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FINITE SOLVABLE GROUPS WITH DISTINCT MONOMIAL CHARACTER DEGREES

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Abstract

In this paper we classify the finite solvable groups in which distinct nonlinear monomial characters have distinct degrees.

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1. Introduction

Providing a detailed classification of group structures under certain arithmetical conditions on the character degree set is a classical theme in group representation theory. Such classification results turn out to have various applications. For example, finite groups in which all the nonlinear irreducible characters have equal degrees were described by Isaacs, Passman, and others (see [9, Ch. 12]). At the other extreme, finite groups in which distinct nonlinear irreducible characters almost have distinct degrees are studied, for example, in [1–4, 6, 7, 10]. In particular, Berkovich *et al.* [2] have classified the finite groups in which distinct nonlinear irreducible characters have distinct degrees (we call them D-groups) and their result is stated below (Theorem 1.1). This can be viewed as a generalization of a result of Seitz [11] which showed that if G has just one nonlinear irreducible character, then either G is an extraspecial 2-group or else G is a Frobenius group with cyclic complement of order $p^n - 1$ and elementary abelian kernel of order p^n, where p is a prime. The result of Berkovich *et al.* has many nice applications including the study of the character degree graph [12] and the Galois conjugate of the characters [6, 7].

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Theorem 1.1. *Let G be a nonabelian D-group. Then one and only one of the following assertions holds.*

(i) *G is an extraspecial 2-group.*
(ii) *G is a 2-transitive Frobenius group with G' as its kernel.*
(iii) *G is a 2-transitive Frobenius group of order 72, where a Frobenius complement is isomorphic to a quaternion group of order 8.*

In this paper G always denotes a finite group, p is always a prime, all characters are complex characters, and we use Isaacs [9] as a source for standard notation and results from character theory. Recall that $\chi \in \mathrm{Irr}(G)$ is monomial if it is induced from a linear character of a subgroup of G.

The aim of this paper is to investigate the structure of solvable DM-groups, which are groups in which distinct nonlinear monomial characters have distinct degrees. We will provide a detailed and explicit classification of this type of groups. Clearly, if G is an M-group, that is, all irreducible characters of G are monomial, then G is a DM-group if and only if G is a D-group.

Remark 1.2. Our result, under the solvable condition, can be viewed as a generalization of the main result of [2]. Since a nonsolvable group may have no nonlinear monomial character (see, for example, [5]), it seems impossible to say something nontrivial for a nonsolvable DM-group.

We denote by $C(k), E(k), Q(k)$ and $ES(k)$, a cyclic group, an elementary abelian group, a generalized quaternion group and an extraspecial group respectively, and each of them has order k. We say that a group A acts Frobeniusly on a group N, provided that $A \ltimes N$ is a Frobenius group with the kernel N and a complement A. A p-group P is called a Camina p-group, provided that $|C_P(x)| = |P/P'|$ for all $x \in P - P'$.

Theorem 1.3. *Let G be a solvable DM-group but not a D-group. Then one and only one of the following assertions holds.*

(i) *$G = A \ltimes P$, where $P \cong ES(3^3)$, $A \cong Q(8)$ acts Frobeniusly on P/P' and acts trivially on P'.*
(ii) *$G = A \ltimes P$, where P is a Camina p-group of order p^{3m}, $P' = \Phi(P) = Z(P) \cong E(p^m)$, $A \cong C(p^{2m} - 1)$ acts Frobeniusly on P/P', $C_A(P') \cong C(p^m + 1)$.*
(iii) *$G = U \ltimes F$, where $U \cong \mathrm{SL}(2, 3)$ acts Frobeniusly on $F \cong E(5^2)$.*
(iv) *$G = U \ltimes F$, where $F \cong ES(5^3)$, $U \cong \mathrm{SL}(2, 3)$ acts Frobeniusly on F/F' and acts trivially on F'.*

Remark 1.4. The following corollary gives a classification of finite solvable groups in which distinct imprimitive characters have distinct degrees. Note that if a solvable group G possesses a primitive character of degree m, then G has at least two primitive characters with degree m. Indeed, let $\chi \in \mathrm{Irr}(G)$ be primitive and let λ be a nonprincipal linear character of G such that $\ker\lambda$ is maximal in G; then χ and $\lambda\chi$ are distinct primitive characters of G.

Corollary 1.5. *Let G be a solvable group in which distinct imprimitive characters have distinct degrees. Then one and only one of the following assertions holds.*

(1) *G is a D-group.*
(2) $G \cong \mathrm{SL}(2,3)$.
(3) *$G = U \ltimes F$, where $U \cong \mathrm{SL}(2,3)$ acts Frobeniusly on $F \cong E(5^2)$.*

Let $G^\sharp = G - \{1\}$ and $\mathrm{Irr}^\sharp(G) = \mathrm{Irr}(G) - \{1_G\}$. For a character χ, let $\mathrm{Irr}(\chi)$ be the set of irreducible constituents of χ.

Assume that $N \trianglelefteq G$, $A \leq G$, $\lambda \in \mathrm{Irr}(N)$. Put $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$ and $\mathrm{I}_A(\lambda) = \{a \in A \mid \lambda^a = \lambda\}$.

2. Lemmas

Lemma 2.1 [8, Ch. 2, Theorem 3.10]. *Suppose that an abelian group A acts faithfully and irreducibly on an elementary abelian group of order p^n. Then A is cyclic, and $|A|$ divides $p^n - 1$ but does not divide $p^k - 1$ for any positive integer $k < n$.*

Lemma 2.2. (1) [8, Ch. 2, Theorem 9.23] *Let V be a nondegenerate symplectic space of degree $2m$ over a finite field of q elements, and C be a cyclic subgroup of $Sp(2m, q)$. If C acts irreducibly on V, then $|C|$ divides $q^m + 1$.*

(2) [8, Ch. 3, Theorem 13.7] *Let P be a nonabelian p-group, where $P/Z(P)$ is elementary abelian and $Z(P)$ is cyclic. Then $P' = \langle c \rangle$ has order p, $P/Z(P) \cong E(p^{2m})$; furthermore, $\overline{P} := P/Z(P)$ becomes a nondegenerate symplectic space of degree $2m$ over a field of p elements, where $(\overline{x}, \overline{y})$ is defined to be a for $\overline{x}, \overline{y} \in \overline{P}$ with $[x, y] = c^a$.*

We will freely use the following fundamental facts.

- Every quotient group of a DM-group is also a DM-group.
- Let $N \leq Z(G)$ and $\lambda \in \mathrm{Irr}(N)$. Then $\lambda^G = \sum \chi_i(1)\chi_i$ for some $\chi_i \in \mathrm{Irr}(G)$.
- Let $G' \leq N \leq G$ and $\lambda \in \mathrm{Irr}(N)$. Then all members in $\mathrm{Irr}(\lambda^G)$ have the same degree; see [9, Problem 6.2].
- Let $N \trianglelefteq G$, $\chi \in \mathrm{Irr}(G)$ and $\lambda \in \mathrm{Irr}(\chi_N)$. Write $\chi_N = e(\lambda_1 + \cdots + \lambda_t)$, where $\lambda_1 = \lambda, \ldots, \lambda_t$ are distinct G-conjugates of λ. Observe that $[\chi_N, \chi_N] = e^2 t$, $|G : N| = (\lambda^G(1)/\lambda(1)) \geq (e\chi(1)/\lambda(1)) = e^2 t$, and χ vanishes on $G - N$ if and only if $[\chi_N, \chi_N] = |G : N|$; see [9, Lemma 2.29]. It follows that χ vanishes on $G - N$ if and only if $\lambda^G(1) = e\chi(1)$, that is, λ^G is a multiple of χ.
- A p-group P is a Camina p-group if and only if all nonlinear irreducible characters of P vanish on $P - P'$.

