This is a ``preproof'' accepted article for *Canadian Journal of Mathematics*
This version may be subject to change during the production process.
DOI: 10.4153/S0008414X25101041

AN EXTENSION-RESTRICTION THEOREM FOR WEIGHTED BESOV SPACES

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ABSTRACT. In this paper, the authors establish an extension-restriction theorem between homogeneous weighted Besov spaces and weighted mixed-Riesz potential spaces. This general frame covers both the classical Besov spaces and their logarithmic analogues.

1. INTRODUCTION AND MAIN RESULTS

1.1. A brief historical background. For $s \in (0,\infty)$ and $p,q \in [1,\infty)$, the homogeneous Besov space $\dot{\Lambda}^s_{p,q}(\mathbb{R}^n)$ is defined to be the set of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\dot{\Lambda}^s_{p,q}(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|\mathrm{D}_h^{\lfloor s\rfloor+1}f(x)\right|^p\,dx\right)^{q/p}\frac{dh}{|h|^{n+sq}}\right]^{1/q} < \infty,$$

where $\lfloor s\rfloor$ denotes the largest integer no more than s and for any $k\in\mathbb{N}$ the symbol $\mathrm{D}_h^k f$ represents the k-th difference of f, that is,

$$\mathrm{D}_h^k f(x) := \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}f(x+mh). \tag{1.1}$$

The inhomogeneous Besov space $\Lambda^s_{p,q}(\mathbb{R}^n)$ is the intersection of $L^p(\mathbb{R}^n)$ and $\dot{\Lambda}^s_{p,q}(\mathbb{R}^n)$, endowed with the norm

$$\|f\|_{\Lambda^s_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{\Lambda}^s_{p,q}(\mathbb{R}^n)}.$$

When $p=q$, the space $\Lambda^s_{p,p}(\mathbb{R}^n)$ is known as the fractional Sobolev space and also called Aronszajn, Gagliardo or Slobodeckij spaces in literature. We refer the readers to [38, 45] and [14] for detailed expositions of Besov spaces.

Besov spaces arise naturally as the trace of the Bessel potential function spaces. For an integrable function f on $\mathbb{R}^n$, its Fourier transform $\hat{f}$ is defined by setting

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx \quad \text{for all } \xi\in\mathbb{R}^n.$$

Recall that, for $s\in(0,\infty)$ and $p\in[1,\infty)$, the Bessel potential space $L^p_s(\mathbb{R}^n)$ is defined by

$$L^p_s(\mathbb{R}^n) = \{G_s * f:\ f\in L^p(\mathbb{R}^n)\},$$

where G_s is the kernel of the Bessel potential whose Fourier transform is

$$\hat{G}_s(x) := (1+|x|^2)^{-\frac{s}{2}} \quad \text{for all } x\in\mathbb{R}^n.$$

2020 *Mathematics Subject Classification.* 46E30, 42B35, 47B38.

Key words and phrases. Besov spaces, mixed-Riesz potential spaces, weight, extension, restriction.

L. Liu was supported by National Natural Science Foundation of China (# 12371102 & # 12271021); S. Wu was supported by the National Natural Science Foundation of China (# 12201098) and the Fundamental Research Funds for the Central Universities (# 3132024199).

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If $u = G_s * f$ for some $f \in L^p(\mathbb{R}^n)$, then define

$$\|u\|_{L^p_s(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)}.$$

In particular, if $p \in (1, \infty)$ and $k \in \mathbb{N}$, then $L^p_k(\mathbb{R}^n)$ coincides to the classical Sobolev space $W^{k,p}(\mathbb{R}^n)$ and

$$\|u\|_{W^{k,p}(\mathbb{R}^n)} := \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)} \approx \|u\|_{L^p_k(\mathbb{R}^n)}.$$

Note that $\Lambda^s_{2,2}(\mathbb{R}^n) = L^2_s(\mathbb{R}^n)$ (see [41]). For $p \in (1, \infty)$, the Besov space $\Lambda^s_{p,p}(\mathbb{R}^n)$ coincides with the trace (restriction to $\mathbb{R}^n$) of the Bessel potential function spaces $L^p_{s+1/p}(\mathbb{R}^{n+1})$ (see, for example, [23, Theorem 11.1] or [45, p. 138, (46)]).

The well-known extension-restriction theorem of Stein (see [42] or [43, p. 193]) says that, under $s \in (0, \infty)$, $p \in (1, \infty)$ and $m > n$, there exist a bounded linear extension operator

$$\mathcal{E} : \ \Lambda^s_{p,p}(\mathbb{R}^n) \to L^p_{s+(m-n)/p}(\mathbb{R}^m)$$

and a bounded linear restriction operator

$$\mathcal{R} : \ L^p_{s+(m-n)/p}(\mathbb{R}^m) \to \Lambda^s_{p,p}(\mathbb{R}^n)$$

such that

$$\mathcal{R} \circ \mathcal{E} = id.$$

This extension-restriction result was proved by Gagliardo [25] when $s = 1$ and $p \in (1, \infty)$, and by Aronszajn–Smith [4] when $s \in (0, \infty)$ and $p = 2$.

Adams [1] established an analogous extension-restriction theorem for general Besov spaces by terms of the mixed-Riesz potential spaces. For $\beta \in \mathbb{R}$ and $p, q \in (1, \infty)$, the mixed-norm Lebesgue space $L^q(L^p)(\mathbb{R}^{2n})$ is consisting of all locally integrable functions f on $\mathbb{R}^{2n}$ such that

$$\|f\|_{L^q(L^p)(\mathbb{R}^{2n})} := \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x, y)|^p \, dx\right)^{q/p} dy\right]^{1/q} < \infty,$$

and the mixed-Riesz potential space $\dot{\mathcal{L}}^{p,q}_\beta(\mathbb{R}^{2n})$ is defined by

$$\dot{\mathcal{L}}^{p,q}_\beta(\mathbb{R}^{2n}) := \left\{(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} f : \ f \in L^q(L^p)(\mathbb{R}^{2n})\right\},$$

where $\Delta_{\mathbb{R}^{2n}} = \sum_{j=1}^{2n} \partial_j^2$ is the Laplace operator on $\mathbb{R}^{2n}$. Adams [1, Theorem 5.2] (see also [2, Theorem A]) show that there exist a bounded linear extension operator

$$\mathcal{E} : \ \dot{\Lambda}^s_{p,q}(\mathbb{R}^n) \to \dot{\mathcal{L}}^{p,q}_{s+n/q}(\mathbb{R}^{2n})$$

and a bounded linear restriction operator

$$\mathcal{R} : \ \dot{\mathcal{L}}^{p,q}_{s+n/q}(\mathbb{R}^{2n}) \to \dot{\Lambda}^s_{p,q}(\mathbb{R}^n)$$

such that

$$\mathcal{R} \circ \mathcal{E} = id.$$

It should be remarked that the study of the mixed-norm Lebesgue spaces can be traced back to [5, 32] and has attracted lots of attention recently (see [8, 9, 10, 33]). Moreover, the aforementioned extension-restriction theorem on Besov spaces can further be applied to establish capacitary inequalities and embedding properties of Besov spaces (see [1, 2]).

Note that functions in the Besov space $\dot{\Lambda}^s_{p,q}(\mathbb{R}^n)$ enjoy a polynomial smoothness $|h|^s$, but function spaces with generalized smoothness have also attracted lots of attention since Gol'dman

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[26, 27] and Kalyabin [34, 35]. We refer the reader to [22] for a review on the earlier development on this topic. In particular, Edmunds and Haroske [19, 20, 30, 31] introduced and investigated the logarithmic Besov spaces with both polynomial smoothness and logarithmic smoothness (see also [6, 7, 21, 36]). Recently, Cobos-Domínguez-Triebel-Tikhonov [11, 12, 13, 16] focus on the limiting case of the logarithmic Besov spaces which has polynomial smoothness zero.

Motivated by the above discussion, our main goal in this paper is to extend the extension-restriction theorem of Stein [42] and Adams [1] to Besov type spaces with generalized smoothness. We will consider a general Besov space with a radial weight w acting on the variable $|h|$ (see Definition 1.1 below), so that to cover both classical Besov spaces and logarithmic Besov spaces. Upon introducing a weighted mixed-Riesz potential space (see Definition 1.4 below), we are aiming to establish an extension-restriction theorem between homogeneous weighted Besov spaces and weighted mixed-Riesz potential spaces, which can recover Stein [42, 43] and Adams [1] (see also Adams and Xiao [2]). Such an extension-restriction theorem is supposed to be useful in the further study of capacitary inequalities involving the weighted Besov capacity.

1.2. **Weighted Besov spaces and weighted mixed-Riesz potential spaces.** A *weight* is a positive locally integrable function on $\mathbb{R}^n$. For a weight w and a number $p \in (0, \infty)$, the *weighted Lebesgue space* $L^p_w(\mathbb{R}^n)$ is defined to be the set of all Lebesgue-measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p\, w(x)\, dx\right)^{1/p} < \infty.$$

If $w \equiv 1$, then we simply write $L^p(\mathbb{R}^n)$ and $\|\cdot\|_{L^p(\mathbb{R}^n)}$. The weighted Besov space is defined as below.

Definition 1.1. Let $p,\ q \in [1, \infty)$, $s \in (0, \infty)$, $k \in \mathbb{N}$ and w be a weight on $\mathbb{R}^n$. For any Lebesgue-measurable function f on $\mathbb{R}^n$, set

$$\|f\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \frac{\|\mathrm{D}^k_h f\|^q_{L^p(\mathbb{R}^n)}}{|h|^{n+sq}} w(h)\, dh\right)^{1/q}.$$

The *inhomogeneous weighted Besov space* $\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)$ is defined to be the collection of all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)} < \infty.$$

The *homogeneous weighted Besov space* $\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)$ is defined to be the completion of $C^\infty_c(\mathbb{R}^n)$ under the semi-norm $\|\cdot\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)}$, where $C^\infty_c(\mathbb{R}^n)$ is the space of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support. In particular, if $k = \lfloor s \rfloor + 1$, then we simply write $\Lambda^s_{p,q,w}(\mathbb{R}^n)$ and $\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$, respectively.

Remark 1.2. The reasonability of Definition 1.1 can be seen from Lemma 2.2 below. Moreover, we present here a four-fold comment on Definition 1.1:

(i) If $w \equiv 1$ and $k = \lfloor s \rfloor + 1$, then Definition 1.1 defines the classical inhomogeneous Besov space $\Lambda^s_{p,q}(\mathbb{R}^n)$ and the homogeneous Besov space $\dot{\Lambda}^s_{p,q}(\mathbb{R}^n)$.

(ii) If $w(h) = (\log(e + 1/|h|))^\gamma$ with $h \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, then $\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$ is the logarithmic Besov space that has been systematically studied in [30, 31, 19, 20, 6, 7, 21, 36].

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(iii) The spaces $\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$ and $\Lambda^s_{p,q,w}(\mathbb{R}^n)$ are known as *Besov spaces of generalized smoothness* in literature; see Section 2.3 below for more details. Let $p,\ q \in [1,\infty)$, $k \in \mathbb{N}$ and $\lambda : (0,1) \to [0,\infty)$ be a non-decreasing, continuous function satisfying $\lim_{t\to 0} \lambda(t) = 0$. Recall that Farkas and Leopold [22] introduced the Besov space $B^\lambda_{p,q}(\mathbb{R}^n)$ with generalized soomthness λ, which is defined to be the set of all $f \in L^p(\mathbb{R}^n)$ such that

$$\left[\int_0^1 \left(\frac{\sup_{|h|<t} \|D^k_h f\|_{L^p(\mathbb{R}^n)}}{\lambda(t)}\right)^q \frac{d\lambda(t)}{\lambda(t)}\right]^{1/q} < \infty.$$

It is worth noting that when λ is differentiable and

$$w(t) = t^{1+sq} \frac{\lambda'(t)}{\lambda(t)^{1+q}} \quad \text{for all } \ t \in (0,1),$$

then the space $B^\lambda_{p,q}(\mathbb{R}^n)$ falls into the scope of Definition 1.1.

(iv) Recall that, for $p,\ q \in [1,\infty]$ and a weight $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, Ansorena-Blasco [3] define the space $\Lambda^{p,q}_\lambda(\mathbb{R}^n)$ to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|^q_{\Lambda^{p,q}_\lambda(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{\|f(\cdot + h) - f(\cdot)\|^q_{L^p(\mathbb{R}^n)}}{\lambda(|h|)^q} \frac{dh}{|h|^n} < \infty,$$

with a usual modification made when $q = \infty$. Note that if $s \in (0,1)$ and

$$\lambda(|h|) = \frac{|h|^s}{[w(h)]^{1/q}} \quad \text{for all } h \in \mathbb{R}^n,$$

then $\Lambda^{p,q}_\lambda(\mathbb{R}^n) = \dot{\Lambda}^{s,1}_{p,q,w}(\mathbb{R}^n)$.

Definition 1.3. Let $p,\ q \in [1,\infty)$ and w be a weight on $\mathbb{R}^n$. Then the *weighted mixed Lebesgue space* $L^q_w(L^p)(\mathbb{R}^{2n})$ is defined to be the collection of all measurable functions f on $\mathbb{R}^{2n}$ satisfying

$$\|f\|_{L^q_w(L^p)(\mathbb{R}^{2n})} := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x,y)|^p \, dx\right)^{q/p} w(y)\, dy\right)^{1/q} < \infty.$$

Let $\mathcal{S}(\mathbb{R}^{2n})$ be the space of Schwartz functions on $\mathbb{R}^{2n}$, consisting of all functions $f \in C^\infty(\mathbb{R}^{2n})$ such that

$$\rho_{M,\alpha}(f) := \sup_{x\in\mathbb{R}^{2n}} (1+|x|)^M |\partial^\alpha f(x)| < \infty \quad \text{for any } M \in \mathbb{Z}_+ \text{ and } \alpha \in \mathbb{Z}^{2n}_+.$$

Denote by $\mathcal{S}'(\mathbb{R}^{2n})$ the dual space of $\mathcal{S}(\mathbb{R}^{2n})$, equipped with the weak-$*$ topology. Let $\mathcal{S}_\infty(\mathbb{R}^{2n})$ be the space of all Schwartz functions φ with the property

$$\int_{\mathbb{R}^{2n}} x^\gamma \varphi(x)\, dx = 0$$

for all multi-indices $\gamma \in \mathbb{Z}^{2n}_+$. Denote by $\mathcal{S}'_\infty(\mathbb{R}^{2n})$ the dual space of $\mathcal{S}_\infty(\mathbb{R}^{2n})$ under the topology inherited from $\mathcal{S}(\mathbb{R}^{2n})$, and it is known that (see [39, §2.4.1.4] or [29, Proposition 1.1.3])

$$\mathcal{S}'_\infty(\mathbb{R}^{2n}) = \mathcal{S}'(\mathbb{R}^{2n})/\mathcal{P}(\mathbb{R}^{2n}),$$

where $\mathcal{P}(\mathbb{R}^{2n})$ is the polynomial space on $\mathbb{R}^{2n}$.

For the moment, we adopt the notation $^\wedge$ and $^\vee$ to denote the Fourier transform and its inverse on $\mathbb{R}^{2n}$. Given any $\beta \in \mathbb{R}$, the fractional Laplace operator $(-\Delta_{\mathbb{R}^{2n}})^{\beta/2}$ can be defined on $\mathcal{S}(\mathbb{R}^{2n})$ as follows: for any $\phi \in \mathcal{S}(\mathbb{R}^{2n})$,

$$(-\Delta_{\mathbb{R}^{2n}})^{\beta/2}\phi = \left((2\pi|\cdot|)^\beta\, \hat{\phi}(\cdot)\right)^\vee.$$

Of course, $(-\Delta_{\mathbb{R}^{2n}})^{\beta/2}\phi \in \mathcal{S}_\infty(\mathbb{R}^{2n})$ when $\phi \in \mathcal{S}_\infty(\mathbb{R}^{2n})$. Moreover, if $f \in \mathcal{S}'_\infty(\mathbb{R}^{2n})$, then we have by duality that

$$\left\langle(-\Delta_{\mathbb{R}^{2n}})^{\beta/2}f,\ \phi\right\rangle = \left\langle f,\ (-\Delta_{\mathbb{R}^{2n}})^{\beta/2}\phi\right\rangle \quad \text{for all } \phi \in \mathcal{S}_\infty(\mathbb{R}^{2n}). \tag{1.2}$$

Definition 1.4. Let $p \in [1, \infty)$, $q \in (1, \infty)$, $\beta \in \mathbb{R}$ and w be a weight on $\mathbb{R}^n$. Then the *weighted mixed-Riesz potential space* is defined by setting

$$\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n}) := \left\{f:\ f = (-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi \ \text{ with } \ \phi \in L^q_w(L^p)(\mathbb{R}^{2n})\right\},$$

equipped with the norm

$$\|f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} := \|\phi\|_{L^q_w(L^p)(\mathbb{R}^{2n})}.$$

If $\beta = 0$, then we take it for granted that $\dot{\mathcal{L}}^{p,q,w}_0(\mathbb{R}^{2n}) = L^q_w(L^p)(\mathbb{R}^{2n})$.

Remark 1.5. Let $p \in [1, \infty)$ and $q \in (1, \infty)$. For general functions $\phi \in L^q_w(L^p)(\mathbb{R}^{2n})$, the element $(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi$ is understood as in (1.2). This is reasonable because of the continuous embedding $L^q_w(L^p)(\mathbb{R}^{2n}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ (see Lemma 2.1 below). If $\beta \in (0, 2n)$, then $(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}$ is known as the Riesz potential operator on $\mathbb{R}^{2n}$ and it has an integration kernel (see [43, p. 117])

$$I^{(2n)}_\beta(x) := \left(\frac{\Gamma(\frac{2n-\beta}{2})}{2^\beta\pi^n\Gamma(\frac{\beta}{2})}\right)|x|^{\beta-2n},$$

thereby leading to that for all $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$ and for all $x \in \mathbb{R}^{2n}$,

$$\left((-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\varphi\right)(x) = I^{(2n)}_\beta * \varphi(x) = \left(\frac{\Gamma(\frac{2n-\beta}{2})}{2^\beta\pi^n\Gamma(\frac{\beta}{2})}\right)\int_{\mathbb{R}^{2n}} |x-y|^{\beta-2n}\varphi(y)\,dy. \tag{1.3}$$

Clearly, if $\phi \in C^\infty_c(\mathbb{R}^{2n})$, then $I^{(2n)}_\beta * \phi \in \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$, which induces (see Corollary 2.4 below)

$$\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n}) = \overline{\left\{I^{(2n)}_\beta * \phi:\ \phi \in C^\infty_c(\mathbb{R}^{2n})\right\}}^{\|\cdot\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})}}.$$

To ensure the validity of the integral expression (1.3) for general locally integrable functions ϕ, one needs to require that

$$\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} I^{(2n)}_\beta(x-y)|\phi(x)\psi(y)|\,dx\,dy < \infty$$

for all $\psi \in \mathcal{S}_\infty(\mathbb{R}^{2n})$, so that the Fubini theorem can be applied to derived that

$$\begin{aligned}\left\langle(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi,\ \psi\right\rangle &= \left\langle\phi,\ (-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\psi\right\rangle\\ &= \int_{\mathbb{R}^{2n}}\phi(x)\left(\int_{\mathbb{R}^{2n}} I^{(2n)}_\beta(x-y)\psi(y)\,dy\right)dx\\ &= \int_{\mathbb{R}^{2n}}\left(\int_{\mathbb{R}^{2n}} I^{(2n)}_\beta(y-x)\phi(x)\,dx\right)\psi(y)\,dy\end{aligned}$$

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$$= \left\langle I_\beta^{(2n)} * \phi,\ \psi \right\rangle$$

and, hence, the integral definition of $(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi$ in (1.3) coincides with the one in (1.2) as a distribution in $\mathcal{S}'_\infty(\mathbb{R}^{2n})$.

1.3. **Main results.** We first recall the classical Muckenhoupt weight class (see [28, Chapter 7]). A weight w is said to be an A_p weight if

$$[w]_{A_p} := \sup_{\text{cube } Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q [w(x)]^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty \quad \text{as } \ p \in (1,\infty)$$

and

$$[w]_{A_1} := \sup_{\text{cube } Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\|w^{-1}\|_{L^\infty(Q)} < \infty \quad \text{as } \ p = 1.$$

Denote

$$A_\infty = \bigcup_{p\in[1,\infty)} A_p.$$

Throughout the whole paper, for a radial weight w, we simply write $w(x)$ as $w(t)$ whenever $|x| = t$.

The main result of this paper is the following extension-restriction theorem between homogeneous weighted Besov spaces and weighted mixed-Riesz potential spaces.

