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A SEQUENTIAL STOPPING PROBLEM WITH COSTLY REVERSIBILITY

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Abstract

We study sequential optimal stopping with partial reversibility. The optimal stopping problem is subject to implementation delay, which is random and exponentially distributed. Once the stopping decision is made, the decision maker can, by incurring a cost, call the decision off and restart the stopping problem. The optimization criterion is to maximize the expected present value of the total payoff. We characterize the value function in terms of a Bellman principle for a wide class of payoff functions and potentially multidimensional strong Markov dynamics. We also analyse the case of linear diffusion dynamics and characterize the value function and the optimal decision rule for a wide class of payoff functions.

Keywords: Optimal stopping; Bellman principle; strong Markov process; diffusion process; exponential distribution; resolvent operator

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Secondary 49K45; 60J60; 93E20

1. Introduction

The purpose of this paper is to study a sequential stopping problem where the stopping decisions are partly reversible. The stopping payoffs are contingent on a stochastic process X. Upon the first stopping decision, the decision maker has to pay a fixed cost K_1. In return, an exponentially distributed time variable, independent of X, is initiated such that the payoff is realized when this time has elapsed. However, during the running time of this time variable, the decision maker can make another stopping decision which stops the time variable from running and restarts the stopping problem. In return, the decision maker receives a fixed sum $K_2 < K_1$. If, on the other hand, the time variable runs until the end, the decision maker gets the payoff g evaluated at the value of X at the time. The objective is then to maximize the expected present value of the total payoff.

The first key aspect of our research problem is costly reversibility, a topic addressed by a number of research papers. One of the early papers is that of Abel and Eberley [1], who studied capacity expansion of a firm under price uncertainty, fixed capital costs, and Cobb–Douglas

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production function. The firm has the option to expand production capacity by capital acquisition. Moreover, when the market deteriorates, the firm has the option to reduce the capacity by selling the capital. Two related papers by Alvarez [4] and Hartman and Hendricksson [15] reconsider the problem of [1] in a more general setting. In [4] more general price dynamics and production functions are considered, whereas in [15] the capital costs are subject to stochastic fluctuations. Adkins and Paxson [2] studied a problem where the investment can be abandoned before the investment event. Similarly to [15], Adkins and Paxson considered a model with three stochastic factors for project present value, investment cost, and abandonment value. In this vein, we also mention Shibata and Wong [27], who considered abandonment options with endogenously determined reversibility costs. On the more mathematical end of the spectrum, we refer to Federico and Pham [13] and Løkka and Zervos [20] for more general analyses of, respectively, singular stochastic control, and stochastic impulse control models in this context.

Another key aspect of our model is the implementation delay (or time-to-build), which has also been extensively studied over recent decades. Aïd et al. [3] considered a singular stochastic control model for capacity expansion with time-to-build, whereas Ø ksendal and Sulem [23] were concerned with stochastic impulse control with implementation delay. Alvarez and Keppo [5] studied a model where the time-to-build depends on the value of the state variable at the time of the investment. Armerin and Song [6] were concerned with the case where the cash flows, resulting from the investment subject to time-to-build, are distributed over time. Liang and Yang [19] studied the optimal exercise boundary of an American put option with fixed delivery lag. Chen and Song [8] considered a delayed optimal stopping model, similar to [21], for investment timing when part of the investment is paid at the time of the investment and the rest at the time of completion. Delayed optimal stopping subject ambiguity (Knightian uncertainty) is the topic of the paper by Delaney [11]. The so-called Parisian implementation delay was considered by Costeniuc et al. [10]; here, the option to invest is not exercised immediately at the boundary of a favourable region but rather when the state process has remained constantly in a favourable region for a sufficiently long period. Haejun [14] and Lempa [17, 18] are concerned with optimal stopping when the implementation delay is stochastic and exogenous. A general approximation approach for optimal stopping with random exercise delay has been developed by Chen and Song [9].

The model of our study can be seen as an extension of the model studied in [17]. In this paper, the exercise payoff is subject to an exponential delay, independent of X. As was mentioned above, we extend the model of [17] by introducing the costly reversibility to the stopping problem. Thus the admissible decision rules in the problem are sequential. Further contributions of our study are twofold. First, we characterize the value function by means of a Bellman principle for a wide class of payoff functions and time-homogeneous strong Markov dynamics; the dynamics are allowed to be multidimensional. Moreover, we analyse the case of linear diffusion dynamics and characterize the value function and the optimal decision rule for a wide class of payoff functions.

