J. Appl. Probab. 1–22 (2025)
*doi:*10.1017/jpr.2025.10049

LARGE DEVIATIONS FOR DYNAMICAL SCHRÖDINGER PROBLEMS

KENGO KATO,* *Cornell University*

Abstract

We establish large deviations for dynamical Schrödinger problems driven by perturbed Brownian motions when the noise parameter tends to zero. Our results show that Schrödinger bridges charge exponentially small masses outside the support of the limiting law that agrees with the optimal solution to the dynamical Monge–Kantorovich optimal transport problem. Our proofs build on mixture representations of Schrödinger bridges and establishing exponential continuity of Brownian bridges with respect to the initial and terminal points.

Keywords: Dynamical Schrödinger problem; entropic optimal transport; large deviation; Schrödinger bridge

2020 Mathematics Subject Classification: Primary 60F10; 60F07
Secondary 49N15

1. Introduction

1.1. Overview

The dynamical Schrödinger problem [19, 27] seeks to find the entropic projection of a reference path measure (such as a Wiener measure) onto the space of path measures with given initial and terminal distributions. Originally motivated by physics, the problem has received increasing interest from other application domains such as statistics and machine learning; see [4, 16, 38, 43] and references therein. From a purely mathematical point of view, the time marginal flow, called entropic interpolation, provides a powerful technique for deriving functional inequalities and analysis of metric measure spaces [5, 20, 21], making the dynamical Schrödinger problem of intrinsic interest. Additionally, the static version of the Schrödinger problem is equivalent to quadratic entropic optimal transport (EOT) [35], the analysis of which has seen extensive research activities. This is in particular due to EOT admitting efficient computation via Sinkhorn's algorithm, which lends itself well to large-scale data analysis [14, 39].

Schrödinger problems can be interpreted as noisy counterparts of Monge–Kantorovich optimal transport (OT) problems. In particular, [26, 31, 32] studied the rigorous connection between the two problems, establishing convergence of optimal solutions for dynamical Schrödinger problems (Schrödinger bridges) toward the dynamical OT problem when the noise level tends to zero. In this work, we study local rates of convergence of Schrödinger

Received 22 August 2025; accepted 22 August 2025.
* Postal address: CIS Building 304, Ithaca, NY 14853, USA. Email: kk976@cornell.edu

bridges toward the limiting law. Specifically, we establish large-deviation principles (LDPs) for Schrödinger bridges on a path space and characterize the rate function.

Our baseline setting goes as follows. Let μ_0, μ_1 be Borel probability measures on $\mathbb{R}^d$ with finite second moments that will be fixed throughout. Let E be the space of continuous maps $[0, 1] \to \mathbb{R}^d$ endowed with the sup norm $\|\omega\|_E = \sup_{t\in[0,1]} |\omega(t)|$ for $\omega = (\omega(t))_{t\in[0,1]} \in E$ (we use $|\cdot|$ to denote the Euclidean norm). For a given $\varepsilon > 0$ (noise level), let R^ε be the law, defined on the Borel σ-field of E, of $\xi + \sqrt{\varepsilon}W$, where $\xi \sim \mu_0$ and $W = (W(t))_{t\in[0,1]}$ is a standard Brownian motion starting at 0 independent of ξ. For $s, t \in [0, 1]$, we denote the projections at t and (s, t) as e_t and e_{st}, respectively, i.e. $e_t(\omega) = \omega(t)$ and $e_{st}(\omega) = (\omega(s), \omega(t))$ for $\omega \in E$. For a given Borel probability measure P on E, we write $P_t = P \circ e_t^{-1}$ and $P_{st} = P \circ e_{st}^{-1}$. Given two endpoint marginals μ_0, μ_1 and a reference measure R^ε, the dynamical Schrödinger problem reads as

$$\min_{P:P_0=\mu_0,P_1=\mu_1} \mathcal{H}(P \mid R^\varepsilon), \tag{1}$$

where $\mathcal{H}(\cdot \mid \cdot)$ denotes the relative entropy (see Section 1.4 for the formal definition). Provided μ_1 has finite entropy relative to the Lebesgue measure (cf. Remark 1), the problem in (1) admits a unique optimal solution P^ε, called the *Schrödinger bridge*. The solution P^ε is given by a mixture of Brownian bridges against a (unique) optimal solution π_ε to the static Schrödinger problem

$$\min_{\pi\in\Pi(\mu_0,\mu_1)} \mathcal{H}(\pi \mid R^\varepsilon_{01}), \tag{2}$$

where $\Pi(\mu_0, \mu_1)$ is the set of couplings with marginals μ_0 and μ_1. The zero-noise limit ($\varepsilon \downarrow 0$) of (2) corresponds to the OT problem with quadratic cost $c(x, y) = |x - y|^2/2$,

$$\min_{\pi\in\Pi(\mu_0,\mu_1)} \int c \, \mathrm{d}\pi, \tag{3}$$

which admits a unique optimal solution (OT plan) π_{o} (as μ_1 is assumed to be absolutely continuous; [7]).

In his influential work [31], Mikami proved, under an additional assumption that μ_0 is absolutely continuous, that P^ε converges weakly to the law P^{o} of the geodesic path connecting two random endpoints following π_{o}, $t \mapsto \sigma^{\xi_0,\xi_1}(t)$ for $\sigma^{xy}(t) = (1 - t)x + ty$ and $(\xi_0, \xi_1) \sim \pi_{\mathrm{o}}$, i.e. $P^{\mathrm{o}} = \int \delta_{\sigma^{xy}} \, \mathrm{d}\pi_{\mathrm{o}}(x, y)$ with $\delta_\cdot$ denoting the Dirac delta ([31] indeed proved convergence with respect to Wasserstein W_2 distance). The limiting law P^{o} can be characterized as an optimal solution to the dynamical OT problem

$$\min_{P:P_0=\mu_0,P_1=\mu_1} \int \left(\frac{1}{2} \int_0^1 |\dot{\omega}(t)|^2 \, \mathrm{d}t \right) \mathrm{d}P(\omega),$$

where $\dot{\omega}(t)$ denotes the time derivative of ω and $\int_0^1 |\dot{\omega}(t)|^2 \, \mathrm{d}t = \infty$ if ω is not absolutely continuous [26]. The marginal laws of the limiting process give rise to a constant-speed geodesic (displacement interpolation; [29]) in the Wasserstein space connecting μ_0 and μ_1.

Our main large deviation results establish that (see Sections 1.4 and 2 for notation and definitions), under regularity conditions, for any sequence $\varepsilon_k \downarrow 0$, the Schrödinger bridges P^{ε_k} satisfy an LDP with rate function $I(h) = \int_0^1 (|\dot{h}(t)|^2/2) \, \mathrm{d}t - \psi^c(h(0)) - \psi(h(1))$, where ψ is an OT (or Kantorovich) potential from μ_1 to μ_0 and ψ^c is its c-transform (the rate function I is set to ∞ if $h(0)$ or $h(1)$ is outside the support of μ_0 or μ_1, respectively). Very roughly, this means $P^{\varepsilon_k}(A) \approx \exp\{-\varepsilon_k^{-1} \inf_{h\in A} I(h)\}$ for large k. The rate function $I(h)$ vanishes as soon

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

as $h \in \Sigma_{\pi_0} := \{\sigma^{xy} : (x, y) \in \mathrm{spt}(\pi_o)\}$, which agrees with the support of P^o, but $I(h)$ is positive outside Σ_{π_0} in many cases. Effectively, our result implies that the Schrödinger bridges P^ε charge exponentially small masses outside the support of the limiting law P^o. Precisely, we establish a weak-type LDP under uniqueness of OT potentials, which allows for marginals with unbounded supports, but induces a full LDP when μ_0, μ_1 are compactly supported.

The proof of the main theorem relies on the expression of P^ε as a π_ε-mixture of Brownian bridges. The main ingredient of the proof is the *exponential continuity* [18] of Brownian bridges, i.e. establishing large-deviation upper and lower bounds for Brownian bridges when the locations of initial and terminal points vary with the noise level. Note that an LDP for Brownian bridges with *fixed* initial and terminal points was derived in [23], but Hsu's proof, which relies on transition density estimates, seems difficult to adapt to establishing the exponential continuity. Instead, we use techniques from abstract Wiener spaces (cf. [45, Chapter 8]) to establish the said result. Given the exponential continuity, the main theorem follows from combining the large-deviation results for π_ε established in [3]. For the compact support case, we provide a more direct proof of the full LDP using the representation of P^ε as an integral of a $(\mu_0 \otimes \mu_1)$-mixture of Brownian bridges. The proof first shows an LDP for the $(\mu_0 \otimes \mu_1)$-mixture of Brownian bridges, and then establishes the full LDP by adapting the (Laplace–)Varadhan lemma (cf. [17, Theorem 4.4.2]) and using the convergence of EOT (or Schrödinger) potentials. The alternative proof can be easily adapted to establish an LDP for the dynamical Schrödinger problem with Langevin diffusion as a reference measure when two marginals are compactly supported; cf. Remark 10.

1.2. Literature review

The literature related to this paper is broad, so we confine ourselves to the references directly related to our work. The most closely related are [3, 36], which established large deviations for static Schrödinger problems in fairly general settings, allowing for marginals on a general Polish space and general continuous costs, and our proofs use several results from their work. [3] derived a weak LDP for EOT via a novel cyclical invariance characterization of EOT plans, while [36] built on convergence of EOT potentials.

The connection between Schrödinger and OT problems has been one of the central problems in the OT literature. We focus here on convergence of Schrödinger problems. The pioneering works in this direction are [26, 31, 32]. Mikami's proof in [31] relies on the fact that the Schrödinger bridge P^ε corresponds to a weak solution of a certain stochastic differential equation (SDE) with diffusion component $\sqrt{\varepsilon}\, \mathrm{d}W(t)$, the special case of which is often referred to as the *Föllmer process* [25, 33]; see Remark 3. The drift function of said SDE being dependent on ε in a nontrivial way (among others) makes the problem of large deviations for dynamical Schrödinger problems fall outside the realm of the Freidlin–Wentzell theory (cf. [17, Chapter 5]). On the other hand, Léonard's proof in [26] relies on the variational representation of the relative entropy and convex analysis techniques to establish Γ-convergence of the Schödinger objective functions, which yields convergence of the optimal solutions. Arguably, recent interest in EOT (static Schrödinger problem) stems from the fact that EOT provides an efficient computational means for unregularized OT [14, 39]. From this perspective, extensive research has been done on convergence and speed of convergence of EOT costs, potentials, plans, and maps toward those of unregularized OT [1, 8, 9, 11, 13, 36, 37, 40].

To the best of the author's knowledge, this is the first paper to establish large deviations for dynamical Schrödinger problems. As noted in the beginning, the dynamical aspect of the Schrödinger bridge has received increasing interest from application domains, which calls for further research on this subject. Our results contribute to the rigorous understanding of the connection between the dynamical Schrödinger and OT problems in the small-noise regime. From a technical perspective, our use of mixture representations to explore large deviations on path spaces might be applied to other problems. Finally, in this work we focus on the Wiener reference measure that corresponds to the quadratic OT problem. Arguably, this setting would be the most basic. Extending our large-deviation results to the dynamical problem in abstract metric spaces [34] would be of interest, but beyond the scope of this paper.

1.3. Organization

The rest of the paper is organized as follows. Section 2 contains background on EOT, Schrödinger, and OT problems, and Section 3 presents the main results. All the proofs are gathered in Section 4.

1.4. Notation and definitions

Let $x \cdot y$ denote the Euclidean inner product for $x, y \in \mathbb{R}^d$. For $x, y \in \mathbb{R}^d$ and a Borel probability measure P on E, let P^{xy} denote the (regular) conditional law of X given $(X(0), X(1)) = (x, y)$ for $X = (X(t))_{t\in[0,1]} \sim P$. For a set A, let $\iota_A(x) = 0$ if $x \in A$ and $= \infty$ if $x \notin A$. On a metric space M, let $B_M(x, r)$ denote the open ball in M with center x and radius r. For a Borel probability measure μ on a metric space, its support is denoted by $\mathrm{spt}(\mu)$. For probability measures α, β on a common measurable space, $\mathcal{H}(\alpha \mid \beta)$ is the relative entropy defined as

$$\mathcal{H}(\alpha \mid \beta) := \begin{cases} \int \log \frac{\mathrm{d}\alpha}{\mathrm{d}\beta}\, \mathrm{d}\alpha & \text{if } \alpha \ll \beta, \\ \infty & \text{otherwise.} \end{cases}$$

A lower semicontinuous function $I\colon M \to [0, \infty]$ defined on a metric space M is called a *rate function*. The rate function I is called *good* if all level sets $\{x\colon I(x) \le \alpha\}$ for $\alpha \in [0, \infty)$ are compact. Given a sequence of positive reals $a_k \to \infty$, a sequence of Borel probability measures $\{P_k\}_{k\in\mathbb{N}}$ on M satisfies a weak *large-deviation principle* with *speed* a_k and rate function I if

(i) for every open set $A \subset M$, $\liminf_{k\to\infty} a_k^{-1} \log P_k(A) \ge -\inf_{x\in A} I(x)$;

(ii) for every compact set $A \subset M$, $\limsup_{k\to\infty} a_k^{-1} \log P_k(A) \le -\inf_{x\in A} I(x)$.