Lemma 2.3. *Let $G = B \ltimes P$, where P is a p-group and $B \cong C(p^a - 1)$ for some positive integer a. Suppose that G has a normal subgroup $E \leq \Phi(P) \cap Z(P)$ such that E is elementary abelian and B acts Frobeniusly on P/E. Assume that $\chi \in \mathrm{Irr}(G)$ has degree p^e, and assume further that e is even when $B \cong C(3)$ and $p = 2$. Then χ is monomial.*

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Proof. Write $\chi_P = \psi$ and $K = \ker\chi$. Clearly $\psi \in \mathrm{Irr}(P)$, $Z(P) \le Z(\psi) = Z(\chi) \cap P$; see [9, Lemma 2.27]. Since all linear characters are monomial, we may assume that χ and ψ are nonlinear.

Assume that $Z(\chi)$ is not a p-group. Take a nonidentity p'-element $z \in Z(\chi) \cap B$. Since $Z(\chi)/K = Z(G/K)$, we have $[z, P] \le P \cap K \le E(P \cap K)$. Thus z centralizes $P/(E(P \cap K))$. As B acts Frobeniusly on P/E, we have $P = E(P \cap K)$. This implies that $P/\ker\psi = P/(P \cap K) \cong E/(P \cap E \cap K)$ is abelian, a contradiction. Hence $Z(\chi)$ is a p-group, and it follows that K is a p-group and $Z(\chi) = Z(\psi)$.

Assume that $K > 1$. Clearly EK/K is an elementary abelian subgroup of $Z(P/K) \cap \Phi(P/K)$, and BK/K acts Frobeniusly on $(P/K)/(EK/K)$. Applying the inductive hypothesis to G/K, we get that χ is monomial. Therefore, we may assume that χ and ψ are faithful. Now $Z(G) = Z(\chi) = Z(\psi) = Z(P)$ is cyclic. Since $E \le Z(P)$ is elementary abelian and B acts Frobeniusly on P/E, it forces that $Z(P) = Z(G) = E \cong C(p)$.

Suppose that χ is primitive. By [9, Corollary 6.13], E is the unique nonidentity normal abelian subgroup of G. Let U/E be a p-chief factor of G. Then $B \cong C(p^a - 1)$ acts faithfully and irreducibly on U/E. Now Lemma 2.1 tells us that $U/E \cong E(p^a)$. As U is nonabelian, it is easy to see that $U \cong ES(p^{a+1})$ where a is even. Considering the action of B on U and applying Lemma 2.2, we get that $(p^a - 1) \mid (p^{a/2} + 1)$. This implies that

$$p^a = 2^2, \quad B \cong C(3).$$

Since $B \cong C(3)$ acts Frobeniusly on P/E, it follows by [8, Ch. 5, Remark 8.8] that the class of P/E is at most 2. This implies that $[P', P, P] = 1$ and therefore $P' = P'/[P', P, P]$ is an abelian normal subgroup of G. Hence

$$P' = E.$$

Assume that P has a unique involution. Then P is a generalized quaternion group, so $P/P' = P/E$ has order 4. This implies that $P \cong Q(8)$ and $\chi(1) = 2^1$, a contradiction. Assume that there is an involution $t \in P - E$. Let $V \trianglelefteq G$ be minimal such that $t \in V$. Since P/E is abelian and $C(3)$ acts Frobeniusly on P/E, it is easy to see that $V/E \cong E(2^2)$ and that V is a dihedral group of order 8. Observe that V has a unique cyclic subgroup of order 4; it follows that B acts reducibly on V/E, and this contradicts the minimality of V. Hence χ is imprimitive.

Since χ is imprimitive, there exist a maximal subgroup G_1 of G and an irreducible character χ_1 of G_1 such that $\chi = \chi_1^G$. Now we need only show that χ_1 is monomial. Since $\chi(1) = p^e$, G_1 contains a Hall p'-subgroup of G. Without loss of generality, we assume that $B \le G_1$. Then $G_1 = B \ltimes P_1$, where P_1 is a maximal B-invariant subgroup of P. Observe that

$$P_1 = P \cap G_1 \ge P \cap \Phi(G) \ge P \cap \Phi(P) = \Phi(P) \ge E$$

and that P/P_1 is a chief factor of G; it follows by Lemma 2.1 that P/P_1 has order p^a. This also implies that $\chi_1(1) = p^{e-a}$. Clearly $E \le Z(P_1)$. Note that if $P_1 = E$, then $\chi_1 \in \mathrm{Irr}(B \ltimes E)$ is linear, and the required result follows. Hence we may assume that $P_1 > E$. Clearly B acts Frobeniusly on P_1/E.

Assume that $E \nleq \Phi(P_1)$. Clearly $E\Phi(P_1)/\Phi(P_1)$ is a proper subgroup of $P_1/\Phi(P_1)$ because $E < P_1$. Observe that B acts completely reducibly on $P_1/\Phi(P_1)$; it follows that there exists a proper B-invariant subgroup P_2 of P_1 such that $P_1/\Phi(P_1) = E\Phi(P_1)/\Phi(P_1) \times P_2/\Phi(P_1)$. This implies that

$$P_1 = E\Phi(P_1)P_2 = EP_2 = E \times P_2, G_1 = E \times (B \ltimes P_2).$$

Thus $\chi_1 = \lambda\chi_2$, where $\lambda \in \mathrm{Irr}(E)$ and $\chi_2 \in \mathrm{Irr}(B \ltimes P_2)$. Since B acts Frobeniusly on P_2 and $\chi_2(1) = \chi_1(1)$ is a power of p, we see that χ_2 and χ_1 are linear, and we are done.

Assume that $E \leq \Phi(P_1)$. Observe that if $B \cong C(3)$ and $p = 2$, then $|P/P_1| = 2^2$ and $\chi_1(1) = 2^{e-2}$, where $e - 2$ is also even. Now induction implies that χ_1 is monomial, as desired. □

Lemma 2.4. *Suppose that P is a Camina p-group of class 2. Then $|P'| < |P/P'|$.*

Proof. Let $x \in P - P'$. Since all nonlinear irreducible characters of P vanish on $P - P'$, we have that $|P'| < |\langle x\rangle P'| \leq |C_P(x)| = \sum_{\xi\in\mathrm{Irr}(P)} |\xi(x)|^2 = |P/P'|$. □

Lemma 2.5. *Let P be a nonabelian p-group. Suppose that $P' = Z(P)$ and P/P' has order p^2. Then $P \cong ES(p^3)$.*

Proof. Observe that all nonlinear irreducible characters of P have degree $p = \sqrt{|P : Z(P)|}$ and so vanish on $P - Z(P)$. It follows that P is a Camina p-group of class 2. Now $P' \cong C(p)$ by Lemma 2.4, and so $P \cong ES(p^3)$. □

Lemma 2.6 [10, Lemma 2.7]. *Let $H \cong C(4)$ act on $N \cong ES(3^3)$. If H acts Frobeniusly on $N/Z(N)$, then H acts trivially on $Z(N)$.*

3. DM-groups with Fitting height 2

In order to classify solvable DM-groups, we have to investigate the groups G satisfying the following hypothesis.