The remainder of the paper is organized as follows. In Section 2 we set up the model for the investment timing problem. The problem is then solved in Section 3. The main results are illustrated with explicit examples in Section 4.

2. The problem

2.1. The dynamics

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$; see [7, p. 2]. We assume that the underlying X is a strong Markov process

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defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and taking values in $E \subseteq \mathbb{R}^d$ for some $d \geq 1$ with the initial state $x \in E$. We further assume that $\mathbb{F}$ is generated by X. We take $E = (a_1, b_1) \times \cdots \times (a_d, b_d)$, where $-\infty \leq a_i < b_i \leq \infty$ for all $i = 1, \ldots, d$. As usual, we augment the state space E with a topologically isolated element Δ if the process X is non-conservative. Then the process X can be made conservative on the augmented state space $E^\Delta := E \cup \{\Delta\}$; see [25]. In what follows, we drop the superscript Δ from the notation. By convention, we augment the definition of functions g on E with $g(\Delta) = 0$. Define the life-time of the process X as $\zeta = \inf\{t \geq 0 \colon X_t = \Delta\}$.

Let $\mathbb{P}_x$ denote the probability measure $\mathbb{P}$ conditioned on the initial state x and let $\mathbb{E}_x$ denote the expectation with respect to $\mathbb{P}_x$. The process X is assumed to evolve under $\mathbb{P}_x$ and the sample paths are assumed to be right-continuous and left-continuous over stopping times, meaning the following: if the sequence of stopping times $\tau_n \uparrow \tau$, then $X_{\tau_n} \to X_\tau$ $\mathbb{P}_x$-almost surely as $n \to \infty$. There is a well-established theory of optimal stopping for this class of processes; see [24].

For $r > 0$, we let L_1^r denote the class of real-valued measurable functions f on E satisfying the integrability condition

$$\mathbb{E}_x\left[\int_0^\zeta \mathrm{e}^{-rt} |f(X_t)| \, \mathrm{d}t\right] < \infty \quad \text{for all } x \in E.$$

For a function $f \in L_1^r$, the *resolvent* $R_r f \colon E \to \mathbb{R}$ is defined as

$$(R_r f)(x) = \mathbb{E}_x\left[\int_0^\zeta \mathrm{e}^{-rs} f(X_s) \, \mathrm{d}s\right]$$

for all $x \in \mathbb{R}_+$. It is well known that the family $(R_\lambda)_{\lambda \geq 0}$ is a strongly continuous contraction resolvent and that it has the following connection to exponentially distributed random times: if $U \sim \mathrm{Exp}(\lambda)$ and independent of X, then $\lambda(R_{r+\lambda} g)(x) = \mathbb{E}_x[\mathrm{e}^{-rU} g(X_U)]$ whenever $g \in L_1^r$; see [25]. Finally, the function h is said to be r-harmonic for X if $h(x) = \mathbb{E}_x[\mathrm{e}^{-r\tau} h(X_\tau)]$ for all $\mathbb{F}$-stopping times τ.

2.2. The timing problem

We define the timing problem inductively and start by considering the case where the stopping decision cannot be reversed. Let $U \sim \mathrm{Exp}(\lambda)$ be independent of X, and define

$$\begin{aligned} V_a^0(x) &= \mathbb{E}_x[\mathrm{e}^{-rU} g(X_U)] = \lambda(R_{r+\lambda} g)(x), \\ V_i^0(x) &= \sup\nolimits_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}(\lambda(R_{r+\lambda} g)(X_\tau) - K_1)]. \end{aligned} \tag{2.1}$$

Here the function g is the payoff function; we assume that this function satisfies the following.

(A1) The payoff $g \colon E \to \mathbb{R}$ is in L_r^1, is lower-bounded, satisfies the condition $S^+ := \{x \colon g(x) > 0\} \neq \emptyset$, and the process X reaches the set S^+ with positive probability for all initial states x,

(A2) There exists an r-harmonic function $h \colon E \to \mathbb{R}_+$ such that the function $x \mapsto g(x)/h(x)$ is bounded.

In (2.1), the function V_a^0 is the value of an active investment (i.e. the investment is initiated) when there is no possibility of calling the investment off. Note that the value function V_i^0 is (essentially) that of [17]. This corresponds to the case where the investment opportunity is inactive (i.e. the investment is not initiated), the investment decision cannot be called off, and initiation of the investment incurs a cost of K_1.