If condition (ii) holds for every closed set $A \subset M$, then we say that $\{P_k\}_{k\in\mathbb{N}}$ satisfies a (full) LDP. We refer the reader to [17] as an excellent reference on large deviations.

2. Preliminaries

2.1. From EOT to Schrödinger problems

We first review EOT and its connection to the Schrödinger problems, which will play a key role in the proofs of the main results. Proofs of the results below can be found in [27] or [36]. Throughout, we set $\mathcal{X} = \mathrm{spt}(\mu_0)$ and $\mathcal{Y} = \mathrm{spt}(\mu_1)$.

Given marginals μ_0, μ_1, the EOT problem for quadratic cost $c(x, y) = |x - y|^2/2$ reads as

$$\min_{\pi\in\Pi(\mu_0,\mu_1)} \int c\, \mathrm{d}\pi + \varepsilon\mathcal{H}(\pi \mid \mu_0 \otimes \mu_1) = \min_{\pi\in\Pi(\mu_0,\mu_1)} \varepsilon\left(\int (c/\varepsilon)\, \mathrm{d}\pi + \mathcal{H}(\pi \mid \mu_0 \otimes \mu_1)\right). \tag{4}$$

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

Setting $\mathrm{d}\nu_\varepsilon = Z_\varepsilon^{-1}\mathrm{e}^{-c/\varepsilon}\ \mathrm{d}(\mu_0 \otimes \mu_1)$ with $Z_\varepsilon = \int \mathrm{e}^{-c/\varepsilon}\ \mathrm{d}(\mu_0 \otimes \mu_1)$, we have

$$\int (c/\varepsilon)\ \mathrm{d}\pi + \mathcal{H}(\pi \mid \mu_0 \otimes \mu_1) = \mathcal{H}(\pi \mid \nu_\varepsilon) - \log Z_\varepsilon,$$

which implies that (4) is equivalent to the static Schrödinger problem

$$\min_{\pi \in \Pi(\mu_0,\mu_1)} \mathcal{H}(\pi \mid \nu_\varepsilon). \tag{5}$$

Recall that $\Pi(\mu_0, \mu_1)$ is compact for the weak topology. Since $\pi \mapsto \mathcal{H}(\pi \mid \nu_\varepsilon)$ is lower semicontinuous with respect to (w.r.t.) the weak topology (which follows from the variational representation of the relative entropy) and strictly convex on the set of π such that $\mathcal{H}(\pi \mid \nu_\varepsilon)$ is finite (which follows from the strict convexity of $x \mapsto x \log x$), the problem in (5) admits a unique optimal solution π_ε, provided $\mathcal{H}(\pi \mid \nu_\varepsilon) < \infty$ for some $\pi \in \Pi(\mu_0, \mu_1)$. Since μ_0 and μ_1 have finite second moments, we have $\mathcal{H}(\mu_0 \otimes \mu_1 \mid \nu_\varepsilon) < \infty$. We will call π_ε the *EOT plan*.

The EOT plan has a density w.r.t. $\mu_0 \otimes \mu_1$ given by

$$\mathrm{d}\pi_\varepsilon(x, y) = \mathrm{e}^{(\varphi_\varepsilon(x)+\psi_\varepsilon(y)-c(x,y))/\varepsilon}\ \mathrm{d}(\mu_0 \otimes \mu_1)(x, y),$$

where $\varphi_\varepsilon \in L^1(\mu_0)$ and $\psi_\varepsilon \in L^1(\mu_1)$ are *EOT potentials* satisfying the Schrödinger system

$$\begin{cases} \int \mathrm{e}^{(\varphi_\varepsilon(x)+\psi_\varepsilon(y)-c(x,y))/\varepsilon}\ \mathrm{d}\mu_1(y) = 1, & \mu_0\text{-almost every } x, \\ \int \mathrm{e}^{(\varphi_\varepsilon(x)+\psi_\varepsilon(y)-c(x,y))/\varepsilon}\ \mathrm{d}\mu_0(x) = 1, & \mu_1\text{-almost every } y. \end{cases} \tag{6}$$

EOT potentials are almost surely (a.s.) unique up to additive constants, i.e. if $(\tilde{\varphi}_\varepsilon, \tilde{\psi}_\varepsilon)$ is another pair of EOT potentials, then there exists a constant $a \in \mathbb{R}$ such that $\tilde{\varphi}_\varepsilon = \varphi_\varepsilon + a$ μ_0-almost everywhere (a.e.) and $\tilde{\psi}_\varepsilon = \psi_\varepsilon - a$ μ_1-a.e. In many cases (e.g. as soon as μ_0, μ_1 are sub-Gaussian), we can choose versions of (finite) EOT potentials for which the Schrödinger system (6) holds for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ (in fact for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$); see [30, Proposition 6]. Whenever possible, we always choose such versions of EOT potentials.

To link EOT to the original static Schrödinger problem (2), we make the following assumption.

Assumption 1. $\mu_1 \ll \mathrm{d}y$ *and* $\mathcal{H}(\mu_1 \mid \mathrm{d}y) < \infty$.

Remark 1. (*On the relative entropy* $\mathcal{H}(\mu_1 \mid \mathrm{d}y)$.) Here, as in [27, Appendix A], we define the relative entropy $\mathcal{H}(\mu_1 \mid \mathrm{d}y)$ against the Lebesgue measure $\mathrm{d}y$ given by

$$\mathcal{H}(\mu_1 \mid \mathrm{d}y) := \int \log(\rho/\mathfrak{g})\ \mathrm{d}\mu_1 + \int (\log \mathfrak{g})\ \mathrm{d}\mu_1 \in (-\infty, \infty],$$

where $\rho = \mathrm{d}\mu_1/\mathrm{d}y$ and $\mathfrak{g}$ is the standard Gaussian density on $\mathbb{R}^d$.

The reference measure $R_{01}^\varepsilon = R^\varepsilon \circ e_{01}^{-1}$ for (2) has a density w.r.t. $\mathrm{d}y\ \mathrm{d}\mu_0(x)$ given by $\mathrm{d}R_{01}^\varepsilon(x, y) = (2\pi\varepsilon)^{-d/2}\mathrm{e}^{-c(x,y)/\varepsilon}\ \mathrm{d}y\, \mathrm{d}\mu_0(x)$, so ν_ε is absolutely continuous w.r.t. R_{01}^ε with density $\mathrm{d}\nu_\varepsilon(x, y) = (2\pi\varepsilon)^{d/2}Z_\varepsilon^{-1}\rho(y)\ \mathrm{d}R_{01}^\varepsilon(x, y)$. Hence,

$$\mathcal{H}(\pi \mid \nu_\varepsilon) = \mathcal{H}(\pi \mid R_{01}^\varepsilon) - \frac{d}{2}\log(2\pi\varepsilon) + \log Z_\varepsilon - \mathcal{H}(\mu_1 \mid \mathrm{d}y),$$

and the unique optimal solution to (2) is given by π_ε.

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

Going back to the dynamical Schrödinger problem (1), by the chain rule for the relative entropy, we have $\mathcal{H}(P \mid R^{\varepsilon}) = \mathcal{H}(P_{01} \mid R^{\varepsilon}_{01}) + \int \mathcal{H}(P^{xy} \mid R^{\varepsilon,xy})\ \mathrm{d}P_{01}(x, y)$, which is minimized by taking $P^{xy} = R^{\varepsilon,xy}$ and $P_{01} = \pi_{\varepsilon}$, i.e.,

$$P^{\varepsilon}(\cdot) = \int R^{\varepsilon,xy}(\cdot)\ \mathrm{d}\pi_{\varepsilon}(x, y) = \int \mathrm{e}^{(\varphi_{\varepsilon}(x)+\psi_{\varepsilon}(y)-c(x,y))/\varepsilon} R^{\varepsilon,xy}(\cdot)\ \mathrm{d}(\mu_0 \otimes \mu_1)(x, y). \tag{7}$$

Alternatively, setting $\bar{R}^{\varepsilon} = \int R^{\varepsilon,xy}\ \mathrm{d}(\mu_0 \otimes \mu_1)$, which is a $(\mu_0 \otimes \mu_1)$-mixture of Brownian bridges, P^{ε} has a density w.r.t. $\bar{R}^{\varepsilon}$ given by

$$\frac{\mathrm{d}P^{\varepsilon}}{\mathrm{d}\bar{R}^{\varepsilon}}(\omega) = \mathrm{e}^{-\phi_{\varepsilon}(\omega(0),\omega(1))/\varepsilon}, \quad \omega = (\omega(t))_{t\in[0,1]} \in E, \tag{8}$$

where $\phi_{\varepsilon} : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a function defined by $\phi_{\varepsilon}(x, y) = c(x, y) - \varphi_{\varepsilon}(x) - \psi_{\varepsilon}(y)$. To see this, for $X = (X(t))_{t\in[0,1]} \sim \bar{R}^{\varepsilon}$ and every Borel set $A \subset E$,

$$\begin{aligned}\mathbb{E}[\mathbf{1}_A(X)\mathrm{e}^{-\phi_{\varepsilon}(X(0),X(1))/\varepsilon}] &= \mathbb{E}[\mathbb{P}(X \in A \mid X(0), X(1))\mathrm{e}^{-\phi_{\varepsilon}(X(0),X(1))/\varepsilon}] \\ &= \mathbb{E}[R^{\varepsilon,(X(0),X(1))}(A)\mathrm{e}^{-\phi_{\varepsilon}(X(0),X(1))/\varepsilon}] = P^{\varepsilon}(A),\end{aligned}$$

where we used $(X(0), X(1)) \sim \mu_0 \otimes \mu_1$.

Remark 2. (*On Assumption 1.*) Assumption 1 is unavoidable to ensure the problem in (2) has a unique optimal solution. On the other hand, the initial distribution μ_0 need not be absolutely continuous, e.g. μ_0 can be discrete.

Remark 3. (*Connnection to Follmer processes.*) The Schrödinder bridge P^{ε} corresponds to the law of a weak solution to a certain SDE, the special case of which is often referred to as the Föllmer process. Let $\mathcal{B}(E)$ be the Borel σ-field on E. Equip $(E, \mathcal{B}(E), R^{\varepsilon})$ with the canonical filtration (augmented, if necessary), and denote by $X = (X(t))_{t\in[0,1]}$ the canonical process, i.e. $X(t, \omega) = \omega(t)$ for $\omega = (\omega(t))_{t\in[0,1]} \in E$. Under R^{ε}, $W = \varepsilon^{-1/2}(X - X(0))$ is a standard Brownian motion starting at 0. Recalling that $\rho = \mathrm{d}\mu_1/\mathrm{d}y$, we set $\tilde{\psi}_{\varepsilon}(y) := \varepsilon((d/2)\log(2\pi\varepsilon) + \log\rho(y)) + \psi_{\varepsilon}(y)$. With this notation, it can be seen that

$$P^{\varepsilon}(\cdot) = \int \mathrm{e}^{(\varphi_{\varepsilon}(x)+\tilde{\psi}_{\varepsilon}(y))/\varepsilon} R^{\varepsilon,xy}(\cdot)\ \mathrm{d}R^{\varepsilon}_{01}(x, y),$$

which implies (cf. the preceding argument) that

$$\frac{\mathrm{d}P^{\varepsilon}}{\mathrm{d}R^{\varepsilon}} = \mathrm{e}^{(\varphi_{\varepsilon}(X(0))+\tilde{\psi}_{\varepsilon}(X(1)))/\varepsilon}.$$