Theorem 1.6. *Let $p \in [1,\infty)$, $q \in (1,\infty)$, $s \in (0,\infty)$, $\beta = s + n/q \in (0,\infty)$ and $w \in A_q(\mathbb{R}^n)$ be a radial weight on $\mathbb{R}^n$ satisfying*

$$\int_0^1 t^{\sigma q}\left(\sup_{\rho\in(0,\infty)} \frac{w(t\rho)}{w(\rho)}\right)\frac{dt}{t} < \infty \quad \textit{for some } \sigma < \lfloor s\rfloor + 1 - s \tag{1.4}$$

and

$$\int_1^\infty t^{\delta q}\left(\sup_{\rho\in(0,\infty)} \frac{w(t\rho)}{w(\rho)}\right)\frac{dt}{t} < \infty \quad \textit{for some } \delta > -s. \tag{1.5}$$

Then, the following hold:

(i) *there exists a bounded linear extension operator*

$$\mathcal{E}:\ \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n) \to \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n}),$$

such that, for any $g \in \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$,

$$\|\mathcal{E}g\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} \le C_1\|g\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)},$$

where C_1 is a positive constant independent of g;

(ii) *there exists a bounded linear restriction operator*

$$\mathcal{R}:\ \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n}) \to \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n),$$

such that, for any $f \in \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$,

$$\|\mathcal{R}f\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \le C_2\|f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})},$$

where C_2 is a positive constant independent of f. Moreover, $\mathcal{R}\mathcal{E} = id$.

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The paper is organized as follows. In Section 2, we establish several auxiliary lemmas, including the continuous embeddings of the Schwartz function spaces into the weighted mixed Lebesgue spaces and then into the Schwartz distribution spaces, the density of $C_c^\infty(\mathbb{R}^n)$ in both the inhomogeneous weighted Besov spaces and the weighted mixed-Riesz potential spaces, as well as the Fourier analytic equivalent characterizations of the weighted Besov spaces. Section 3 focuses on the proof of Theorem 1.6 by considering the case $\beta \in (0, 2n)$ and $\beta \in [2n, \infty)$, respectively. Finally, in Section 4, we consider the homogeneous logarithmic Besov spaces as an example.

Notation. In the previous and forthcoming discussions, we adopt the following notion:

- Suppose $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.
- For any $a, b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.
- For any $p \in [1, \infty]$, denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.
- The symbol 0 may denote the real number zero, or the origin of the Euclidean space $\mathbb{R}^n$, depending on the context in which it is used.
- If E is a subset of $\mathbb{R}^n$, then $|E|$ denotes the Lebesgue measure of E, and $\mathbf{1}_E$ is the characteristic function of E.
- We always use $|x|$ to denote the Euclidean norm of a vector x, no matter it is in $\mathbb{R}^n$ or $\mathbb{R}^{2n}$. The reader can distinguish what it really means from the context.
- A multi-index α can be an n-tuple or a $2n$-tuple of nonnegative integers. For example, if α is an n-tuple multi-index, then $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ and $\partial^\alpha f := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$, where $\partial_i = \partial_{x_i}$ for $i = 1, 2, \dots, n$.
- For $m = n$ or $2n$, the symbol $C^\infty(\mathbb{R}^m)$ denotes the set of infinitely differentiable functions on $\mathbb{R}^m$, while $C_c^\infty(\mathbb{R}^m)$ denotes the set of $C^\infty(\mathbb{R}^m)$-functions with compact support.
- The letters C and c are used to denote positive constants that are independent of the variables in question, but may vary at each occurrence. The relation $u \lesssim v$ (resp., $u \gtrsim v$) between functions u and v means that $u \le Cv$ (resp., $u \ge Cv$) for a positive constant C and for a specified range of the variables. We write $u \approx v$ if $u \lesssim v \lesssim u$.

2. Preliminaries

2.1. **Embeddings.** Now we establish the following continuous embedding results, so that Definition 1.4 makes sense.

Lemma 2.1. *Let $p \in [1, \infty)$, $q \in (1, \infty)$ and $w \in A_q(\mathbb{R}^n)$. Then $\mathcal{S}(\mathbb{R}^{2n}) \hookrightarrow L_w^q(L^p)(\mathbb{R}^{2n}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2n})$.*

Proof. We first show $\mathcal{S}(\mathbb{R}^{2n}) \hookrightarrow L_w^q(L^p)(\mathbb{R}^{2n})$. To this end, for any Lebesgue measurable set $E \subset \mathbb{R}^n$, define

$$w(E) := \int_E w\, dx.$$

By [28, Proposition 7.1.5(9)], we know that if $w \in A_q(\mathbb{R}^n)$ then w is doubling and satisfies

$$w(\lambda B) \le \lambda^{nq} [w]_{A_q(\mathbb{R}^n)} w(B) \tag{2.1}$$

uniformly for all $\lambda \in (1, \infty)$ and all balls $B \subset \mathbb{R}^n$, where λB denotes the ball with the same center as that of B but of radius λ times of B. Then, for any $f \in \mathcal{S}(\mathbb{R}^{2n})$, we have

$$|f(x, y)| \le \left(1 + \sqrt{|x|^2 + |y|^2}\right)^{-N} \rho_{N,0}(f)$$

https://doi.org/10.4153/S0008414X25101041 Published online by Cambridge University Press

for all $x, y \in \mathbb{R}^n$ and for some large constant N to be determined later. So,

$$\|f\|_{L^q_w(L^p)(\mathbb{R}^{2n})} \le \rho_{N,0}(f)\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}(1+\sqrt{|x|^2+|y|^2})^{-Np}\,dx\right)^{q/p} w(y)\,dy\right)^{1/q}$$
$$\le \rho_{N,0}(f)\left(\int_{\mathbb{R}^n}(1+|x|)^{-Np/2}\,dx\right)^{1/p}\left(\int_{\mathbb{R}^n}(1+|y|)^{-Nq/2}w(y)\,dy\right)^{1/q}. \tag{2.2}$$

If $N > 2n/p$, then $\int_{\mathbb{R}^n}(1+|x|)^{-Np/2}\,dx < \infty$. Moreover, if $N > 2n$, then by (2.1) we have

$$\int_{\mathbb{R}^n}(1+|y|)^{-Nq/2}w(y)\,dy = \left(\int_{|y|<1}+\sum_{j=1}^{\infty}\int_{2^{j-1}\le|y|<2^j}\right)(1+|y|)^{-Nq/2}w(y)\,dy$$
$$\le w(B(0,1)) + \sum_{j=1}^{\infty}2^{-(j-1)Nq/2}w(B(0,2^j))$$
$$\lesssim w(B(0,1))\left(1+\sum_{j=1}^{\infty}2^{-j(N/2-n)q}\right) < \infty. \tag{2.3}$$

Altogether, we obtain $f \in L^q_w(L^p)(\mathbb{R}^{2n})$ and $\|f\|_{L^q_w(L^p)(\mathbb{R}^{2n})} \lesssim \rho_{N,0}(f)$, which implies

$$\mathcal{S}(\mathbb{R}^{2n}) \hookrightarrow L^q_w(L^p)(\mathbb{R}^{2n}).$$

Now, we show $L^q_w(L^p)(\mathbb{R}^{2n}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2n})$. For any $f \in L^q_w(L^p)(\mathbb{R}^{2n})$ and $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, from the Hölder inequality, we deduce

$$|\langle f,\varphi\rangle| \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x,y)\varphi(x,y)|\,dx\,dy$$
$$\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x,y)|^p\,dx\right)^{1/p} w(y)^{1/q}\left(\int_{\mathbb{R}^n}|\varphi(x,y)|^{p'}\,dx\right)^{1/p'} w(y)^{-1/q}\,dy$$
$$\le \|f\|_{L^q_w(L^p)(\mathbb{R}^{2n})}\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\varphi(x,y)|^{p'}\,dx\right)^{q'/p'} w(y)^{1-q'}\,dy\right)^{1/q'}. \tag{2.4}$$

When $q \in (1,\infty)$, by the fact that $w \in A_q(\mathbb{R}^n)$ if and only if $w^{1-q'} \in A_{q'}(\mathbb{R}^n)$ (see, for instance, [17, Proposition 7.2(2)]), and the already obtained result $\mathcal{S}(\mathbb{R}^{2n}) \hookrightarrow L^q_w(L^p)(\mathbb{R}^{2n})$, we find that

$$|\langle f,\varphi\rangle| \le \|f\|_{L^q_w(L^p)(\mathbb{R}^{2n})}\|\varphi\|_{L^{q'}_{w^{1-q'}}(L^{p'})(\mathbb{R}^{2n})} \lesssim \|f\|_{L^q_w(L^p)(\mathbb{R}^{2n})}\rho_{N,0}(\varphi)$$

holds when $N \in \mathbb{N}$ is sufficiently large, thereby leading to $f \in \mathcal{S}'(\mathbb{R}^{2n})$. Thus, we obtain the continuous embedding [1]

$$L^q_w(L^p)(\mathbb{R}^{2n}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

and, hence, complete the proof of Lemma 2.1. □

[1]Consider the case $q = 1$ and $p \in [1,\infty)$. For a general weight w, by (2.4), if we assume in addition that for all $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$ there is

$$\operatorname*{esssup}_{y\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\varphi(x,y)|^{p'}\,dx\right)^{1/p'} w(y)^{-1} < \infty,$$

then the embedding $L^q_w(L^p)(\mathbb{R}^{2n}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ remains valid. For example, this is the case when $w = 1$ or $w(y) = (\log \frac{e}{|y|\wedge 1})^b$ with $b > 0$.

2.2. **Density lemmas.** The forthcoming lemma shows the density of $C_c^\infty(\mathbb{R}^n)$ in the inhomogeneous weighted Besov space $\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)$. This explains the reason of defining the homogeneous space $\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under $\|\cdot\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)}$.

Lemma 2.2. *Let $p,\ q \in [1,\infty)$, $s \in (0,\infty)$ and $k \geq \lfloor s \rfloor + 1$. Assume that w is a radial weight satisfying* (1.4) *and* (1.5)*. Then, the following hold:*

(i) *any function in $C_c^\infty(\mathbb{R}^n)$ has finite semi-norm $\|\cdot\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)}$;*

(ii) *for any $f \in \Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)$, there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that*

$$\lim_{j\to\infty} \|\phi_j - f\|_{\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)} = 0. \tag{2.5}$$

Proof. We first show (i). Suppose that $f \in C_c^\infty(\mathbb{R}^n)$. From (1.1), it follows that

$$\|\mathrm{D}_h^k f\|_{L^p(\mathbb{R}^n)} \leq \sum_{m=0}^{k} \binom{k}{m} \|f(\cdot + mh)\|_{L^p(\mathbb{R}^n)} = 2^k \|f\|_{L^p(\mathbb{R}^n)} \lesssim 1.$$

Moreover, by the fact that

$$\mathrm{D}_h^k f(x) = \int_{[0,1]^k} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} h_{j_1} \cdots h_{j_k} (\partial_{j_1} \cdots \partial_{j_k} f)(x + (t_1 + \cdots + t_k)h)\, dt_1 \cdots dt_k \tag{2.6}$$

and the Minkowski inequality, we have

$$\|\mathrm{D}_h^k f\|_{L^p(\mathbb{R}^n)} \lesssim |h|^k.$$

Therefore,

$$\begin{aligned}
\|f\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)} &\lesssim \left(\int_{\mathbb{R}^n} \frac{\min\{1, |h|^{kq}\}}{|h|^{n+sq}} w(h)\, dh \right)^{\frac{1}{q}} \\
&\approx \left(\int_0^\infty \min\{1, t^{kq}\} w(t) t^{-sq-1}\, dt \right)^{\frac{1}{q}} \\
&= \left(\int_0^1 t^{(k-s)q-1} w(t)\, dt + \int_1^\infty t^{-sq-1} w(t)\, dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Note that $k \geq \lfloor s \rfloor + 1$, $\sigma < \lfloor s \rfloor + 1 - s$ and $\delta > -s$. On the one hand, condition (1.4) implies

$$\begin{aligned}
\int_0^1 t^{(k-s)q-1} w(t)\, dt &\leq \int_0^1 t^{(\lfloor s \rfloor + 1 - s)q-1} w(t)\, dt \\
&= \int_0^1 t^{(\lfloor s \rfloor + 1 - s)q} \left(\frac{w(t \cdot 1)}{w(1)} \right) w(1) \frac{dt}{t} \\
&\lesssim \int_0^1 t^{\sigma q} \left(\sup_{\rho \in (0,\infty)} \frac{w(t\rho)}{w(\rho)} \right) \frac{dt}{t} < \infty.
\end{aligned} \tag{2.7}$$

On the other hand, condition (1.5) yields

$$\begin{aligned}
\int_1^\infty t^{-sq-1} w(t)\, dt &\leq \int_1^\infty t^{\delta q-1} w(t)\, dt \\
&= \int_1^\infty t^{\delta q} \left(\frac{w(t \cdot 1)}{w(1)} \right) w(1) \frac{dt}{t}
\end{aligned} \tag{2.8}$$

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$$\lesssim \int_1^\infty t^{\delta q}\left(\sup_{\rho\in(0,\infty)}\frac{w(t\rho)}{w(\rho)}\right)\frac{dt}{t}<\infty.$$

This shows that any $f\in C_c^\infty(\mathbb{R}^n)$ satisfies $\|f\|_{\dot\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)}<\infty$.

Next, we show (ii). Fix $f\in\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)$. Choose a function $0\le\phi\in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\phi(x)\,dx=1$. For any $\epsilon\in(0,\infty)$, set

$$\phi_\epsilon(\cdot):=\epsilon^{-n}\phi(\epsilon^{-1}\,\cdot).$$

and

$$f_\epsilon:=\phi_\epsilon * f.$$

Clearly $f_\epsilon\in C^\infty(\mathbb{R}^n)$. From $f\in\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)\subset L^p(\mathbb{R}^n)$, it follows that

$$\lim_{\epsilon\to0}\|f_\epsilon-f\|_{L^p(\mathbb{R}^n)}=0. \tag{2.9}$$

From the Young inequality and the fact $\int_{\mathbb{R}^n}\phi(x)\,dx=1$, we deduce

$$\left\|\mathrm{D}^k_h f_\epsilon\right\|_{L^p(\mathbb{R}^n)}=\left\|\phi_\epsilon*\left(\mathrm{D}^k_h f\right)\right\|_{L^p(\mathbb{R}^n)}\le\left\|\mathrm{D}^k_h f\right\|_{L^p(\mathbb{R}^n)}$$

and, hence,

$$\left\|\mathrm{D}^k_h(f_\epsilon-f)\right\|_{L^p(\mathbb{R}^n)}\le\left\|\mathrm{D}^k_h f_\epsilon\right\|_{L^p(\mathbb{R}^n)}+\left\|\mathrm{D}^k_h f\right\|_{L^p(\mathbb{R}^n)}\le 2\left\|\mathrm{D}^k_h f\right\|_{L^p(\mathbb{R}^n)}.$$

Also, note that (2.9) and (1.1) imply

$$\begin{aligned}\lim_{\epsilon\to0}\left\|\mathrm{D}^k_h(f_\epsilon-f)\right\|_{L^p(\mathbb{R}^n)}&\le\lim_{\epsilon\to0}\sum_{m=0}^k\binom{k}{m}\|(f_\epsilon-f)(\cdot+mh)\|_{L^p(\mathbb{R}^n)}\\&=\sum_{m=0}^k\binom{k}{m}\lim_{\epsilon\to0}\|f_\epsilon-f\|_{L^p(\mathbb{R}^n)}\\&=0.\end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem, we have

$$\lim_{\epsilon\to0}\left(\int_{\mathbb{R}^n}\frac{\|\mathrm{D}^k_h(f_\epsilon-f)\|^q_{L^p(\mathbb{R}^n)}}{|h|^{n+sq}}w(h)\,dh\right)^{\frac1q}=\left(\int_{\mathbb{R}^n}\frac{\lim_{\epsilon\to0}\|\mathrm{D}^k_h(f_\epsilon-f)\|^q_{L^p(\mathbb{R}^n)}}{|h|^{n+sq}}w(h)\,dh\right)^{\frac1q}=0.$$

This proves

$$\lim_{\epsilon\to0}\|f_\epsilon-f\|_{\dot\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)}=0.$$

So, we have find a sequence $\{\phi_j\}_{j\in\mathbb{N}}$ in $C^\infty(\mathbb{R}^n)$ such that (2.5) holds.

Now, we are left to show that $C_c^\infty(\mathbb{R}^n)$ is dense in $\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$ under $\|\cdot\|_{\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)}$. Fix $f\in\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$. Let $\eta\in C_c^\infty(\mathbb{R}^n)$ satisfy $\eta=1$ on $B(0,1)$, $\operatorname{supp}\eta\subset B(0,2)$ and $0\le\eta\le1$. For any $N\in(0,\infty)$, define

$$\eta_N(\cdot):=\eta(N^{-1}\cdot)\qquad\text{and}\qquad f_N:=\eta_N f.$$

It is obvious that $f_N\in C_c^\infty(\mathbb{R}^n)$ and $f_N\to f$ in $L^p(\mathbb{R}^n)$ as $N\to\infty$. To validate that $\{f_N\}_{N\in\mathbb{N}}$ converges to f with respect to the semi-norm $\|\cdot\|_{\dot\Lambda^{s,k}_{p,q,w}(\mathbb{R}^n)}$, by (1.1), we write that for all $x,\ h\in$

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$\mathbb{R}^n$,

$$\begin{aligned} D_h^k(f_N - f)(x) &= \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} [\eta_N(x+mh) - 1]\, f(x+mh) \\ &= \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} [\eta_N(x+mh) - \eta_N(x)]\, f(x+mh) + [\eta_N(x) - 1]\, D_h^k f(x), \end{aligned}$$

which implies

$$\begin{aligned} |D_h^k(f_N - f)(x)| &\lesssim \sum_{m=0}^{k} \min\left\{ \frac{|mh|}{N} \|\nabla\eta\|_{L^\infty(\mathbb{R}^n)},\ \|\eta\|_{L^\infty(\mathbb{R}^n)} \right\} |f(x+mh)| + |D_h^k f(x)| \\ &\lesssim \min\{|h|,\ 1\} \sum_{m=0}^{k} |f(x+mh)| + |D_h^k f(x)| \end{aligned} \tag{2.10}$$

by terms of the mean value theorem. Note that (1.4) implies that (see (2.7))

$$\int_0^1 t^{(1-s)q-1} w(t)\, dt < \infty.$$

Meanwhile, by the fact that (1.5) leads to (see (2.8))

$$\int_1^\infty t^{-sq-1} w(t)\, dt < \infty.$$

With these last two estimates and (2.10), we then derive

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \frac{\| \min\{|h|,\ 1\} \sum_{m=0}^{k} |f(\cdot + mh)| + |D_h^k f| \|_{L^p(\mathbb{R}^n)}^q}{|h|^{n+sq}} w(h)\, dh \right)^{\frac{1}{q}} \\ &\le \left(\int_{|h|\le 1} \frac{\| |h| \sum_{m=0}^{k} |f(\cdot + mh)| \|_{L^p(\mathbb{R}^n)}^q}{|h|^{n+sq}} w(h)\, dh \right)^{\frac{1}{q}} \\ &\quad + \left(\int_{|h|>1} \frac{\| \sum_{m=0}^{k} |f(\cdot + mh)| \|_{L^p(\mathbb{R}^n)}^q}{|h|^{n+sq}} w(h)\, dh \right)^{\frac{1}{q}} + \|f\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^n)} \left(\int_{|h|\le 1} |h|^{q-sq-n} w(h)\, dh \right)^{\frac{1}{q}} + \|f\|_{L^p(\mathbb{R}^n)} \left(\int_{|h|>1} |h|^{-sq-n} w(h)\, dh \right)^{\frac{1}{q}} + \|f\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)} \\ &\approx \|f\|_{L^p(\mathbb{R}^n)} \left(\int_0^1 t^{(1-s)q-1} w(t)\, dt \right)^{\frac{1}{q}} + \|f\|_{L^p(\mathbb{R}^n)} \left(\int_1^\infty t^{-sq-1} w(t)\, dt \right)^{\frac{1}{q}} + \|f\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)} \\ &< \infty. \end{aligned}$$

Meanwhile, we also know that for all $h \in \mathbb{R}^n$,

$$\lim_{N\to\infty} \|D_h^k(f_N - f)\|_{L^p(\mathbb{R}^n)} \le \sum_{m=0}^{k} \binom{k}{m} \|[\eta_N(\cdot) - 1] f(\cdot)\|_{L^p(\mathbb{R}^n)} = 0.$$

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From these and the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{N\to\infty}\left(\int_{\mathbb{R}^n}\frac{\|D_h^k(f_N-f)\|_{L^p(\mathbb{R}^n)}^q}{|h|^{n+sq}}w(h)\,dh\right)^{\frac{1}{q}}=\left(\int_{\mathbb{R}^n}\frac{\lim_{N\to\infty}\|D_h^k(f_N-f)\|_{L^p(\mathbb{R}^n)}^q}{|h|^{n+sq}}w(h)\,dh\right)^{\frac{1}{q}}=0,$$

as desired. Altogether, we conclude the proof of Lemma 2.2. □

Next, we show the density of $C_c^\infty(\mathbb{R}^{2n})$ in weighted mixed-norm Lebesgue spaces.

