d\mu(\omega, x).$$

The Brin–Katok's entropy formula for RDS is given by Zhu in [40, theorem 2.1][39, theorem 2.1]. When $\mu \in M_{\mathbb{P}}(\mathcal{E})$, they stated that

$$\lim_{\epsilon\to 0} \overline{h}^{BK}_{\mu}(T, \epsilon) = \lim_{\epsilon\to 0} \underline{h}^{BK}_{\mu}(T, \epsilon) = h^{\mathbf{r}}_{\mu}(T).$$

In particular, when μ is ergodic, $\lim_{\epsilon\to 0} \overline{h}^{BK}_{\mu_\omega}(T, x, \epsilon) = \lim_{\epsilon\to 0} \underline{h}^{BK}_{\mu_\omega}(T, x, \epsilon) = h^{\mathbf{r}}_{\mu}(T)$. We give the following equalities for given $\epsilon > 0$.

Proposition 3.2. *Let* T *be a continuous bundle RDS on* $\mathcal{E}$ *over a measure-preserving system* $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$*. If* $\mu \in E_{\mathbb{P}}(\mathcal{E})$*, then for every* $\epsilon > 0$*,*

$$\overline{h}^{BK}_{\mu_\omega}(T, x, \epsilon) = \overline{h}^{BK}_{\mu}(T, \epsilon) \text{ and } \underline{h}^{BK}_{\mu_\omega}(T, x, \epsilon) = \underline{h}^{BK}_{\mu}(T, \epsilon) \tag{3.12}$$

for μ*-a.e* (ω, x)*.*

Proof. Let $\mu \in E_{\mathbb{P}}(\mathcal{E})$. By [5, proposition 3.6], μ can be disintegrated as $d\mu(\omega, x) = d\mu_\omega(x)d\mathbb{P}(\omega)$. Here the definition of μ_ω can see [5, defintion 3.1]. Let $F(\omega, x) := \overline{h}^{BK}_{\mu_\omega}(T, x, \epsilon)$. Fix $n \in \mathbb{N}$ and $\omega \in \Omega$, we have

$$\begin{aligned} B_{d_n^\omega}(x, \epsilon) &= \cap_{j=0}^{n-1}(T^j_\omega)^{-1}(B(T^j_\omega x, \epsilon) \cap \mathcal{E}_{\theta^j\omega}) \\ &= T^{-1}_\omega\big(\cap_{j=1}^{n-1}(T^{j-1}_{\theta\omega})^{-1}(B(T^{j-1}_{\theta\omega}(T_\omega x), \epsilon) \cap \mathcal{E}_{\theta^{j-1}\theta\omega})\big) \cap (B(x, \epsilon) \cap \mathcal{E}_\omega) \\ &\subseteq T^{-1}_\omega B_{d^{\theta\omega}_{n-1}}(T_\omega x, \epsilon), \end{aligned}$$

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and hence $\mu_\omega(B_{d_n^\omega}(x,\epsilon)) \le \mu_\omega(T_\omega^{-1}B_{d_{n-1}^{\theta\omega}}(T_\omega x,\epsilon)) = \mu_{\theta\omega}(B_{d_{n-1}^{\theta\omega}}(T_\omega x,\epsilon))$ for $\mathbb{P}$-a.e ω by using the fact $T_\omega\mu_\omega = \mu_{\theta\omega}$. This shows for μ-a.e (ω,x)

$$F(\omega,x) = \overline{h}_{\mu_\omega}^{BK}(T,x,\epsilon) \ge \overline{h}_{\mu_{\theta\omega}}^{BK}(T,T_\omega x,\epsilon) = F\circ\Theta(\omega,x).$$

Since μ is ergodic, this shows for μ-a.e (ω,x) $\overline{h}_{\mu_\omega}^{BK}(T,x,\epsilon) = \overline{h}_{\mu}^{BK}(T,\epsilon)$. □

Theorem 3.6. *Let* T *be a homeomorphic bundle RDS on* $\mathcal{E}$ *over an ergodic measure-preserving* $(\Omega,\mathscr{F},\mathbb{P},\theta)$*. Then*

$$\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T,\mathcal{E},d) = \limsup_{\epsilon\to 0}\frac{1}{|\log\epsilon|}\sup_{\mu\in M_{\mathbb{P}}(\mathcal{E})}\overline{h}_{\mu}^{BK}(T,\epsilon),$$
$$\mathbb{E}\underline{\mathrm{mdim}}_{\mathrm{M}}(T,\mathcal{E},d) = \liminf_{\epsilon\to 0}\frac{1}{|\log\epsilon|}\sup_{\mu\in M_{\mathbb{P}}(\mathcal{E})}\overline{h}_{\mu}^{BK}(T,\epsilon).$$