We write

$$\mathfrak{h}_{\varepsilon}(t, y) := \begin{cases} (2\pi\varepsilon(1-t))^{-d/2} \displaystyle\int \exp\left\{-\frac{1}{\varepsilon}\left(\frac{c(y, y')}{1-t} - \tilde{\psi}_{\varepsilon}(y')\right)\right\}\ \mathrm{d}y' & \text{if } t \in [0, 1), \\ \mathrm{e}^{\tilde{\psi}_{\varepsilon}(y)/\varepsilon} & \text{if } t = 1, \end{cases}$$

which satisfies $(\partial_t + \varepsilon\Delta_y/2)\mathfrak{h}_{\varepsilon} = 0$ under regularity conditions (cf. the heat equation). Applying Itô's formula (cf. [24, Theorem 3.3.6]), we have

$$\log\mathfrak{h}_{\varepsilon}(1, X(1)) = \underbrace{\log\mathfrak{h}_{\varepsilon}(0, X(0))}_{=-\varphi_{\varepsilon}(X(0))/\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\int_0^1 b_{\varepsilon}(t, X(t)) \cdot\ \mathrm{d}W(t) - \frac{1}{2\varepsilon}\int |b_{\varepsilon}(t, X(t))|^2\ \mathrm{d}t,$$

where we define $b_\varepsilon(t, y) = \varepsilon \nabla_y \log \mathfrak{h}_\varepsilon(t, y)$. We conclude that, R^ε-a.s.,

$$\frac{\mathrm{d}P^\varepsilon}{\mathrm{d}R^\varepsilon} = \exp\left\{\frac{1}{\sqrt{\varepsilon}} \int_0^1 b_\varepsilon(t, X(t)) \cdot \,\mathrm{d}W(t) - \frac{1}{2\varepsilon} \int |b_\varepsilon(t, X(t))|^2 \,\mathrm{d}t\right\}.$$

By Girsanov's theorem, under P^ε the new process $\tilde{W} = W - \varepsilon^{-1/2} \int_0^\cdot b_\varepsilon(t, X(t)) \,\mathrm{d}t$ is a standard Brownian motion, and the process X solves the SDE

$$\mathrm{d}X(t) = b_\varepsilon(t, X(t)) \,\mathrm{d}t + \sqrt{\varepsilon} \,\mathrm{d}\tilde{W}(t), \quad X(0) \sim \mu_0 \tag{9}$$

(see, e.g., [24, Proposition 5.3.6]; see also [15]). When $\varepsilon = 1$ and $\mu_0 = \delta_0$, we can take $\varphi_1(x) = 0$ and $\psi_1(y) = |y|^2/2$, so $\mathrm{e}^{\tilde{\psi}_1}$ is the density of μ_1 w.r.t. the standard Gaussian. Hence, the SDE in (9) corresponds to the Föllmer process in [25, (12)] and [33]. Abusing terminology, we call P^ε with $\varepsilon > 0$ and $\mu_0 = \delta_0$ a (perturbed) Föllmer process.

2.2. OT potentials

The rate function for Schrödinger bridges involves OT potentials. For duality theory of OT, we refer the reader to [2, 41, 47]. The OT problem (3) admits a dual problem that reads as

$$\max_{\substack{(\varphi,\psi)\in L^1(\mu_0)\times L^1(\mu_1)\\ \varphi+\psi\le c}} \int \varphi \,\mathrm{d}\mu_0 + \int \psi \,\mathrm{d}\mu_1. \tag{10}$$

By restricting to the respective support, it is assumed without loss of generality that φ and ψ are functions defined on $\mathcal{X}$ and $\mathcal{Y}$, respectively. One of φ and ψ can be replaced with the *c-transform* of the other. Recall that the c-transform of $\psi\colon \mathcal{Y} \to [-\infty, \infty)$ with $\psi \not\equiv -\infty$ is a function $\psi^c\colon \mathcal{X} \to [-\infty, \infty)$ defined by $\psi^c(x) := \inf_{y\in\mathcal{Y}}\{c(x, y) - \psi(y)\}$, $x \in \mathcal{X}$. The c-transform of $\varphi\colon \mathcal{X} \to [-\infty, \infty)$ with $\varphi \not\equiv -\infty$ is defined analogously. The dual problem (10) then reduces to

$$\max_{\psi\in L^1(\mu_1)} \int \psi^c \,\mathrm{d}\mu_0 + \int \psi \,\mathrm{d}\mu_1, \tag{11}$$

whose maximum is attained at some c-concave function $\psi \in L^1(\mu_1)$ with $\psi^c \in L^1(\mu_0)$ (a function on $\mathcal{Y}$ is called *c-concave* if it is the c-transform of a function on $\mathcal{X}$); see, e.g., [47, Theorem 5.9] or [2, Theorem 6.1.5]. We call such a ψ an *OT potential* from μ_1 to μ_0. An OT potential from μ_0 to μ_1 is defined analogously.

For any OT potential ψ and any OT plan π, the support of π is contained in the *c-superdifferential* $\partial^c\psi$ of ψ, $\partial^c\psi := \{(x, y)\colon \psi^c(x) + \psi(y) = c(x, y)\}$. Indeed, $\partial^c\psi$ is a closed set (as c-concave functions are upper semicontinuous) on which π has full measure by duality, so $\mathrm{spt}(\pi) \subset \partial^c\psi$. In particular, for $(x, y) \in \mathrm{spt}(\pi)$, $\psi^c(x)$ and $\psi(y)$ are finite.

Observe that Assumption 1 ensures that *the OT problem (3) admits a unique OT plan π_o*. Let $\mathcal{X}_\mathrm{o}$ and $\mathcal{Y}_\mathrm{o}$ denote the projections of $\mathrm{spt}(\pi_\mathrm{o})$ onto $\mathcal{X}$ and $\mathcal{Y}$, respectively, i.e.,

$$\mathcal{X}_\mathrm{o} = \{x\colon (x, y) \in \mathrm{spt}(\pi_\mathrm{o}) \text{ for some } y\},$$

and $\mathcal{Y}_\mathrm{o}$ is defined analogously. As π_o is a coupling for μ_0 and μ_1, the sets $\mathcal{X}_\mathrm{o}$ and $\mathcal{Y}_\mathrm{o}$ have full μ_0- and μ_1-measure, respectively. As in [3], we assume uniqueness of OT potentials (from μ_1 to μ_0) on $\mathcal{Y}_\mathrm{o}$ to derive our large-deviation results.

Assumption 2. *The dual problem (11) admits a unique OT potential ψ on $\mathcal{Y}_\mathrm{o}$, i.e. if $\tilde{\psi}$ is another OT potential, then $\psi - \tilde{\psi}$ is constant on $\mathcal{Y}_\mathrm{o}$.*

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

Appendix B in [3] and [42] provide various sufficient conditions under which uniqueness of OT potential holds. For example, Assumption 2 holds under each of the following cases:

(A) $\mathcal{X}$ and $\mathcal{Y}$ are compact, and one of them agrees with the closure of a connected open set [24, Theorem 7.18].

(B) The interior $\mathrm{int}(\mathcal{Y})$ is connected, μ_1 is absolutely continuous with positive Lebesgue density on $\mathrm{int}(\mathcal{Y})$, and $\mu_1(\partial\mathcal{Y}) = 0$ [3, Proposition B.2].

Case (A) does not require μ_0 or μ_1 to have a Lebesgue density (although Assumption 1 requires $\mu_1 \ll \mathrm{d}y$). We provide a self-contained proof of Case (A) in Lemma 3 for completeness. Case (B) imposes no restrictions on μ_0, so it allows μ_0 to be discrete.

Remark 4. Often, regularity conditions are imposed on the input measure μ_0 to ensure uniqueness or regularity of OT potentials from μ_0 to μ_1. For the (static) EOT case, the role of μ_0 and μ_1 is symmetric, so it is possible, without loss of generality, to focus on the forward ($\mu_0 \to \mu_1$) case. However, in our dynamical setting, the roles of μ_0 and μ_1 are asymmetric because of Assumption 1. Since Assumption 1 already imposes absolute continuity on μ_1, we treat OT potentials for the backward direction ($\mu_1 \to \mu_0$), contrary to the convention in the literature.

3. Main results

We first recall the weak convergence of P^ε toward $P^{\mathrm{o}} = \int \delta_{\sigma^{xy}}\, \mathrm{d}\pi_{\mathrm{o}}(x, y)$ with $\sigma^{xy}(t) = (1 - t)x + ty$. Recall that the cost function is $c(x, y) = |x - y|^2/2$.

Proposition 1. *Under Assumption 1, $P^\varepsilon \to P^{\mathrm{o}}$ weakly as $\varepsilon \downarrow 0$. The support of P^{o} agrees with $\Sigma_{\pi_{\mathrm{o}}} := \{\sigma^{xy} \colon (x, y) \in \mathrm{spt}(\pi_{\mathrm{o}})\}$.*

Remark 5. (*On Proposition 1.*) A version of this proposition was proved in [31] under the extra assumption that μ_0 is absolutely continuous. [26, Theorem 3.7] implies the proposition but the proof is somewhat involved (as it covers more general settings). We provide a simple proof in Section 4.

We are now in a position to state our main results. Let H denote the space of absolutely continuous maps $h \colon [0, 1] \to \mathbb{R}^d$ with $\int_0^1 |\dot{h}(t)|^2\, \mathrm{d}t < \infty$, where $\dot{h}(t) = \mathrm{d}h(t)/\,\mathrm{d}t$. We endow H with the (semi-)inner product $(g, h)_H = \int_0^1 \dot{g}(t) \cdot \dot{h}(t)\, \mathrm{d}t$. Set $\|\cdot\|_H = \sqrt{(\cdot, \cdot)_H}$. Formally, define $\|h\|_H = \infty$ for $h \in E \setminus H$. We first state the weak-type LDP for Schrödinger bridges, which allows for marginals with unbounded supports.

Theorem 1. (Weak-type LDP for Schrodinger bridges.) *Suppose Assumptions 1 and 2 hold. Pick any $\varepsilon_k \downarrow 0$. Then the following hold:*

(i) *For every open set $A \subset e_{01}^{-1}(\mathcal{X}_o \times \mathcal{Y}_o)$ (w.r.t. the relative topology),*

$$\liminf_{k\to\infty} \varepsilon_k \log P^{\varepsilon_k}(A) \ge -\inf_{h\in A} I(h)$$

for the rate function $I(h) = \|h\|_H^2/2 - \psi^c(h(0)) - \psi(h(1))$.

(ii) *For every closed set $A \subset E$ of the form $A = e_{01}^{-1}(C)$ for some compact set $C \subset \mathcal{X}_{\mathrm{o}} \times \mathcal{Y}_{\mathrm{o}}$,*

$$\limsup_{k\to\infty} \varepsilon_k \log P^{\varepsilon_k}(A) \le -\inf_{h\in A} I(h).$$

Theorem 1 is not precisely a weak LDP since (ii) holds for every compact set $C \subset e_{01}^{-1}(\mathcal{X}_o \times \mathcal{Y}_o)$ but also for some noncompact closed sets. As such, we call Theorem 1 a weak-type LDP. If the marginals have compact supports, then a full LPD holds, subject to one technical condition essential to guarantee the uniqueness of OT potentials.

Corollary 1. (Full LDP for Schrodinger bridges.) *Suppose Assumption 1 holds. Pick any $\varepsilon_k \downarrow 0$. If $\mathcal{X}$ and $\mathcal{Y}$ are compact and one of them agrees with the closure of a connected open set, then the sequence $\{P^{\varepsilon_k}\}_{k\in\mathbb{N}}$ satisfies a (full) LDP on E with speed ε_k^{-1} and good rate function I, where I is set to ∞ outside $e_{01}^{-1}(\mathcal{X} \times \mathcal{Y})$.*

We leave several remarks on the preceding results.

Remark 6. (*On Corollary 1.*) The sets $\mathcal{X}$, $\mathcal{Y}$ being compact implies $\mathcal{X}_o = \mathcal{X}$ and $\mathcal{Y}_o = \mathcal{Y}$, as the projections from $\mathcal{X} \times \mathcal{Y}$ onto $\mathcal{X}$ and $\mathcal{Y}$ are then closed maps. The assumption of Corollary 1 guarantees the uniqueness of OT potentials; see the discussion after Assumption 2. Since P^ε charges no mass outside $e_{01}^{-1}(\mathcal{X} \times \mathcal{Y})$, the full LDP is indeed deduced from the preceding theorem. Connectedness of the support of one of the marginals is essential for the uniqueness of OT potentials (see the discussion before Proposition B.2 in [3]). The full LPD (more specifically, establishing exponential tightness) for Schrödinger bridges requires both marginals to be compactly supported, since exponential tightness implies the limiting law P^o is concentrated on a compact set, which fails to hold if one of the marginals has unbounded support. See [36, Remark 4.2(b)] for a relevant discussion in the static case.

Remark 7. (*On the rate function I(h).*) Since $\psi^c(x) + \psi(y) \le c(x, y)$ by construction, the rate function $I(h)$ is positive as soon as $h \neq \sigma^{h(0),h(1)}$. Even when $h = \sigma^{h(0),h(1)}$, which entails $\|h\|_H^2/2 = c(h(0), h(1))$, the rate function $I(h) = c(h(0), h(1)) - \psi^c(h(0)) - \psi(h(1))$ can be positive provided $(h(0), h(1)) \notin \mathrm{spt}(\pi_o)$. [3, Section 5] provides several conditions under which the rate function for the static case, $\phi(x, y) = c(x, y) - \psi^c(x) - \psi(y)$, is positive outside $\mathrm{spt}(\pi_o)$. Considering the characterization of the support of P_o, our large-deviation results essentially imply that the Schrödinger bridges P^ε charge exponentially small masses outside $\mathrm{spt}(P_o)$ when $\varepsilon \downarrow 0$.