Lemma 2.3. *Let $p,\ q\in[1,\infty)$ and $w\in A_q(\mathbb{R}^n)$. Then $C_c^\infty(\mathbb{R}^{2n})$ is dense in $L_w^q(L^p)(\mathbb{R}^{2n})$.*

Proof. It is obvious that $C_c^\infty(\mathbb{R}^{2n})\subseteq L_w^q(L^p)(\mathbb{R}^{2n})$. Since w is locally integrable, we know that

$$d\mu(x,y):=w(y)\,dx\,dy$$

defines a σ-finite Radon-measure on $\mathbb{R}^{2n}$. Given any $f\in L_w^q(L^p)(\mathbb{R}^{2n})$, we have by [5, p. 313] that, for any $\epsilon>0$, there exists a simple function

$$\varphi(x,y):=\sum_{i=1}^N C_i\,\mathbf{1}_{E_i}(x)\,\mathbf{1}_{F_i}(y),$$

where $|E_i|<\infty$ and $w(F_i)<\infty$ for all $i\in\{1,2,\dots,N\}$, such that

$$\|f-\varphi\|_{L_w^q(L^p)(\mathbb{R}^{2n})}<\epsilon. \tag{2.11}$$

We claim that there exists a function $g\in C_c^\infty(\mathbb{R}^{2n})$ such that

$$\|\varphi-g\|_{L_w^q(L^p)(\mathbb{R}^{2n})}<\epsilon. \tag{2.12}$$

Once we have (2.12), then applying (2.11) gives

$$\|f-g\|_{L_w^q(L^p)(\mathbb{R}^{2n})}\le\|f-\varphi\|_{L_w^q(L^p)(\mathbb{R}^{2n})}+\|\varphi-g\|_{L_w^q(L^p)(\mathbb{R}^{2n})}<\epsilon+\epsilon=2\epsilon,$$

which indicates that any $f\in L_w^q(L^p)(\mathbb{R}^{2n})$ can be approximated by functions in $C_c^\infty(\mathbb{R}^{2n})$.

To show (2.12), it suffices to consider the case when $\varphi(x,y)=\mathbf{1}_E(x)\mathbf{1}_F(y)$, where E,F are measurable subsets in $\mathbb{R}^n$ with $|E|<\infty$ and $w(F)<\infty$. To see this, by the fact that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, for any $\eta>0$, there exists a function $\psi_1\in C_c^\infty(\mathbb{R}^n)$ such that

$$\|\mathbf{1}_E-\psi_1\|_{L^p(\mathbb{R}^n)}<\eta\,w(F)^{-1/q}.$$

Since $w(x)\,dx$ is a doubling measure, by an argument similar to the unweighted case, we obtain that if

$$\begin{cases}\phi\in C_c^\infty(\mathbb{R}^n);\\ \int_{\mathbb{R}^n}\phi(y)w(y)\,dy=1;\\ \phi_t(\cdot):=t^{-n}\phi(t^{-1}\,\cdot)\text{ for all }t\in(0,\infty),\end{cases}$$

then any $h\in L_w^q(\mathbb{R}^n)$ satisfies that

$$\int_{\mathbb{R}^n}\phi_t(\cdot-y)h(y)\mathbf{1}_{B(0,R)}(y)w(y)\,dy\to h\ \text{ in }\ L_w^q(\mathbb{R}^n)$$

as $R\to\infty$ and $t\to 0$. In other words, $C_c^\infty(\mathbb{R}^n)$ is dense in $L_w^q(\mathbb{R}^n)$. Consequently, for any $\eta>0$, there exists a function $\psi_2\in C_c^\infty(\mathbb{R}^n)$, such that

$$\|\mathbf{1}_F-\psi_2\|_{L_w^q(\mathbb{R}^n)}<\eta\|\psi_1\|_{L^p(\mathbb{R}^n)}^{-1}.$$

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Let $g(x,y) := \psi_1(x)\psi_2(y)$. Then,

$$\begin{aligned}\|\mathbf{1}_{E\times F} - g\|_{L^q_w(L^p)(\mathbb{R}^{2n})} &= \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\mathbf{1}_E(x)\mathbf{1}_F(y) - \psi_1(x)\psi_2(y)|^p\,dx\right)^{q/p} w(y)\,dy\right)^{1/q}\\ &\le \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\mathbf{1}_E(x)\mathbf{1}_F(y) - \psi_1(x)\mathbf{1}_F(y)|^p\,dx\right)^{q/p} w(y)\,dy\right)^{1/q}\\ &\quad + \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\psi_1(x)\mathbf{1}_F(y) - \psi_1(x)\psi_2(y)|^p\,dx\right)^{q/p} w(y)\,dy\right)^{1/q}\\ &< 2\eta.\end{aligned}$$

This proves (2.12). □

Corollary 2.4. *For any $p \in [1,\infty)$, $q \in (1,\infty)$, $\beta \in \mathbb{R}$ and $w \in A_q(\mathbb{R}^n)$, the space*

$$\left\{(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi : \phi \in C_c^\infty(\mathbb{R}^{2n})\right\}$$

is dense in $\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$.

Proof. Let $f \in \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$. By Definition 1.4, there exists a function $\phi_0 \in L^q_w(L^p)(\mathbb{R}^{2n})$ such that

$$f = (-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi_0 \quad\text{and}\quad \|f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} = \|\phi_0\|_{L^q_w(L^p)(\mathbb{R}^{2n})}.$$

From Lemma 2.3, for any $\epsilon > 0$, we can find a function $\phi \in C_c^\infty(\mathbb{R}^{2n})$ such that

$$\|\phi_0 - \phi\|_{L^q_w(L^p)(\mathbb{R}^{2n})} < \epsilon,$$

which implies

$$\left\|f - (-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi\right\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} = \left\|(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}(\phi_0 - \phi)\right\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} = \|\phi_0 - \phi\|_{L^q_w(L^p)(\mathbb{R}^{2n})} < \epsilon.$$

This ends the proof. □

Now, we are at the point to show the density of $C_c^\infty(\mathbb{R}^{2n})$ in weighted mixed-Riesz potential spaces.

Lemma 2.5. *Let $p \in [1,\infty)$, $q \in (1,\infty)$, $w \in A_q(\mathbb{R}^n)$ and $\beta \in (0,\infty)$. Then $C_c^\infty(\mathbb{R}^{2n})$ is dense in $\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$.*

Proof. Let $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfy $\operatorname{supp}\widehat{\Phi} \subset \{x \in \mathbb{R}^{2n} : c^{-1} \le |x| \le c\}$ for some constant $c > 1$ and

$$\sum_{j\in\mathbb{Z}} \widehat{\Phi}(2^{-j}\xi) = 1 \quad \text{for all } \xi \ne 0.$$

For any $j \in \mathbb{Z}$ and $x \in \mathbb{R}^{2n}$, let $\Phi_j(x) := 2^{2jn}\Phi(2^j x)$. For any $\phi \in C_c^\infty(\mathbb{R}^{2n})$ and $N \in \mathbb{N}$,

$$\phi_N := \sum_{|j|\le N} \Phi_j * \phi \in \mathcal{S}_\infty(\mathbb{R}^{2n}).$$

Moreover, the sequence $\{\phi_N\}$ converges to ϕ uniformly on $\mathbb{R}^{2n}$ (and, hence, in $L^q_w(L^p)(\mathbb{R}^{2n})$) as $N \to \infty$. From this and Corollary 2.4, it follows that $\{(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi : \phi \in \mathcal{S}_\infty(\mathbb{R}^{2n})\}$ is dense in $\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$. Thus, to show the density of $C_c^\infty(\mathbb{R}^{2n})$ in $\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$, we may as well assume that

$$f = (-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g$$

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for some $g \in \mathcal{S}_\infty(\mathbb{R}^{2n})$, then our aim is to find a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ in $C_c^\infty(\mathbb{R}^{2n})$ such that

$$\lim_{k\to\infty} \|\varphi_k - f\|_{\dot{\mathcal{L}}_\beta^{p,q,w}(\mathbb{R}^{2n})} = \lim_{k\to\infty} \|(-\Delta_{\mathbb{R}^{2n}})^{\beta/2}\varphi_k - g\|_{L_w^q(L^p)(\mathbb{R}^{2n})} = 0. \tag{2.13}$$

Choose $\eta \in C_c^\infty(\mathbb{R}^{2n})$ satisfying $0 \le \eta \le 1$, $\eta = 1$ if $|x| \le 1$ and $\eta = 0$ if $|x| \ge 2$. For any $k \in \mathbb{N}$, set

$$\varphi_k := \eta_k(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g,$$

where $\eta_k(x) = \eta(x/k)$ for all $x \in \mathbb{R}^{2n}$. Clearly, each $\varphi_k \in C_c^\infty(\mathbb{R}^{2n})$. In order to show that such $\{\varphi_k\}_{k\in\mathbb{Z}}$ satisfies (2.13), we write $\beta = 2m + s$, where $m \in \mathbb{N}$ and $s \in [0, 2)$. Notice that

$$(-\Delta_{\mathbb{R}^{2n}})^{\beta/2}\varphi_k - g = (-\Delta_{\mathbb{R}^{2n}})^{s/2}(-\Delta_{\mathbb{R}^{2n}})^m \left(\eta_k(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g\right) - g$$

and, if $m \neq 0$, then

$$\begin{aligned}
(-\Delta_{\mathbb{R}^{2n}})^m \left(\eta_k(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g\right) &= \left(-\sum_{j=1}^m \partial_{x_j}^2\right)^m \left(\eta_k(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g\right)\\
&= (-1)^m \sum_{|\alpha|=m} \partial^{2\alpha}\left(\eta_k(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g\right)\\
&= \eta_k(-\Delta_{\mathbb{R}^{2n}})^m \left((-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g\right) + \sum_{\substack{|\alpha|=m\\ 0<|\gamma|\le 2m}} c_{m,\alpha,\gamma} \left(\partial^\gamma \eta_k\right) \partial^{2\alpha-\gamma}\left((-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g\right),
\end{aligned}$$

where α, γ are $2n$-tuples of multi-indices and $c_{m,\alpha,\gamma}$ are constants. By the Minkowski inequality, we find that the norm $\|\cdot\|_{L_w^q(L^p)(\mathbb{R}^{2n})}$ in (2.13) can be controlled by the $\|\cdot\|_{L_w^q(L^p)(\mathbb{R}^{2n})}$-norm of

$$g_k := (-\Delta_{\mathbb{R}^{2n}})^{s/2}(\eta_k(-\Delta_{\mathbb{R}^{2n}})^{(2m-\beta)/2} g) - g = (-\Delta_{\mathbb{R}^{2n}})^{s/2}(\eta_k(-\Delta_{\mathbb{R}^{2n}})^{-s/2} g) - g$$

and a finite linear combination of the $\|\cdot\|_{L_w^q(L^p)(\mathbb{R}^{2n})}$-norm of the following type of functions

$$h_k := (-\Delta_{\mathbb{R}^{2n}})^{s/2}((\partial^\gamma \eta_k)h),$$

where $h \in \mathcal{S}_\infty(\mathbb{R}^{2n})$ is of the form $\partial^{2\alpha-\gamma}((-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} g)$ and γ is a non-zero multi-index. Thus, to obtain (2.13), we are left to prove that the $L_w^q(L^p)(\mathbb{R}^{2n})$-norms of g_k and h_k tend to 0 as $k \to \infty$.

If $s = 0$, then $g_k = \eta_k g - g$ and $|h_k| \le k^{-|\gamma|}\|\partial^\gamma \eta\|_{L^\infty(\mathbb{R}^{2n})}|h|$, whose $L_w^q(L^p)(\mathbb{R}^{2n})$-norms obviously go to 0 as $k \to \infty$. Next, we consider the case when $s \in (0, 2)$.

Part 1: estimate of the $L_w^q(L^p)(\mathbb{R}^{2n})$-norm of g_k. To simplify the notation, we set

$$u = (-\Delta_{\mathbb{R}^{2n}})^{(2m-\beta)/2} g = (-\Delta_{\mathbb{R}^{2n}})^{-s/2} g,$$

which belongs to $\mathcal{S}(\mathbb{R}^{2n})$ by using the Fourier transform and the fact that $g \in \mathcal{S}(\mathbb{R}^{2n})$. By [40, Section 2], we have

$$(-\Delta_{\mathbb{R}^{2n}})^{s/2}(\eta_k u) = C_{n,s}\,\text{p.v.}\int_{\mathbb{R}^{2n}} \frac{\eta_k(x)u(x) - \eta_k(y)u(y)}{|x-y|^{2n+s}}\,dy,$$

with $C_{n,s}$ being a positive constant depending only on n and s. Via writing

$$\eta_k(x)u(x) - \eta_k(y)u(y) = \eta_k(x)[u(x) - u(y)] + [\eta_k(x) - \eta_k(y)][u(y) - u(x)] + u(x)[\eta_k(x) - \eta_k(y)],$$

we then have

$$g_k(x) = \left(C_{n,s}\eta_k(x)\text{p.v.}\int_{\mathbb{R}^{2n}} \frac{u(x)-u(y)}{|x-y|^{2n+s}}\,dy - g(x)\right) \tag{2.14}$$

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$$+ C_{n,s}\text{p.v.} \int_{|x-y|<\frac{1+|x|}{2}} \frac{[\eta_k(x) - \eta_k(y)][u(y) - u(x)]}{|x-y|^{2n+s}}\, dy$$
$$+ C_{n,s}u(x)\text{p.v.} \int_{|x-y|<\frac{1+|x|}{2}} \frac{\eta_k(x) - \eta_k(y)}{|x-y|^{2n+s}}\, dy$$
$$+ C_{n,s}\text{p.v.} \int_{|x-y|\geq\frac{1+|x|}{2}} \frac{[\eta_k(x) - \eta_k(y)]u(y)}{|x-y|^{2n+s}}\, dy$$
$$=: G_k^1(x) + G_k^2(x) + G_k^3(x) + G_k^4(x).$$

When $k \to \infty$, it is obvious that

$$\|G_k^1\|_{L_w^q(L^p)(\mathbb{R}^{2n})} = \|\eta_k(-\Delta_{\mathbb{R}^n})^{s/2}u - g\|_{L_w^q(L^p)(\mathbb{R}^{2n})} = \|\eta_k g - g\|_{L_w^q(L^p)(\mathbb{R}^{2n})} \to 0. \tag{2.15}$$

For G_k^2, note that the mean value theorem implies that

$$|\eta_k(x) - \eta_k(y)| \leq k^{-1}|x-y|\|\nabla\eta\|_{L^\infty(\mathbb{R}^{2n})}$$

and

$$|u(y) - u(x)| \leq |x-y||\nabla u(x + \theta(y-x))|$$

hold for some constant $\theta \in (0,1)$. If $|x-y| < \frac{1+|x|}{2}$, then $1 + |x + \theta(y-x)| \approx 1 + |x|$, which, along with the fact that $u \in \mathcal{S}_\infty$, implies that

$$|\nabla u(x + \theta(y-x))| \lesssim (1+|x|)^{-(2n+2)}.$$

Thus, we obtain

$$|G_k^2(x)| \leq C_{n,s} \int_{|x-y|<\frac{1+|x|}{2}} \frac{|\eta_k(x) - \eta_k(y)||u(y) - u(x)|}{|x-y|^{2n+s}}\, dy$$
$$\lesssim k^{-1}(1+|x|)^{-(2n+2)} \int_{|x-y|<\frac{1+|x|}{2}} |x-y|^{2-s-2n}\, dy$$
$$\approx k^{-1}(1+|x|)^{-(2n+s)}.$$

According to the arguments in (2.2) and (2.3) (by taking N therein to be $2n+s$), we see that

$$\|(1+|\cdot|)^{-(2n+s)}\|_{L_w^q(L^p)(\mathbb{R}^{2n})} < \infty \tag{2.16}$$

and, hence,

$$\lim_{k\to\infty} \|G_k^2\|_{L_w^q(L^p)(\mathbb{R}^{2n})} = 0. \tag{2.17}$$

Next, we consider G_k^3. From the Taylor expansion formula,

$$\eta_k(x) - \eta_k(y) = \left(\frac{x-y}{k}\right) \cdot \nabla\eta\left(\frac{x}{k}\right) + O\left(\left|\frac{x-y}{k}\right|^2\right).$$

Observe that

$$\text{p.v.} \int_{|x-y|<\frac{1+|x|}{2}} \frac{(\frac{x-y}{k}) \cdot \nabla\eta(\frac{x}{k})}{|x-y|^{2n+s}}\, dy = 0,$$

which induces

$$|G_k^3(x)| \lesssim k^{-2}|u(x)| \int_{|x-y|<\frac{1+|x|}{2}} |x-y|^{2-s-2n}\, dy \lesssim k^{-2}(1+|x|)^{2-s}|u(x)|,$$

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thereby leading to

$$\lim_{k\to\infty}\|G_k^3\|_{L_w^q(L^p)(\mathbb{R}^{2n})}\lesssim\lim_{k\to\infty}k^{-2}\|(1+|\cdot|)^{2-s}u(\cdot)\|_{L_w^q(L^p)(\mathbb{R}^{2n})}=0. \tag{2.18}$$

Next, observe that the integrand of G_k^4 can be controlled by $\|\eta\|_{L^\infty(\mathbb{R}^{2n})}\|u\|_{L^\infty(\mathbb{R}^{2n})}|x-y|^{-(2n+s)}$ and

$$\int_{|x-y|\geq\frac{1+|x|}{2}}|x-y|^{-(2n+s)}\,dy\lesssim(1+|x|)^{-s}<\infty,$$

which, together with the Lebesgue dominated convergence theorem, shows that

$$\lim_{k\to\infty}G_k^4(x)=\int_{|x-y|\geq\frac{1+|x|}{2}}\lim_{k\to\infty}\frac{[\eta_k(x)-\eta_k(y)]u(y)}{|x-y|^{2n+s}}\,dy=0.$$

Moreover, note that

$$\int_{|x-y|\geq\frac{1+|x|}{2}}\frac{|u(y)|}{|x-y|^{2n+s}}\,dy\lesssim\int_{|x-y|\geq\frac{1+|x|}{2}}\frac{|u(y)|}{(1+|x|)^{2n+s}}\,dy\lesssim(1+|x|)^{-(2n+s)}\|u\|_{L^1(\mathbb{R}^{2n})},$$

which, combined with (2.16) and the Lebesgue dominated convergence theorem, again yields

$$\lim_{k\to\infty}\|G_k^4\|_{L_w^q(L^p)(\mathbb{R}^{2n})}=\left\|\lim_{k\to\infty}G_k^4\right\|_{L_w^q(L^p)(\mathbb{R}^{2n})}=0. \tag{2.19}$$

Substituting (2.15)-(2.17)-(2.18)-(2.19) into (2.14) yields that $\|g_k\|_{L_w^q(L^p)(\mathbb{R}^{2n})}\to 0$ as $k\to\infty$.

Part 2: estimate of the $L_w^q(L^p)(\mathbb{R}^{2n})$-norm of h_k. Just like (2.14), we now write

$$\begin{aligned}h_k(x)&=C_{n,s}\partial^\gamma\eta_k(x)\text{p.v.}\int_{\mathbb{R}^{2n}}\frac{h(x)-h(y)}{|x-y|^{2n+s}}\,dy\\&\quad+C_{n,s}\text{p.v.}\int_{|x-y|<\frac{1+|x|}{2}}\frac{[\partial^\gamma\eta_k(x)-\partial^\gamma\eta_k(y)][h(y)-h(x)]}{|x-y|^{2n+s}}\,dy\\&\quad+C_{n,s}h(x)\text{p.v.}\int_{|x-y|<\frac{1+|x|}{2}}\frac{\partial^\gamma\eta_k(x)-\partial^\gamma\eta_k(y)}{|x-y|^{2n+s}}\,dy\\&\quad+C_{n,s}\text{p.v.}\int_{|x-y|\geq\frac{1+|x|}{2}}\frac{[\partial^\gamma\eta_k(x)-\partial^\gamma\eta_k(y)]h(y)}{|x-y|^{2n+s}}\,dy\\&=:H_k^1(x)+H_k^2(x)+H_k^3(x)+H_k^4(x).\end{aligned}$$

For $i=2,3,4$, the estimate of G_k^i also implies that (with η_k and u therein replaced by $\partial^\gamma\eta_k$ and h, respectively)

$$\lim_{k\to\infty}\|H_k^i\|_{L_w^q(L^p)(\mathbb{R}^{2n})}=0.$$

For H_k^1, it follows from $h\in\mathcal{S}_\infty$ that $(-\varDelta_{\mathbb{R}^{2n}})^{s/2}h\in\mathcal{S}_\infty$, thereby leading to

$$\|H_k^1\|_{L_w^q(L^p)(\mathbb{R}^{2n})}=\left\|(\partial^\gamma\eta_k)\left((-\varDelta_{\mathbb{R}^{2n}})^{s/2}h\right)\right\|_{L_w^q(L^p)(\mathbb{R}^{2n})}\leq k^{-|\gamma|}\|\partial^\gamma\eta\|_{L^\infty(\mathbb{R}^{2n})}\left\|(-\varDelta_{\mathbb{R}^{2n}})^{s/2}h\right\|_{L_w^q(L^p)(\mathbb{R}^{2n})}\to 0$$

as $k\to\infty$. This induces that $\|h_k\|_{L_w^q(L^p)(\mathbb{R}^{2n})}\to 0$ when $k\to\infty$, as desired.