Proceeding inductively, for the case where a stopping decision can be reversed k times, we write

$$\begin{aligned} V_a^k(x) &= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-rU} g(X_U)\mathbf{1}(U<\tau) + \mathrm{e}^{-r\tau}(V_i^{k-1}(X_\tau)+K_2)\mathbf{1}(U>\tau)], \\ V_i^k(x) &= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}(V_a^k(X_\tau)-K_1)]. \end{aligned} \tag{2.2}$$

An alternative expression for the function V_a above can be found. To find it, we first note that the following holds for any measurable function f:

$$\begin{aligned} \mathbb{E}_x[\mathrm{e}^{-r\tau} f(X_\tau)\mathbf{1}(U>\tau)] &= \mathbb{E}_x[\mathbb{E}_x[\mathrm{e}^{-r\tau} f(X_\tau)\mathbf{1}(U>\tau)|\mathcal{F}_\tau]] \\ &= \mathbb{E}_x[\mathrm{e}^{-r\tau} f(X_\tau)\mathbb{E}_x[\mathbf{1}(U>\tau)|\mathcal{F}_\tau]] \\ &= \mathbb{E}_x[\mathrm{e}^{-r\tau} f(X_\tau)\mathrm{e}^{-\lambda\tau}] \\ &= \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\tau} f(X_\tau)]. \end{aligned}$$

Hence

$$\begin{aligned} V_a^k(x) &= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-rU} g(X_U)\mathbf{1}(U<\tau) + \mathrm{e}^{-r\tau}(V_i^{k-1}(X_\tau)+K_2)\mathbf{1}(U>\tau)] \\ &= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-rU} g(X_U)) + \mathrm{e}^{-r\tau}(V_i^{k-1}(X_\tau) - \mathrm{e}^{-rU} g(X_U)) + K_2)\mathbf{1}(U>\tau)] \\ &= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-rU} g(X_U)) + \mathrm{e}^{-r\tau}(V_i^{k-1}(X_\tau) - \lambda(R_{r+\lambda} g)(X_\tau) + K_2)\mathbf{1}(U>\tau)] \\ &= \lambda(R_{r+\lambda} g)(x) + \sup_\tau \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\tau}(V_i^{k-1}(X_\tau) - \lambda(R_{r+\lambda} g)(X_\tau) + K_2)]. \end{aligned}$$

Our main problem can then be written as the limiting case

$$V_a(x) = V_a^\infty(x) = \lim_{k\to\infty} V_a^k(x), \qquad V_i(x) = V_i^\infty(x) = \lim_{k\to\infty} V_i^k(x). \tag{2.3}$$

The corresponding decision variables are then increasing sequences of stopping times denoted as $\bar{\tau} = (\tau_n)_{n\geq 1}$. The following proposition provides us with sufficient conditions for the main problem (2.3) to be well-defined.

Proposition 2.1. *The problem* (2.3) *is well-defined, i.e. the limiting functions* $\lim_{k\to\infty} V_a^k$ *and* $\lim_{k\to\infty} V_i^k$ *exist.*

Proof. We prove the result for V_i; the function V_a is handled similarly. For the purpose of the argument, we write the value function $V_i^k = V_i^k(\cdot, K_1, K_2)$; here, K_i are the cost parameters in the definition of the problem. Then it is straightforward to show by induction that $V_i^k(\cdot, K_1, K_2) \leq V_i^k(\cdot, 0, 0)$.

Our task is to show that the function $V_i^k(\cdot, 0, 0)$ can be represented as the value of an optimal stopping problem. To this end, we first note that by [24, Theorem 1 and Corollary, p. 124], the optimal stopping problem $V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x[\mathrm{e}^{-r\tau} g(X_\tau)]$, where $\mathcal{T}$ is the set of $\mathbb{F}$-stopping times, has a finite solution under assumptions (A1) and (A2). Now let N be a Poisson process, independent of X, with rate λ. Furthermore, let $\mathbb{F}_N$ be the filtration generated by N and (X_t) and let $\mathcal{T}_N$ be the set of $\mathbb{F}_N$-stopping times. Let $V_N(x) = \sup_{\tau\in\mathcal{T}_N} \mathbb{E}_x[\mathrm{e}^{-r\tau} g(X_\tau)]$. Since X is $\mathbb{F}_N$-adapted and the payoff is independent of N, we find that $V = V_N$.