Hypothesis 3.1. *Let $G = A \ltimes P$ be a DM-group and assume the following hold:*

(1) *A is either cyclic or isomorphic to $Q(8)$;*
(2) *P is a nonabelian p-group with $P' = \Phi(P)$;*
(3) *$|A| = p^c - 1$, $P/P' \cong E(p^c)$, and we always replace c by $2m$ if c is even;*
(4) *A acts Frobeniusly (and irreducibly) on P/P'.*

Lemma 3.2. *Let G satisfy Hypothesis 3.1 and $E \leq P'$ be a minimal normal subgroup of G. Set $B = C_A(E)$. Then the following assertions hold.*

(1) *$B = \mathrm{I}_A(\lambda) = C_A(e)$ for all $\lambda \in \mathrm{Irr}^\sharp(E)$ and for all $e \in E^\sharp$.*
(2) *If $B = 1$, then every $\psi \in \mathrm{Irr}(P|E)$ vanishes on $P - E$.*

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Proof. (1) Assume that $a \in A$ fixes some $\lambda \in \mathrm{Irr}^\sharp(E)$ (or some $e \in E^\sharp$). By [9, Theorem 13.24], a centralizes an element of $E^\sharp$. Observe that $C_E(a)$ is A-invariant because $\langle a\rangle$ is normal in A. Since $E \leq Z(P)$, $C_E(a) \trianglelefteq G$. Now the minimal normality of E yields that $C_E(a) = E$, that is, $a \in C_A(E) = B$. Thus $B = \mathrm{I}_A(\lambda) = C_A(e)$.

(2) Assume that $B = 1$ and let $\lambda \in \mathrm{Irr}^\sharp(E)$. Clearly $\mathrm{I}_G(\lambda) = P$ by (1). We need only show that $|\mathrm{Irr}(\lambda^P)| = 1$. Write $\lambda^P = \sum_{1\leq i\leq s} \psi_i(1)\psi_i$, where $\mathrm{Irr}(\lambda^P) = \{\psi_1, \dots, \psi_s\}$. By [9, Theorem 6.11], $\psi_1^G, \dots, \psi_s^G$ are distinct irreducible characters of G. Observe that all ψ_i^G are monomial because ψ_i are monomial; it follows that $\psi_1, \dots, \psi_s$ have distinct degrees. Since $\sum_{1\leq i\leq s} \psi_i(1)^2$ is a power of p, we conclude easily that $s = 1$, and the required result follows. □

Lemma 3.3. *Let G satisfy Hypothesis 3.1 and $E \trianglelefteq G$ with $E \leq P' \cap Z(P)$. Assume that $C := C_A(E) > 1$ acts Frobeniusly on P/E. Let $\lambda \in \mathrm{Irr}^\sharp(E)$ be such that $C = \mathrm{I}_A(\lambda)$ and let $P_1 \trianglelefteq G$ with $E \leq P_1 \leq P$. Then there exist $\xi_0 \in \mathrm{Irr}(\lambda^{P_1})$ and $\psi_0 \in \mathrm{Irr}((\xi_0)^P)$ such that $\mathrm{I}_A(\xi_0) = \mathrm{I}_A(\psi_0) = C$, $\mathrm{I}_A(\xi) = \mathrm{I}_A(\psi) = 1$ for all $\xi \in \mathrm{Irr}(\lambda^{P_1}) - \{\xi_0\}$ and all $\psi \in \mathrm{Irr}(\lambda^P) - \{\psi_0\}$. Furthermore, the following hold.*

(1) *$\Omega := \mathrm{Irr}(\lambda^P) - \{\psi_0\}$ is a union of some C-orbits of size $|C|$; ψ^G is monomial and of degree $|A|\psi(1)$ for every $\psi \in \Omega$; if $\psi_i, \psi_j \in \Omega$ lie in distinct C-orbits, then ψ_i and ψ_j have distinct degrees.*

(2) *If $C \cong C(p^a - 1)$ for some integer a and E is elementary abelian, then $|C| = 3$, $p = 2$, and $\psi_0(1) = 2^e$ for some odd integer e.*

(3) *If $E = P'$ and $|\mathrm{Irr}(\lambda^P)| \geq 2$, then $|C| = p^a - 1$ for some integer a.*

Proof. Let $1 < C_1 \leq C$. We claim that C_1 fixes a unique member of $\mathrm{Irr}(\lambda^{P_1})$. Since C_1 acts Frobeniusly on P_1/E and acts trivially on E, C_1 fixes exactly $|E|$ conjugacy classes of P_1. This implies by [9, Theorem 13.24] that C_1 fixes exactly $|E|$ irreducible characters of P_1. Since C_1 fixes μ for every $\mu \in \mathrm{Irr}(E)$ and thus fixes at least one irreducible constituent of μ^{P_1} (see [9, Theorem 13.28]), C_1 fixes a unique member, say ξ_0, of $\mathrm{Irr}(\lambda^{P_1})$.

By the randomness of C_1, C also fixes a unique member ξ^* in $\mathrm{Irr}(\lambda^{P_1})$. Since ξ^* is also C_1-invariant, we have that $\xi^* = \xi_0$ and $C \leq \mathrm{I}_A(\xi_0)$. Assume that $a \in A$ fixes ξ_0. Then a fixes λ because $(\xi_0)_E = \xi_0(1)\lambda$. Thus $a \in \mathrm{I}_A(\lambda) = C$. So $\mathrm{I}_A(\xi_0) = C$.

Let $\xi \in \mathrm{Irr}(\lambda^{P_1}) - \{\xi_0\}$ and $a \in \mathrm{I}_A(\xi)$. Then a fixes λ and so $a \in \mathrm{I}_A(\lambda) = C$. Now $\langle a\rangle$ fixes two irreducible constituents ξ_0 and ξ of λ^{P_1}. This implies by the preceding claim that $a = 1$. So $\mathrm{I}_A(\xi) = 1$.

Taking $P_1 = P$ and applying the established statements, we may find some $\psi_0 \in \mathrm{Irr}(\lambda^P)$ such that $\mathrm{I}_A(\psi_0) = C$ and $\mathrm{I}_A(\psi) = 1$ for all $\psi \in \mathrm{Irr}(\lambda^P) - \{\psi_0\}$. Since ξ_0 is C-invariant, C must fix some $\psi^* \in \mathrm{Irr}(\xi_0^P)$ by [9, Theorem 13.28]. Now the uniqueness of ψ_0 yields that $\psi_0 = \psi^*$, hence $\psi_0 \in \mathrm{Irr}((\xi_0)^P)$.

(1) Clearly C acts on Ω, and thus Ω is a union of some C-orbits.

Since $\mathrm{I}_C(\psi) = 1$ for all $\psi \in \Omega$, all C-orbits of Ω have size $|C|$. Since $\mathrm{I}_G(\lambda) = CP$ and ψ^{CP} is an irreducible constituent of λ^{CP}, ψ^G is irreducible. This implies that ψ^G is monomial and of degree $|A|\psi(1)$.