Altogether, we conclude the proof of Lemma 2.5. □

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2.3. **Fourier analytic characterizations.**

Definition 2.6. Let $p,\ q \in [1,\infty)$, $s \in (0,\infty)$ and w be a radial weight on $\mathbb{R}^n$. Denote by $\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)$ the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ satisfying

$$\|f\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)} := \left(\int_0^\infty \|\varphi_t * f\|^q_{L^p(\mathbb{R}^n)} t^{-sq-1} w(t)\, dt\right)^{1/q} < \infty,$$

where $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\hat{\varphi} \subset \{x \in \mathbb{R}^n :\ c^{-1} \le |x| \le c\}$ for some constant $c \in (1,\infty)$ and $\varphi_t(\cdot) = t^{-n}\varphi(t^{-1}\cdot)$ for all $t \in (0,\infty)$.

Remark 2.7. Note that $\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)$ contains equivalence classes of tempered distributions modulo all polynomials. If we define $\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)$ as the space of locally integrable functions f satisfying $\|f\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)} < \infty$, then such homogeneous weighted Besov spaces are equivalence classes of locally integrable functions modulo polynomials of degree at most k. Let us remark that this definition is different from the one we adopted in Definition 1.1, in which the space $\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)$ is defined to be the completion of $C^\infty_c(\mathbb{R}^n)$ under $\|\cdot\|_{\dot{\Lambda}^{s,k}_{p,q,w}(\mathbb{R}^n)}$. We will take care of this difference in our arguments below.

Lemma 2.8. *Let $p,\ q \in [1,\infty)$, $s \in (0,\infty)$ and $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ such that*

$$\begin{cases} \operatorname{supp}\hat{\varphi} \subset \{x \in \mathbb{R}^n :\ c_1^{-1} \le |x| \le c_1\}; \\ \operatorname{supp}\hat{\psi} \subset \{x \in \mathbb{R}^n :\ c_2^{-1} \le |x| \le c_2\}, \end{cases}$$

where $c_1, c_2 \in (1,\infty)$. If w is a radial weight satisfying (1.4) *and* (1.5)*, then*

$$\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n) = \dot{\Lambda}^s_{p,q,w,\psi}(\mathbb{R}^n)$$

with equivalent norms.

Proof. By symmetry, it suffices to show that

$$\|f\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{\Lambda}^s_{p,q,w,\psi}(\mathbb{R}^n)}.$$

Based on the argument in [37, Proposition 2.3], there exists a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\hat{\phi} \subset \{x \in \mathbb{R}^n :\ c_0^{-1} \le |x| \le c_0\}$$

for some constant $c_0 \in (1,\infty)$ and, moreover,

$$\int_0^\infty \overline{\hat{\phi}(t\xi)}\, \hat{\psi}(t\xi)\, \frac{dt}{t} = 1 \ \text{ for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Consequently, for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, we know from [24, p.122, Theorem 3] that

$$f = \int_0^\infty \phi_t * \psi_t * f\, \frac{dt}{t} \quad \text{in } \mathcal{S}'_\infty(\mathbb{R}^n). \tag{2.20}$$

Given any $r \in (0,\infty)$, since $\varphi_r \in \mathcal{S}_\infty(\mathbb{R}^n)$, it follows from this last equality that

$$\varphi_r * f = \int_0^\infty \varphi_r * \phi_t * \psi_t * f\, \frac{dt}{t}$$

holds pointwisely. Consequently, by the Young inequality and the Hölder inequality,

$$\|\varphi_r * f\|_{L^p(\mathbb{R}^n)} \le \int_0^\infty \|\varphi_r * \phi_t * \psi_t * f\|_{L^p(\mathbb{R}^n)}\, \frac{dt}{t}$$

$$
\begin{aligned}
&\le \int_0^\infty \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \|\psi_t * f\|_{L^p(\mathbb{R}^n)} \frac{dt}{t} \\
&\le \left(\int_0^\infty \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \frac{dt}{t}\right)^{1/q'} \left(\int_0^\infty \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \|\psi_t * f\|^q_{L^p(\mathbb{R}^n)} \frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

Given any $K, M \in \mathbb{Z}_+$, we derive from [24, p.121, Lemma 2; p.122, Lemma 4] that

$$
|\varphi * \phi_{t/r}(x)| \lesssim \begin{cases} \frac{t}{r}(1+|x|)^{-M}, & 0 < t \le r; \\ \left(\frac{r}{t}\right)^{n+1+K} \left(1 + \frac{r|x|}{t}\right)^{-M} \lesssim \left(\frac{r}{t}\right)^N (1+|x|)^{-M}, & 0 < r \le t, \end{cases}
$$

where $N = n + 1 + K - M$. If we choose $M > n$ and $K > M - n - 1$, then $N > 0$ and

$$
\|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \lesssim \min\left\{\frac{t}{r}, \left(\frac{r}{t}\right)^N\right\}, \tag{2.21}
$$

thereby leading to

$$
\int_0^\infty \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \frac{dt}{t} \lesssim \int_0^r \frac{t}{r} \frac{dt}{t} + \int_r^\infty \left(\frac{r}{t}\right)^N \frac{dt}{t} \lesssim 1.
$$

Thus, we obtain

$$
\|\varphi_r * f\|_{L^p(\mathbb{R}^n)} \lesssim \left(\int_0^\infty \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \|\psi_t * f\|^q_{L^p(\mathbb{R}^n)} \frac{dt}{t}\right)^{1/q},
$$

which further implies

$$
\begin{aligned}
\|f\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)} &= \left(\int_0^\infty \|\varphi_r * f\|^q_{L^p(\mathbb{R}^n)} r^{-sq-1} w(r)\, dr\right)^{1/q} \\
&\lesssim \left(\int_0^\infty r^{-sq} \left(\int_0^\infty \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \|\psi_t * f\|^q_{L^p(\mathbb{R}^n)} \frac{dt}{t}\right) w(r) \frac{dr}{r}\right)^{1/q} \\
&= \left(\int_0^\infty t^{-sq} \|\psi_t * f\|^q_{L^p(\mathbb{R}^n)} w(t) \left(\int_0^\infty \left(\frac{r}{t}\right)^{-sq} \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \frac{w(r)}{w(t)} \frac{dr}{r}\right) \frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

We may choose K sufficiently large such that

$$
N = n + 1 + K - M > (s+1)q.
$$

This implies $\sigma q < N - sq$ due to $\sigma < 1$. Further, by (2.21), (1.4) and (1.5), together with a change of variables $u = r/t$, we deduce

$$
\begin{aligned}
\int_0^t \left(\frac{r}{t}\right)^{-sq} \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \frac{w(r)}{w(t)} \frac{dr}{r} &\lesssim \int_0^t \left(\frac{r}{t}\right)^{-sq+N} \frac{w(r)}{w(t)} \frac{dr}{r} \\
&\le \int_0^1 u^{-sq+N} \left(\sup_{\rho \in (0,\infty)} \frac{w(u\rho)}{w(\rho)}\right) \frac{du}{u} \\
&\le \int_0^1 u^{\sigma q} \left(\sup_{\rho \in (0,\infty)} \frac{w(u\rho)}{w(\rho)}\right) \frac{du}{u} < \infty
\end{aligned}
$$

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and

$$\begin{aligned}\int_t^\infty \left(\frac{r}{t}\right)^{-sq} \|\varphi * \phi_{t/r}\|_{L^1(\mathbb{R}^n)} \frac{w(r)}{w(t)} \frac{dr}{r} &\lesssim \int_t^\infty \left(\frac{r}{t}\right)^{-sq-1} \frac{w(r)}{w(t)} \frac{dr}{r} \\ &\leq \int_1^\infty u^{-sq-1} \left(\sup_{\rho\in(0,\infty)} \frac{w(u\rho)}{w(\rho)}\right) \frac{du}{u} \\ &\leq \int_1^\infty u^{\delta q} \left(\sup_{\rho\in(0,\infty)} \frac{w(u\rho)}{w(\rho)}\right) \frac{du}{u} < \infty,\end{aligned}$$

where we used $N > (\sigma + s)q$ and $\delta > -s - 1/q$. Altogether, we obtain

$$\begin{aligned}\|f\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)} &= \left(\int_0^\infty \|\varphi_r * f\|^q_{L^p(\mathbb{R}^n)} r^{-sq-1} w(r)\, dr\right)^{1/q} \\ &\lesssim \left(\int_0^\infty \|\psi_t * f\|^q_{L^p(\mathbb{R}^n)} t^{-sq-1} w(t)\, dt\right)^{1/q} = \|f\|_{\dot{\Lambda}^s_{p,q,w,\psi}(\mathbb{R}^n)},\end{aligned}$$

as desired. □

Next, we introduce a technical lemma, whose proof is essentially given in [44, Lemma (2.1)].

Lemma 2.9. *Let $q \in [1,\infty)$, w be a radial weight, and K be a nonnegative function defined on $(0,\infty)\times(0,\infty)$ which is homogeneous of degree $-n$ and satisfies*

$$J := \int_0^\infty K(1,t) t^{\frac{n}{q'}-1} \left(\sup_{\rho\in(0,\infty)} \frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}} dt < \infty. \tag{2.22}$$

Then, for any $f \in L^q_w(\mathbb{R}^n)$, the function

$$Tf(x) := \int_{\mathbb{R}^n} K(|x|, |y|) f(y)\, dy$$

satisfies

$$\|Tf\|_{L^q_w(\mathbb{R}^n)} \leq J\, \omega_{n-1} \|f\|_{L^q_w(\mathbb{R}^n)},$$

where ω_{n-1} denotes the surface area of the unit ball in $\mathbb{R}^n$.

Proof. Observe that Tf is radial, which implies

$$\|Tf\|_{L^q_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |Tf(x)|^q w(x)\, dx\right)^{1/q} = \omega_{n-1}^{1/q} \left(\int_0^\infty |Tf(R)|^q w(R) R^{n-1}\, dR\right)^{1/q}.$$

Upon writing

$$\begin{aligned}Tf(R) &= \int_0^\infty \int_{\mathbb{S}^{n-1}} K(R,r) f(r\eta) r^{n-1}\, d\sigma(\eta)\, dr \\ &= \int_0^\infty \int_{\mathbb{S}^{n-1}} R^{-n} K(1,t)\, f(tR\eta)\, (tR)^{n-1} R\, d\sigma(\eta)\, dt \\ &= \int_{\mathbb{S}^{n-1}} \left(\int_0^\infty K(1,t)\, f(tR\eta)\, t^{n-1}\, dt\right) d\sigma(\eta) \\ &=: \int_{\mathbb{S}^{n-1}} T_\eta f(R)\, d\sigma(\eta),\end{aligned}$$

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we then apply the Minkowski inequality and deduce

$$\begin{aligned}\|Tf\|_{L^q_w(\mathbb{R}^n)} &= \omega_{n-1}^{1/q}\left(\int_0^\infty \left|\int_{\mathbb{S}^{n-1}} T_\eta f(R)\,d\sigma(\eta)\right|^q w(R)R^{n-1}\,dR\right)^{1/q} \\ &\le \omega_{n-1}^{1/q}\int_{\mathbb{S}^{n-1}}\left(\int_0^\infty |T_\eta f(R)|^q w(R)R^{n-1}\,dR\right)^{1/q} d\sigma(\eta).\end{aligned} \tag{2.23}$$

By duality, there exists a function h such that

$$\int_0^\infty |h(R)|^{q'} w(R)R^{n-1}\,dR = 1$$

and

$$\left(\int_0^\infty |T_\eta f(R)|^q w(R)R^{n-1}\,dR\right)^{1/q} = \int_0^\infty T_\eta f(R)h(R)\,w(R)R^{n-1}\,dR. \tag{2.24}$$

Notice that (2.22), the Fubini theorem and the Hölder inequality imply

$$\begin{aligned}&\int_0^\infty T_\eta f(R)h(R)w(R)R^{n-1}\,dR \\ &\le \int_0^\infty \left(\int_0^\infty K(1,t)\,|f(tR\eta)|\,t^{n-1}\,dt\right)|h(R)|w(R)R^{n-1}\,dR \\ &= \int_0^\infty K(1,t)\,t^{n-1}\left(\int_0^\infty |f(tR\eta)\,h(R)|w(R)R^{n-1}\,dR\right)dt \\ &\le \int_0^\infty K(1,t)\,t^{n-1}\left(\int_0^\infty |f(tR\eta)|^q w(R)R^{n-1}\,dR\right)^{1/q} dt \\ &= \int_0^\infty K(1,t)\,t^{n-1-\frac{n}{q}}\left(\int_0^\infty |f(\rho\eta)|^q w(t^{-1}\rho)\rho^{n-1}\,d\rho\right)^{1/q} dt \\ &\le \int_0^\infty K(1,t)t^{\frac{n}{q'}-1}\left(\sup_{\rho\in(0,\infty)}\frac{w(t^{-1}\rho)}{w(\rho)}\right)^{1/q} dt\left(\int_0^\infty |f(\rho\eta)|^q w(\rho)\rho^{n-1}\,d\rho\right)^{1/q} \\ &= J\left(\int_0^\infty |f(\rho\eta)|^q w(\rho)\rho^{n-1}\,d\rho\right)^{1/q}.\end{aligned}$$

This, together with (2.23), (2.24), and the Hölder inequality, further implies

$$\begin{aligned}\|Tf\|_{L^q_w(\mathbb{R}^n)} &\le J\,\omega_{n-1}^{1/q}\int_{\mathbb{S}^{n-1}}\left(\int_0^\infty |f(\rho\eta)|^q w(\rho)\rho^{n-1}\,d\rho\right)^{1/q} d\sigma(\eta) \\ &\le J\,\omega_{n-1}^{1/q}\left(\int_{\mathbb{S}^{n-1}}\int_0^\infty |f(\rho\eta)|^q w(\rho)\rho^{n-1}\,d\rho d\sigma(\eta)\right)^{1/q}\omega_{n-1}^{1/q'} \\ &= J\,\omega_{n-1}\|f\|_{L^q_w(\mathbb{R}^n)}.\end{aligned}$$

Thus, we complete the proof of Lemma 2.9. □

Theorem 2.10. *Let* $p,\ q\in[1,\infty)$, $s\in(0,\infty)$. *Suppose that* $\varphi\in\mathcal{S}_\infty(\mathbb{R}^n)$ *such that*

$$\operatorname{supp}\hat{\varphi}\subset\{x : c^{-1}\le|x|\le c\}$$

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for some constant $c \in (1,\infty)$*. Assume that* $w \in A_q(\mathbb{R}^n)$ *is a radial weight satisfying* (1.4) *and* (1.5)*. Then, there exists a positive constant* C *such that the following hold:*

(i) *for any* $f \in \dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)$*, there exists a Lebesgue measurable function* $f_0 \in \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$ *such that* $f_0 = f$ *in* $\mathcal{S}'_\infty(\mathbb{R}^n)$ *and*

$$\|f_0\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \le C\|f\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)};$$

(ii) *for any* $f \in \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$*,*

$$\|f\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)} \le C\|f\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)}.$$

Proof. We first prove (i). As in (2.20), there exists a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\hat{\phi} \subset \{x \in \mathbb{R}^n : \ c_0^{-1} \le |x| \le c_0\}$$

and

$$f = \int_0^\infty \phi_t * \varphi_t * f \,\frac{dt}{t} \quad \text{in } \mathcal{S}'_\infty(\mathbb{R}^n), \tag{2.25}$$

where $c_0 \in (1,\infty)$ is a constant. Set

$$f_0 := \int_0^\infty \phi_t * \varphi_t * f \,\frac{dt}{t}.$$

Clearly, f_0 is a Lebesgue measurable function, but may be infinite on a set of positive Lebesgue measure.

Let us calculate the semi-norm $\|f_0\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)}$. To this end, let $k = \lfloor s \rfloor + 1$. Then, we have

$$\mathrm{D}^k_h f_0 = \int_0^\infty \mathrm{D}^k_h(\phi_t * \varphi_t * f)\,\frac{dt}{t},$$

which implies

$$\|\mathrm{D}^k_h f_0\|_{L^p(\mathbb{R}^n)} \le \int_0^\infty \|\mathrm{D}^k_h(\phi_t * \varphi_t * f)\|_{L^p(\mathbb{R}^n)}\,\frac{dt}{t}. \tag{2.26}$$

On the one hand, by (1.1) and the Young inequality, we obtain

$$\begin{aligned}\|\mathrm{D}^k_h(\phi_t * \varphi_t * f)\|_{L^p(\mathbb{R}^n)} &= \left\|\sum_{m=0}^k (-1)^{k-m}\binom{k}{m}(\phi_t * \varphi_t * f)(\cdot + mh)\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|\phi_t * \varphi_t * f\|_{L^p(\mathbb{R}^n)} \le \|\phi_t\|_{L^1(\mathbb{R}^n)}\|\varphi_t * f\|_{L^p(\mathbb{R}^n)} \lesssim \|\varphi_t * f\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{2.27}$$

On the other hand, by

$$\mathrm{D}^k_h g(x) = \int_{[0,1]^k} \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n h_{j_1}\cdots h_{j_k}(\partial_{j_1}\cdots\partial_{j_k} g)(x + (t_1 + \cdots + t_k)h)\,dt_1\cdots dt_k$$

and the Minkowski inequality, we have

$$\begin{aligned}\|\mathrm{D}^k_h(\phi_t * \varphi_t * f)\|_{L^p(\mathbb{R}^n)} &\lesssim |h|^k \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n \|\partial_{i_1}\cdots\partial_{i_k}(\phi_t * \varphi_t * f)\|_{L^p(\mathbb{R}^n)} \\ &= |h|^k \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n \|(\partial_{i_1}\cdots\partial_{i_k}\phi_t) * (\varphi_t * f)\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

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$$\le |h|^k \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \|\partial_{i_1} \cdots \partial_{i_k} \phi_t\|_{L^1(\mathbb{R}^n)} \|\varphi_t * f\|_{L^p(\mathbb{R}^n)}.$$

Since

$$\|\partial_{i_1} \cdots \partial_{i_k} \phi_t\|_{L^1(\mathbb{R}^n)} = t^{-k} \|(\partial_{i_1} \cdots \partial_{i_k} \phi)_t(x)\|_{L^1(\mathbb{R}^n)} \lesssim t^{-k},$$

it follows that

$$\|\mathrm{D}_h^k(\phi_t * \varphi_t * f)\|_{L^p(\mathbb{R}^n)} \lesssim t^{-k}|h|^k \|\varphi_t * f\|_{L^p(\mathbb{R}^n)}. \tag{2.28}$$

Inserting (2.27) and (2.28) into (2.26) yields

$$\|\mathrm{D}_h^k f_0\|_{L^p(\mathbb{R}^n)} \lesssim \int_0^\infty \min\left\{1, \frac{|h|}{t}\right\}^k \|\varphi_t * f\|_{L^p(\mathbb{R}^n)} \frac{dt}{t}, \tag{2.29}$$

thereby leading to

$$\begin{aligned} \|f_0\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \frac{\|\mathrm{D}_h^k f_0\|^q_{L^p(\mathbb{R}^n)}}{|h|^{n+sq}} w(h)\, dh \right)^{1/q} \\ &\lesssim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \min\left\{1, \frac{|h|}{t}\right\}^k \|\varphi_t * f\|_{L^p(\mathbb{R}^n)} \frac{dt}{t} \right)^q \frac{w(h)}{|h|^{n+sq}}\, dh \right)^{1/q} \\ &= \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \min\left\{1, \frac{|h|}{t}\right\}^k \left(\frac{t}{|h|}\right)^{s+n/q} t^{-n} \frac{\|\varphi_t * f\|_{L^p(\mathbb{R}^n)}}{t^{s+n/q}} t^{n-1}\, dt \right)^q w(h)\, dh \right)^{1/q}. \end{aligned}$$