Remark 8. (*Proofs of Theorem 1 main and Corollary 1.*) The proof of Theorem 1 uses the expression $P^\varepsilon(A) = \int R^{\varepsilon,xy}(A)\ \mathrm{d}\pi_\varepsilon(x, y)$ from (7). The main ingredient is the exponential continuity of $\{R^{\varepsilon_k,xy}\}$, i.e. establishing large-deviation upper and lower bounds for $\{R^{\varepsilon_k,(x_k,y_k)}\}_{k\in\mathbb{N}}$ when $(x_k, y_k) \to (x, y)$, which will be proved in Proposition 4. The proof then directly evaluates $P^\varepsilon(A)$ by combining the large-deviation results for the static case from [3]. As noted in Remark 6, Corollary 1 is a special case of Theorem 1. Nonetheless, we provide a separate, more direct proof for the compact support case. It relies on the expression $P^\varepsilon(A) = \int_A \mathrm{e}^{-\phi_\varepsilon \circ e_{01}(\omega)/\varepsilon}\ \mathrm{d}\bar{R}^\varepsilon(\omega)$ from (8). Then the proof proceeds by (i) proving an LDP for $\bar{R}^\varepsilon$, which follows directly from the exponential continuity [18], and then (ii) adapting the (Laplace–)Varadhan lemma (cf. [17, Theorem 4.4.2]) to evaluate $P^\varepsilon(A)$. Step (ii) is relatively simple, because, while the function ϕ_ε depends on ε, so the Varadhan lemma is not directly applicable, the assumption of Corollary 1 ensures uniform convergence of the EOT potentials.

Remark 9. (*On uniqueness of OT potentials.*) Inspection of the proof of Corollary 1 reveals that, as long as Assumption 1 holds and $\mathcal{X}$ and $\mathcal{Y}$ are compact (but without assuming uniqueness of OT potentials), the conclusion of Corollary 1 continues to hold, provided that

$$\lim_{k\to\infty} \varphi_{\varepsilon_k} = \bar{\varphi} \quad \text{and} \quad \lim_{k\to\infty} \psi_{\varepsilon_k} = \bar{\psi} \quad \text{uniformly on } \mathcal{X} \text{ and } \mathcal{Y}\text{, respectively} \tag{12}$$

for some (continuous) functions $\bar{\varphi}$ and $\bar{\psi}$ on $\mathcal{X}$ and $\mathcal{Y}$, respectively (necessarily, $(\bar{\varphi}, \bar{\psi})$ are dual potentials for (μ_0, μ_1)). The rate function I needs to be modified so that (ψ^c, ψ) are replaced with $(\bar{\varphi}, \bar{\psi})$. A similar comment applies to Proposition 3. Conversely, the uniform convergence of the EOT potentials in (12) is necessary for the LDP for the Schrödinger bridges $\{P^{\varepsilon_k}\}_{k\in\mathbb{N}}$ to hold, by [36, Proposition 4.5] in the static case and the contraction principle.

We now look at a few special cases.

Example 1. (*Föllmer process.*) When $\mu_0 = \delta_0$, we have $\mathcal{Y}_o = \mathcal{Y}$ and $\psi(y) = |y|^2/2$, so the rate function reduces to $I(h) = \|h\|_H^2/2 - |h(1)|^2/2 + \iota_{\{0\}}(h(0)) + \iota_{\mathcal{Y}}(h(1))$, which vanishes if and only if $h(t) = ty$ for $t \in [0, 1]$ for some $y \in \mathcal{Y}$, i.e. if and only if $h \in \mathrm{spt}(P^o)$.

Example 2. (*Two-point marginal.*) The LDP in Corollary 1 directly yields an LDP for $P^{\varepsilon_k} \circ f^{-1}$ for any continuous function f from E into another metric space by the contraction principle (cf. [17, Theorem 4.2.1]). We consider the case where $f = e_{st}$ for $0 \le s < t \le 1$. Note that $P_{st}^{\varepsilon_k}$ is a coupling for $P_s^{\varepsilon_k}$ and $P_t^{\varepsilon_k}$. Recall that the marginal flow $(P_t^{\varepsilon})_{t\in[0,1]}$ is called an entropic interpolation, and its limiting analog $(P_t^o)_{t\in[0,1]}$ is a displacement interpolation connecting μ_0 and μ_1. To characterize the rate function for $P_{st}^{\varepsilon_k}$, we need additional notation.

For a function $f\colon \mathbb{R}^d \to (-\infty, \infty]$ and $t \ge 0$, define

$$\mathcal{Q}_t(f)(y) = \inf_{x\in\mathbb{R}^d} \left\{ \frac{c(x, y)}{t} + f(x) \right\}, \quad t > 0, \ \mathcal{Q}_0(f) = f.$$

The family of operators $\{\mathcal{Q}_t\}_{t\ge 0}$ is called the *Hopf–Lax semigroup*; cf. [47, Chapter 7]. Assuming Case (A) after Assumption 2, we set $\varphi = \psi^c$ and extend φ and ψ to the whole $\mathbb{R}^d$ by setting $\varphi = -\infty$ and $\psi = -\infty$ outside $\mathcal{X}$ and $\mathcal{Y}$, respectively. For $0 \le s < t \le 1$, consider the rescaled cost $c^{s,t}(x, t) = c(x, y)/(t - s)$.

Proposition 2. (LDP for two-point marginal.) *Suppose Assumption 1 holds. Pick any $0 \le s < t \le 1$ and $\varepsilon_k \downarrow 0$. If $\mathcal{X}$ and $\mathcal{Y}$ are compact and one of them agrees with the closure of a connected open set, then the sequence $\{P_{st}^{\varepsilon_k}\}_{k\in\mathbb{N}}$ satisfies an LDP on $\mathbb{R}^{2d}$ with speed ε_k^{-1} and good rate function $I_{st}(x, y) = c^{s,t}(x, y) - \varphi_s(x) - \psi_t(y)$, where $(\varphi_s, \psi_t) := (-\mathcal{Q}_s(-\varphi), -\mathcal{Q}_{1-t}(-\psi))$ are dual potentials for (P_s^o, P_t^o) w.r.t. $c^{s,t}$, i.e. optimal solutions to (10) with (μ_0, μ_1, c) replaced by $(P_s^o, P_t^o, c^{s,t})$.*

Finally, we point out that the direct proof for Corollary 1 can be easily adapted to cover the dynamical Schrödinger problem with Langevin diffusion as a reference measure.

Remark 10. (*Langevin diffusion as reference measure.*) For a bounded smooth potential $V\colon \mathbb{R}^d \to \mathbb{R}$ with bounded derivatives, consider the Langevin diffusion $X = (X(t))_{t\ge 0}$ defined by the unique (strong) solution to the SDE $\mathrm{d}X(t) = -\nabla V(X(t))\ \mathrm{d}t + \mathrm{d}W(t)$, $X(0) \sim \mu_0$, where $(W(t))_{t\ge 0}$ is a standard Brownian motion starting at 0 independent of X(0). Let $p_t(x, y)$ denote the transition density of the Langevin diffusion X and $\check{R}^{\varepsilon}$ be the law of $X^{\varepsilon} := (X(\varepsilon t))_{t\in[0,1]}$ defined on $\mathcal{B}(E)$. Instead of the Wiener reference measure as in (1), we consider the dynamical Schrödinger problem with reference measure $\check{R}^{\varepsilon}$:

$$\min_{P:P_0=\mu_0, P_1=\mu_1} \mathcal{H}(P \mid \check{R}^{\varepsilon}). \tag{13}$$

Under Assumption 1, arguing as in Section 2 (see also [27, Proposition 2.3]), we can see that the unique optimal solution to (13) is given by

$$\check{P}^{\varepsilon}(\cdot) = \int \check{R}^{\varepsilon,xy}(\cdot)\ \mathrm{d}\check{\pi}^{\varepsilon}(x, y),$$

where $\check{R}^{\varepsilon,xy}$ is the conditional law of X^ε given $(X^\varepsilon(0), X^\varepsilon(1)) = (x, y)$ and $\check{\pi}^\varepsilon$ is the unique optimal solution to the static EOT problem

$$\min_{\pi\in\Pi(\mu_0,\mu_1)} \int c_\varepsilon \, d\pi + \varepsilon\mathcal{H}(\pi \mid \mu_0 \otimes \mu_1)$$

with $c_\varepsilon(x, y) := -\varepsilon \log p_\varepsilon(x, y)$. Recall that the transition density $p_t(x, y)$ is everywhere positive (cf. [44, Chapter 4]) and the conditional laws $\check{R}^{\varepsilon,xy}$ are defined for all $(x, y) \in \mathbb{R}^{2d}$ [6]. The classical Varadhan asymptotics implies that $\lim_{\varepsilon\downarrow 0} c_\varepsilon(x, y) = |x - y|^2/2 = c(x, y)$ (cf. [44, Chapter 4]), so we can expect that the Schrödinger bridges $\{\check{P}^\varepsilon\}_{\varepsilon>0}$ satisfy the LDP with the same rate function I as in the Brownian case. The next proposition confirms this under a similar setting to Corollary 1.

Proposition 3. (Full LDP for Schrodinger bridges: Langevin case.) *Suppose Assumption 1 holds. Pick any $\varepsilon_k \downarrow 0$. If $\mathcal{X}$ and $\mathcal{Y}$ are compact and one of them agrees with the closure of a connected open set, then the sequence $\{\check{P}^{\varepsilon_k}\}_{k\in\mathbb{N}}$ satisfies a (full) LDP on E with speed ε_k^{-1} and good rate function I, where I is given in Corollary 1.*

The condition on the potential V appears to be stronger than needed, but is imposed for the sake of simplicity. As announced, the proof follows similar arguments to the direct proof for Corollary 1. To establish exponential continuity for the Langevin bridge $\check{R}^{\varepsilon,xy}$, we use the explicit expression for the Radon–Nikodym density of the Langevin bridge against the Brownian bridge; cf. [28].

4. Proofs for Section 3

Recall that $R^{\varepsilon,xy}$ is the (regular) conditional law of $x + \sqrt{\varepsilon}W$ given $x + \sqrt{\varepsilon}W(1) = y$ for a standard Brownian motion $W = (W(t))_{t\in[0,1]}$ starting at 0. Alternatively, $R^{\varepsilon,xy}$ can be characterized as the law of $\sqrt{\varepsilon}W^\circ + \sigma^{xy}$ with $W^\circ = (W(t) - tW(1))_{t\in[0,1]}$ a standard Brownian bridge. For simplicity of notation, let $z = (x, y) \in \mathbb{R}^{2d}$ and write $R^{\varepsilon,z} = R^{\varepsilon,xy}$.

4.1. Proof of Proposition 1

Proof of Proposition 1. By the uniqueness of the OT plan, we have $\pi_\varepsilon \to \pi_o$ weakly by [3, Proposition 3.2], which implies that

$$\eta_\varepsilon := \sup_{\substack{g:\mathbb{R}^{2d}\to[-1,1] \\ g \text{ 1-Lipschitz}}} \left| \int g \, d(\pi_\varepsilon - \pi_o) \right| \to 0$$

(see [46, Chapter 1.12]. Pick any 1-Lipschitz function $f : E \to [-1, 1]$. We have

$$\int f \, dP^\varepsilon = \int \left(\int f \, dR^{\varepsilon,z} \right) d\pi_\varepsilon(z) = \int \underbrace{\mathbb{E}[f(\sqrt{\varepsilon}W^\circ + \sigma^z)]}_{=:g_\varepsilon(z)} \, d\pi_\varepsilon(z).$$

By construction, g_ε is bounded by 1, $|g_\varepsilon(z) - g_\varepsilon(z')| \le \|\sigma^z - \sigma^{z'}\|_E \le 2|z - z'|$, and $\lim_{\varepsilon\downarrow 0} g_\varepsilon(z) = f(\sigma^z) = \int f \, d\delta_{\sigma^z}$. Hence,

$$\int g_\varepsilon \, d\pi_\varepsilon \le \int g_\varepsilon \, d\pi_o + 2\eta_\varepsilon = \underbrace{\int \left(\int f \, d\delta_{\sigma^z} \right) d\pi_o}_{=\int f \, dP^o} + o(1),$$

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

where we used the dominated convergence theorem. The reverse inequality follows similarly, and we conclude that $\lim_{\varepsilon\downarrow 0}\int f\ \mathrm{d}P^{\varepsilon}=\int f\ \mathrm{d}P^{\mathrm{o}}$, which yields $P^{\varepsilon}\to P^{\mathrm{o}}$ weakly. The second claim follows from Lemma 1, which follows. □

Lemma 1. *For any Borel probability measure γ on $\mathbb{R}^{2d}$, the mixture $P=\int\delta_{\sigma^{xy}}\ \mathrm{d}\gamma(x,y)$ has support $\Sigma_{\gamma}:=\{\sigma^{xy}\colon (x,y)\in\mathrm{spt}(\gamma)\}$.*

Proof. The set Σ_{γ} is closed in E. Pick any $(x,y)\in\mathrm{spt}(\gamma)$ and any open set U containing σ^{xy}. Since $O=\{(x',y')\colon \sigma^{x'y'}\in U\}$ is open in $\mathbb{R}^{2d}$ (as $(x',y')\mapsto\sigma^{x'y'}$ is continuous), we have, for $(\xi_0,\xi_1)\sim\gamma$, $P(U)=\mathbb{P}(\sigma^{\xi_0,\xi_1}\in U)=\mathbb{P}((\xi_0,\xi_1)\in O)=\gamma(O)>0$, which yields $\mathrm{spt}(P)=\Sigma_{\gamma}$. □

4.2. Exponential continuity of Brownian bridges

For given $x,y\in\mathbb{R}^d$, [23] showed that the sequence $\{R^{\varepsilon,xy}\}_{\varepsilon>0}$ satisfies an LDP with rate function

$$J_{xy}(h)=\frac{\|h\|_H^2}{2}-c(x,y)+\iota_{\{(x,y)\}}(h(0),h(1)).$$

Write $J_z(h)=J_{xy}(h)$ for $z=(x,y)$. Additionally, set $H_z:=\{h\in H\colon (h(0),h(1))=z\}$. Pick any $\varepsilon_k\downarrow 0$.