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Assume that $\psi_i, \psi_j \in \Omega$ lie in different C-orbits. Then $(\psi_i)^{CP} \neq (\psi_j)^{CP}$, so $(\psi_i)^G$ and $(\psi_j)^G$ are distinct monomial characters of G. Since G is a DM-group, ψ_i and ψ_j have distinct degrees.

(2) Clearly ψ_0 is nonlinear and extends to CP. Let $\chi_1, \ldots, \chi_{p^a-1}$ be different extensions of ψ_0 to CP; see [9, Corollary 6.17]. Since $\mathrm{I}_G(\lambda) = CP$ and $\chi_i \in \mathrm{Irr}(\lambda^{CP})$, we see that $(\chi_1)^G, \ldots, (\chi_{p^a-1})^G$ are distinct irreducible characters of G. Since G is a DM-group, all χ_i are nonmonomial. Now the required result follows by Lemma 2.3.

(3) Assume that $E = P'$ and $|\mathrm{Irr}(\lambda^P)| \geq 2$. Write $\lambda^P = p^e(\psi_0 + \psi_1 + \cdots + \psi_k)$, where $\psi_0(1) = \psi_1(1) = \cdots = \psi_k(1)$. Since $k = |C|$ by (1), we have $|P/P'| = \lambda^P(1) = p^e(1 + |C|)\psi_0(1)$. This implies that $|C| = p^a - 1$ for some integer a. □

Let a, m be positive integers where $a \geq 2$. A Zsigmondy prime for $a^m - 1$ is a prime divisor p of $a^m - 1$ such that $p \nmid a^i - 1$ for all $i = 1, \ldots, m - 1$. A well-known theorem of Zsigmondy asserts that there exists at least one Zsigmondy prime for $a^m - 1$ unless $(a, m) = (2, 6)$ or $(2^k - 1, 2)$.

Lemma 3.4. *Let G satisfy Hypothesis 3.1 and assume that P' is minimal normal in G.*

(1) *Suppose that $A \cong Q(8)$. Then $P \cong ES(p^3)$ and A centralizes P'.*
(2) *Suppose that A is cyclic. Then $P' = \Phi(P) = Z(P)$, $P/P' \cong E(p^{2m})$, P is a Camina p-group, $P' \cong E(p^m)$, and $C_A(P') \cong C(p^m + 1)$.*

Proof. Since P/P' and P' are chief factors of G, we have $P' = \Phi(P) = Z(P)$. Suppose that $A \cong Q(8)$. Then $P/P' \cong E(3^2)$ and so $P \cong ES(3^3)$ by Lemma 2.5. Applying Lemma 2.6, we get that A centralizes P'. Now suppose that A is cyclic. Write $B = C_A(P')$ and $|P'| = p^e$.

We claim that P is a Camina p-group. To see this, we need only show that $|\mathrm{Irr}(\lambda^P)| = 1$ for all $\lambda \in \mathrm{Irr}^\sharp(P')$. Assume this is not true and let $\lambda_0 \in \mathrm{Irr}^\sharp(P')$ be such that

$$(\lambda_0)^P = p^f(\psi_0 + \psi_1 + \cdots + \psi_k), \quad k \geq 1, \tag{3.1}$$

where $\psi_0, \psi_1, \ldots, \psi_k \in \mathrm{Irr}(P)$ are distinct and have the same degree p^f. Note that

$$B > 1, \quad B = \mathrm{I}_A(\lambda_0), \tag{3.2}$$

by Lemma 3.2. By Lemma 3.3, we may assume that $\mathrm{I}_A(\psi_0) = B$ and $\mathrm{I}_A(\psi_i) = 1$ for all $i \geq 1$; also

$$|B| = k$$

by Lemma 3.3(1). Calculating the degrees on both sides of (3.1), we get that

$$p^c = |P/P'| = p^{2f}(1 + |B|).$$

Therefore

$$B \cong C(p^{c-2f} - 1). \tag{3.3}$$

By (3.2), (3.3) and Lemma 3.3(2), we get that

$$B \cong C(3), \quad p = 2, \quad |P/P'| = 2^c = 2^{2f+2}, \quad \psi_0(1) = 2^f \quad \text{where } f \text{ is odd}, \tag{3.4}$$

and that λ_0^G has a monomial constituent χ_1 of degree $2^f|A| = \sqrt{2^{c-2}}|A|$.

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Suppose that $\{(\lambda_0)^g, g \in G\} = \mathrm{Irr}^\sharp(P')$. Then $2^e = |P'| = 1 + |G : \mathrm{I}_G(\lambda_0)| = 1 + (2^c - 1/3)$. It follows that $c = 2$, and this contradicts (3.4).

Suppose that there exists $\lambda \in \mathrm{Irr}^\sharp(P') - \{(\lambda_0)^g, g \in G\}$. Assume that $|\mathrm{Irr}(\lambda^P)| \geq 2$. Using the same arguments as above, we get that λ^G has a monomial constituent χ_2 of degree $\sqrt{2^{c-2}}|A|$. As $\chi_1 \neq \chi_2$, we get a contradiction. Assume that $|\mathrm{Irr}(\lambda^P)| = 1$. Then $\lambda^P = \psi(1)\psi$ for some $\psi \in \mathrm{Irr}(P)$, where $\psi(1) = \sqrt{|P/P'|} = 2^{f+1}$. Note that $\mathrm{I}_A(\lambda) = C_A(P')$ by Lemma 3.2 and that $C_A(P') = B \cong C(p^{c-2f} - 1)$ by (3.3). Now Lemma 3.3(2) yields that $f + 1$ is odd. However, f is odd by (3.4), a contradiction.

Hence P is a Camina p-group, as claimed. Now it is easy to check that $c = \log_p |P/P'|$ is even. Hence $P/P' \cong E(p^{2m})$ and $A \cong C(p^{2m} - 1)$.

In what follows, we will show that $C_A(P') \cong C(p^m + 1)$ and $P' \cong E(p^m)$. Since P is a Camina p-group, by Lemma 2.4 we see that

$$|P'| = p^e \leq p^{2m-1}. \tag{3.5}$$

Clearly if $A > B$, then A/B acts faithfully and irreducibly on P'. Suppose that $B = 1$. Since $A \cong C(p^{2m} - 1)$ acts faithfully and irreducibly on P', we get by Lemma 2.1 that $P' \cong E(p^{2m})$. This contradicts (3.5). Hence $B > 1$.

Suppose that $(p, 2m) = (2^r - 1, 2)$. Then $P \cong ES(p^3)$ by (3.5). We claim that $(p + 1) \mid |B|$. To see this, we may assume that $B < A$. Since A/B acts faithfully and irreducibly on $P' \cong C(p)$, we have that $|A/B|$ divides $p - 1$, and the claim follows. Now $B \geq C(p + 1)$ acts irreducibly on $P/P' \cong E(p^2)$. Investigating the action of B on P and applying Lemma 2.2, we get that $|B| \mid p + 1$, therefore $B \cong C(p + 1)$ as desired.