Let $((\tau_i, \sigma_i))_{i=1}^k$ be a vector of pairs of $\mathbb{F}$-stopping times such that $\tau_i \leq \sigma_i \leq \tau_{i+1} \leq \sigma_{i+1}$ for all $i = 1, \ldots, k-1$ with $\sigma_k = \infty$. For such a vector, let τ be the first arrival of N such that its

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arrival time $T \in (\tau_i, \sigma_i)$ for some $i = 1, \ldots, k$; if this does not occur, set $\tau = \infty$. Then $\tau \in \mathcal{T}_N$; denote the set of such τ as $\mathcal{S}_k$. Moreover, since $\tau \geq \tau_1$, we find that

$$\begin{aligned}\mathbb{E}_x[\mathrm{e}^{-r\tau} g(X_\tau)] &= \mathbb{E}_x[\mathrm{e}^{-r\tau_1} \mathbb{E}_{X_{\tau_1}}[\mathrm{e}^{-r\tau} g(X_\tau)]] \\ &= \mathbb{E}_x[\mathrm{e}^{-r\tau_1} \mathbb{E}_{X_{\tau_1}}[\mathrm{e}^{-r\tau} g(X_\tau) \mathbf{1}(\tau < \sigma_1) + \mathrm{e}^{-r\tau} g(X_\tau) \mathbf{1}(\tau > \sigma_1)]] \\ &= \mathbb{E}_x[\mathrm{e}^{-r\tau_1} \mathbb{E}_{X_{\tau_1}}[\mathrm{e}^{-rU} g(X_U) \mathbf{1}(U < \sigma_1) + \mathrm{e}^{-r\sigma_1} \mathbb{E}_{X_{\sigma_1}}[\mathrm{e}^{-r\tau} g(X_\tau)] \mathbf{1}(U > \sigma_1)]],\end{aligned}$$

where $U \sim \mathrm{Exp}(\lambda)$ is independent of X. In the last equality, we used the memoryless property of the exponential distribution. Now we can proceed with the term $\mathbb{E}_{X_{\sigma_1}}[\mathrm{e}^{-r\tau} g(X_\tau)]$ in the same way and eventually recover the objective functional in (2.2) with costs equal to zero. Finally, since the stopping times τ are indexed by the vectors $((\tau_i, \sigma_i))_{i=1}^k$, we find that $V_i^k(x, 0, 0) = \sup_{\tau \in \mathcal{S}_k} \mathbb{E}_x[\mathrm{e}^{-r\tau} g(X_\tau)]$. Thus

$$V_i^k(x) = V_i^k(x, K_1, K_2) \leq V_i^k(x, 0, 0) \leq V_N(x) = V(x) \quad \text{for all } k,$$

since $\mathcal{S}_k \subset \mathcal{T}_N$.

To conclude, we observe that the sequence (V_i^k) is increasing; indeed, we can augment any k-vector $((\tau_i, \sigma_i))_{i=1}^k$ to a $(k+1)$-vector with $\tau_{k+1} = \sigma_{k+1} = \infty$, which yields the same payoff as the original k-vector. Consequently the limit $V_i = \lim_{k \to \infty} V_i^k$ exists and is finite. □

Remark 2.1. The limiting case $K_2 \to -\infty$ corresponds to the case where abandonment of the project becomes prohibitively expensive. In this case the problem is reduced to $V_i^0(X) = \sup_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}(\lambda(R_{r+\lambda} g)(X_\tau) - K_1)]$.

The limiting case $\lambda \to \infty$ corresponds to the case where the implementation delay vanishes and the payoff is realized immediately at the exercise. In this case the problem is reduced to the standard optimal stopping problem $V(x) = \sup_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}(g(X_\tau) - K_1)]$.

3. Bellman principle

The purpose of this section is prove the Bellman principle of optimality for the problem (2.3). More precisely, we define the Bellman operator on an appropriate function space and prove that the value function is the unique fixed point of this operator. We study the problem (2.3) under the following assumption.

Assumption 3.1. The process X is time-homogeneous.

Since the process X is time-homogeneous, it is reasonable to expect that after a single 'inactive–active' cycle has been completed, the problem starts afresh with the same remaining value. Thus we will look for a Bellman operator over a single 'inactive–active' cycle.

Next we set up the function space on which we define our Bellman operator. Let $\mathbf{B}$ be the set of functions $f: E \to \mathbb{R}$ satisfying the conditions

(B1) f is continuous,

(B2) the function f/h is bounded, where h is the harmonic function in assumption (A2).