Proposition 4. (Exponential continuity of Brownian bridges)

(i) *For every open set $A\subset E$, $\liminf_{k\to\infty}\varepsilon_k\log R^{\varepsilon_k,z_k}(A)\geq-\inf_{h\in A}J_z(h)$ whenever $z_k\to z$ in $\mathbb{R}^{2d}$.*

(ii) *For every closed set $A\subset E$,*

$$\limsup_{k\to\infty}\varepsilon_k\log R^{\varepsilon_k,z_k}(A)\leq-\inf_{h\in A}J_z(h)\tag{14}$$

whenever $z_k\to z$ in $\mathbb{R}^{2d}$.

Proof. Hsu's proof in [23] that relies on transition function estimates seems difficult to adapt to establishing the exponential continuity. Instead, we adapt the proof of large deviations for abstract Wiener spaces; cf. [45, Chapter 8]. For the sake of completeness, we provide a self-contained proof.

For (i), it suffices to show that for every $h\in H$ such that $J_z(h)<\infty$,

$$\liminf_{r\downarrow 0}\liminf_{k\to\infty}\varepsilon_k\log R^{\varepsilon_k,z_k}(B_E(h,r))\geq -J_z(h).$$

Set $\bar{h}\in H_0$ by $\bar{h}=h-\sigma^{xy}$ and $h_k\in H_{z_k}$ by $h_k=\bar{h}+\sigma^{x_k,y_k}$. Since $\|h_k-h\|_E\to 0$, $B_E(h_k,r/2)\subset B_E(h,r)$ for large k. Observe that

$$R^{\varepsilon_k,z_k}(B_E(h_k,r/2))=R^{\varepsilon_k,0}(B_E(\bar{h},r/2))=\mathbb{P}\big(W^{\circ}\in B_E(\bar{h}/\sqrt{\varepsilon_k},r/(2\sqrt{\varepsilon_k}))\big).$$

Recall that $(H_0,(\cdot,\cdot)_H)$ is a reproducing kernel Hilbert space for W° (cf. [22, Exercise 2.6.16]), whose closure in E agrees with $E_0:=\{\omega\in E\colon \omega(0)=\omega(1)=0\}$. Hence, the pair of spaces (H_0,E_0) coupled with the law of W° constitutes an abstract Wiener space; cf. [45, Chapter 8]. Let E_0^* denote the topological dual of E_0 with dual norm $\|\cdot\|_{E_0^*}$, and $\langle\omega,\omega^*\rangle$ denote the duality

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

pairing for $\omega \in E_0$ and $\omega^* \in E_0^*$. Since H_0 is continuously embedded as a dense subspace of E_0 (as $\|\cdot\|_E \le \|\cdot\|_H$ on H_0), for each $\omega^* \in E_0^*$, there exists a unique $h_{\omega^*} \in H_0$ with the property that $(h, h_{\omega^*})_H = \langle h, \omega^* \rangle$ for all $h \in H_0$, and the map $\omega^* \mapsto h_{\omega^*}$ is continuous, linear, one-to-one, and onto a dense subspace of H_0 (cf. [45, Lemma 8.2.3]). Let $0 < \delta < r/2$ and $\omega^* \in E_0^*$ be such that $B_{E_0}(h_{\omega^*}, \delta) \subset B_{E_0}(\bar{h}, r/2)$. Now, an application of the Cameron–Martin formula (cf. [45, Theorem 8.2.9]) yields

$$\begin{aligned} R^{\varepsilon_k,0}(B_E(\bar{h}, r/2)) &= R^{\varepsilon_k,0}(B_{E_0}(\bar{h}, r/2)) \ge R^{\varepsilon_k,0}(B_{E_0}(h_{\omega^*}, \delta)) \\ &= \mathbb{P}\big(W^\circ - \varepsilon_k^{-1/2} h_{\omega^*} \in B_{E_0}\big(0, \varepsilon_k^{-1/2}\delta\big)\big) \\ &= \mathbb{E}\big[\exp\big\{-\varepsilon_k^{-1/2}\langle W^\circ, \omega^*\rangle - \varepsilon_k^{-1}\|h_{\omega^*}\|_H^2/2\big\} \mathbf{1}_{B_{E_0}(0, \varepsilon_k^{-1/2}\delta)}(W^\circ)\big] \\ &\ge \exp\big\{-\delta\varepsilon_k^{-1}\|\omega^*\|_{E_0^*} - \varepsilon_k^{-1}\|h_{\omega^*}\|_H^2/2\big\} \mathbb{P}\big(W^\circ \in B_{E_0}\big(0, \varepsilon_k^{-1/2}\delta\big)\big), \end{aligned}$$

so that, by taking $k \to \infty$,

$$\liminf_{k\to\infty} \varepsilon_k \log R^{\varepsilon_k,0}(B_E(\bar{h}, r/2)) \ge -\delta\|\omega^*\|_{E_0^*} - \frac{\|h_{\omega^*}\|_H^2}{2}.$$

Choosing $\delta = r/4$ and $\omega^* \in E_0^*$ with $\|\bar{h} - h_{\omega^*}\|_H < r/4$, and then taking $r \downarrow 0$, we have

$$\liminf_{r\downarrow 0} \liminf_{k\to\infty} \varepsilon_k \log R^{\varepsilon_k,0}(B_E(\bar{h}, r/2)) \ge -\frac{\|\bar{h}\|_H^2}{2} = -\frac{1}{2}(\|h\|_H^2 - |x-y|^2) = -J_z(h).$$

For (ii), we first show that for every $h \in E$,

$$\limsup_{r\downarrow 0} \limsup_{k\to\infty} \varepsilon_k \log R^{\varepsilon_k,z_k}(B_E(h, r)) \le -J_z(h). \tag{15}$$

Using the same notation as in (i), we have $B_E(h, r) \subset B_E(h_k, 2r)$ for large k and

$$\begin{aligned} R^{\varepsilon_k,z_k}(B_E(h_k, 2r)) &= \mathbb{P}(W^\circ \in B_E(\bar{h}/\sqrt{\varepsilon_k}, 2r/\sqrt{\varepsilon_k})) \\ &= \mathbb{E}\big[\exp\big\{-\varepsilon_k^{-1/2}\langle W^\circ, \omega^*\rangle + \varepsilon_k^{-1/2}\langle W^\circ, \omega^*\rangle\big\} \mathbf{1}_{B_{E_0}(\bar{h}/\sqrt{\varepsilon_k}, 2r/\sqrt{\varepsilon_k})}(W^\circ)\big] \\ &\le \exp\big\{-\varepsilon_k^{-1}(\langle \bar{h}, \omega^*\rangle - 2r\|\omega^*\|_{E_0^*})\big\} \mathbb{E}\big[e^{\varepsilon_k^{-1/2}\langle W^\circ, \omega^*\rangle}\big] \\ &= \exp\big\{-\varepsilon_k^{-1}(\langle \bar{h}, \omega^*\rangle - \|h_{\omega^*}\|_H^2/2 - 2r\|\omega^*\|_{E_0^*})\big\} \end{aligned}$$

for all $\omega^* \in E_0^*$, where we used the fact that $\langle W^\circ, \omega^*\rangle \sim N(0, \|h_{\omega^*}\|_H^2)$. This yields

$$\begin{aligned} \limsup_{r\downarrow 0} \limsup_{k\to\infty} \varepsilon_k \log R^{\varepsilon_k,z_k}(B_E(h_k, 2r)) &\le -\sup_{\omega^* \in E_0^*} \left(\langle \bar{h}, \omega^*\rangle - \frac{\|h_{\omega^*}\|_H^2}{2}\right) \\ &= \begin{cases} -\dfrac{\|\bar{h}\|_H^2}{2} & \text{if } \bar{h} \in H_0, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Now, $\bar{h} \in H_0$ if and only if $h \in H_z$, and $\|\bar{h}\|_H^2 = \|h\|_H^2 - |x-y|^2$, which leads to (15).

Given (15), it is standard to show that (14) holds for every compact set $A \subset E$. It remains to verify exponential tightness for $\{R^{\varepsilon_k,z_k}\}_{k\in\mathbb{N}}$ (cf. [17, Lemma 1.2.18]), i.e. for every $\alpha < \infty$,

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

there exists a compact set $K \subset E$ such that $\limsup_{k\to\infty} \varepsilon_k \log R^{\varepsilon_k, z_k}(K^c) < -\alpha$. We first note that the exponential tightness holds for $\{R^{\varepsilon_k,0}\}_{k\in\mathbb{N}}$. Indeed, by [45, Corollary 8.3.10], we can construct a separable Banach space F that is continuously embedded in E_0 as a measurable subset with the properties that $\mathbb{P}(W^\circ \in F) = 1$, bounded subsets of F are totally bounded in E_0, and (H_0, F) coupled with the restriction of the law of W° on F is another abstract Wiener space. Then, choosing K_0 to be the E_0-closure of a ball in F with large enough radius satisfies $\limsup_{k\to\infty} \varepsilon_k \log R^{\varepsilon_k,0}(K_0^c) < -\alpha$ by Fernique's theorem (cf. [45, Theorem 8.2.1]), and K_0 is compact in E_0 by construction.

Now, for an arbitrary bounded neighborhood $O \subset \mathbb{R}^{2d}$ of z, set $K_1 = \{\sigma^{x'y'} : (x', y') \in O\}$. By the Ascoli–Arzelà theorem, the set $K = \{\omega + \omega' : \omega \in K_0, \omega' \in K_1\}$ is relatively compact in E, and such that $R^{\varepsilon_k,z_k}(K) \geq R^{\varepsilon_k,0}(K_0)$ for large k. Indeed, $\sigma^{z_k} \in K_1$ for large k, so if $\omega \in K_0$, then $\omega + \sigma^{z_k} \in K$, which implies $R^{\varepsilon_k,0}(K_0) \leq R^{\varepsilon_k,0}(\omega + \sigma^{z_k} \in K) = R^{\varepsilon_k,z_k}(K)$. This yields exponential tightness for $\{R^{\varepsilon_k,z_k}\}_{k\in\mathbb{N}}$. □

Given the exponential continuity, the following corollary concerning large deviations of mixtures of Brownian bridges follows immediately from [18, Theorems 2.1 and 2.2]. The result might be of independent interest.

Corollary 2. (Large deviations for mixtures of Brownian bridges.) *Let γ be a Borel probability measure on $\mathbb{R}^{2d}$. Consider the mixture distribution $Q^\varepsilon = \int R^{\varepsilon,xy}\, \mathrm{d}\gamma(x, y)$.*

(i) *The function*

$$J(h) := \inf_{(x,y)\in \mathrm{spt}(\gamma)} J_{xy}(h) = \frac{\|h\|_H^2}{2} - c(h(0), h(1)) + \iota_{\mathrm{spt}(\gamma)}(h(0), h(1)) \qquad (16)$$

is lower semicontinuous from E into $[0, \infty]$.

(ii) *For every open set $A \subset E$, $\liminf_{k\to\infty} \varepsilon_k \log Q^{\varepsilon_k}(A) \geq -\inf_{h\in A} J(h)$.*

(iii) *If γ is compactly supported, then for every closed set $A \subset E$,*

$$\limsup_{k\to\infty} \varepsilon_k \log Q^{\varepsilon_k}(A) \leq -\inf_{h\in A} J(h),$$

and J is a good rate function.