Suppose that $(p, 2m) = (2, 6)$. Clearly $A \cong C(63)$ and $|B| = 3, 7, 9, 21$ or 63. Assume that $|B| \in \{21, 63\}$. Then B acts irreducibly on P/P' (see Lemma 2.1). Let P'/F be a BP-chief factor. It is easy to see that $P'/F \cong C(2)$ and $P/F \cong ES(2^7)$. Investigating the action of B on P/F and applying Lemma 2.2, we get that $|B|$ divides $2^3 + 1$, a contradiction. Assume that $|B| \in \{3, 7\}$. Considering the action of A/B on P', we get by Lemma 2.1 that $|P'| = 2^6$, and this contradicts (3.5). Hence $|B| = 9$. Considering the action of A/B on P', we get that $P' \cong E(2^3)$, and the required result follows.

Now let us consider the remaining cases. The set π of Zsigmondy primes for $p^{2m} - 1$ is nonempty. Let R be a Hall π-subgroup of A. Since A/B acts irreducibly on P' and $|P'| \leq p^{2m-1}$, we get by Lemma 2.1 that $R \leq B$. Applying Lemma 2.1 again, we see that B acts irreducibly on P/P'. Let P'/F be a BP-chief factor. Investigating the action of B on P/F, we get that $P/F \cong ES(p^{2m+1})$, and that $|B|$ divides $p^m + 1$ by Lemma 2.2. Set $|A/B| = (p^m - 1)k$, where k divides $p^m + 1$. Investigating the action of A/B on P', we conclude that

$$(p^m - 1)k \mid (p^e - 1).$$

This implies that $m \mid e$ and thus $e = m$ by (3.5). Thus $k = 1$ and the required result follows. □

Lemma 3.5. *Let G satisfy Hypothesis 3.1. Then P' is minimal normal in G.*

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PROOF. Assume the result is not true and let P'/E be a chief factor of G. To see a contradiction, we may assume that E is minimal normal in G. Set $B = C_A(P'/E)$. By Lemma 3.4, we have that P/E is a Camina p-group, $P'/E = \Phi(P/E) = Z(P/E) \cong E(p^m)$, $P/P' \cong E(p^{2m})$, that $B \cong C(p^m + 1)$ if A is cyclic and $B = A$ if $A \cong Q(8)$.

Case 1. Suppose that $P' = Z(P)$.

Note that if $A \cong Q(8)$, then $P/P' = P/Z(P) \cong E(3^2)$, and Lemma 2.5 yields $|P'| = 3$, a contradiction. Hence A is cyclic, so $B \cong C(p^m + 1)$.

We claim that $C_A(P') = B = \mathrm{I}_A(\lambda)$ for all $\lambda \in \mathrm{Irr}^\sharp(P')$.

Assume first that there is a minimal A-invariant subgroup F of P' such that $F \neq E$. By Lemma 3.4 we get that $C_A(P'/F) \cong C(p^m + 1)$. Therefore $C_A(P'/E) = B = C_A(P'/F)$, whence $B = C_A(P')$. Let $x \in \mathrm{I}_A(\lambda)$. Then x centralizes a nonidentity element $y \in P'$. We may assume $y \in P - E$. Since $C_A(yE) = B$ by Lemma 3.2, we have $x \in B$. Thus $\mathrm{I}_A(\lambda) = B$, and the claim follows.

Assume that E is the unique minimal A-invariant subgroup of P'. Let $x \in \mathrm{I}_A(\lambda)$. Observe that $[P', \langle x\rangle]$ and $C_{P'}(x) > 1$ are G-invariant and that $P' = [P', \langle x\rangle] \times C_{P'}(x)$. Now the uniqueness of E implies that $[P', \langle x\rangle] = 1$. Hence $x \in C_A(P')$, and thus $C_A(P') = \mathrm{I}_A(\lambda)$ for all $\lambda \in \mathrm{Irr}^\sharp(P')$. In particular, $C_A(P') = \mathrm{I}_A(\lambda_0)$ for some $\lambda_0 \in \mathrm{Irr}^\sharp(P'/E)$. Since $\mathrm{I}_A(\lambda_0) = B$ by Lemma 3.2, we get that $C_A(P') = B = \mathrm{I}_A(\lambda)$, and the claim follows.

By [9, Theorem 13.14], we get by the claim that $C_A(x) = B$ for all nonidentity elements $x \in P'$. This also implies that $C_A(E) = B$. Now $A/B \cong C(p^m - 1)$ acts faithfully and irreducibly on E; it follows by Lemma 2.1 that $E \cong E(p^m)$.

Suppose that $|\mathrm{Irr}(\lambda^P)| = 1$ for all $\lambda \in \mathrm{Irr}^\sharp(P')$. Then P is a Camina p-group of class 2. This implies by Lemma 2.4 that $|P'| \leq p^{2m-1} < p^{2m} = |P'|$, a contradiction.

Suppose that $|\mathrm{Irr}(\lambda^P)| \geq 2$ for some $\lambda \in \mathrm{Irr}^\sharp(P')$. By Lemma 3.3(3), we get that $|B| = p^a - 1$ for some integer a. Now $p^m + 1 = p^a - 1$, and this implies that $p = 2$, $m = 1$ and $P/P' = P/Z(P) \cong E(2^2)$. Then Lemma 2.5 yields $|P'| = p$, a contradiction.

Case 2. Suppose that $P' \neq Z(P)$.

In this case, we have $E = Z(P) = [P', P]$. Clearly P' is abelian because P has class 3. Let $C = C_A(E)$ and $|C| = c$. We will work toward a contradiction via several steps.

Step 1. $B \cap C = 1$, $C = \mathrm{I}_A(\lambda)$ for all $\lambda \in \mathrm{Irr}^\sharp(E)$, $E \cong E(p^{2m})$.

Assume that $B \cap C > 1$. Observe that $B \cap C$ centralizes P', $C_{(B\cap C)P}(P') \trianglelefteq (B \cap C)P$, and $O^p((B \cap C)P) = (B \cap C)P$ because $B \cap C$ acts Frobeniusly on P/P'; it follows that $C_{(B\cap C)P}(P') = (B \cap C)P$. Then $P' \leq Z(P)$, a contradiction, hence $B \cap C = 1$.

By Lemma 3.2(1), $C = \mathrm{I}_A(\lambda)$ for all $\lambda \in \mathrm{Irr}^\sharp(E)$.

Since $B \cap C = 1$, A/C has a subgroup isomorphic to B where $B \cong C(p^m + 1)$ or $B = A$. Observe that A/C acts faithfully and irreducibly on E; it follows by Lemma 2.1 that $E \cong E(p^{2m})$.

Step 2. $c \geq 3$, A is cyclic, $B \cong C(p^m + 1)$, C acts Frobeniusly on P/E.

Assume that $c = 1$ and let $x \in P - P'$. Observe that all $\psi \in \mathrm{Irr}(P|E)$ vanish on $P - E$ by Lemma 3.2(2) and that P/E is a Camina p-group. We get that $p^{2m+1} \leq |\langle x\rangle E| \leq |C_P(x)| = |C_{P/E}(xE)| = |P/P'| = p^{2m}$, a contradiction.

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Assume that A is not cyclic. Then $A = B \cong Q(8)$. Since $B \cap C = 1$ by Step 1, we get that $c = 1$, a contradiction. Hence A is cyclic, so $B \cong C(p^m + 1)$ by Lemma 3.4.

Assume that $c = 2$. Then p is odd, and thus B and C have even order. Since $A \cong C(p^{2m} - 1)$ contains a unique involution, it follows that $B \cap C > 1$, a contradiction.