If we set

$$F_1(y) := \frac{\|\varphi_{|y|} * f\|_{L^p(\mathbb{R}^n)}}{|y|^{s+n/q}}$$

and

$$K_1(|h|, |y|) := \min\left\{1, \frac{|h|}{|y|}\right\}^k \left(\frac{|y|}{|h|}\right)^{s+n/q} |y|^{-n},$$

then

$$\int_0^\infty \min\left\{1, \frac{|h|}{t}\right\}^k \left(\frac{t}{|h|}\right)^{s+n/q} t^{-n} \frac{\|\varphi_t * f\|_{L^p(\mathbb{R}^n)}}{t^{s+n/q}} t^{n-1}\, dt = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} K_1(|h|, |y|) F_1(y)\, dy$$

and, hence,

$$\|f_0\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \lesssim \left(\int_{\mathbb{R}^n} \left(\frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} K_1(|h|, |y|) F_1(y)\, dy \right)^q w(h)\, dh \right)^{1/q}. \tag{2.30}$$

Clearly, K_1 is homogeneous of degree $-n$. Moreover, by (1.4) and (1.5), we have

$$\begin{aligned} &\int_0^\infty K_1(1, t)\, t^{\frac{n}{q'}-1} \left(\sup_{\rho\in(0,\infty)} \frac{w(t^{-1}\rho)}{w(\rho)} \right)^{\frac{1}{q}} dt \\ &= \int_0^1 t^s \left(\sup_{\rho\in(0,\infty)} \frac{w(t^{-1}\rho)}{w(\rho)} \right)^{\frac{1}{q}} \frac{dt}{t} + \int_1^\infty t^{s-k} \left(\sup_{\rho\in(0,\infty)} \frac{w(t^{-1}\rho)}{w(\rho)} \right)^{\frac{1}{q}} \frac{dt}{t} \\ &= \int_1^\infty t^{-s} \left(\sup_{\rho\in(0,\infty)} \frac{w(t\rho)}{w(\rho)} \right)^{\frac{1}{q}} \frac{dt}{t} + \int_0^1 t^{k-s} \left(\sup_{\rho\in(0,\infty)} \frac{w(t\rho)}{w(\rho)} \right)^{\frac{1}{q}} \frac{dt}{t} \end{aligned}$$

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$$
\begin{aligned}
&\le \left(\int_1^\infty t^{\delta q}\left(\sup_{\rho\in(0,\infty)}\frac{w(t\rho)}{w(\rho)}\right)\frac{dt}{t}\right)^{\frac{1}{q}}\left(\int_1^\infty t^{-(s+\delta)q'}\frac{dt}{t}\right)^{\frac{1}{q'}}\\
&\quad+\left(\int_0^1 t^{\sigma q}\left(\sup_{\rho\in(0,\infty)}\frac{w(t\rho)}{w(\rho)}\right)\frac{dt}{t}\right)^{\frac{1}{q}}\left(\int_0^1 t^{(k-s-\sigma)q'}\frac{dt}{t}\right)^{\frac{1}{q'}}\\
&<\infty,
\end{aligned}
$$

which, together with Lemma 2.9, implies

$$
\begin{aligned}
\|f_0\|_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)} \lesssim \|F_1\|_{L^q_w(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}\frac{\|\varphi_{|y|}*f\|^q_{L^p(\mathbb{R}^n)}}{|y|^{sq+n}}w(y)\,dy\right)^{1/q}\\
&\approx \left(\int_0^\infty \|\varphi_t*f\|^q_{L^p(\mathbb{R}^n)}t^{-sq-1}w(t)\,dt\right)^{1/q} = \|f\|_{\dot\Lambda^s_{p,q,w,\varphi}(\mathbb{R}^n)}.
\end{aligned}
\tag{2.31}
$$

We still need to show that $f_0 \in \dot\Lambda^s_{p,q,w}(\mathbb{R}^n)$. To achieve this, we only need to validate that f_0 can be approximated by $C_c^\infty(\mathbb{R}^n)$-functions under $\|\cdot\|_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)}$. Indeed, for any $\epsilon\in(0,1)$, define

$$
f_\epsilon := \int_\epsilon^{1/\epsilon}\phi_t*\varphi_t*f\,\frac{dt}{t}.
$$

After a revisit of the above proof of $\|f_0\|_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)}<\infty$, we derive from (2.26)-(2.29)-(2.30)-(2.31) that

$$
\|f_0\|^q_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)} \le \int_{\mathbb{R}^n}\frac{\left(\int_0^\infty \|\mathrm{D}^k_h(\phi_t*\varphi_t*f)\|_{L^p(\mathbb{R}^n)}\frac{dt}{t}\right)^q}{|h|^{n+sq}}w(h)\,dh<\infty,
$$

which directly gives

$$
\|f_\epsilon\|^q_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)} \le \int_{\mathbb{R}^n}\frac{\left(\int_\epsilon^{1/\epsilon} \|\mathrm{D}^k_h(\phi_t*\varphi_t*f)\|_{L^p(\mathbb{R}^n)}\frac{dt}{t}\right)^q}{|h|^{n+sq}}w(h)\,dh<\infty.
\tag{2.32}
$$

Meanwhile, observing that

$$
\|\mathrm{D}^k_h(f_0-f_\epsilon)\|_{L^p(\mathbb{R}^n)} \le \int_0^\epsilon \|\mathrm{D}^k_h(\phi_t*\varphi_t*f)\|_{L^p(\mathbb{R}^n)}\frac{dt}{t}+\int_{1/\epsilon}^\infty \|\mathrm{D}^k_h(\phi_t*\varphi_t*f)\|_{L^p(\mathbb{R}^n)}\frac{dt}{t},
$$

and applying the Lebesgue dominated convergence theorem to

$$
\int_{\mathbb{R}^n}\frac{\left(\int_0^\epsilon \|\mathrm{D}^k_h(\phi_t*\varphi_t*f)\|_{L^p(\mathbb{R}^n)}\frac{dt}{t}+\int_{1/\epsilon}^\infty \|\mathrm{D}^k_h(\phi_t*\varphi_t*f)\|_{L^p(\mathbb{R}^n)}\frac{dt}{t}\right)^q}{|h|^{n+sq}}w(h)\,dh,
$$

we arrive at

$$
\lim_{\epsilon\to0}\|f_0-f_\epsilon\|_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)}=0.
\tag{2.33}
$$

Moreover, by the Young inequality and the Hölder inequality, we have

$$
\begin{aligned}
\|f_\epsilon\|_{L^p(\mathbb{R}^n)} &\le \int_\epsilon^{1/\epsilon}\|\phi_t*\varphi_t*f\|_{L^p(\mathbb{R}^n)}\frac{dt}{t}\\
&\le \int_\epsilon^{1/\epsilon}\|\phi_t\|_{L^1(\mathbb{R}^n)}\|\varphi_t*f\|_{L^p(\mathbb{R}^n)}\frac{dt}{t}
\end{aligned}
$$

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$$\leq \left(\int_\epsilon^{1/\epsilon} \|\varphi_t * f\|_{L^p(\mathbb{R}^n)}^q t^{-sq-1} w(t)\, dt\right)^{1/q} \left(\int_\epsilon^{1/\epsilon} \|\phi_t\|_{L^1(\mathbb{R}^n)}^{q'} t^{sq'-1} w(t)^{1-q'}\, dt\right)^{1/q'}$$

$$\leq \|f\|_{\dot\Lambda^s_{p,q,w,\varphi}(\mathbb{R}^n)} \|\phi\|_{L^1(\mathbb{R}^n)} \left(\int_\epsilon^{1/\epsilon} t^{sq'-1} w(t)^{1-q'}\, dt\right)^{1/q'}.$$

Note that $w \in A_q(\mathbb{R}^n)$ means $w^{1-q'} \in A_{q'}(\mathbb{R}^n)$. This, together with the radial property of w, implies

$$\int_\epsilon^{1/\epsilon} t^{sq'-1} w(t)^{1-q'}\, dt \approx \int_{\epsilon\leq|x|\leq\frac{1}{\epsilon}} |x|^{sq'-n} w(x)^{1-q'}\, dx < \infty$$

and, hence,

$$f_\epsilon \in L^p(\mathbb{R}^n).$$

By this and (2.32), we see that $f_\epsilon \in \Lambda^s_{p,q,w}(\mathbb{R}^n)$. Further, it follows from Lemma 2.2 that there exists a sequence $\{f_{\epsilon,j}\}_j \subset C_c^\infty(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} \|f_{\epsilon,j} - f_\epsilon\|_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)} \leq \lim_{j\to\infty} \|f_{\epsilon,j} - f_\epsilon\|_{\Lambda^s_{p,q,w}(\mathbb{R}^n)} = 0. \tag{2.34}$$

Combining (2.33) and (2.34), we deduce that f_0 can be approximated by $C_c^\infty(\mathbb{R}^n)$-functions $\{f_{\epsilon,j}\}_{\epsilon,j}$ under the semi-norm $\|\cdot\|_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)}$. This finishes the proof of (i).

Next, we show (ii). To this end, we choose a radial function $\psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp}\hat\psi \subset \{x \in \mathbb{R}^n : c_0^{-1} \leq |x| \leq c_0\}$ for some constant $c_0 \in (1,\infty)$. For any $z \in \mathbb{R}^n$, define

$$\Psi(z) = \sum_{m=1}^k (-1)^{k-m}\binom{k}{m} m^{-n}\psi\left(\frac{z}{m}\right).$$

Clearly, $\Psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ and $\operatorname{supp}\hat\Psi \subset \{x \in \mathbb{R}^n : (kc_0)^{-1} \leq |x| \leq c_0\}$. Due to Lemma 2.8, it suffices to show that for any $f \in \dot\Lambda^s_{p,q,w}(\mathbb{R}^n)$ there is

$$\|f\|_{\dot\Lambda^s_{p,q,w,\Psi}(\mathbb{R}^n)} \lesssim \|f\|_{\dot\Lambda^s_{p,q,w}(\mathbb{R}^n)}.$$

Indeed, by the cancellation condition of ψ, we write

$$\begin{aligned}
\Psi_t * f(x) &= \int_{\mathbb{R}^n} f(x-z)\Psi_t(z)\, dz \\
&= \int_{\mathbb{R}^n} f(x-z)\sum_{m=1}^k (-1)^{k-m}\binom{k}{m} m^{-n}\psi_t\left(\frac{z}{m}\right) dz \\
&= \int_{\mathbb{R}^n} \sum_{m=1}^k (-1)^{k-m}\binom{k}{m} f(x+my)\psi_t(y)\, dy \\
&= \int_{\mathbb{R}^n} \sum_{m=0}^k (-1)^{k-m}\binom{k}{m} f(x+my)\psi_t(y)\, dy \\
&= \int_{\mathbb{R}^n} \mathrm{D}_y^k f(x)\psi_t(y)\, dy,
\end{aligned}$$

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which, together with the Minkowski inequality, further implies

$$\begin{aligned}\|\Psi_t * f\|_{L^p(\mathbb{R}^n)}^q &= \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \mathrm{D}_y^k f(x)\psi_t(y)\,dy\right|^p\,dx\right)^{q/p}\\ &\leq \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\mathrm{D}_y^k f(x)|^p|\psi_t(y)|^p\,dx\right)^{1/p}\,dy\right)^q\\ &= \left(\int_{\mathbb{R}^n} |\psi_t(y)|\,\|\mathrm{D}_y^k f\|_{L^p(\mathbb{R}^n)}\,dy\right)^q.\end{aligned}$$

Thus,

$$\begin{aligned}\|f\|_{\dot{\Lambda}^s_{p,q,w,\Psi}(\mathbb{R}^n)} &= \left(\int_0^\infty \|\Psi_t * f\|_{L^p(\mathbb{R}^n)}^q t^{-sq-1}w(t)\,dt\right)^{1/q}\\ &\leq \left(\int_0^\infty \left(\int_{\mathbb{R}^n} |\psi_t(y)|\,\|\mathrm{D}_y^k f\|_{L^p(\mathbb{R}^n)}\,dy\right)^q t^{-sq-1}w(t)\,dt\right)^{1/q}\\ &= \left(\frac{1}{\omega_{n-1}}\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\psi_{|x|}(y)|\,\|\mathrm{D}_y^k f\|_{L^p(\mathbb{R}^n)}\,dy\right)^q |x|^{-sq-n}w(x)\,dx\right)^{1/q}\\ &\approx \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\psi_{|x|}(y)|\left(\frac{|y|}{|x|}\right)^{s+\frac{n}{q}} \frac{\|\mathrm{D}_y^k f\|_{L^p(\mathbb{R}^n)}}{|y|^{s+\frac{n}{q}}}\,dy\right)^q w(x)\,dx\right)^{1/q}.\end{aligned}$$

Since $\psi \in \mathcal{S}_\infty(\mathbb{R}^n)$, it follows that for all $x, y \in \mathbb{R}^n$,

$$\left|\psi_{|x|}(y)\right|\left(\frac{|y|}{|x|}\right)^{s+n/q} \lesssim |x|^{-n}\left(\frac{|y|}{|x|}\right)^{s+n/q}\left(1+\frac{|y|}{|x|}\right)^{-N},$$

where $N > s + n + \sigma$. For any $x,\ y \in \mathbb{R}^n$, upon setting

$$F_2(y) := \frac{\|\mathrm{D}_y^k f\|_{L^p(\mathbb{R}^n)}}{|y|^{s+n/q}}$$

and

$$K_2(|x|,|y|) := |x|^{-n}\left(\frac{|y|}{|x|}\right)^{s+n/q}\left(1+\frac{|y|}{|x|}\right)^{-N},$$

we then arrive at the estimate

$$\|f\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \lesssim \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K_2(|x|,|y|)F_2(y)\,dy\right)^q w(x)\,dx\right)^{1/q}.$$

Clearly, K_2 is homogeneous of degree $-n$. Moreover, from (1.4) and (1.5), it follows that

$$\begin{aligned}&\int_0^\infty K_2(1,t)\,t^{\frac{n}{q'}-1}\left(\sup_{\rho\in(0,\infty)}\frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}}dt\\ &\leq \int_0^1 t^{s+n}\left(\sup_{\rho\in(0,\infty)}\frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dt}{t} + \int_1^\infty t^{s+n-N}\left(\sup_{\rho\in(0,\infty)}\frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dt}{t}\\ &= \int_1^\infty t^{-s-n}\left(\sup_{\rho\in(0,\infty)}\frac{w(t\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dt}{t} + \int_0^1 t^{N-s-n}\left(\sup_{\rho\in(0,\infty)}\frac{w(t\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dt}{t}\end{aligned}$$

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$$
\begin{aligned}
&\leq \left(\int_1^\infty t^{\delta q}\left(\sup_{\rho\in(0,\infty)}\frac{w(t\rho)}{w(\rho)}\right)\frac{dt}{t}\right)^{\frac{1}{q}}\left(\int_1^\infty t^{-(\delta+s+n)q'}\,\frac{dt}{t}\right)^{\frac{1}{q'}}\\
&\quad+\left(\int_0^1 t^{\sigma q}\left(\sup_{\rho\in(0,\infty)}\frac{w(t\rho)}{w(\rho)}\right)\frac{dt}{t}\right)^{\frac{1}{q}}\left(\int_0^1 t^{(N-s-n-\sigma)q'}\,\frac{dt}{t}\right)^{\frac{1}{q'}}\\
&<\infty,
\end{aligned}
$$

where in the last step we used $N > s+n+\sigma$ and $\delta > -s$. We then have by Lemma 2.9 that

$$
\|f\|_{\dot{\Lambda}^s_{p,q,w,\Psi}(\mathbb{R}^n)} \lesssim \left(\int_{\mathbb{R}^n}\frac{\|\mathrm{D}^k_y f\|^q_{L^p(\mathbb{R}^n)}}{|y|^{n+sq}}w(y)\,dy\right)^{\frac{1}{q}} = \|f\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)}.
$$

This finishes the proof of (ii) and, hence, Theorem 2.10. □

Remark 2.11. In Theorem 2.10(i), if $f\in L^r(\mathbb{R}^n)$ for some $r\in[1,\infty)$, then we know that the Calderón reproducing formula (2.25) holds almost everywhere on $\mathbb{R}^n$ (see, for example, [24]), so that $f_0 = f$ almost everywhere on $\mathbb{R}^n$.

3. Proof of Theorem 1.6

The main goal of this section is to show Theorem 1.6. In Sections 3.1 and 3.2, we restrict β to $(0,2n)$ and deal with the extension and restriction part, respectively. The case $\beta\in[2n,\infty)$ is proved in Section 3.3.

3.1. Extension part: the case $\beta\in(0,2n)$.

We begin with the following estimate associated with the Bessel kernel.

Lemma 3.1. *For $\alpha\in(0,\infty)$, let G_α be the Bessel kernel on $\mathbb{R}^n$, that is, for all $x\in\mathbb{R}^n$,*

$$
\hat{G}_\alpha(x) = (1+|x|^2)^{-\frac{\alpha}{2}}.
$$

If $k\in\mathbb{N}$ satisfying $k>n-\alpha$, then for all $\xi\in\mathbb{R}^n$ the integral

$$
\int_{\mathbb{R}^n} G_\alpha(|h|\xi)\left(e^{2\pi i h\cdot\xi}-1\right)^k\frac{dh}{|h|^n}
$$

absolutely converges to the same constant c_α independent of ξ.

Proof. Let $e_1 := (1,0,\ldots,0)$ be the unit vector in $\mathbb{R}^n$. After a rotation, we then have

$$
\int_{\mathbb{R}^n} G_\alpha(|h|\xi)\left(e^{2\pi i h\cdot\xi}-1\right)^k\frac{dh}{|h|^n} = \int_{\mathbb{R}^n} G_\alpha(|h|e_1)\left(e^{2\pi i h\cdot e_1}-1\right)^k\frac{dh}{|h|^n},
$$

which is independent of ξ if the integral converges. For all $h\in\mathbb{R}^n$, observe that

$$
\left|\left(e^{2\pi i h\cdot e_1}-1\right)^k\right| \lesssim \min\{1,|h|^k\}.
$$

Note that $\hat{G}_\alpha$ is radial, so does G_α. Due to this reason, we may write $G_\alpha(x)$ as $G_\alpha(r)$ whenever $|x|=r$. Thus,

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n} G_\alpha(|h|e_1)\left(e^{2\pi i h\cdot e_1}-1\right)^k\frac{dh}{|h|^n}\right| &\lesssim \int_{\mathbb{R}^n}|G_\alpha(|h|e_1)|\min\{1,|h|^k\}\,\frac{dh}{|h|^n}\\
&\lesssim \int_0^\infty G_\alpha(r)\min\{1,r^k\}\,\frac{dr}{r}.
\end{aligned}
$$

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According to [29, (1.2.11) and (1.2.12)], we know that when $|\xi| < 1$,

$$G_\alpha(\xi) \approx \begin{cases} |\xi|^{\alpha-n} + 1 + O(|\xi|^{\alpha-n+2}) \approx |\xi|^{\alpha-n} & \text{if } \alpha \in (0,n); \\ \log|\xi|^{-1} + 1 + O(|\xi|^2) \lesssim \log|\xi|^{-1} & \text{if } \alpha = n; \\ 1 + O(|\xi|^{\alpha-n}) \lesssim 1 & \text{if } \alpha \in (n,\infty), \end{cases}$$

and when $|\xi| \geq 1$,

$$G_\alpha(\xi) \lesssim e^{-\pi|\xi|}.$$

From these, we deduce that

$$\int_0^1 G_\alpha(r)\min\{1, r^k\}\,\frac{dr}{r} \lesssim \begin{cases} \int_0^1 r^{\alpha-n}\, r^{k-1}\, dr \lesssim 1 & \text{if } \alpha \in (0,n); \\ \int_0^1 \log\frac{1}{r}\, r^{k-1}\, dr \lesssim 1 & \text{if } \alpha = n; \\ \int_0^1 r^{k-1}\, dr \lesssim 1 & \text{if } \alpha \in (n,\infty), \end{cases}$$

and

$$\int_1^\infty G_\alpha(r)\min\{1, r^k\}\,\frac{dr}{r} \lesssim \int_1^\infty e^{-\pi r}\,\frac{dr}{r} \lesssim 1.$$

Combining the last two formulae derives that $\int_{\mathbb{R}^n} G_\alpha(|h|\xi)(e^{2\pi i h\cdot\xi} - 1)^k \frac{dh}{|h|^n}$ converges to a constant which independent of ξ. This constant will be denoted by c_α. □

Next, we prove Theorem 1.6(i) under $\beta \in (0, 2n)$.