Proof. For (i), set $F = \{(h, z) \in E \times \mathbb{R}^{2d} : (h(0), h(1)) = z\}$. The rate function $J_z(h)$ can be expressed as

$$J_z(h) = \frac{\|h\|_H^2}{2} - c(x, y) + \iota_F(h, z) = \frac{\|h\|_H^2}{2} - c(h(0), h(1)) + \iota_F(h, z).$$

This yields the second expression for the J function in (16). Since $\mathrm{spt}(\gamma)$ is closed by definition, what remains is to verify that the mapping $E \ni h \mapsto \|h\|_H^2/2$ is lower semicontinuous. It suffices to show that the set $\{h \in H : \|h\|_H \leq 1\}$ is closed in E. Let $\{h_n\}_{n\in\mathbb{N}} \subset H$ be a sequence with $\|h_n\|_H \leq 1$ for all $n \in \mathbb{N}$ and $h_n \to h_\infty$ in E. We may assume without loss of generality that $h_n(0) = h_\infty(0) = 0$. Since $\tilde{H} = \{h \in H : h(0) = 0\}$ endowed with inner product $(\cdot, \cdot)_H$ is a Hilbert space, by the Banach–Alaoglu theorem, there exists a subsequence $h_{n'}$ such that $h_{n'} \to \tilde{h}$ weakly in $\tilde{H}$ for some $\tilde{h} \in \tilde{H}$ with $\|\tilde{h}\|_H \leq 1$, i.e. $\lim_{n'} (h_{n'}, g)_H = (\tilde{h}, g)_H$ for all $g \in \tilde{H}$. This implies that $h_\infty = \tilde{h}$ (choose appropriate g) and $\|h_\infty\|_H \leq 1$, as desired.

Part (ii) follows from Proposition 4(i) and [18, Theorem 2.1].

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

For (iii), the large-deviation upper bound follows from Proposition 4(ii) and [18, Theorem 2.2]. Finally, we verify that J has compact level sets, but this follows from [17, Lemma1.2.18], since the argument in Proposition 4(ii) indeed shows that $\{Q^{\varepsilon_k}\}_{k\in\mathbb{N}}$ is exponentially tight (replace O by $\mathrm{spt}(\gamma)$). □

4.3. Proof of Theorem 1

Proof of Theorem 1. Set $\phi(z) = c(x, y) - \psi^c(x) - \psi(y)$ for $z = (x, y) \in \mathcal{X}_\mathrm{o} \times \mathcal{Y}_\mathrm{o}$.

For (i), it suffices to show that for any $h \in e_{01}^{-1}(\mathcal{X}_\mathrm{o} \times \mathcal{Y}_\mathrm{o}) \cap H$ and $r > 0$,

$$\liminf_{k\to\infty} \varepsilon_k \log P^{\varepsilon_k}(B_E(h, r)) \geq -I(h).$$

Set $z = (h(0), h(1)) \in \mathcal{X}_o \times \mathcal{Y}_o$. By the exponential continuity of $\{R^{\varepsilon_k,z}\}$ established in Proposition 4, for every $\delta > 0$ we can choose an open neighborhood $O_z \subset \mathcal{X}_\mathrm{o} \times \mathcal{Y}_\mathrm{o}$ of z and a positive integer k_z such that, for every $z' \in O_z$,

$$\varepsilon_k \log R^{\varepsilon_k,z'}(B_E(h, r)) \geq -\inf_{h'\in B_E(h,r)} J_z(h') - \delta, \quad k \geq k_z.$$

For if not, for the open ball O_i in $\mathcal{X}\circ \times \mathcal{Y}\circ$ with center z and radius i^{-1}, we can find $z_i' \in O_i$ and a large enough positive integer k_i (with $k_i > k_{i-1}$) such that

$$\varepsilon_{k_i} \log R^{\varepsilon_{k_i},z_i'}(B_E(h, r)) < -\inf_{h'\in B_E(h,r)} J_z(h') - \delta,$$

but this contradicts the exponential continuity (as $z_i' \to z$). Hence,

$$\begin{aligned} P^{\varepsilon_k}(B_E(h, r)) &\geq \int_{O_z} \exp\left\{\varepsilon_k^{-1} \cdot \varepsilon_k \log R^{\varepsilon_k,z'}(B_E(h, r))\right\} \, \mathrm{d}\pi_{\varepsilon_k}(z') \\ &\geq \exp\left\{-\varepsilon_k^{-1}\left(\inf_{h'\in B_E(h,r)} J_z(h') + \delta\right)\right\}\pi_{\varepsilon_k}(O_z). \end{aligned}$$

Invoking [3, Corollary 4.7], we arrive at

$$\begin{aligned} \liminf_{k\to\infty} \varepsilon_k \log P^{\varepsilon_k}(B_E(h, r)) &\geq -\inf_{h'\in B_E(h,r)} J_z(h') - \delta - \inf_{z'\in O_z} \phi(z') \\ &\geq -(J_z(h) + \phi(z)) - \delta = -I(h) - \delta, \end{aligned}$$

establishing the desired claim.

For (ii), we first observe that for $A = e_{01}^{-1}(C)$ with $C \subset \mathcal{X}_\mathrm{o} \times \mathcal{Y}_\mathrm{o}$ compact, $P^\varepsilon(A) = \int_C R^{\varepsilon,z}(A) \, \mathrm{d}\pi_\varepsilon(z)$. Taking into account [3, Proposition 4.5], extend ϕ to $\mathcal{X} \times \mathcal{Y}$ as

$$\phi(x, y) = \sup_{\ell\geq 2} \sup_{\{(x_i,y_i)\}_{i=1}^{\ell}\subset \mathrm{spt}(\pi_\mathrm{o})} \sup_\tau \sum_{i=1}^{\ell} c(x_i, y_i) - \sum_{i=1}^{\ell} c(x_i, y_{\tau(i)}),$$

where $\sup_\tau$ is taken over all permutations of $\{1, \ldots, \ell\}$ and $(x_1, y_1) = (x, y)$. The function $\phi\colon \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ is lower semicontinuous [3, Lemma 4.2]) and agrees with the previous definition of ϕ on $\mathcal{X}_\mathrm{o} \times \mathcal{Y}_\mathrm{o}$. Let $\delta > 0$ be given. For every $z \in C$, by the exponential continuity of $\{R^{\varepsilon_k,z}\}$ established in Proposition 4 we can choose a bounded open neighborhood $O_z \subset \mathcal{X} \times \mathcal{Y}$ of z and a positive integer k_z such that, for every $z' \in O_z$,

$$\varepsilon_k \log R^{\varepsilon_k,z'}(A) \leq -\inf_{h\in A} J_z(h) + \delta, \quad k \geq k_z.$$

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

Furthermore, since ϕ is lower semicontinuous, by choosing O_z smaller if necessary, we have $\inf_{z'\in\bar{O}_z}\phi(z')\ge\phi(z)-\delta$, where $\bar{O}_z$ denotes the closure of O_z in $\mathcal{X}\times\mathcal{Y}$. By the compactness of C, we can find $z_1,\ldots,z_N\in C$ such that $C\subset\bigcup_{i=1}^N O_{z_i}$, so

$$P^{\varepsilon_k}(A)\le\sum_{i=1}^N\int_{O_{z_i}}\mathrm{e}^{\varepsilon_k^{-1}\cdot\varepsilon_k\log R^{\varepsilon_k,z}(A)}\ \mathrm{d}\pi_{\varepsilon_k}(z)\le\sum_{i=1}^N\mathrm{e}\Big\{\varepsilon_k^{-1}\Big(-\inf_{h\in A}J_{z_i}(h)+\delta+\varepsilon_k\log\pi_{\varepsilon_k}(\bar{O}_{z_i})\Big)\Big\}.$$

We invoke the following elementary result, whose proof follows from Jensen's inequality [10].

Lemma 2. (Smooth max function.) *For $\beta>0$ and $v=(v_1,\ldots,v_N)\in\mathbb{R}^N$, consider a smooth max function $m_\beta(v)=\beta^{-1}\log\big(\sum_{i=1}^N\mathrm{e}^{\beta v_i}\big)$. Then, for every $v\in\mathbb{R}^N$, we have $\max_{1\le i\le N}v_i\le m_\beta(v)\le\max_{1\le i\le N}v_i+\beta^{-1}\log N$.*

Using Lemma 1 combined with [3, Corollary 4.3], we have

$$\begin{aligned}
\varepsilon_k\log P^{\varepsilon_k}(A)&\le\max_{1\le i\le N}\Big\{-\inf_{h\in A}J_{z_i}(h)+\delta+\varepsilon_k\log\pi_{\varepsilon_k}(\bar{O}_{z_i})\Big\}+\varepsilon_k\log N\\
&\le\max_{1\le i\le N}\Big\{-\inf_{h\in A}J_{z_i}(h)-\inf_{z\in\bar{O}_{z_i}}\phi(z)\Big\}+\delta+o(1)\\
&\le\max_{1\le i\le N}\Big\{-\inf_{h\in A}J_{z_i}(h)-\phi(z_i)\Big\}+2\delta+o(1)\\
&\le-\inf_{h\in A}\inf_{z\in C}\{J_z(h)+\phi(z)\}+2\delta+o(1)\\
&=-\inf_{h\in A}\inf_{z\in C}\{I(h)+\iota_{\{z\}}(h(0),h(1))\}+2\delta+o(1)\\
&=-\inf_{h\in A}I(h)+2\delta+o(1),
\end{aligned}$$

where we used the fact that $(h(0),h(1))\in C$ whenever $h\in A$ by our choice of A. This completes the proof. □

4.4. Direct proof of Corollary 1

We first prove the following technical result concerning convergence of EOT potentials.

Lemma 3. (Convergence of EOT potentials.) *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are compact and one of them agrees with the closure of a connected open set. Then, under normalization $\int\psi^c\ \mathrm{d}\mu_0=\int\psi\ \mathrm{d}\mu_1$, the OT potential ψ from μ_1 to μ_0 is everywhere unique, and (ψ^c,ψ) are bounded and Lipschitz on $\mathcal{X}\times\mathcal{Y}$. Furthermore, let $(\varphi_\varepsilon,\psi_\varepsilon)$ be the unique EOT potentials under normalization $\int\varphi_\varepsilon\ \mathrm{d}\mu_0=\int\psi_\varepsilon\ \mathrm{d}\mu_1$. Then, for any sequence $\varepsilon_k\downarrow0$, $\varphi_{\varepsilon_k}\to\psi^c$ and $\psi_{\varepsilon_k}\to\psi$ uniformly on $\mathcal{X}$ and $\mathcal{Y}$, respectively.*

Proof. The lemma follows from [41, Proposition 7.18] and [36, Proposition 3.2]. We include a self-contained proof for completeness. First, under the current assumption, we observe that any OT potential ψ is bounded and Lipschitz on $\mathcal{Y}$. We have seen that the support of any OT plan π is contained in $\partial^c\psi$, so any $(x_0,y_0)\in\mathrm{spt}(\pi)$ satisfies $\psi(y_0)>-\infty$ and $\psi^c(x_0)>-\infty$, which entails $\psi=\psi^{cc}\le\sup_{\mathcal{X}\times\mathcal{Y}}c-\psi^c(x_0)$ and $\psi\ge-\sup_{\mathcal{X}}\psi^c\ge-\sup_{\mathcal{X}\times\mathcal{Y}}c+\psi(y_0)$. Lipschitz continuity follows from c-concavity. For the uniqueness, suppose $\mathrm{int}(\mathcal{Y})$ is connected. Recall that the projections of $\mathrm{spt}(\pi)$ onto $\mathcal{X}$ and $\mathcal{Y}$ agree with $\mathcal{X}$ and $\mathcal{Y}$, respectively (cf. Remark 6). For any OT potential ψ and any $y_0\in\mathrm{int}(\mathcal{Y})$, we can find $x_0\in\mathcal{X}$ such that

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

$\psi^c(x_0) + \psi(y_0) = c(x_0, y_0)$, i.e. $c(x_0, \cdot) - \psi(\cdot)$ is minimized at y_0, which entails $\nabla\psi(y_0) = \nabla_y c(x_0, y_0)$ as long as ψ is differentiable at y_0. We have shown that $\nabla\psi$ is uniquely determined Lebesgue-a.e. on $\mathrm{int}(\mathcal{Y})$. As $\mathrm{int}(\mathcal{Y})$ is connected, ψ is uniquely determined on $\mathrm{int}(\mathcal{Y})$ up to additive constants. By continuity, ψ is uniquely determined on $\mathcal{Y}$ up to additive constants. If $\mathrm{int}(\mathcal{X})$ is connected, then the OT potential φ from μ_0 to μ_1 is unique up to additive constants. If ψ is an OT potential from μ_1 to μ_0, then by the definition of the c-transform, we must have $\int (\psi - \varphi^c)\ \mathrm{d}\mu_1 = 0$, which yields $\psi = \varphi^c$ μ_1-a.e. By continuity, we have $\psi = \varphi^c$ on $\mathcal{Y}$.