Assume that $y \in C^\sharp$ centralizes a nonidentity element of P/E. Then y centralizes a nonidentity element of P'/E because A acts Frobeniusly on P/P'. This implies by Lemma 3.2(1) that $y \in C_A(P'/E) = B$, and then $y \in B \cap C = 1$, a contradiction. Hence C acts Frobeniusly on P/E.

Step 3. We fix some notation.

For a given $\lambda \in \mathrm{Irr}^\sharp(E)$, by Lemma 3.3 there exist $\xi_0 \in \mathrm{Irr}(\lambda^{P'})$, $\psi_0 \in \mathrm{Irr}(\xi_0)^P$ such that

$$\mathrm{I}_A(\xi_0) = \mathrm{I}_A(\psi_0) = C, \quad \mathrm{I}_A(\xi) = \mathrm{I}_A(\psi) = 1,$$

for all $\xi \in \mathrm{Irr}(\lambda^{P'}) - \{\xi_0\}$ and all $\psi \in \mathrm{Irr}(\lambda^P) - \{\psi_0\}$. Let γ be a P-orbit which ξ_0 lies in and let $\omega = \mathrm{Irr}(\lambda^{P'}) - \{\gamma\}$. Clearly P acts on γ and ω. Observe that C also acts on γ and ω, and thus acts on $\bigcup_{\xi\in\gamma} \mathrm{Irr}(\xi^P)$ and $\bigcup_{\xi\in\omega} \mathrm{Irr}(\xi^P)$. Clearly

$$\bigcup_{\xi\in\gamma} \mathrm{Irr}(\xi^P) = \mathrm{Irr}((\xi_0)^P), \quad \bigcup_{\xi\in\gamma} \mathrm{Irr}(\xi^P) \cap \bigcup_{\xi\in\omega} \mathrm{Irr}(\xi^P) = \emptyset.$$

Since P' is abelian, we have

$$\lambda^{P'} = \sum_{\xi\in\gamma} \xi + \sum_{\xi\in\omega} \xi,$$

where $|\gamma| + |\omega| = |P'/E| = p^m$. This implies that

$$\lambda^P = \sum_{\xi\in\gamma} \xi^P + \sum_{\xi\in\omega} \xi^P. \tag{3.6}$$

Write

$$T = \mathrm{I}_P(\xi_0), \quad |\gamma| = |P : T| = p^d.$$

Write $(\xi_0)^T = p^a(\mu_0 + \mu_1 + \cdots + \mu_k)$ where $k \geq 0$, and set $\mu_j^P = \psi_j$. We have

$$p^a = \mu_j(1), \quad (\xi_0)^P = p^a(\psi_0 + \psi_1 + \cdots + \psi_k).$$

Observe that

$$\sum_{\xi\in\gamma} \xi^P = |\gamma|(\xi_0)^P = p^d(\xi_0)^P \tag{3.7}$$

and that $[\lambda^P, \tau] = \tau(1)$ for all $\tau \in \mathrm{Irr}(\lambda^P)$; it follows that

$$\sum_{\xi\in\gamma} \xi^P = p^{a+d}(\psi_0 + \psi_1 + \cdots + \psi_k) \quad \text{where } p^{a+d} = \psi_0(1) = \cdots = \psi_k(1). \tag{3.8}$$

Since C acts on $\bigcup_{\xi\in\gamma} \mathrm{Irr}(\xi^P) = \{\psi_0, \psi_1, \ldots, \psi_k\}$, we conclude by Lemma 3.3(1) that if $k \geq 1$, then $\{\psi_1, \ldots, \psi_k\}$ is a C-orbit of size

$$k = |C| = c. \tag{3.9}$$

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Let us consider the following cases: (i) $\omega = \emptyset$ and $k = 0$; (ii) $\omega = \emptyset$ and $k = c$; (iii) $\omega \neq \emptyset$.

Step 4. Assume that case (i) holds. Then ψ_0 vanishes on $P - E$, $\psi_0(1) = \sqrt{p^{3m}}$, and if in addition $c = 3$, $p = 2$, then $m \equiv 2 \bmod 4$.

Clearly $\lambda^P = \psi_0(1)\psi_0$. Hence ψ_0 has degree $\sqrt{|P/E|} = \sqrt{p^{3m}}$ and vanishes on $P - E$. Assume further that $c = 3$ and $p = 2$. Then Lemma 3.3(2) yields that $3m/2$ is odd, that is, $m \equiv 2 \bmod 4$.

Step 5. Assume that case (ii) holds. Then $c = 3$, $p = 2$, $m \equiv 0 \bmod 4$, and λ^G has a monomial character of degree $(2^{2m} - 1)\sqrt{2^{3m-2}}$.

By (3.6), (3.8) and (3.9), we have that $\lambda^P = p^{a+d}(\psi_0 + \psi_1 + \cdots + \psi_c)$. Then $p^{3m} = \lambda^P(1) = (1 + c)\psi_0(1)^2$, so $c = p^a - 1$ for some $a \in \mathbb{Z}^+$. This implies by Lemma 3.3(2) that $c = 3$, $p = 2$, $\psi_0(1) = \sqrt{(p^{3m}/1 + c)} = \sqrt{2^{3m-2}}$ where $3m - 2/2$ is odd (that is, $m \equiv 0 \bmod 4$). Also, Lemma 3.3(1) guarantees that λ^G possesses a monomial constituent of degree $(2^{2m} - 1)\sqrt{2^{3m-2}}$.

Step 6. Assume that case (iii) holds. Then $c = 3$, $p = 2$, λ^G has a monomial constituent of degree $(2^{2m} - 1)\sqrt{2^{3m-2}}$, $P' \leq T := \mathrm{I}_P(\xi_0) < P$ and each $\psi \in \mathrm{Irr}(\lambda^P)$ vanishes on $P - T$.

Let $\xi \in \omega$. Since ξ_0 is an extension of λ to P', there exists $\nu \in \mathrm{Irr}(P'/E)$ such that $\xi = \nu\xi_0$. Observe that ν is P-invariant because $P'/E = Z(P/E)$; it follows that $\mathrm{I}_P(\xi) = \mathrm{I}_P(\xi_0) = T$. If $\mathrm{I}_G(\xi)$ is not a p-group, then a nonidentity element of A fixes some P-conjugate ξ^y of ξ. However, $\xi^y \neq \xi_0$ because $\xi^y \in \omega$, and we have $\mathrm{I}_A(\xi^y) = 1$, a contradiction. Thus

$$\mathrm{I}_G(\xi) = \mathrm{I}_P(\xi) = \mathrm{I}_P(\xi_0) = T. \tag{3.10}$$

By (3.10), all members in $\mathrm{Irr}(\lambda^P) = \bigcup_{\xi\in\omega} \mathrm{Irr}(\xi^P) \cup \mathrm{Irr}(\xi_0^P)$ vanish on $P - T$. Assume $T = P$, that is, $\xi_0 \in \mathrm{Irr}(P')$ is P-invariant. Then $P'/\ker\xi_0 \leq Z(P/\ker\xi_0)$; this implies that $\ker\xi_0 \geq [P', P] = E$. However, $(\xi_0)_E = \lambda \neq 1_E$, a contradiction. Thus $P' \leq T < P$.