Proof of Theorem 1.6(i) under $\beta \in (0, 2n)$. Since $\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$ is defined as the completeness of $C_c^\infty(\mathbb{R}^n)$ under the semi-norm $\|\cdot\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)}$, we may as well assume that $g \in C_c^\infty(\mathbb{R}^n) \cap \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$. Let $k := \lfloor s \rfloor + 1$. For any $x, y \in \mathbb{R}^n$, define

$$f(x,y) := |y|^{-\beta} \mathrm{D}_y^k g(x) \tag{3.1}$$

and

$$F(x,y) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(|x-z|^2 + |y-h|^2\right)^{\frac{\beta-2n}{2}} |h|^{-\beta} \mathrm{D}_h^k g(z)\, dz\, dh. \tag{3.2}$$

Before going further, we are about to show that for all $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |F(x,y)||\varphi(x,y)|\, dx\, dy < \infty. \tag{3.3}$$

Because $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, the integral in (3.3) is bounded by a constant multiple of

$$\mathcal{J} := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(|z| + |y-h|)^{\beta-2n}\, |h|^{-\beta}|\mathrm{D}_h^k g(x-z)|}{(1+|x|)^N(1+|y|)^N}\, dz\, dh\, dx\, dy,$$

where $N \in \mathbb{N}$ can be any large integer. To obtain (3.3), it suffices to prove that $\mathcal{J} < \infty$ holds for sufficiently large N.

Since $g \in C_c^\infty(\mathbb{R}^n)$, we take a large number $R \in (1,\infty)$ such that $\operatorname{supp} g \subset B(0,R)$. Via splitting the integral domain of $\mathcal{J}$ into annulus, we find that

$$\mathcal{J} \leq \iint_{|x|+|y|<R} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(|z| + |y-h|)^{\beta-2n}\, |h|^{-\beta}|\mathrm{D}_h^k g(x-z)|}{(1+|x|)^N(1+|y|)^N}\, dz\, dh\, dx\, dy$$

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$$+\sum_{j=1}^{\infty}\iint_{2^{j-1}R\le|x|+|y|<2^jR}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(|z|+|y-h|)^{\beta-2n}\,|h|^{-\beta}|\mathrm{D}_h^kg(x-z)|}{(1+|x|)^N(1+|y|)^N}\,dz\,dh\,dx\,dy$$
$$=:\mathrm{I}_0+\sum_{j=1}^{\infty}\mathrm{I}_j.$$

If $2^{j-1}R\le|x|+|y|$ for some $j\ge1$, then either $|x|\ge 2^{j-2}R$ or $|y|\ge 2^{j-2}R$, whatever the cases we always have

$$\mathrm{I}_j\le(2^{j-2}R)^{-N/2}\iint_{|x|+|y|<2^jR}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(|z|+|y-h|)^{\beta-2n}\,|h|^{-\beta}|\mathrm{D}_h^kg(x-z)|}{(1+|x|)^{N/2}(1+|y|)^{N/2}}\,dz\,dh\,dx\,dy.$$

Fix $j\in\{0,1,2,\dots\}$. We now consider the cases $|h|\ge 2^{j+1}R$ and $|h|<2^{j+1}R$, respectively.

- If $|x|+|y|<2^jR$ and $|h|\ge2^{j+1}R$, then $|y-h|\approx|h|$, which, together with (1.1) and $g\in C_c^\infty(\mathbb{R}^n)$, implies that

$$\begin{aligned}\mathrm{Z}_1&:=\int_{|h|\ge2^{j+1}R}\int_{\mathbb{R}^n}(|z|+|y-h|)^{\beta-2n}\,|h|^{-\beta}\left|\mathrm{D}_h^kg(x-z)\right|\,dz\,dh\\&\lesssim\sum_{m=0}^{k}\binom{k}{m}\int_{|h|\ge2^{j+1}R}\int_{\mathbb{R}^n}|h|^{-2n}|g(x-z+mh)|\,dz\,dh\\&\lesssim\|g\|_{L^1(\mathbb{R}^n)}\int_{|h|\ge2^{j+1}R}|h|^{-2n}\,dh\\&\lesssim1.\end{aligned}\tag{3.4}$$

- If $|x|+|y|<2^jR$ and $|h|<2^{j+1}R$, then by (1.1), we have $\operatorname{supp}(\mathrm{D}_h^kg)\subseteq B(0,k2^{j+2}R)$, which implies that $\mathrm{D}_h^kg(x-z)\neq0$ only if $|x-z|<k2^{j+2}R$, thereby leading to $|z|<k2^{j+3}R$. Moreover, by (2.6) and $g\in C_c^\infty(\mathbb{R}^n)$, we see that

$$\begin{aligned}\left|\mathrm{D}_h^kg(x)\right|&\le\sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n}|h|^k\int_{[0,1]^k}\left|(\partial_{j_1}\cdots\partial_{j_k}g)(x+(t_1+\cdots+t_k)h)\right|\,dt_1\cdots dt_k\\&\lesssim|h|^k\end{aligned}$$

holds uniformly in $x\in\mathbb{R}^n$. Thus, we obtain

$$\begin{aligned}\mathrm{Z}_1&:=\int_{|h|<2^{j+1}R}\int_{\mathbb{R}^n}(|z|+|y-h|)^{\beta-2n}\,|h|^{-\beta}\left|\mathrm{D}_h^kg(x-z)\right|\,dz\,dh\\&\lesssim\int_{|z|<k2^{j+3}R}\left(\int_{\substack{|h|<2^{j+1}R\\|y-h|\ge|h|/2}}(|z|+|h|)^{\beta-2n}|h|^{k-\beta}\,dh+\int_{\substack{|h|<2^{j+1}R\\|y-h|<|h|/2}}(|z|+|y-h|)^{\beta-2n}|h|^{k-\beta}\,dh\right)dz.\end{aligned}$$

Denote these last two double-integrals by $\mathrm{Z}_{2,1}$ and $\mathrm{Z}_{2,2}$, respectively. Clearly,

$$\mathrm{Z}_{2,1}\lesssim\int_0^{2^{j+1}R}\left(\int_0^{k2^{j+3}R}(s+t)^{\beta-2n}s^{n-1}t^{k+n-\beta-1}\,ds\right)dt.$$

Considering the integrals over the domain $s\le t$ and $s>t$, we then get

$$\mathrm{Z}_{2,1}\lesssim\int_0^{2^{j+1}R}\left(\int_0^{t}s^{n-1}t^{k-n-1}\,ds\right)dt+\int_0^{2^{j+1}R}\left(\int_t^{k2^{j+3}R}s^{\beta-n-1}t^{k+n-\beta-1}\,ds\right)dt$$

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$$\lesssim \int_0^{2^{j+1}R} t^{k-1}\,dt + \begin{cases} \int_0^{2^{j+1}R} t^{k-1}\,dt & \text{as } 0<\beta<n \\ \int_0^{2^{j+1}R} \left(\log \frac{k2^{j+3}R}{t}\right) t^{k-1}\,dt & \text{as } \beta = n \\ [k2^{j+3}R]^{\beta-n} \int_0^{2^{j+1}R} t^{k+n-\beta-1}\,dt & \text{as } n<\beta<2n \end{cases}$$

$$\lesssim (2^jR)^k,$$

where for the case $n<\beta<2n$ the integral converges because of the fact

$$k+n-\beta = \lfloor s\rfloor + 1 + n - \left(s+\frac{n}{q}\right) > 0.$$

Concerning $Z_{2,2}$, observe that if $|y-h|<|h|/2$ then $|y|/2<|h|<2|y|$, which induces

$$\begin{aligned} Z_{2,2} &\lesssim |y|^{k-\beta} \int_{|z|<k2^{j+3}R} \int_{|y-h|<|y|} (|z|+|y-h|)^{\beta-2n}\,dh\,dz \\ &\approx |y|^{k-\beta} \int_{|z|<k2^{j+3}R} \int_{|w|<|y|} (|z|+|w|)^{\beta-2n}\,dw\,dz \\ &\approx |y|^{k-\beta} \int_0^{2^{j+1}R} \left(\int_0^{k2^{j+3}R} (s+\tau)^{\beta-2n} s^{n-1}\tau^{n-1}\,ds \right) d\tau. \end{aligned}$$

In a way similar to the estimate of $Z_{2,2}$, it holds that

$$Z_{2,2} \lesssim |y|^{k-\beta}(2^jR)^{\beta}.$$

A combination of the estimates of $Z_{2,1}$ and $Z_{2,2}$ gives

$$Z_2 \lesssim (2^jR)^k + |y|^{k-\beta}(2^jR)^{\beta}. \tag{3.5}$$

Next, upon taking $N>2(n+k+\beta)$, we utilize (3.4) and (3.5) to derive

$$I_j \lesssim (2^{j-2}R)^{-N/2} \int_{|x|<2^jR} \int_{|y|<2^jR} \frac{(2^jR)^k + |y|^{k-\beta}(2^jR)^{\beta}}{(1+|x|)^{N/2}(1+|y|)^{N/2}}\,dy\,dx \lesssim (2^jR)^{k-N/2} + (2^jR)^{\beta-N/2}.$$

Note that this last estimate is still true when $j=0$. Summing in j, we arrive at the conclusion

$$\mathcal{J} \le \sum_{j=0}^{\infty} I_j \lesssim \sum_{j=0}^{\infty} \left((2^jR)^{k-N/2} + (2^jR)^{\beta-N/2} \right) \lesssim 1.$$

Thus, we obtain the desired estimate in (3.3).

Next, we continue with the proof of Theorem 1.6(i). Observe that the proof of (3.3) (especially the estimate of I_0) also implies that the restricted function $F(\cdot,0)$ is locally integrable on $\mathbb{R}^n$. Given any $\psi \in C_c^\infty(\mathbb{R}^n)$, it follows from the Fubini theorem that

$$\begin{aligned} \langle F(\cdot,0),\psi\rangle &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(|x-z|^2+|h|^2\right)^{\frac{\beta-2n}{2}} |h|^{-\beta} D_h^k g(z)\,dz\,dh \right) \psi(x)\,dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left[\left(\frac{|x-z|}{|h|}\right)^2 + 1 \right]^{\frac{\beta-2n}{2}} |h|^{-n} D_h^k g(z)\,\psi(x)\,dz\,dx \right) \frac{dh}{|h|^n}. \end{aligned}$$

Let $G_{2n-\beta}$ be the kernel of the Bessel potential on $\mathbb{R}^n$ as in Lemma 3.1. For any $x\in\mathbb{R}^n$, set

$$\phi(x) = \hat{G}_{2n-\beta}(x) = (1+|x|^2)^{-\frac{2n-\beta}{2}}.$$

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Via using the dilation $\phi_t(x) = t^{-n}\phi(t^{-1}x)$ for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, we then have

$$\int_{\mathbb{R}^n} \left[\left(\frac{|x-z|}{|h|}\right)^2 + 1\right]^{\frac{\beta-2n}{2}} |h|^{-n} \mathrm{D}_h^k g(z)\, dz = \left(\phi_{|h|} * \mathrm{D}_h^k g\right)(x)$$

and, hence,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left[\left(\frac{|x-z|}{|h|}\right)^2 + 1\right]^{\frac{\beta-2n}{2}} |h|^{-n} \mathrm{D}_h^k g(z)\, dz \right) \psi(x)\, dx \\
&= \left\langle \phi_{|h|} * \mathrm{D}_h^k g,\, \psi \right\rangle \\
&= \left\langle \left(\phi_{|h|} * \mathrm{D}_h^k g\right)^{\wedge},\, \psi^{\vee} \right\rangle \\
&= \left\langle \widehat{\phi_{|h|}}\widehat{\mathrm{D}_h^k g},\, \psi^{\vee} \right\rangle \\
&= \left\langle \widehat{\phi}(|h|\cdot)\left(e^{2\pi i h\cdot(\cdot)} - 1\right)^k \hat{g}(\cdot),\, \psi^{\vee}(\cdot) \right\rangle \\
&= \int_{\mathbb{R}^n} G_{2n-\beta}(|h|\xi) \left(e^{2\pi i h\cdot\xi} - 1\right)^k \hat{g}(\xi)\, \psi^{\vee}(\xi)\, d\xi,
\end{aligned}$$

where the last step is due to $\hat{\phi}(\xi) = G_{2n-\beta}(-\xi)$ so that $\hat{\phi}(|h|\xi) = G_{2n-\beta}(-|h|\xi) = G_{2n-\beta}(|h|\xi)$. Further, an application of Lemma 3.1 yields

$$\begin{aligned}
\langle F(\cdot, 0),\, \psi \rangle &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G_{2n-\beta}(|h|\xi) \left(e^{2\pi i h\cdot\xi} - 1\right)^k \hat{g}(\xi)\, \psi^{\vee}(\xi)\, d\xi \right) \frac{dh}{|h|^n} \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G_{2n-\beta}(|h|\xi) \left(e^{2\pi i h\cdot\xi} - 1\right)^k \hat{g}(\xi)\, \psi^{\vee}(\xi)\, \frac{dh}{|h|^n} \right) d\xi \\
&= c_{2n-\beta} \int_{\mathbb{R}^n} \hat{g}(\xi)\, \psi^{\vee}(\xi)\, d\xi \\
&= c_{2n-\beta} \langle g,\, \psi \rangle,
\end{aligned}$$

where $c_{2n-\beta}$ is the constant determined in Lemma 3.1. Thus,

$$F(x, 0) = c_{2n-\beta}\, g(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{3.6}$$

Define the extension operator $\mathcal{E}$ via

$$\mathcal{E}(g) := \left(c_{2n-\beta}\right)^{-1} \left(\frac{\Gamma(\frac{2n-\beta}{2})}{2^\beta \pi^n \Gamma(\frac{\beta}{2})} \right)^{-1} I_\beta^{(2n)} * f, \tag{3.7}$$

which is an element in $\mathcal{S}'_\infty(\mathbb{R}^{2n})$ by using (3.3) and (1.3) in Remark 1.5. Applying (3.1)-(3.2) and (3.6)-(3.7) yields that for a. e. $x \in \mathbb{R}^n$,

$$\mathcal{E}(g)(x, 0) = \left(c_{2n-\beta}\right)^{-1} \left(\frac{\Gamma(\frac{2n-\beta}{2})}{2^\beta \pi^n \Gamma(\frac{\beta}{2})} \right)^{-1} (I_\beta^{(2n)} * f)(x, 0) = \left(c_{2n-\beta}\right)^{-1} F(x, 0) = g(x). \tag{3.8}$$

Since $g \in C_c^\infty(\mathbb{R}^n)$, it follows that f in (3.1) belongs to $L_w^q(L^p)(\mathbb{R}^{2n})$, which implies both

$$\mathcal{E}g \in \dot{\mathcal{L}}_\beta^{p,q,w}(\mathbb{R}^{2n})$$

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and

$$\|\mathcal{E}g\|_{\dot{\mathcal{L}}_{\beta}^{p,q,w}(\mathbb{R}^{2n})} = \left(c_{2n-\beta}\right)^{-1}\left(\frac{\Gamma(\frac{2n-\beta}{2})}{2^{\beta}\pi^{n}\Gamma(\frac{\beta}{2})}\right)^{-1}\|f\|_{L_w^q(L^p)(\mathbb{R}^{2n})} = \left(c_{2n-\beta}\right)^{-1}\left(\frac{\Gamma(\frac{2n-\beta}{2})}{2^{\beta}\pi^{n}\Gamma(\frac{\beta}{2})}\right)^{-1}\|g\|_{\dot{\Lambda}^{s}_{p,q,w}(\mathbb{R}^{n})}.$$

Summarizing all, we conclude that Theorem 1.6(i) holds under $\beta \in (0, 2n)$. □

3.2. **Restriction part: the case $\beta \in (0, 2n)$.** The aim of this subsection is to show Theorem 1.6(ii) under the case $\beta \in (0, 2n)$. To this end, we need the following proposition.

Proposition 3.2. *Let $p \in [1, \infty)$ and $\beta \in (0, 2n)$. Given any measurable function $\phi : \mathbb{R}^{2n} \to \mathbb{R}$, define for all $\xi, x \in \mathbb{R}^n$ that*

$$g_{\xi}(x) := \int_{\mathbb{R}^n} \phi(z, \xi) I_{\beta}^{(2n)}(x - z, \xi)\, dz.$$

If $k \in \mathbb{N}$ such that $k > \beta - n$, then the following hold:

(i) *when $\beta \in (0, n)$,*

$$\left\|\mathrm{D}_h^k g_{\xi}\right\|_{L^p(\mathbb{R}^n)} \le C\|\phi(\cdot, \xi)\|_{L^p(\mathbb{R}^n)}|\xi|^{\beta-n} \min\left\{1, \frac{|h|}{|\xi|}\right\}^k;$$

(ii) *when $\beta = n$,*

$$\left\|\mathrm{D}_h^k g_{\xi}\right\|_{L^p(\mathbb{R}^n)} \le C\|\phi(\cdot, \xi)\|_{L^p(\mathbb{R}^n)} \min\left\{\log\left(e + \frac{|h|}{|\xi|}\right), \left(\frac{|h|}{|\xi|}\right)^k\right\};$$

(iii) *when $\beta \in (n, 2n)$,*

$$\left\|\mathrm{D}_h^k g_{\xi}\right\|_{L^p(\mathbb{R}^n)} \le C\|\phi(\cdot, \xi)\|_{L^p(\mathbb{R}^n)}|h|^{\beta-n} \min\left\{1, \frac{|h|}{|\xi|}\right\}^{n+k-\beta},$$

where C in (i)-(ii)-(iii) is a positive constant independent of ξ, h and ϕ.

Proof. Assume without loss of generality that the norm $\|\phi(\cdot, \xi)\|_{L^p(\mathbb{R}^n)}$ is finite; otherwise items (i)-(ii)-(iii) hold trivially.

Given a locally integrable function f on $\mathbb{R}^{2n}$, we denote by $\widetilde{\mathrm{D}_h^k} f$ the k-order difference for the first n variables of f, that is, for all $x, y, h \in \mathbb{R}^n$,

$$\widetilde{\mathrm{D}_h^k} f(x, y) := \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(x + mh, y). \tag{3.9}$$

With this notation, we apply the Minkowski inequality to write

$$\begin{aligned}
\left\|\mathrm{D}_h^k g_{\xi}\right\|_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} \phi(z, \xi)\, \widetilde{\mathrm{D}_h^k}(I_{\beta}^{(2n)})(x - z, \xi)\, dz\right|^p dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} \phi(x - z, \xi)\, \widetilde{\mathrm{D}_h^k}(I_{\beta}^{(2n)})(z, \xi)\, dz\right|^p dx\right)^{1/p} \\
&\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |\phi(x - z, \xi)|^p\, dx\right)^{1/p} \left|\widetilde{\mathrm{D}_h^k}(I_{\beta}^{(2n)})(z, \xi)\right| dz \\
&= \|\phi(\cdot, \xi)\|_{L^p(\mathbb{R}^n)} \int_{\mathbb{R}^n}\left|\widetilde{\mathrm{D}_h^k}(I_{\beta}^{(2n)})(z, \xi)\right| dz.
\end{aligned} \tag{3.10}$$

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In order to obtain the desired estimates in (i)-(ii)-(iii), we only need to deal with

$$\mathrm{G} := \int_{\mathbb{R}^n} \left| \widetilde{\mathrm{D}_h^k}(I_\beta^{(2n)})(z,\xi) \right| \, dz.$$

Before further continuation, observe that for all $z, \xi, h \in \mathbb{R}^n$ the identities (3.9) and (2.6) imply not only

$$\left| \widetilde{\mathrm{D}_h^k}(I_\beta^{(2n)})(z,\xi) \right| = \left| \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} I_\beta^{(2n)}(z+mh,\xi) \right| \lesssim \sum_{m=0}^{k} (|z+mh| + |\xi|)^{\beta-2n}, \tag{3.11}$$

but also

$$\begin{aligned} &\left| \widetilde{\mathrm{D}_h^k}(I_\beta^{(2n)})(z,\xi) \right| \\ &\quad \le |h|^k \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \int_{[0,1]^k} \left| \left(\partial_{x_{j_1}} \cdots \partial_{x_{j_k}} I_\beta^{(2n)} \right) (z + (t_1 + \cdots + t_k)h, \xi) \right| dt_1 \cdots dt_k \\ &\quad \lesssim |h|^k \int_{[0,1]^k} (|z + (t_1 + \cdots + t_k)h| + |\xi|)^{\beta-2n-k} \, dt_1 \cdots dt_k. \end{aligned} \tag{3.12}$$

Next, upon utilizing (3.11) and (3.12), we estimate G.