For the second result, by the Schrödinger system (6) and Jensen's inequality, we have $\psi_\varepsilon^c \le \varphi_\varepsilon \le \sup_{\mathcal{X}\times\mathcal{Y}} c$ and $\varphi_\varepsilon^c \le \psi_\varepsilon \le \sup_{\mathcal{X}\times\mathcal{Y}} c$, so the EOT potentials are uniformly bounded by $\sup_{\mathcal{X}\times\mathcal{Y}} c$. Furthermore, under our assumption, the EOT potentials extend to smooth functions on $\mathbb{R}^d$ by the Schrödinger system, and directly calculating derivatives shows that $|\nabla\varphi_\varepsilon| \vee |\nabla\psi_\varepsilon| \le C$ on $\mathcal{X} \times \mathcal{Y}$ for some constant C independent of ε. Hence, the Ascoli–Arzelà theorem applies, and after passing to a subsequence, $\varphi_{\varepsilon_k} \to \bar{\varphi}$ and $\psi_{\varepsilon_k} \to \bar{\psi}$ uniformly on $\mathcal{X}$ and $\mathcal{Y}$, respectively. By the identity $\int \mathrm{e}^{(\varphi_\varepsilon+\psi_\varepsilon-c)/\varepsilon}\ \mathrm{d}(\mu_0 \otimes \mu_1) = 1$ and Fatou's lemma, we have $\bar{\varphi} + \bar{\psi} \le c$ $(\mu_0 \otimes \mu_1)$-a.e. By continuity, $\bar{\varphi} + \bar{\psi} \le c$ on $\mathcal{X} \times \mathcal{Y}$, but $\bar{\psi}^c \le \bar{\varphi}$ and $\bar{\varphi}^c \le \bar{\psi}$ by construction, $\bar{\varphi} = \bar{\psi}^c$ and $\bar{\psi} = \bar{\varphi}^c$, i.e. $(\bar{\varphi}, \bar{\psi})$ are c-concave. Now, using duality, for any OT plan π, $\int c\ \mathrm{d}\pi \le \lim_k \big(\int c\ \mathrm{d}\pi_{\varepsilon_k} + \varepsilon_k H(\pi_{\varepsilon_k} \mid \mu_0 \otimes \mu_1)\big) = \int \bar{\varphi}\ \mathrm{d}\mu_0 + \int \bar{\psi}\ \mathrm{d}\mu_1 \le \int c\ \mathrm{d}\pi$, so $(\bar{\varphi}, \bar{\psi})$ are OT potentials. Since $\int \bar{\varphi}\ \mathrm{d}\mu_0 = \int \bar{\psi}\ \mathrm{d}\mu_1$ by construction, by the uniqueness result, $\bar{\varphi} = \psi^c$ and $\bar{\psi} = \psi$. Finally, by the uniqueness of the limits, along the original sequence, $\varphi_{\varepsilon_k} \to \psi^c$ and $\psi_{\varepsilon_k} \to \psi$ uniformly on $\mathcal{X}$ and $\mathcal{Y}$, respectively. □

Direct proof of Corollary 1. Set $S = e_{01}^{-1}(\mathcal{X} \times \mathcal{Y}) \subset E$. Recall that $\bar{R}^\varepsilon = \int R^{\varepsilon,xy}\ \mathrm{d}(\mu_0 \otimes \mu_1)$. By construction, $\bar{R}^\varepsilon(S) = 1$ for all $\varepsilon > 0$. Abusing notation, we shall write $\phi_\varepsilon(\omega) = \phi_\varepsilon(\omega(0), \omega(1))$. With this convention, we have $P^\varepsilon(A) = \int_A \mathrm{e}^{-\phi_\varepsilon/\varepsilon}\ \mathrm{d}\bar{R}^\varepsilon$. Set $J(h) = \inf_{(x,y)\in\mathcal{X}\times\mathcal{Y}} J_{xy}(h)$ and $\phi(h) = \phi(h(0), h(1)) = c(h(0), h(1)) - \psi^c(h(0)) - \psi(h(1))$ for $h \in S$.

Step 1. Let $A \subset E$ be open and pick any $h \in A$ such that $I(h) < \infty$ (if no such h exists then the conclusion is trivial). By Lemma 3, for every $\delta > 0$ there exists an open neighborhood $G \subset A$ of h such that $\sup_{\omega\in G\cap S} \phi_{\varepsilon_k}(\omega) \le \phi(h) + \delta$ for all large k. Hence,

$$P^{\varepsilon_k}(A) \ge P^{\varepsilon_k}(G) \ge \mathrm{e}^{-(\phi(h)+\delta)/\varepsilon_k} \bar{R}^{\varepsilon_k}(G).$$

Corollary 2 implies that $\varepsilon_k \log P^{\varepsilon_k}(A) \ge -\phi(h) - \delta - J(h) + o(1)$ as $k \to \infty$. Noting that $\phi(h) + J(h) = I(h)$ yields the desired lower bound.

Step 2. For the upper bound, we first note that by Lemma 3, ϕ_ε are uniformly lower bounded on S, and $\phi_\varepsilon(\omega) \ge -M$ for all $\omega \in S$ and $\varepsilon > 0$ for some $M > 0$. Let $A \subset E$ be closed. Pick any $\alpha < \infty$ and $\delta > 0$. Set $\Psi_J(\alpha) = \{h \colon J(h) \le \alpha\} \cap A$, which is a compact subset of E as J is a good rate function and A is closed. By Lemma 3 and the lower semicontinuity of the function J, for every $h \in \Psi_J(\alpha)$ (which entails $h \in S$), we can find an open neighborhood U_h of h such that $\inf_{\omega\in\bar{U}_h} J(\omega) \ge J(h) - \delta$ and $\inf_{\omega\in\bar{U}_h\cap S} \phi_{\varepsilon_k}(\omega) \ge \phi(h) - \delta$ for large k, where $\bar{U}_h$ denotes the closure of U_h in E. By the compactness of $\Psi_J(\alpha)$, we can find $h_1, \ldots, h_N \in \Psi_J(\alpha)$ such that $\Psi_J(\alpha) \subset \bigcup_{i=1}^N U_{h_i}$. Now, setting $F = \big(\bigcup_{i=1}^N U_{h_i}\big)^c \cap A$ (which is a closed subset of E), we observe that

$$\begin{aligned} P^{\varepsilon_k}(A) &= \int_A \mathrm{e}^{-\phi_{\varepsilon_k}/\varepsilon_k}\ \mathrm{d}\bar{R}^\varepsilon \\ &\le \sum_{i=1}^N \exp\big\{(\varepsilon_k \log \bar{R}^{\varepsilon_k}(\bar{U}_{h_i}) - \phi(h_i) + \delta)/\varepsilon_k\big\} + \mathrm{e}^{(M+\varepsilon_k \log \bar{R}^{\varepsilon_k}(F))/\varepsilon_k}. \end{aligned}$$

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

Using Lemma 2, and combining Lemma 3 and Corollary 2, we have

$$\begin{aligned}\varepsilon_k \log P^{\varepsilon_k}(A) &\leq \max\Big\{\varepsilon_k \log \bar{R}^{\varepsilon_k}(\bar{U}_{h_1}) - \phi(h_1) + \delta, \ldots, \varepsilon_k \log \bar{R}^{\varepsilon_k}(\bar{U}_{h_N}) - \phi(h_N) + \delta,\\ &\qquad M + \varepsilon_k \log \bar{R}^{\varepsilon_k}(F)\Big\} + \varepsilon_k \log (N+1)\\ &\leq \max\Big\{-\inf_{\omega\in\bar{U}_{h_1}} J(\omega) - \phi(h_1) + \delta, \ldots, -\inf_{\omega\in\bar{U}_{h_N}} J(\omega) - \phi(h_N) + \delta,\\ &\qquad M - \inf_{\omega\in F} J(\omega)\Big\} + o(1)\\ &\leq \max\{-I(h_1) + 2\delta, \ldots, -I(h_N) + 2\delta, M - \alpha\} + o(1)\\ &\leq \max\Big\{-\inf_{h\in A} I(h) + 2\delta, M - \alpha\Big\} + o(1),\end{aligned}$$

where we used $J(h) + \phi(h) = I(h)$. Since $\alpha < \infty$ and $\delta > 0$ are arbitrary, we obtain the desired upper bound. Finally, the rate function I being good follows from an argument similar to the proof of Corollary 2(iii). This completes the proof. □

4.5. Proof of Proposition 2

Proof of Proposition 2. The fact that the sequence $\{P_{st}^{\varepsilon_k}\}_{k\in\mathbb{N}}$ satisfies an LDP having a good rate function follows from Corollary 1 and the contraction principle. The rate function is given by

$$I_{st}(x, y) = \inf_{h:h(s)=x,h(t)=y} \frac{\|h\|_H^2}{2} - \varphi(h(0)) - \psi(h(1)).$$

First, fix two endpoints $h(0) = x'$ and $h(1) = y'$ and optimize $\|h\|_H^2$ under the constraint $(h(s), h(t)) = (x, y)$. The optimal h is given by

$$h(u) = \begin{cases} \left(1 - \dfrac{u}{s}\right)x' + \dfrac{u}{s}x & \text{if } u \in [0, s],\\ \left(1 - \dfrac{u-s}{t-s}\right)x + \dfrac{u-s}{t-s}y & \text{if } u \in [s, t],\\ \left(1 - \dfrac{u-t}{1-t}\right)y + \dfrac{u-s}{1-t}y' & \text{if } u \in [t, 1],\end{cases}$$

which gives $\|h\|_H^2/2 = c^{0,s}(x', x) + c^{s,t}(x, y) + c^{t,1}(y, y')$. Hence,

$$\begin{aligned}I_{st}(x, y) &= \inf_{x',y'} \left\{c^{0,s}(x', x) + c^{s,t}(x, y) + c^{t,1}(y, y') - \varphi(x') - \psi(y')\right\}\\ &= c^{st}(x, y) + \mathcal{Q}_s(-\varphi)(x) + \mathcal{Q}_{1-t}(-\psi)(x) = c^{s,t}(x, y) - \varphi_s(x) - \psi_t(y).\end{aligned}$$

The final claim follows from [47, Theorem 7.35] after adjusting the signs. □

4.6. Proof of Proposition 3

Proof of Proposition 3. The EOT plan $\check{\pi}^\varepsilon$ is of the form

$$d\check{\pi}^\varepsilon(x, y) = \exp\{(\check{\varphi}_\varepsilon(x) + \check{\psi}_\varepsilon(y) - c_\varepsilon(x, y))/\varepsilon\}\ d(\mu_0 \otimes \mu_1)(x, y),$$

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

where $(\check{\varphi}_\varepsilon, \check{\psi}_\varepsilon)$ are EOT potentials satisfying the Schrödinger system (6) with c replaced by c_ε. For uniqueness, we assume without loss of generality that $\int \check{\varphi}_\varepsilon \, \mathrm{d}\mu_0 = \int \check{\psi}_\varepsilon \, \mathrm{d}\mu_1$. Consider the mixture distribution $Q^\varepsilon = \int \check{R}^{\varepsilon,xy} \, \mathrm{d}(\mu_0 \otimes \mu_1)(x, y)$; then

$$\frac{\mathrm{d}\check{P}^\varepsilon}{\mathrm{d}Q^\varepsilon}(\omega) = \exp\left\{\frac{1}{\varepsilon}(\check{\varphi}_\varepsilon(\omega(0)) + \check{\psi}_\varepsilon(\omega(1)) - c_\varepsilon(\omega(0), \omega(1)))\right\}, \quad \omega = (\omega(t))_{t\in[0,1]} \in E.$$

Furthermore, by [44, Theorems 4.4.6 and 4.4.12], we have $\lim_{\varepsilon\downarrow 0} c_\varepsilon(x, y) = |x - y|^2/2 = c(x, y)$ uniformly over $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Hence, in view of the direct proof of Corollary 1, the desired claim follows once we verify the following:

- the mixture distributions $\{Q^{\varepsilon_k}\}_{k\in\mathbb{N}}$ satisfy the LDP with good rate function $J(h) = \inf_{(x,y)\in\mathcal{X}\times\mathcal{Y}} J_{xy}(h)$;
- as $k \to \infty$, $\check{\varphi}_{\varepsilon_k} \to \psi^c$ and $\check{\psi}_{\varepsilon_k} \to \psi$ uniformly on $\mathcal{X}$ and $\mathcal{Y}$, respectively.