By (3.10), all irreducible constituents of ξ^G are monomial, hence distinct irreducible constituents of ξ^T have distinct degrees. Since T/P' is abelian, all irreducible constituents of ξ^T have the same degree. Therefore ξ^P is a multiple of an irreducible character ψ of P. This implies that

$$p^{2m+d} = |P/P'||P : T| = \xi^P(1)|P : T| = \psi(1)^2.$$

Now all members in $\bigcup_{\xi\in\omega} \mathrm{Irr}(\xi^P)$ have the same degree $\sqrt{p^{2m+d}}$. By Lemma 3.3(1), $\bigcup_{\xi\in\omega} \mathrm{Irr}(\xi^P)$ is a C-orbit of size c. Since $[\lambda^P, \tau] = \tau(1)$ for all $\tau \in \mathrm{Irr}(\lambda^P)$, we have

$$\sum_{\xi\in\omega} \xi^P = \sqrt{p^{2m+d}}(\psi_1 + \cdots + \psi_c) \quad \text{where } \sqrt{p^{2m+d}} = \psi_1(1) = \cdots = \psi_c(1). \tag{3.11}$$

By (3.6), (3.7) and (3.11), we have

$$p^{3m} = \lambda^P(1) = \sum_{\xi\in\gamma} \xi^P(1) + \sum_{\xi\in\omega} \xi^P(1) = p^d p^{2m} + cp^{d+2m}.$$

So

$$c = p^{m-d} - 1, \quad p^d = \frac{p^m}{1+c}.$$

By Lemma 3.3, we get that $c = 3$, $p = 2$, and that λ^G has a monomial character of degree $|A|\sqrt{p^{2m+d}} = (2^{2m} - 1)\sqrt{2^{3m-2}}$.

Step 7. Final contradiction.

By Step 1, we have that $|\mathrm{Irr}^\sharp(E)| = p^{2m} - 1$ and that every G-orbit of $\mathrm{Irr}^\sharp(E)$ has size $|A/C| = p^{2m} - 1/c$. Thus $\mathrm{Irr}^\sharp(E)$ is a union of c orbits of G. Let $\lambda_1, \ldots, \lambda_c \in \mathrm{Irr}^\sharp(E)$ belong to different G-orbits. Observe that case (ii) holds for at most one λ_i by Step 5, and that case (iii) holds for at most one λ_j by Step 6.

Suppose that case (i) does not occur for any λ_i. Since $c \geq 3$ by Step 2, either case (ii) or case (iii) occurs for different λ_i and λ_j, a contradiction.

Suppose that case (i) holds for some λ_i, say λ_1. By Steps 4 and 5, case (ii) cannot occur for any λ_i. Assume that case (iii) does not occur for any i, then let $x_0 \in P - P'$. Assume that case (iii) occurs for some λ_i, say λ_2. Then case (i) occurs for all λ_i with $i \geq 3$, and let $x_0 \in P - \mathrm{I}_P(\xi_0)$ where ξ_0 corresponds to λ_2. By Steps 4 and 6, all $\psi \in \mathrm{Irr}(P|E)$ vanish on x_0. This leads to a contradiction that $p^{2m+1} \leq |\langle x_0\rangle E| \leq |C_P(x_0)| = |C_{P/E}(x_0E)| = |C_{P/P'}(x_0P')| = p^{2m}$. □

PROPOSITION 3.6. *Let G be a solvable DM-group with Fitting height 2. Then one and only one of the following holds:*

(1) *$G = A \ltimes P$, where $A \cong C(p^n - 1)$ acts Frobeniusly on $P \cong E(p^n)$;*
(2) *$G = A \ltimes P$, where $A \cong Q(8)$ acts Frobeniusly on $P \cong E(3^2)$;*
(3) *$G = A \ltimes P$, where $P \cong ES(3^3)$, and $A \cong Q(8)$ acts Frobeniusly on P/P' and acts trivially on P';*
(4) *$G = A \ltimes P$, where $P' = \Phi(P) = Z(P) \cong E(p^m)$, $P/P' \cong E(p^{2m})$, P is a Camina p-group, and $A \cong C(p^{2m} - 1)$ acts Frobeniusly on P/P', $C_A(P') \cong C(p^m + 1)$.*

PROOF. Suppose that G is a solvable DM-group with Fitting height 2. Since $G/\mathrm{F}(G)'$ is an M-group (see [9, Theorems 6.22 and 6.23]), $G/\mathrm{F}(G)'$ is a D-group with Fitting height 2. If $\mathrm{F}(G)$ is abelian, then Theorem 1.1 implies that G is of type (1) or (2). In particular, $\mathrm{F}(G)/\mathrm{F}(G)'$ is a chief factor of G, and $\mathrm{F}(G)$ is a normal Sylow p-subgroup of G for some prime p. If $\mathrm{F}(G)' > 1$, then G satisfies Hypothesis 3.1, so G is of type (3) or (4) by Lemmas 3.4 and 3.5.

Suppose conversely that G is of type (1) or (2) or (3). It is easy to see that G is a solvable DM-group with Fitting height 2. Assume that G is of type (4). To see that G is a DM-group, we need only show that every $\chi \in \mathrm{Irr}(G|P')$ is nonmonomial. Let $\lambda \in \mathrm{Irr}^\sharp(P')$ and write $B := C_A(P') \cong C(p^m + 1)$. Using the same arguments as in Lemma 3.2, we get that $\mathrm{I}_A(\lambda) = B$. Now $\mathrm{I}_G(\lambda) = B \ltimes P$. Observe that $\lambda^P = p^m\phi$, where $\phi \in \mathrm{Irr}(P)$ has degree p^m because P is a Camina p-group. We see that ϕ is B-invariant and thus extends to BP. This implies that every $\chi \in \mathrm{Irr}(G|P')$ has degree $(p^m - 1)p^m$. Notice that B acts irreducibly on P/P' by Lemma 2.1. It is easy to see that G has no subgroup of order $(p^m + 1)p^{2m} = |G|/\chi(1)$. Thus χ is nonmonomial, and we are done. □

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4. Main theorem

PROPOSITION 4.1. *Let G be a solvable DM-group with Fitting height* 3. *Then one and only one of the following holds:*

(i) $G = U \ltimes F$, *where* $U \cong \mathrm{SL}(2,3)$ *acts Frobeniusly on* $F \cong E(5^2)$*;*
(ii) $G = U \ltimes F$, *where* $F \cong ES(5^3)$, *and* $U \cong \mathrm{SL}(2,3)$ *acts Frobeniusly on* F/F' *and acts trivially on* F'.

PROOF. Assume that G is of type (i). It is routine to check that G is a solvable DM-group with Fitting height 3. Assume that G is of type (ii). To see that G is a solvable DM-group with Fitting height 3, we need only show that every $\chi \in \mathrm{Irr}(G|F')$ is nonmonomial. Let ψ be an irreducible constituent of χ_F. Then ψ has degree 5, and $5\psi = \lambda^F$ for some nonprincipal $\lambda \in \mathrm{Irr}(F')$. Clearly λ is G-invariant because $F' \leq Z(G)$. Since ψ is a unique irreducible constituent of λ^F, we see that ψ is G-invariant. As ψ is extendible to G, $\chi(1) = 5m$ where $m \in \mathrm{cd}(U) = \{1, 2, 3\}$. It is easy to see that G has no subgroup with index $5m$, and hence χ is nonmonomial.