- If $|\xi| > k|h|$, then the integrand in (3.12) satisfies

$$(|z + (t_1 + \cdots + t_k)h| + |\xi|)^{\beta-2n-k} \approx \begin{cases} |\xi|^{\beta-2n-k} & \text{as } |z| < 2|\xi|; \\ |z|^{\beta-2n-k} & \text{as } |z| \ge 2|\xi|, \end{cases}$$

  and, hence,

$$\begin{aligned} \mathrm{G} &\lesssim |h|^k \left(\int_{|z|<2|\xi|} |\xi|^{\beta-2n-k} \, dz + \int_{|z|\ge 2|\xi|} |z|^{\beta-2n-k} \, dz \right) \\ &\approx |h|^k |\xi|^{\beta-n-k} \\ &\approx \begin{cases} |\xi|^{\beta-n} \min\left\{1, \frac{|h|}{|\xi|}\right\}^k & \text{as } \beta \in (0,n); \\ \min\left\{\log\left(e + \frac{|h|}{|\xi|}\right), \left(\frac{|h|}{|\xi|}\right)^k\right\} & \text{as } \beta = n; \\ |h|^{\beta-n} \min\left\{1, \frac{|h|}{|\xi|}\right\}^{n+k-\beta} & \text{as } \beta \in (n, 2n). \end{cases} \end{aligned}$$

  From this and (3.10), it follows that (i)-(ii)-(iii) hold when $|\xi| > k|h|$.

- If $\beta \in (0,n)$ and $|\xi| \le k|h|$, then applying (3.11) yields

$$\begin{aligned} \mathrm{G} &\lesssim \sum_{m=0}^{k} \int_{\mathbb{R}^n} (|z+mh| + |\xi|)^{\beta-2n} \, dz \\ &\lesssim \sum_{m=0}^{k} \left(\int_{|z+mh|\le|\xi|} |\xi|^{\beta-2n} \, dz + \int_{|z+mh|>|\xi|} |z+mh|^{\beta-2n} \, dz \right) \\ &\approx |\xi|^{\beta-n} \\ &\approx |\xi|^{\beta-n} \min\left\{1, \frac{|h|}{|\xi|}\right\}^k. \end{aligned}$$

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This, together with (3.10), induces (i) under the situation $|\xi| \le k|h|$.

- Now, let $\beta \in [n, 2n)$ and $|\xi| \le k|h|$. On the one hand, if we assume further that $|z| \ge 2k|h|$, then for $t_1, \dots, t_k \in [0,1]$, we have

$$|z + (t_1 + \cdots + t_k)h| + |\xi| \approx |z| + |\xi| \approx |z|,$$

which, along with (3.12) and the fact $\beta - n - k < 0$, implies

$$\int_{|z|\ge 2k|h|} \left|\widetilde{\mathrm{D}_h^k}(I_\beta^{(2n)})(z,\xi)\right| dz \lesssim |h|^k \int_{|z|\ge 2k|h|} |z|^{\beta-2n-k}\, dz \lesssim \begin{cases} |h|^{\beta-n} & \text{as } \beta \in (n,2n); \\ 1 & \text{as } \beta = n. \end{cases} \tag{3.13}$$

If $|z| < 2k|h|$, then for all $m \in \{0,1,\dots,k\}$ we have $|z + mh| < 3k|h|$, so that applying (3.11) and a change of variables $u = (z+mh)/|\xi|$ derives

$$\begin{aligned} \int_{|z|<2k|h|} \left|\widetilde{\mathrm{D}_h^k}(I_\beta^{(2n)})(z,\xi)\right| dz &\lesssim \sum_{m=0}^{k} \int_{|z|<2k|h|} (|z+mh| + |\xi|)^{\beta-2n}\, dz \\ &\lesssim |\xi|^{\beta-n} \int_{|u|<\frac{3k|h|}{|\xi|}} (|u|+1)^{\beta-2n}\, du \\ &\lesssim \begin{cases} |\xi|^{\beta-n} \left(\frac{|h|}{|\xi|}\right)^{\beta-n} & \text{as } \beta \in (n,2n); \\ \log\left(\frac{3k|h|}{|\xi|}\right) & \text{as } \beta = n. \end{cases} \end{aligned} \tag{3.14}$$

From (3.13) and (3.14), we deduce that (ii) and (iii) hold when $|\xi| \le k|h|$.

Altogether, we conclude the proof of Proposition 3.2. □

Proof of Theorem 1.6(ii) under $\beta \in (0, 2n)$. Let $f \in \dot{\mathcal{L}}_\beta^{p,q,w}(\mathbb{R}^{2n})$, where $p \in [1,\infty)$ and $q \in (1,\infty)$. Since $\beta \in (0,2n)$, we consider first the case $f = (-\Delta)_{\mathbb{R}^{2n}}^{-\beta/2}\phi$ for some $\phi \in L_w^q(L^p)(\mathbb{R}^{2n})$ such that

$$I_\beta^{(2n)} * \phi \in L^1_{\mathrm{loc}}(\mathbb{R}^{2n}) \cap \mathcal{S}'_\infty(\mathbb{R}^{2n})$$

and

$$(-\Delta)_{\mathbb{R}^{2n}}^{-\beta/2}\phi = I_\beta^{(2n)} * \phi \quad \text{in } \mathcal{S}'_\infty(\mathbb{R}^{2n}).$$

By modifying f at a null set, we may assume that f is pointwisely defined on $\mathbb{R}^{2n}$. In this case, we can write

$$f(x,0) = I_\beta^{(2n)} * \phi(x,0) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(z,\xi) I_\beta^{(2n)}(x-z,\xi)\, dz\, d\xi \quad \text{for all } x \in \mathbb{R}^n.$$

Assume for the moment that we have proved

$$\|f(\cdot,0)\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{\mathcal{L}}_\beta^{p,q,w}(\mathbb{R}^{2n})}. \tag{3.15}$$

After we have obtained (3.15), then we immediately have $f(\cdot,0) \in \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$. Indeed, by Lemma 2.5, there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^{2n})$ such that $\varphi_j \to f$ in $\dot{\mathcal{L}}_\beta^{p,q,w}(\mathbb{R}^{2n})$, which, together with (3.15), implies that $\varphi_j(\cdot,0) \to f(\cdot,0)$ in $\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$. Then, upon defining the restriction operator $\mathcal{R}$ via

$$\mathcal{R}f(\cdot) := f(\cdot,0) = I_\beta^{(2n)} * \phi(\cdot,0), \tag{3.16}$$

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we find that

$$\|\mathcal{R}f\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} = \|f(\cdot,0)\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})}. \tag{3.17}$$

For a general $f \in \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$, by Corollary 2.4 there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^{2n})$ such that $\{I^{(2n)}_\beta * \phi_j\}_{j\in\mathbb{N}}$ converges to f in $\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$. For any $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$g_j(x) := (I^{(2n)}_\beta * \phi_j)(x,0).$$

For any $k, j \in \mathbb{N}$, by (3.16) and (3.17), we have

$$\|g_j - g_k\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} = \|(I^{(2n)}_\beta * (\phi_j - \phi_k))(\cdot,0)\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \lesssim \|I^{(2n)}_\beta * (\phi_j - \phi_k)\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})},$$

which tends to 0 as $k, j \to \infty$ since $\{I^{(2n)}_\beta * \phi_j\}_{j\in\mathbb{N}}$ converges to f in $\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$. Thus $\{g_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$. So, it is reasonable to define

$$\mathcal{R}f := \lim_{j\to\infty} g_j \quad \text{in} \quad \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n). \tag{3.18}$$

We need to explain that $\mathcal{R}f$ in (3.18) is a uniquely defined element in $\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$. Indeed, if $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^{2n})$ is another sequence such that $\{I^{(2n)}_\beta * \psi_j\}_{j\in\mathbb{N}}$ converges to f in $\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$, then we use (3.17) to deduce

$$\begin{aligned}
&\lim_{j\to\infty} \|(I^{(2n)}_\beta * \phi_j)(\cdot,0) - (I^{(2n)}_\beta * \psi_j)(\cdot,0)\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \\
&\quad\lesssim \lim_{j\to\infty} \|I^{(2n)}_\beta * \phi_j - I^{(2n)}_\beta * \psi_j\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} \\
&\quad\le \lim_{j\to\infty} \left(\|I^{(2n)}_\beta * \phi_j - f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} + \|f - I^{(2n)}_\beta * \psi_j\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})}\right) \\
&\quad= 0.
\end{aligned}$$

Moreover, for $\mathcal{R}f$ in (3.18), we have

$$\begin{aligned}
\|\mathcal{R}f\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} = \lim_{j\to\infty} \|g_j\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} &= \lim_{j\to\infty} \|(I^{(2n)}_\beta * \phi_j)(\cdot,0)\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \\
&\lesssim \lim_{j\to\infty} \|I^{(2n)}_\beta * \phi_j\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} = \|f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})}.
\end{aligned}$$

Further, by (3.2) and (3.8), we see that $\mathcal{R}\mathcal{E} = id$ on $\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$.

To finish the proof of Theorem 1.6(ii) under $\beta \in (0, 2n)$, we still need to validate (3.15). To simplify the notation, for any $\xi, x \in \mathbb{R}^n$, set $g(x) = f(x,0)$ and

$$g_\xi(x) := \int_{\mathbb{R}^n} \phi(z,\xi) I^{(2n)}_\beta(x - z, \xi)\, dz.$$

Let $k := \lfloor s \rfloor + 1$. Then, for any $h \in \mathbb{R}^n$, by the expression of g and the Minkowski inequality, we have

$$\left\|\mathrm{D}^k_h g\right\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \mathrm{D}^k_h g_\xi(x)\, d\xi\right|^p dx\right)^{1/p} \le \int_{\mathbb{R}^n} \left\|\mathrm{D}^k_h g_\xi\right\|_{L^p(\mathbb{R}^n)} d\xi.$$

Consequently,

$$\|g\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left\|\mathrm{D}^k_h g\right\|^q_{L^p(\mathbb{R}^n)} |h|^{-n-sq} w(h)\, dh\right)^{1/q} \tag{3.19}$$

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$$\leq \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left\|\mathrm{D}_h^k g_\xi\right\|_{L^p(\mathbb{R}^n)}\,d\xi\right)^q |h|^{-n-sq}w(h)\,dh\right)^{1/q}$$
$$=\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|h|^{-\beta}\left\|\mathrm{D}_h^k g_\xi\right\|_{L^p(\mathbb{R}^n)}\,d\xi\right)^q w(h)\,dh\right)^{1/q},$$

where in the last step we used the fact $\beta = s + n/q$. Observe that the integrand $|h|^{-\beta}\|\mathrm{D}_h^k g_\xi\|_{L^p(\mathbb{R}^n)}$ can be estimated by utilizing Proposition 3.2. Indeed, upon setting

$$\begin{cases} K_1(|h|,|\xi|) := |h|^{-n}\left(\frac{|h|}{|\xi|}\right)^{n-\beta}\min\left\{1,\frac{|h|}{|\xi|}\right\}^k & \text{as } \beta\in(0,n);\\ K_2(|h|,|\xi|) := |h|^{-n}\min\left\{\log\left(e+\frac{|h|}{|\xi|}\right),\left(\frac{|h|}{|\xi|}\right)^k\right\} & \text{as } \beta = n;\\ K_3(|h|,|\xi|) := |h|^{-n}\min\left\{1,\frac{|h|}{|\xi|}\right\}^{n+k-\beta} & \text{as } \beta\in(n,2n),\end{cases}$$

we apply Proposition 3.2 and find that the last line of (3.19) can be controlled by a constant multiple of

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}K_i(|h|,|\xi|)\,\|\phi(\cdot,\xi)\|_{L^p(\mathbb{R}^n)}\,d\xi\right)^q w(h)\,dh\right)^{1/q} \quad \text{for } i\in\{1,2,3\}.$$

Thus, the proof of (3.15) falls into validating that for $i\in\{1,2,3\}$,

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}K_i(|h|,|\xi|)\,\|\phi(\cdot,\xi)\|_{L^p(\mathbb{R}^n)}\,d\xi\right)^q w(h)\,dh\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^n}\|\phi(\cdot,\xi)\|^q_{L^p(\mathbb{R}^n)}w(\xi)\,d\xi\right)^{1/q} = \|\phi\|_{L^q_w(L^p)(\mathbb{R}^{2n})}. \tag{3.20}$$

Below we prove (3.20) for $i = 1,2,3$, respectively.

- *Proof of* (3.20) *for* $i = 1$. Since K_1 is homogeneous of degree $-n$, it follows from Lemma 2.9 that (3.20) holds provided that K_1 satisfies (2.22).

  Let us show that K_1 satisfies (2.22). Note that $\beta = s+\frac{n}{q}$. So, after a change of variables $r = t^{-1}$, we find that

$$\int_0^\infty K_1(1,t)t^{\frac{n}{q'}-1}\left(\sup_{\rho\in(0,\infty)}\frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}}dt$$
$$=\int_0^\infty t^s\min\left\{1,t^{-1}\right\}^k\left(\sup_{\rho\in(0,\infty)}\frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dt}{t}$$
$$=\int_0^\infty r^{-s}\min\{1,r\}^k\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r}$$
$$=\int_0^1 r^{k-s}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r}+\int_1^\infty r^{-s}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r}.$$

  By (1.4) and (1.5), together with $\sigma < \lfloor s\rfloor + 1 - s = k - s$ and $\delta > -s$, we obtain

$$\int_0^1 r^{k-s}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r} \tag{3.21}$$

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$$\leq \left(\int_0^1 r^{\sigma q}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)\frac{dr}{r}\right)^{\frac{1}{q}}\left(\int_0^1 r^{(k-s-\sigma)q'}\,\frac{dr}{r}\right)^{\frac{1}{q'}}<\infty$$

and

$$\int_1^\infty r^{-s}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r} \tag{3.22}$$
$$\leq \left(\int_1^\infty r^{\delta q}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)\frac{dr}{r}\right)^{\frac{1}{q}}\left(\int_1^\infty r^{-(s+\delta)q'}\,\frac{dr}{r}\right)^{\frac{1}{q'}}<\infty.$$

This proves that K_1 satisfies (2.22) and, hence, (3.20) holds.

- *Proof of* (3.20) *for* $i = 2$. The proof is similar to the case $i = 1$. Indeed, observing that K_2 is homogeneous of degree $-n$, we only need to validate that K_2 satisfies (2.22), so that Lemma 2.9 can be applied to show that (3.20) holds. To this end, using $\beta = n$ and $\frac{n}{q'} = \beta - \frac{n}{q} = s$, we then write

$$\int_0^\infty K_2(1,t)t^{\frac{n}{q'}-1}\left(\sup_{\rho\in(0,\infty)}\frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}}dt$$
$$=\int_0^\infty t^s\min\left\{\log(e+t^{-1}),\ t^{-k}\right\}\left(\sup_{\rho\in(0,\infty)}\frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dt}{t}$$
$$=\int_0^\infty r^{-s}\min\left\{\log(e+r),\ r^{k}\right\}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r}$$
$$\approx\int_0^1 r^{k-s}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r}+\int_1^\infty r^{-s}\log(e+r)\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r}.$$

Clearly, (3.21) remains valid. Instead of (3.22), we now also have

$$\int_1^\infty r^{-s}\log(e+r)\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}}\frac{dr}{r}$$
$$\leq \left(\int_1^\infty r^{\delta q}\left(\sup_{\rho\in(0,\infty)}\frac{w(r\rho)}{w(\rho)}\right)\frac{dr}{r}\right)^{\frac{1}{q}}\left(\int_1^\infty r^{-(s+\delta)q'}\log(e+r)\,\frac{dr}{r}\right)^{\frac{1}{q'}}<\infty,$$

by terms of (1.5) and the fact $\delta > -s$. Thus, K_2 satisfies (2.22). We obtain (3.20).

- *Proof of* (3.20) *for* $i = 3$. In this case, write the left hand side of (3.20) as

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|h|^n}\int_{|\xi|\leq|h|}\|\phi(\cdot,\xi)\|_{L^p(\mathbb{R}^n)}\,d\xi\right)^q w(h)\,dh\right)^{1/q}$$
$$+\left(\int_{\mathbb{R}^n}\left(\int_{|\xi|>|h|}|h|^{-n}\left(\frac{|h|}{|\xi|}\right)^{n+k-\beta}\|\phi(\cdot,\xi)\|_{L^p(\mathbb{R}^n)}\,d\xi\right)^q w(h)\,dh\right)^{1/q}=:\mathrm{I}+\mathrm{II}.$$

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Let $\Phi(\xi) := \|\phi(\cdot,\xi)\|_{L^p(\mathbb{R}^n)}$ and $\mathcal{M}$ be the classical Hardy-Littlewood maximal function on $\mathbb{R}^n$, that is,

$$\mathcal{M}\Phi(z) = \sup_{B\ni z} \frac{1}{|B|} \int_B |\Phi(x)|\,dx \quad \text{for all } z \in \mathbb{R}^n.$$

Then, using $q \in (1,\infty)$, $w \in A_q(\mathbb{R}^n)$ and the boundedness of $\mathcal{M}$ on $L^q_w(\mathbb{R}^n)$, we derive

$$\mathrm{I} \lesssim \left(\int_{\mathbb{R}^n} [\mathcal{M}\Phi(h)]^q w(h)\,dh\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^n} [\Phi(h)]^q w(h)\,dh\right)^{1/q} = \|\phi\|_{L^q_w(L^p)(\mathbb{R}^{2n})}.$$

For II, upon setting

$$K_3'(|h|,|\xi|) := |h|^{-n} \min\left\{1, \frac{|h|}{|\xi|}\right\}^{n+k-\beta} \mathbf{1}_{|\xi|>|h|}$$

and using $\beta = s + \frac{n}{q}$, we find that

$$\begin{aligned}
&\int_1^\infty K_3'(1,t)t^{\frac{n}{q'}-1} \left(\sup_{\rho\in(0,\infty)} \frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}} dt \\
&= \int_1^\infty t^{n-\beta+s} t^{-(n+k-\beta)} \left(\sup_{\rho\in(0,\infty)} \frac{w(t^{-1}\rho)}{w(\rho)}\right)^{\frac{1}{q}} \frac{dt}{t} \\
&= \int_0^1 r^{k-s} \left(\sup_{\rho\in(0,\infty)} \frac{w(r\rho)}{w(\rho)}\right)^{\frac{1}{q}} \frac{dr}{r},
\end{aligned}$$

which is finite by terms of (3.21). From this, the fact that K_3' is homogeneous of degree $-n$ and Lemma 2.9, it follows that

$$\mathrm{II} \lesssim \left(\int_{\mathbb{R}^n} \|\phi(\cdot,\xi)\|^q_{L^p(\mathbb{R}^n)} w(\xi)\,d\xi\right)^{1/q} = \|\phi\|_{L^q_w(L^p)(\mathbb{R}^{2n})}.$$

Combining the estimates of I and II shows that (3.20) holds for $i = 3$.

Summarizing all, we conclude the proof of Theorem 1.6(ii) under $\beta \in (0, 2n)$. □

3.3. **The case $\beta \in [2n,\infty)$.** In order to prove Theorem 1.6 under $\beta \in [2n,\infty)$, we need the following two lemmas.