The first item follows by establishing the exponential continuity of $\{\check{R}^{\varepsilon_k,xy}\}_{k\in\mathbb{N}}$ w.r.t. (x,y). To this end, we invoke the Radon–Nikodym derivative of the Langevin bridge $\check{R}^{\varepsilon,xy}$ against the Brownian bridge $R^{\varepsilon,xy}$:

$$\frac{\mathrm{d}\check{R}^{\varepsilon,xy}}{\mathrm{d}R^{\varepsilon,xy}}(\omega) = Z_{\varepsilon,xy}^{-1} \exp\left\{-\frac{\varepsilon}{2}\int_0^1 \left(|\nabla V(\omega(t))|^2 - \Delta V(\omega(t))\right) \, \mathrm{d}t\right\}, \tag{17}$$

where ΔV is the Laplacian of V and $Z_{\varepsilon,xy}$ is the normalizing constant. See [28, Section 5] and the proof of [12, Theorem 2.1]; see also Remark 11. Heuristically, this follows from the following observation. The Langevin diffusion X^ε follows the SDE

$$\mathrm{d}X^\varepsilon(t) = -\varepsilon\nabla V(X^\varepsilon(t)) \, \mathrm{d}t + \sqrt{\varepsilon} \, \mathrm{d}W(t).$$

The Girsanov theorem yields that

$$\frac{\mathrm{d}\check{R}^\varepsilon}{\mathrm{d}R^\varepsilon}(\omega) = \exp\left\{-\int_0^1 \nabla V(\omega(t)) \cdot \, \mathrm{d}\omega(t) - \frac{\varepsilon}{2}\int_0^1 |\nabla V(\omega(t))|^2 \, \mathrm{d}t\right\}$$

under R^ε. An application of Itô's formula yields

$$\int_0^1 \nabla V(\omega(t)) \cdot \, \mathrm{d}\omega(t) = V(\omega(1)) - V(\omega(0)) - \frac{\varepsilon}{2}\int_0^1 \Delta V(\omega(t)) \, \mathrm{d}t$$

under R^ε. The bridge case is obtained by canceling $V(\omega(1)) - V(\omega(0))$, which is to be expected since it depends only on the endpoints. Now, since the potential V has bounded derivatives, the desired exponential continuity follows from Proposition 4.

For the second item, by the Schrödinger system and Jensen's inequality, we have

$$\begin{aligned}|\check{\varphi}_\varepsilon(x) - \check{\varphi}_\varepsilon(x')| &\le \sup_{y\in\mathcal{Y}} |c_\varepsilon(x, y) - c_\varepsilon(x', y)| \\ &\le \sup_{y\in\mathcal{Y}} |c(x, y) - c(x', y)| + 2 \sup_{\mathcal{X}\times\mathcal{Y}} |c_\varepsilon - c|.\end{aligned}$$

By the generalized Ascoli–Arzelà theorem (cf. [36, Lemma 2.2]), the sequence of functions $\{\check{\varphi}_{\varepsilon_k}\}_{k\in\mathbb{N}}$ converges uniformly on $\mathcal{X}$ along a subsequence. A similar result holds for $\check{\psi}_{\varepsilon_k}$. The

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

rest of the proof is analogous to the second part of the proof of Lemma 3. This completes the proof. □

Remark 11 (*Derivation of (17).*) Formally, the Radon–Nikodym derivative (17) follows by reducing to the $\varepsilon = 1$ case via reparameterization and [12, (25)]. Indeed, the process $Y^\varepsilon(t) = X^\varepsilon(t)/\sqrt{\varepsilon}$ satisfies $\mathrm{d}Y^\varepsilon(t) = -\nabla V^\varepsilon(Y^\varepsilon(t))\ \mathrm{d}t + \mathrm{d}W(t)$, where $V^\varepsilon(x) = V(\sqrt{\varepsilon}x)$. By [12, (25)], denoting by $Y^\varepsilon_\# \mathbb{P}$ the law of the process $Y^\varepsilon = (Y^\varepsilon(t))_{t\in[0,1]}$, we have

$$\frac{\mathrm{d}(Y^\varepsilon_\#\mathbb{P})^{xy}}{\mathrm{d}R^{1,xy}}(\omega) = Z_{xy}^{-1} \exp\left\{-\frac{1}{2}\int_0^1 \left(|\nabla V^\varepsilon(\omega(t))|^2 - \Delta V^\varepsilon(\omega(t))\right)\ \mathrm{d}t\right\}$$
$$= Z_{xy}^{-1} \exp\left\{-\frac{\varepsilon}{2}\int_0^1 \left(|\nabla V(\sqrt{\varepsilon}\omega(t))|^2 - \Delta V(\sqrt{\varepsilon}\omega(t))\right)\ \mathrm{d}t\right\},$$

where Z_{xy} is the normalizing constant. Now, the formula (17) follows by a simple reparameterization.

Acknowledgement

The author would like to thank the editor and two anonymous referees for their careful reading and constructive comments that helped improve the quality of this paper.

Funding information.

K. Kato is partially supported by NSF grants DMS-1952306, DMS-2210368, and DMS-2413405.

Competing interests.

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

[1] Altschuler, J. M., Niles-Weed, J. and Stromme, A. J. (2022). Asymptotics for semidiscrete entropic optimal transport. *SIAM J. Math. Anal.* **54**, 1718–1741.
[2] Ambrosio, L., Gigli, N. and Savare, G. (2008). *Gradient Flows: In Metric Spaces and in the Space of Probability Measures*. Springer, Berlin.
[3] Bernton, E., Ghosal, P. and Nutz, M. (2022). Entropic optimal transport: Geometry and large deviations. *Duke Math. J.* **171**, 3363–3400.
[4] Bernton, E., Heng, J., Doucet, A. and Jacob, P. E. (2019). Schrödinger bridge samplers. Preprint, arXiv:1912.13170.
[5] Boissard, E., Gozlan, N., Lehec, J., Léonard, C., Menz, G. and Schlichting, A. (2014). Some recent developments in functional inequalities. *ESAIM: Proc.* **44**, 338–354.
[6] Bravo, G. U. and Chaumont, L. (2011). Markovian bridges: Weak continuity and pathwise constructions. *Ann. Prob.* **39**, 609–647.
[7] Brenier, Y. (1991). Polar factorization and monotone rearrangement of vector-valued functions. *Commun. Pure Appl. Math.* **44**, 375–417.
[8] Carlier, G., Duval, V., Peyré, G. and Schmitzer, B. (2017). Convergence of entropic schemes for optimal transport and gradient flows. *SIAM J. Math. Anal.* **49**, 1385–1418.
[9] Carlier, G., Pegon, P. and Tamanini, L. (2023). Convergence rate of general entropic optimal transport costs. *Calc. Var. Partial Differ. Equ.* **62**, 116.
[10] Chatterjee, S. (2005). An error bound in the Sudakov–Fernique inequality. Preprint, arXiv:0510424.
[11] Chizat, L., Roussillon, P., Léger, F., Vialard, F.-X. and Peyré, G. (2020). Faster Wasserstein distance estimation with the Sinkhorn divergence. In *Proc. 34th Int. Conf. Neural Information Processing Systems*, 2257–2269.

[12] CONFORTI, G. (2018). Fluctuations of bridges, reciprocal characteristics and concentration of measure. *Ann. Inst. H. Poincaré Prob. Statist.* **54**, 1432–1463.

[13] CONFORTI, G. AND TAMANINI, L. (2021). A formula for the time derivative of the entropic cost and applications. *J. Funct. Anal.* **280**, 108964.

[14] CUTURI, M. (2013). Sinkhorn distances: Lightspeed computation of optimal transport. In *Proc. 27th Int. Conf. Neural Information Processing Systems*, 2292–2300.

[15] DAI PRA, P. (1991). A stochastic control approach to reciprocal diffusion processes. *Appl. Math. Optim.* **23**, 313–329.

[16] DE BORTOLI, V., THORNTON, J., HENG, J. AND DOUCET, A. (2021). Diffusion Schrödinger bridge with applications to score-based generative modeling. In *Proc. 35th Int. Conf. Neural Information Processing Systems*, 17695–17709.

[17] DEMBO, A. AND ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd edn. Springer, New York.

[18] DINWOODIE, I. H. AND ZABELL, S. L. (1992). Large deviations for exchangeable random vectors. *Ann. Prob.* **20**, 1147–1166.

[19] FÖLLMER, H. (1988). Random fields and diffusion processes. In *Ecole d'Ete de Probabilites de Saint-Flour XV–XVII, 1985–87*, eds P. Diaconis, D. Elworthy, H. Föllmer, E. Nelson, G. Papanicolaou and Srinivasa R. S. Varadhan. Springer, New York, pp. 101–203.

[20] GENTIL, I., LÉONARD, C., RIPANI, L. AND TAMANINI, L. (2020). An entropic interpolation proof of the HWI inequality. *Stoch. Process. Appl.* **130**, 907–923.

[21] GIGLI, N. AND TAMANINI, L. (2021). Second order differentiation formula on $RCD^*(K, N)$ spaces. *J. Europ. Math. Soc.* **23**, 1727–1795.

[22] GINÉ, E. AND NICKL, R. (2021). *Mathematical Foundations of Infinite-Dimensional Statistical Models*. Cambridge University Press.

[23] HSU, P. (1990). Brownian bridges on Riemannian manifolds. *Prob. Theory Relat. Fields* **84**, 103–118.

[24] KARATZAS, I. AND SHREVE, S. E. (1998). *Brownian Motion and Stochastic Calculus*. Springer, New York.

[25] LEHEC, J. (2013). Representation formula for the entropy and functional inequalities. *Ann. Inst. H. Poincaré Prob. Statist.* **49**, 885–899.

[26] LÉONARD, C. (2012). From the Schrödinger problem to the Monge–Kantorovich problem. *J. Funct. Anal.* **262**, 1879–1920.

[27] LÉONARD, C. (2014). A survey of the Schrödinger problem and some of its connections with optimal transport. *Discrete Continuous Dynam. Syst.* **34**, 1533–1574.

[28] LEVY, B. C. AND KRENER, A. J. (1993). Dynamics and kinematics of reciprocal diffusions. *J. Math. Phys.* **34**, 1846–1875.

[29] MCCANN, R. J. (1997). A convexity principle for interacting gases. *Adv. Math.* **128**, 153–179.

[30] MENA, G. AND NILES-WEED, J. (2019). Statistical bounds for entropic optimal transport: Sample complexity and the central limit theorem. In *Proc. 33rd Int. Conf. Neural Information Processing Systems*, 4541–4551.

[31] MIKAMI, T. (2004). Monge's problem with a quadratic cost by the zero-noise limit of h-path processes. *Prob. Theory Relat. Fields* **129**, 245–260.

[32] MIKAMI, T. AND THIEULLEN, M. (2008). Optimal transportation problem by stochastic optimal control. *SIAM J. Control Optim.* **47**, 1127–1139.

[33] MIKULINCER, D. AND SHENFELD, Y. (2021). The Brownian transport map. Preprint, arXiv:2111.11521.

[34] MONSAINGEON, L., TAMANINI, L. AND VOROTNIKOV, D. (2023). The dynamical Schrödinger problem in abstract metric spaces. *Adv. Math.* **426**, 109100.

[35] NUTZ, M. (2021). Introduction to entropic optimal transport. *Lecture Notes*, Columbia University, New York.

[36] NUTZ, M. AND WIESEL, J. (2022). Entropic optimal transport: Convergence of potentials. *Prob. Theory Relat. Fields* **184**, 401–424.

[37] PAL, S. (2024). On the difference between entropic cost and the optimal transport cost. *Ann. Appl. Prob.* **34**, 1003–1028.

[38] PAVON, M., TRIGILA, G. AND TABAK, E. G. (2021). The data-driven Schrödinger bridge. *Commun. Pure Appl. Math.* **74**, 1545–1573.

[39] PEYRÉ, G. AND CUTURI, M. (2019). Computational optimal transport: With applications to data science. *Foundations and Trends in Machine Learning* **11**, 355–607.

[40] POOLADIAN, A.-A. AND NILES-WEED, J. (2021). Entropic estimation of optimal transport maps. Preprint, arXiv:2109.12004.

[41] SANTAMBROGIO, F. (2015). *Optimal Transport for Applied Mathematicians*. Springer, New York.

[42] STAUDT, T., HUNDRIESER, S. AND MUNK, A. (2022). On the uniqueness of Kantorovich potentials. Preprint, arXiv:2201.08316.

[43] STROMME, A. (2023). Sampling from a Schrödinger bridge. *Proc. Mach. Learn. Res*. **26**, 4058–4067.

[44] STROOCK, D. W. (2008). *Partial Differential Equations for Probabalists*. Cambridge University Press.

https://doi.org/10.1017/jpr.2025.10049 Published online by Cambridge University Press

[45] STROOCK, D. W. (2011). *Probability Theory: An Analytic View*, 2nd edn. Cambridge University Press.
[46] VAN DER VAART, A. W. AND WELLNER, J. A. (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer, New York.
[47] VILLANI, C. (2009). *Optimal Transport: Old and New*. Springer, New York.