Moreover, both $M_{\mathbb{P}}(\mathcal{E})$ *in the above equalities can be replaced by* $E_{\mathbb{P}}(\mathcal{E})$*.*

Proof. Given $\epsilon>0$ and $\mu\in M_{\mathbb{P}}(\mathcal{E})$, choose a finite Borel partition ξ of X with $\mathrm{diam}\xi \le \epsilon$. By Lemma 3.3, we have

$$\int \lim_{n\to\infty} -\frac{1}{n}\log\mu_\omega(A^n_{(\Omega\times\xi)_{\mathcal{E}},\omega}(x))d\mu(\omega,x) = h^{\mathbf{r}}_\mu(T,(\Omega\times\xi)_{\mathcal{E}}).$$

It is clear that $A^n_{(\Omega\times\xi)_{\mathcal{E}},\omega}(x)\subset B_{d_n^\omega}(x,\epsilon)$ for every $n\in\mathbb{N}$. Then

$$\int \limsup_{n\to\infty} -\frac{1}{n}\log\mu_\omega(B_{d_n^\omega}(x,\epsilon))d\mu(\omega,x) \le \int \lim_{n\to\infty} -\frac{1}{n}\log\mu_\omega(A^n_{(\Omega\times\xi)_{\mathcal{E}},\omega}(x))d\mu(\omega,x).$$

Moreover

$$\overline{h}_{\mu}^{BK}(T,\epsilon) \le h^{\mathbf{r}}_\mu(T,(\Omega\times\xi)_{\mathcal{E}}).$$

Therefore,

$$\begin{aligned}\mathbb{E}\overline{\mathrm{mdim}}_{\mathrm{M}}(T,\mathcal{E},d) &= \limsup_{\epsilon\to 0}\frac{1}{|\log\epsilon|}\sup_{\mu\in M_{\mathbb{P}}(\mathcal{E})}\inf_{\substack{\mathrm{diam}\xi\le\epsilon\\ \xi\in\mathcal{P}_X}} h^{\mathbf{r}}_\mu(T,(\Omega\times\xi)_{\mathcal{E}}), \text{by Theorem 3.2}\\ &\ge \limsup_{\epsilon\to 0}\frac{1}{|\log\epsilon|}\sup_{\mu\in M_{\mathbb{P}}(\mathcal{E})}\overline{h}_{\mu}^{BK}(T,\epsilon).\end{aligned}$$

We next verify the converse direction. By (3.12), there exists a μ-full measure set $E\subset \Omega\times X$ so that for $(\omega,x)\in E$,

$$\limsup_{n\to\infty} -\frac{1}{n}\log\mu_\omega(B_{d_n^\omega}(x,\epsilon)) = \overline{h}_{\mu}^{BK}(T,\epsilon).$$

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Note that $\mathbb{P}(\pi_\Omega E) = 1$ and $\mu(E) = \int_{\pi_\Omega E} \mu_\omega(E(\omega))d\mathbb{P}(\omega) = 1$, where $E(\omega) = \{x \in \mathcal{E}_\omega : (\omega, x) \in E\}$. Then we can obtain that $\mu_\omega(E(\omega)) = 1$ for all $\omega \in \pi_\Omega E$. Given $\omega \in \pi_\Omega E$, $\rho > 0$ and $n \in \mathbb{N}$, set

$$G^\omega_{n,\rho} = \left\{ x \in E(\omega) : -\frac{1}{n} \log \mu_\omega(B_{d^\omega_n}(x, \epsilon)) < \overline{h}^{BK}_\mu(T, \epsilon) + \rho \right\}.$$

Let $0 < \delta < 1$. Then for all sufficiently large $n \in \mathbb{N}$ (depending on δ, ω, ρ), one has $\mu_\omega(G^\omega_{n,\rho}) > 1 - \delta$. Let H_n be a maximal $(\omega, 2\epsilon, n)$-separated subset of $G^\omega_{n,\rho}$. Therefore it is also an $(\omega, 2\epsilon, n)$-spanning subset of $G^\omega_{n,\rho}$ and the family $\{B_{d^\omega_n}(x, \epsilon) : x \in H_n\}$ is pairwise disjoint. It follows that $\mu_\omega(\bigcup_{x\in H_n} B_{d^\omega_n}(x, 2\epsilon)) \geq \mu_\omega(G^\omega_{n,\rho}) > 1 - \delta$ and

$$\# H_n \cdot e^{-n(\overline{h}^{BK}_\mu(T,\epsilon)+\rho)} \leq \sum_{x \in H_n} \mu_\omega(B_{d^\omega_n}(x, \epsilon)) = \mu_\omega(\bigcup_{x\in H_n} B_{d^\omega_n}(x, \epsilon)) \leq 1.$$