Suppose conversely that G is a solvable DM-group with Fitting height 3. Let $F \trianglelefteq G$ be minimal such that G/F has Fitting height 2. Clearly F is nilpotent. By Proposition 3.6, we see that the Fitting subgroup $\mathrm{F}(G/F)$ is a p-group and $(G/F)/\mathrm{F}(G/F)$ is a p'-group. Write $\pi = \pi(G/F) - \{p\}$ and let F/E be a chief factor of G. Then G/E has Fitting height 3 and G/E splits over F/E.

(1) We claim that G/E is of type (i).

To prove the claim, we may assume that $E = 1$. Now $G = (A \ltimes P) \ltimes F$, where A is a nilpotent π-group, and P is a p-group. Let $\lambda \in \mathrm{Irr}^\sharp(F)$. Clearly λ extends to $\mathrm{I}_G(\lambda)$. Let $\lambda_1, \dots, \lambda_k$ be all distinct extensions of λ to $\mathrm{I}_G(\lambda)$. By [9, Corollary 6.17], we have

$$k = |\mathrm{I}_G(\lambda)/\mathrm{I}_G(\lambda)' F|.$$

Observe that every λ_i^G is monomial and of degree $|G : \mathrm{I}_G(\lambda)|$; it follows that either $k = 1$ or $G = \mathrm{I}_G(\lambda)$.

Note that if $G = \mathrm{I}_G(\lambda)$ for some $\lambda \in \mathrm{Irr}^\sharp(F)$, then $F \leq Z(G)$, and this implies that G and G/F have the same Fitting height, a contradiction. Hence $k = 1$ and so $\mathrm{I}_G(\lambda) = F$ for all $\lambda \in \mathrm{Irr}^\sharp(F)$. This implies that G is a Frobenius group with F as its kernel.

Assume that P is abelian. Then A acts Frobeniusly on P by Proposition 3.6, and therefore acts Frobeniusly on PF. This implies that PF is nilpotent, so G has Fitting height 2, a contradiction. Hence P is nonabelian and thus is a generalized quaternion group. Applying Proposition 3.6 and investigating the structure of $A \ltimes P$, we get that $P \cong Q(8)$, $A \cong C(3)$, and $A \ltimes P \cong \mathrm{SL}(2,3)$. Since every $\chi \in \mathrm{Irr}(G|F)$ is monomial and of degree $|A \ltimes P| = 24$, we get that $|\mathrm{Irr}(G|F)| = 1$. So $F \cong E(5^2)$, and (i) holds.

(2) We claim that if $E > 1$, then G is of type (ii).

By (1), we see that F is a 5-group and that $G = U \ltimes F$, where $U = A \ltimes P \cong \mathrm{SL}(2,3)$, $A \cong C(3)$, $P \cong Q(8)$. Now every $\lambda \in \mathrm{Irr}(F)$ extends to $\mathrm{I}_G(\lambda)$. Using the same arguments as in (1), we conclude that U acts Frobeniusly on F/F' and that $|F/F'| = 5^2$. Hence

$$F/F' \cong E(5^2), \quad E = F' = \Phi(F).$$

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Let $D = [F', F]$. By Lemma 2.5, we get that $|F'/D| = 5$ and $F/D \cong ES(5^3)$. Since $U/C_U(F'/D)$ is a cyclic group of order divisible by 4, we conclude that $C_U(F'/D) = U$. Thus G/D is of type (ii).

Now it suffices to show that $D = 1$. Suppose that $D > 1$. To see a contradiction, we may assume that D is minimal normal in G. Clearly F has class 3 and $Z(F) = D = [F', F]$. Let us consider $\mathrm{I}_U(\lambda)$ where $\lambda \in \mathrm{Irr}^\sharp(D)$.

Assume that $\mathrm{I}_U(\lambda)$ has even order. Then $Z(P)$ fixes λ because $Z(P)$ is the unique subgroup of U of order 2. Since $C_D(Z(P))$ is normal in G and $Z(P)$ centralizes an element of $D^\sharp$, we have $C_D(Z(P)) = D$. Now $Z(P)$ centralizes F'/D and D, and so centralizes F'. If $C_{Z(P)F}(F') < Z(P)F$, then $O^5(Z(P)F) < Z(P)F$, which is impossible because $Z(P)$ acts Frobeniusly on F/F'. If $C_{Z(P)F}(F') = Z(P)F$, then $F' \leq Z(F)$, and we get a contradiction.

Assume that $\mathrm{I}_U(\lambda) = 1$. Then $\mathrm{I}_G(\lambda) = F$. Observe that all irreducible constituents of λ^F have degree 5 because $|F : Z(F)| = 5^3$. We have $\lambda^F = 5(\phi_1 + \cdots + \phi_5)$. Now ϕ_i^G, $i = 1, \ldots, 5$, are distinct nonlinear monomial characters of G, a contradiction.

Hence $\mathrm{I}_U(\lambda)$ is exactly a Sylow 3-subgroup of U for every $\lambda \in \mathrm{Irr}^\sharp(D)$. Let us consider the action of U on the abelian group $\mathrm{Irr}(F) \cong F$. We get that

$$\mathrm{Irr}(D) = \bigcup_{X \in \mathrm{Syl}_3(U)} C_{\mathrm{Irr}(D)}(X),$$

and that

$$C_{\mathrm{Irr}(D)}(X) \cap C_{\mathrm{Irr}(D)}(Y) = \{1_D\}$$

whenever $X, Y \in \mathrm{Syl}_3(U)$ are different. Set $|D| = 5^m$, $|C_{\mathrm{Irr}(D)}(A)| = 5^e$. We get that

$$5^m - 1 = |\mathrm{Irr}^\sharp(D)| = |\mathrm{Syl}_3(U)|(5^e - 1) = 4 \cdot (5^e - 1),$$

which is impossible. Thus $D = 1$ and the proof is compete. □

Proof of Theorem 1.3. By Propositions 3.6 and 4.1, we need only show that all solvable DM-groups have Fitting height at most 3. Assume that this is not true and let G be a counterexample of minimal order. Then $G = H \ltimes V$, where H has Fitting height 3 and acts faithfully and irreducibly on V. Arguing as in Proposition 4.1, we get that H acts Frobeniusly on V. In particular, all 5-subgroups of H are cyclic. However, $\mathrm{F}(H) \cong E(5^2)$ or $ES(5^3)$ by Proposition 4.1, we get a contradiction, and the proof is complete. □

Proof of Corollary 1.5. Note that all linear characters are primitive. It is easy to verify that if G is one of the groups listed in the corollary, then distinct imprimitive characters of G have distinct degrees.

Suppose conversely that G is a solvable group in which distinct imprimitive characters have distinct degrees. Then G is a solvable DM-group and Theorem 1.3 applies.

Assume that G is of type (i) in Theorem 1.3. Then G has two imprimitive characters of degree 6, a contradiction.

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Assume that G is of type (ii) in Theorem 1.3. Observe that if $p^m - 1 > 1$, then $|\mathrm{Irr}(G|P')| > 1$ and all members in $\mathrm{Irr}(G|P')$ are imprimitive and have the same degree $p^m(p^m - 1)$. This implies that $p^m - 1 = 1$, so $G \cong \mathrm{SL}(2, 3)$.

Assume that G is of type (iv) in Theorem 1.3. Then there are five imprimitive members in $\mathrm{Irr}(G|F')$ with the same degree 15, a contradiction. Now the proof is complete. □

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