Lemma 3.3. *Let $p \in [1,\infty)$, $q \in (1,\infty)$, $\beta \in \mathbb{R}$ and w be a weight on $\mathbb{R}^n$. Then, for any $\gamma \in \mathbb{R}$, $f \in \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$ if and only if $(-\Delta_{\mathbb{R}^{2n}})^{\gamma/2} f \in \dot{\mathcal{L}}^{p,q,w}_{\beta-\gamma}(\mathbb{R}^{2n})$. Moreover, for any $f \in \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$,*

$$\|f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})} = \|(-\Delta_{\mathbb{R}^{2n}})^{\gamma/2} f\|_{\dot{\mathcal{L}}^{p,q,w}_{\beta-\gamma}(\mathbb{R}^{2n})}.$$

Proof. For any $f \in \dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$, by Definition 1.4, there exists a function $\phi \in L^q_w(L^p)(\mathbb{R}^{2n})$ such that $f = (-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi$. From this and

$$(-\Delta_{\mathbb{R}^{2n}})^{\gamma/2} f = (-\Delta_{\mathbb{R}^{2n}})^{\gamma/2} (-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} \phi = (-\Delta_{\mathbb{R}^{2n}})^{-(\beta-\gamma)/2}\phi,$$

we derive

$$(-\Delta_{\mathbb{R}^{2n}})^{\gamma/2} f \in \dot{\mathcal{L}}^{p,q,w}_{\beta-\gamma}(\mathbb{R}^{2n})$$

and

$$\|(-\Delta_{\mathbb{R}^{2n}})^{\gamma/2} f\|_{\dot{\mathcal{L}}^{p,q,w}_{\beta-\gamma}(\mathbb{R}^{2n})} = \|\phi\|_{L^q_w(L^p)(\mathbb{R}^{2n})} = \|f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})},$$

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as desired. □

Lemma 3.4. *Let* $p, q \in [1,\infty)$*,* $s \in (0,\infty)$*,* $\gamma \in (0,s)$ *and* w *be a radial weight satisfying* (1.4) *and* (1.5)*. Suppose that* $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$ *such that* $\operatorname{supp}\hat{\varphi} \subset \{x \in \mathbb{R}^n : c^{-1} \le |x| \le c\}$ *for some constant* $c \in (1,\infty)$*. Then,* $g \in \dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)$ *if and only if* $(-\Delta_{\mathbb{R}^n})^{\gamma/2} g \in \dot{\Lambda}^{s-\gamma}_{p,q,w,\varphi}(\mathbb{R}^n)$*. Moreover, for all* $g \in \dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)$*, it holds that*

$$\|g\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)} \approx \|(-\Delta_{\mathbb{R}^n})^{\gamma/2} g\|_{\dot{\Lambda}^{s-\gamma}_{p,q,w,\varphi}(\mathbb{R}^n)}$$

with implicit constants independent of g*.*

Proof. Associated to the function φ, define a function ψ via

$$\hat{\psi}(\xi) = |2\pi\xi|^\gamma \hat{\varphi}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n.$$

Clearly, $\psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ and $\operatorname{supp}\hat{\psi} \subset \{x \in \mathbb{R}^n : c^{-1} \le |x| \le c\}$. Note that the Fourier transform gives

$$\left(\varphi_t * \left((-\Delta_{\mathbb{R}^n})^{\gamma/2} g\right)\right)^\wedge(\xi) = \widehat{\varphi_t}(\xi)\,(2\pi|\xi|)^\gamma\,\hat{g}(\xi) = t^{-\gamma}\,\widehat{\psi_t}(\xi)\,\hat{g}(\xi) = t^{-\gamma}(\psi_t * g)^\wedge(\xi),$$

which implies

$$\varphi_t * \left((-\Delta_{\mathbb{R}^n})^{\gamma/2} g\right) = t^{-\gamma}(\psi_t * g).$$

By this and Lemma 2.8, we obtain

$$\begin{aligned}\|(-\Delta_{\mathbb{R}^n})^{\gamma/2} g\|_{\dot{\Lambda}^{s-\gamma}_{p,q,w,\varphi}(\mathbb{R}^n)} &= \left(\int_0^\infty \left\|\varphi_t * \left((-\Delta_{\mathbb{R}^n})^{\gamma/2} g\right)\right\|^q_{L^p(\mathbb{R}^n)} t^{-(s-\gamma)q-1} w(t)\,dt\right)^{1/q}\\ &= \left(\int_0^\infty \|\psi_t * g\|^q_{L^p(\mathbb{R}^n)} t^{-sq-1} w(t)\,dt\right)^{1/q} = \|g\|_{\dot{\Lambda}^s_{p,q,w,\psi}(\mathbb{R}^n)} \approx \|g\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)}.\end{aligned}$$

This ends the proof. □

Now, we prove Theorem 1.6 under $\beta \in [2n,\infty)$.

Proof of Theorem 1.6 under $\beta \in [2n,\infty)$. If $\beta \ge 2n$, then there exists a unique $k \in \mathbb{N}$ such that $2n + k - 1 \le \beta < 2n + k$, which implies $\beta - k \in [2n-1, 2n)$. Moreover, the facts $\beta - k \ge 2n - 1$ and $\beta = s + n/q$ imply that $s > k$.

First, we show (i). Suppose that $g \in \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$. By Definiton 1.1, we may as well assume that $g \in C_c^\infty(\mathbb{R}^n)$. From Lemma 3.4 and Theorem 2.10(ii), it follows that

$$\|(-\Delta_{\mathbb{R}^n})^{k/2} g\|_{\dot{\Lambda}^{s-k}_{p,q,w,\varphi}(\mathbb{R}^n)} \approx \|g\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)}, \tag{3.23}$$

where φ is as in Lemma 3.4.

If k is even, say $k = 2m$ for some $m \in \mathbb{N}$, then we have

$$(-\Delta_{\mathbb{R}^n})^{k/2} g = (-\Delta_{\mathbb{R}^n})^m g \in C_c^\infty(\mathbb{R}^n),$$

so that Theorem 2.10(i) and Remark 2.11 imply

$$\|(-\Delta_{\mathbb{R}^n})^{k/2} g\|_{\dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)} \lesssim \|(-\Delta_{\mathbb{R}^n})^{k/2} g\|_{\dot{\Lambda}^{s-k}_{p,q,w,\varphi}(\mathbb{R}^n)} \tag{3.24}$$

and, hence, $(-\Delta_{\mathbb{R}^n})^{k/2} g \in \dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)$.

If k is odd, then we write $k = 2m + 1$ and, hence,

$$(-\Delta_{\mathbb{R}^n})^{k/2} g = (-\Delta_{\mathbb{R}^n})^{1/2}\left((-\Delta_{\mathbb{R}^n})^m g\right) \in (-\Delta_{\mathbb{R}^n})^{1/2}\left(C_c^\infty(\mathbb{R}^n)\right).$$

For any $f \in \mathcal{S}(\mathbb{R}^n)$, it is known that (see [40, Section 2]) the function $(-\Delta_{\mathbb{R}^n})^{1/2} f$ satisfies

$$|(-\Delta_{\mathbb{R}^n})^{1/2} f(x)| \lesssim (1+|x|)^{-(n+1)}$$

uniformly in $x \in \mathbb{R}^n$. This in turn gives that $(-\Delta_{\mathbb{R}^n})^{k/2} g \in L^p(\mathbb{R}^n)$, which, along with Remark 2.11 and Theorem 2.10(i), again implies

$$\|(-\Delta_{\mathbb{R}^n})^{k/2} g\|_{\dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)} \lesssim \|(-\Delta_{\mathbb{R}^n})^{k/2} g\|_{\dot{\Lambda}^{s-k}_{p,q,w,\varphi}(\mathbb{R}^n)}. \tag{3.25}$$

From this and Lemma 2.2, we deduce that $(-\Delta_{\mathbb{R}^n})^{k/2} g$ can be approximated by $C_c^\infty(\mathbb{R}^n)$-functions under the semi-norm $\|\cdot\|_{\dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)}$, which induces $(-\Delta_{\mathbb{R}^n})^{k/2} g \in \dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)$.

Moreover, since $\beta - k \in (0, 2n)$, it follows from the already proved argument of Theorem 1.6 in Section 3.1 that

$$\mathcal{E}\left((-\Delta_{\mathbb{R}^n})^{k/2} g\right) \in \dot{\mathcal{L}}^{p,q,w}_{\beta-k}(\mathbb{R}^{2n}),$$

where $\mathcal{E}$ is the extension operator defined in Section 3.1. Thus, it makes sense to define the extension operator $\widetilde{\mathcal{E}}$ by

$$\widetilde{\mathcal{E}} g := (-\Delta_{\mathbb{R}^{2n}})^{-k/2} \mathcal{E}\left((-\Delta_{\mathbb{R}^n})^{k/2} g\right).$$

By this, together with Lemma 3.3 and (3.23)-(3.24)-(3.25), we derive $\widetilde{\mathcal{E}} g \in \dot{\mathcal{L}}^{p,q,w}_{\beta}(\mathbb{R}^{2n})$ with

$$\begin{aligned} \|\widetilde{\mathcal{E}} g\|_{\dot{\mathcal{L}}^{p,q,w}_{\beta}(\mathbb{R}^{2n})} &= \left\|(-\Delta_{\mathbb{R}^{2n}})^{-k/2} \mathcal{E}\left((-\Delta_{\mathbb{R}^n})^{k/2} g\right)\right\|_{\dot{\mathcal{L}}^{p,q,w}_{\beta}(\mathbb{R}^{2n})} \\ &= \left\|\mathcal{E}\left((-\Delta_{\mathbb{R}^n})^{k/2} g\right)\right\|_{\dot{\mathcal{L}}^{p,q,w}_{\beta-k}(\mathbb{R}^{2n})} = \left\|(-\Delta_{\mathbb{R}^n})^{k/2} g\right\|_{\dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\Lambda}^{s}_{p,q,w}(\mathbb{R}^n)}. \end{aligned}$$

This proves that (i) holds for $\beta \in [2n, \infty)$.

Now, we show (ii). Let $f \in \dot{\mathcal{L}}^{p,q,w}_{\beta}(\mathbb{R}^{2n})$. Then $f = (-\Delta_{\mathbb{R}^{2n}})^{-\beta/2} \phi$ for some $\phi \in L^q_w(L^p)(\mathbb{R}^{2n})$. Moreover, it follows from Lemma 3.3 that

$$(-\Delta_{\mathbb{R}^{2n}})^{-(\beta-k)/2} \phi = (-\Delta_{\mathbb{R}^{2n}})^{k/2} f \in \dot{\mathcal{L}}^{p,q,w}_{\beta-k}(\mathbb{R}^{2n}).$$

Since $\beta - k \in (0, 2n)$, by the already proved argument of Theorem 1.6(ii) in Section 3.2, we obtain

$$\mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right) = \mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{-(\beta-k)/2} \phi\right) \in \dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)$$

and

$$\begin{aligned} \left\|\mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right)\right\|_{\dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)} &= \left\|\mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{-(\beta-k)/2} \phi\right)\right\|_{\dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)} \\ &\lesssim \left\|(-\Delta_{\mathbb{R}^{2n}})^{-(\beta-k)/2} \phi\right\|_{\dot{\mathcal{L}}^{p,q,w}_{\beta-k}(\mathbb{R}^{2n})} \\ &= \left\|(-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right\|_{\dot{\mathcal{L}}^{p,q,w}_{\beta-k}(\mathbb{R}^{2n})} \approx \|f\|_{\dot{\mathcal{L}}^{p,q,w}_{\beta}(\mathbb{R}^{2n})}. \end{aligned} \tag{3.26}$$

Further, from Lemma 3.4 and Theorem 2.10(ii), we derive

$$\begin{aligned} \left\|(-\Delta_{\mathbb{R}^n})^{-k/2} \mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right)\right\|_{\dot{\Lambda}^{s}_{p,q,w,\varphi}(\mathbb{R}^n)} &\lesssim \left\|\mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right)\right\|_{\dot{\Lambda}^{s-k}_{p,q,w,\varphi}(\mathbb{R}^n)} \\ &\lesssim \left\|\mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right)\right\|_{\dot{\Lambda}^{s-k}_{p,q,w}(\mathbb{R}^n)} \end{aligned} \tag{3.27}$$

and, hence,

$$(-\Delta_{\mathbb{R}^n})^{-k/2} \mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right) \in \dot{\Lambda}^{s}_{p,q,w,\varphi}(\mathbb{R}^n).$$

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Next, by Theorem 2.10(i), there exists a Lebesgue measurable function $g_0 \in \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$ such that

$$g_0 = (-\Delta_{\mathbb{R}^n})^{-k/2}\mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right) \text{ in } \mathcal{S}'_\infty(\mathbb{R}^n)$$

and

$$\|g_0\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \lesssim \left\|(-\Delta_{\mathbb{R}^n})^{-k/2}\mathcal{R}\left((-\Delta_{\mathbb{R}^{2n}})^{k/2} f\right)\right\|_{\dot{\Lambda}^s_{p,q,w,\varphi}(\mathbb{R}^n)}. \tag{3.28}$$

Thus, when $\beta \in [2n+k-1, 2n+k)$, it makes sense to define the restriction operator $\widetilde{\mathcal{R}}$ on $\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})$ by

$$\widetilde{\mathcal{R}} f := g_0.$$

By (3.26)-(3.27)-(3.28), we know that $\widetilde{\mathcal{R}} f \in \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$ and

$$\|\widetilde{\mathcal{R}} f\|_{\dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{\mathcal{L}}^{p,q,w}_\beta(\mathbb{R}^{2n})}.$$

Furthermore, by the fact that $\mathcal{R}\mathcal{E} = id$ under $\beta \in (0, 2n)$, we deduce that, for any $g \in \dot{\Lambda}^s_{p,q,w}(\mathbb{R}^n)$ with $2n+k-1 \le \beta < 2n+k$,

$$\widetilde{\mathcal{R}}\widetilde{\mathcal{E}} g = \widetilde{\mathcal{R}}\left((-\Delta_{\mathbb{R}^{2n}})^{-k/2}\mathcal{E}\left((-\Delta_{\mathbb{R}^n})^{k/2} g\right)\right) = (-\Delta_{\mathbb{R}^n})^{-k/2}\,\mathcal{R}\,\mathcal{E}\,(-\Delta_{\mathbb{R}^n})^{k/2} g = g,$$

where all the equalities hold in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Altogether, we conclude the proof of Theorem 1.6 under $\beta \in [2n, \infty)$. □

4. An example: logarithmic Besov space

In this section, we show that logarithmic Besov spaces from [30, 31, 19, 20, 15] (see also [6, 7, 21, 36]) fall into the scope of weighted Besov spaces in the present article and, hence, the results in Theorem 1.6 are valid for these spaces.

Definition 4.1. Let $p,\ q \in [1, \infty)$, $s \in (0, \infty)$ and $b \in \mathbb{R}$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, set

$$\|f\|_{\dot{\Lambda}^{(s,b)}_{p,q}(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} \left(\log \frac{e}{|h| \wedge 1}\right)^{bq} \left\|\mathrm{D}^{\lfloor s\rfloor+1}_h f\right\|^q_{L^p(\mathbb{R}^n)} \frac{dh}{|h|^{n+sq}}\right]^{1/q}.$$

The logarithmic Besov space $\dot{\Lambda}^{(s,b)}_{p,q}(\mathbb{R}^n)$ is defined to be the completion of $C^\infty_c(\mathbb{R}^n)$ under the semi-norm $\|\cdot\|_{\dot{\Lambda}^{(s,b)}_{p,q}(\mathbb{R}^n)}$.

Lemma 4.2. *Suppose that $q \in [1, \infty)$ and $b \in \mathbb{R}$. For any $y \in \mathbb{R}^n$, let*

$$w(y) := \left(\log \frac{e}{|y| \wedge 1}\right)^{bq}.$$

Then, $w \in A_r(\mathbb{R}^n)$ whenever $r \in (1, \infty)$ and, moreover, w satisfies (1.4) *and* (1.5).

Proof. For any $t, \rho \in (0, \infty)$, observe that

$$\frac{\log \frac{e}{(t\rho)\wedge 1}}{\log \frac{e}{\rho \wedge 1}} = \frac{\log \frac{e}{\rho\wedge 1} + \log \frac{\rho\wedge 1}{(t\rho)\wedge 1}}{\log \frac{e}{\rho\wedge 1}} \le 1 + \left|\log \frac{\rho \wedge 1}{(t\rho)\wedge 1}\right| \le 1 + |\log t|.$$

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Since w is a radial function, we write $w(y)$ as $w(t)$ whenever $|y| = t$. Given any $\sigma > 0$, we have

$$\int_0^1 t^{\sigma q} \left(\sup_{\rho \in (0,\infty)} \frac{w(t\rho)}{w(\rho)} \right) \frac{dt}{t} \lesssim \int_0^1 t^{\sigma q - 1} \left(1 + \log \frac{1}{t} \right)^{|b|q} dt < \infty,$$

which shows that w satisfies (1.4). Moreover, given any $\delta < 0$, we also have

$$\int_1^\infty t^{\delta q} \left(\sup_{\rho \in (0,\infty)} \frac{w(t\rho)}{w(\rho)} \right) \frac{dt}{t} \lesssim \int_1^\infty t^{\delta q - 1} (1 + \log t)^{|b|q} \, dt < \infty,$$

thereby leading to (1.5).

It remains to validate that $w \in A_r(\mathbb{R}^n)$, where $r \in (1, \infty)$. To this end, for any ball $B \subset \mathbb{R}^n$, we need to show that

$$\left(\frac{1}{|B|} \int_B \left(\log \frac{e}{|x| \wedge 1} \right)^{bq} dx \right) \left(\frac{1}{|B|} \int_B \left(\log \frac{e}{|x| \wedge 1} \right)^{-\frac{bq}{r-1}} dx \right)^{r-1} \leq C \tag{4.1}$$

holds for some positive constant C independent of B.

Suppose that $B = B(x_0, R)$ for some $x_0 \in \mathbb{R}^n$ and $R \in (0, \infty)$. Then, we show (4.1) by considering the following two cases:

- *Case 1:* $|x_0| \geq 2R$. In this case, for any $x \in B$, we have

$$|x| \geq |x_0| - |x_0 - x| > |x_0| - R \geq |x_0|/2$$

and

$$|x| \leq |x_0| + |x_0 - x| < |x_0| + R < 2|x_0|,$$

which in turn gives

$$\log \frac{e}{|x| \wedge 1} \approx \log \frac{e}{|x_0| \wedge 1}$$

and, hence, (4.1) holds.

- *Case 2:* $|x_0| < 2R$. For any $x \in B$, we now have

$$|x| \leq |x_0| + |x_0 - x| < |x_0| + R < 3R,$$

so that $B \subset B(0, 3R)$. Consequently,

$$\begin{aligned}
\frac{1}{|B|} \int_B \left(\log \frac{e}{|x| \wedge 1} \right)^{bq} dx &\leq \frac{1}{|B|} \int_{B(0,3R)} \left(\log \frac{e}{|x| \wedge 1} \right)^{bq} dx \\
&= \frac{n}{R^n} \int_0^{3R} \left(\log \frac{e}{\rho \wedge 1} \right)^{bq} \rho^{n-1} \, d\rho \\
&\approx \left(\log \frac{e}{(3R) \wedge 1} \right)^{bq},
\end{aligned}$$

where the last step follows from a direct calculation (see also [18, Proposition 3.4.33] and [15, Lemma 2.4]). In a similar manner,

$$\begin{aligned}
\frac{1}{|B|} \int_B \left(\log \frac{e}{|x| \wedge 1} \right)^{-\frac{bq}{r-1}} dx &\leq \frac{1}{|B|} \int_{B(0,3R)} \left(\log \frac{e}{|x| \wedge 1} \right)^{-\frac{bq}{r-1}} dx \\
&= \frac{n}{R^n} \int_0^{3R} \left(\log \frac{e}{\rho \wedge 1} \right)^{-\frac{bq}{r-1}} \rho^{n-1} \, d\rho
\end{aligned}$$

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$$\approx \left(\log \frac{e}{(3R)\wedge 1}\right)^{-\frac{bq}{r-1}}.$$

Combining the last two formulae yields (4.1).

Summarizing the arguments in the above two cases, we arrived at the conclusion that $w \in A_r(\mathbb{R}^n)$ for all $r \in (1,\infty)$. □

For $p \in [1,\infty)$, $q \in (1,\infty)$ and $b \in \mathbb{R}$, we denote by $L^{p,(q,b)}(\mathbb{R}^{2n})$ the collection of all measurable functions f on $\mathbb{R}^{2n}$ satisfying

$$\|f\|_{L^{p,(q,b)}(\mathbb{R}^{2n})} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x,y)|^p\,dx\right)^{q/p}\left(\log\frac{e}{|y|\wedge 1}\right)^{bq}dy\right)^{1/q} < \infty.$$

The corresponding mixed-Riesz potential space is defined by

$$\dot{\mathcal{L}}^{p,q,b}_{\beta}(\mathbb{R}^{2n}) := \left\{f:\ f=(-\Delta_{\mathbb{R}^{2n}})^{-\beta/2}\phi \ \text{ with } \ \phi\in L^{p,(q,b)}(\mathbb{R}^{2n})\right\}.$$

Then, an application of Lemma 4.2 and Theorem 1.6 yields the following result.

Theorem 4.3. *Let $p \in [1,\infty)$, $q \in (1,\infty)$, $s \in (0,\infty)$, $\beta := s + n/q \in (0,\infty)$ and $b \in \mathbb{R}$. Then, the following hold:*

(i) *there exists a bounded linear extension operator*

$$\mathcal{E}:\ \dot{\Lambda}^{(s,b)}_{p,q}(\mathbb{R}^n) \to \dot{\mathcal{L}}^{p,q,b}_{\beta}(\mathbb{R}^{2n}),$$

such that, for any $g \in \dot{\Lambda}^{(s,b)}_{p,q}(\mathbb{R}^n)$,

$$\|\mathcal{E}g\|_{\dot{\mathcal{L}}^{p,q,b}_{\beta}(\mathbb{R}^{2n})} \le C_1\|g\|_{\dot{\Lambda}^{(s,b)}_{p,q}(\mathbb{R}^n)},$$

where C_1 is a positive constant independent of g;

(ii) *there exists a bounded linear restriction operator*

$$\mathcal{R}:\ \dot{\mathcal{L}}^{p,q,b}_{\beta}(\mathbb{R}^{2n}) \to \dot{\Lambda}^{(s,b)}_{p,q}(\mathbb{R}^n),$$

such that, for any $f \in \dot{\mathcal{L}}^{p,q,b}_{\beta}(\mathbb{R}^{2n})$,

$$\|\mathcal{R}f\|_{\dot{\Lambda}^{(s,b)}_{p,q}(\mathbb{R}^n)} \le C_2\|f\|_{\dot{\mathcal{L}}^{p,q,b}_{\beta}(\mathbb{R}^{2n})},$$

where C_2 is a positive constant independent of f. Moreover, $\mathcal{R}\mathcal{E} = id$.

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