Then $N^\delta_{\mu_\omega}(n, 2\epsilon) \leq \# H_n \leq e^{n(\overline{h}^{BK}_\mu(T,\epsilon)+\rho)}$. This yields that

$$\begin{aligned}
\overline{h}^{BK}_\mu(T, \epsilon) + \rho &\geq \int_{\pi_\Omega E} \limsup_{n\to\infty} \frac{1}{n} \log N^\delta_{\mu_\omega}(n, 2\epsilon) d\mathbb{P}(\omega) \\
&= \int \limsup_{n\to\infty} \frac{1}{n} \log N^\delta_{\mu_\omega}(n, 2\epsilon) d\mathbb{P}(\omega) \\
&\geq \limsup_{n\to\infty} \frac{1}{n} \int \log N^\delta_{\mu_\omega}(n, 2\epsilon) d\mathbb{P}(\omega), \text{by Fatou's lemma.}
\end{aligned}$$

Letting $\delta \to 0$ and then letting $\rho \to 0$, we obtain $\overline{h}^{K}_\mu(T, 2\epsilon) \leq \overline{h}^{BK}_\mu(T, \epsilon)$ for every $\mu \in E_{\mathbb{P}}(\mathcal{E})$. Then by Theorem 3.5, we have

$$\mathbb{E}\overline{\text{mdim}}_{\text{M}}(T, \mathcal{E}, d) = \limsup_{\epsilon\to 0} \frac{1}{|\log \epsilon|} \sup_{\mu\in E_{\mathbb{P}}(\mathcal{E})} \overline{h}^{K}_\mu(T, \epsilon) \leq \limsup_{\epsilon\to 0} \frac{1}{|\log \epsilon|} \sup_{\mu\in E_{\mathbb{P}}(\mathcal{E})} \overline{h}^{BK}_\mu(T, \epsilon).$$

To illustrate our main theorem, we discuss the following example.

Example 3. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic measure-preserving system. Define the standard metric on torus $\mathbb{T}$ as follows:

$$||x - y||_{\mathbb{T}} = \min\{|x - y - n| : n \in \mathbb{N}\}$$

for each $x, y \in \mathbb{T}$. Set $\mathbb{T}^{\mathbb{Z}}$ equipped with the metric

$$d(x, y) = \sum_{n\in\mathbb{Z}} 2^{-|n|} ||x_n - y_n||_{\mathbb{T}}$$

and let $\sigma : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ be the shift on $\mathbb{T}$. Assume that $h : \Omega \to \mathbb{T}^{\mathbb{Z}}$ is a measurable map. We consider the $T_\omega(x)$ generated by $T_\omega(x) = \sigma(x) + h(\omega)$ for $\omega \in \Omega$, $x \in \mathbb{T}^{\mathbb{Z}}$. Observer

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that for any $0 \leq j < n$,

$$T_\omega^j(x) = \sigma^j(x) + \sum_{k=0}^{j-1} \sigma^{j-1-m} h(\theta^k \omega).$$

The skew product is defined by $\Theta^n(w, x) \to (\theta^n w, T_\omega^n x)$ for $n \in \mathbb{Z}$. Let $0 < \epsilon < \frac{1}{2}$, set $l = \lceil \log_2(4/\varepsilon) \rceil$. Then $\sum_{|n|>l} 2^{-|n|} \leq \varepsilon/2$. Consider the cover of $\mathbb{T}$ by

$$I_k = \left(\frac{(k-1)\varepsilon}{12}, \frac{(k+1)\varepsilon}{12} \right), \quad 0 \leq k \leq \lfloor 12/\varepsilon \rfloor.$$

I_k has length $\varepsilon/6$. For $n \geq 1$, consider

$$\mathbb{T}^{\mathbb{Z}} = \bigcup_{0 \leq k_{-l}, \ldots, k_{n+l} \leq \lfloor 12/\varepsilon \rfloor} \left\{ x \mid ||x_{-l} - \frac{k_\ell \epsilon}{12}||_{\mathbb{T}} < \frac{\epsilon}{12} \ldots, ||x_{n+\ell} - \frac{k_{n+\ell}\epsilon}{12}||_{\mathbb{T}} < \frac{\epsilon}{12} \right\}.$$

Each open set in the right-hand side has diameter less than ε with respect to the distance d_n. Hence

$$\#(\mathbb{T}^{\mathbb{Z}}, d_n, \varepsilon) \leq (1 + \lfloor 12/\varepsilon \rfloor)^{n+2l+1} = (1 + \lfloor 12/\varepsilon \rfloor)^{n+2\lceil \log_2(4/\varepsilon) \rceil + 1} \tag{3}$$

On the other hand, any two distinct points in the sets

$$\{x \in \mathbb{T}^{\mathbb{Z}} \mid x_m \in \{0, \varepsilon, 2\varepsilon, \ldots, \lfloor 1/\varepsilon \rfloor \varepsilon\} \text{for all } 0 \leq m < n\}$$

have distance $\geq \varepsilon$ with respect to d_n. It follows that

$$\#(\mathbb{T}^{\mathbb{Z}}, d_n, \varepsilon) \geq (1 + \lfloor 1/\varepsilon \rfloor)^n.$$

Thus

$$\mathrm{mdim_M}(\mathbb{T}^{\mathbb{Z}}, \sigma, \mathrm{d}) = 1.$$

Let d_Ω be the metric on Ω. Take the metric d' on $\Omega \times [0,1]^{\mathbb{Z}}$ as follows:

$$d'((\omega_1, x), (\omega_2, y)) = d(x, y) + d_\Omega(w_1, w_2), \quad \forall \omega_1, \omega_2 \in \Omega, \; x, y \in \mathbb{T}^{\mathbb{Z}}.$$

Note that for any $\omega \in \Omega$, $k \geq 0$ and $x, y \in \mathbb{T}^{\mathbb{Z}}$,

$$\begin{aligned}
&d'(\Theta^k(\omega, x), \Theta^k(\omega, y)) \\
&= d'((\theta^k \omega, T_\omega^n x), (\theta^k \omega, T_\omega^n y)) \\
&= d'((\theta^k \omega, \sigma^j(x) + \sum_{k=0}^{j-1} \sigma^{j-1-m} h(\theta^k \omega)), (\theta^k \omega, \sigma^j(y) + \sum_{k=0}^{j-1} \sigma^{j-1-m} h(\theta^k \omega)) \\
&= d(\sigma^k x, \sigma^k y).
\end{aligned}$$

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Hence for any $\epsilon > 0$, and $n \geq 1$, one has

$$\mathrm{sep}(\Omega \times \mathbb{T}^{\mathbb{Z}}, \omega, \epsilon, n) = \mathrm{sep}(\mathbb{T}^{\mathbb{Z}}, \epsilon, n), \quad \forall \omega \in \Omega.$$

Thus

$$\begin{aligned}
&\mathbb{E}\mathrm{dim}_M(T, \Omega \times \mathbb{T}^{\mathbb{Z}}, d) \\
&= \lim_{\epsilon \to 0} \int_{\Omega} \frac{1}{|\log \epsilon|} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(\Omega \times \mathbb{T}^{\mathbb{Z}}, \omega, \epsilon, n) d\mathbb{P}(\omega) \\
&= \lim_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(\mathbb{T}^{\mathbb{Z}}, \epsilon, n) \\
&= \mathrm{mdim}_M(\mathbb{T}^{\mathbb{Z}}, \sigma, d) = 1.
\end{aligned}$$

Let $\mathcal{L}$ be the Lebesgue measure on $\mathbb{T}$ and $\mu_\omega = \mathcal{L}^{\bigotimes \mathbb{Z}}$ for $\mathbb{P}$ a.e ω. Set $d\mu(\omega, x) = d\mu_\omega(x) d\mathbb{P}(\omega)$. Let $0 < \epsilon < \frac{1}{2}$, $x \in \mathbb{T}^{\mathbb{Z}}$. Take $\ell = \lceil \log_2 \frac{4}{\epsilon} \rceil$ such that $\sum_{|n| > \ell} \frac{1}{2^{|n|}} \leq \frac{\epsilon}{2}$. Let

$$I_n(x, \epsilon) = \left\{ y \in \mathbb{T}^{\mathbb{Z}} : ||y_k - x_k||_{\mathbb{T}} \leq \frac{\epsilon}{6}, \ \forall -\ell \leq k \leq n + \ell \right\},$$

and

$$J_n(x, \epsilon) = \left\{ y \in \mathbb{T}^{\mathbb{Z}} : ||y_k - x_k|_{\mathbb{T}} \leq \epsilon, \ \forall 0 \leq k < n \right\}.$$

It is easy to see that for any $\omega \in \Omega$,

$$I_n(x, \epsilon) \subset B_{d_n^\omega}(x, \epsilon) \subset J_n(x, \epsilon).$$

Since $\mu_\omega(I_n(x, \epsilon)) \geq \left(\frac{\epsilon}{6}\right)^{n+\ell}$ and $\mu_\omega(J_n(x, \epsilon)) \leq (4\epsilon)^n$, we obtain that

$$\log \frac{1}{4\epsilon} \leq h_{\mu_\omega}^{BK}(x, \epsilon) \leq \log \frac{3}{\epsilon}$$

for $\forall \omega \in \Omega$. Therefore

$$\lim_{\epsilon \to 0} \frac{\int h_{\mu_\omega}^{BK}(x, \epsilon) d\mu(\omega, x)}{|\log \epsilon|} = 1 = \mathbb{E}\mathrm{dim}_M(T, \Omega \times \mathbb{T}^{\mathbb{Z}}, d).$$

□

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