Define the norm

$$\|f\|_{\mathbf{B}} = \left\| \frac{f}{h} \right\|_u$$

on $\mathbf{B}$; under the metric induced by this norm, the space $\mathbf{B}$ is a complete metric space.

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We elaborate the definition of the function space by setting up auxiliary stopping problems. For $f \in \mathbf{B}$, define the first auxiliary stopping problem

$$\begin{aligned} W_a(x;f) &= \sup_\sigma \mathbb{E}_x[\mathrm{e}^{-rU} g(X_U)\mathbf{1}(U<\sigma) + \mathrm{e}^{-r\sigma}(f(X_\sigma)+K_2)\mathbf{1}(U \geq \sigma)] \\ &= \lambda(R_{r+\lambda}g)(x) + \sup_\sigma \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\sigma}(f(X_\sigma)+K_2-\lambda(R_{r+\lambda}g)(X_\sigma))]. \end{aligned} \tag{3.1}$$

Assumption 3.2. The function $W_a(\cdot, f)$ is continuous for all $f \in \mathbf{B}$.

Under this assumption, we know from the general theory of optimal stopping (see [24, Corollary I.2.9]) that there is an optimal stopping time σ_f^* in (3.1) which can be identified as the first hitting time for the closed set

$$S_f^a = \{x \mid W_a(x;f) = f(x) + K_2 - \lambda(R_{r+\lambda}g)(x)\}.$$

For $f \in \mathbf{B}$, the second auxiliary stopping problem is defined as

$$W_i(x;f) = \sup_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}(W_a(x;f) - K_1)]. \tag{3.2}$$

Assumption 3.3. The function $W_i(\cdot, f)$ is continuous for all $f \in \mathbf{B}$.

Under this assumption, we find (again by [24, Corollary I.2.9]) that there is an optimal stopping time τ_f^* in (3.2) which can be identified as the first hitting time for the closed set

$$S_f^i = \{x \mid W_i(x;f) = W_a(x;f) - K_1\}.$$

Assumption 3.4. The resolvent $(R_{r+\lambda}g)(x)$ is continuous.

Assumption 3.4 is required to show that the value function is sufficiently well-behaved.

Remark 3.1. Assumptions 3.1–3.4 are satisfied for linear diffusion dynamics for payoffs satisfying assumptions (A1) and (A2), if the payoff is also continuous. See Section 4.1 for details.

To work in the space $\mathbf{B}$, using Doob's excessive transform (see [7]), we rewrite the optimal stopping problems (3.1) and (3.2) as

$$\begin{aligned} W_a(x;f) &= h(x)\mathbb{E}_x^h\left[\frac{g(X_U)}{h(X_U)}\mathbf{1}\big(U<\sigma_f^*\big) + \left(\frac{f\big(X_{\sigma_f^*}\big)+K_2}{h\big(X_{\sigma_f^*}\big)}\right)\mathbf{1}\big(U>\sigma_f^*\big)\right], \\ W_i(x;f) &= h(x)\mathbb{E}_x^h\left[\frac{W_a\big(X_{\tau_f^*};f\big)-K_1}{h\big(X_{\tau_f^*}\big)}\right]. \end{aligned} \tag{3.3}$$

Using these formulations, we define the Bellman operator

$$(\Lambda f) = W_i(\cdot, f), \tag{3.4}$$

for all $f \in \mathbf{B}$.

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We narrow down the considered function space by the two additional conditions

(B3) $(\Lambda f) \geq f$,

(B4) Λf is continuous.

The function space satisfying assumptions (B1)–(B4) is denoted by $\hat{\mathbf{B}}$; we point out that $\hat{\mathbf{B}} \neq \emptyset$, since the function $f = 0$ is in $\hat{\mathbf{B}}$. Intuitively, assumption (B3) describes the fact that increased flexibility has to yield additional value.

Lemma 3.1. *The function space* $\hat{\mathbf{B}}$ *is a complete metric space under the metric induced by the norm* $\|\cdot\|_{\mathbf{B}}$ *and* Λ *is a mapping from* $\hat{\mathbf{B}}$ *to* $\hat{\mathbf{B}}$.

Proof. Since $\mathbf{B}$ is a complete metric space, it is enough to show that $\hat{\mathbf{B}}$ is closed. For stopping times $\tau \leq \sigma$, function f, and point x, let us define the J operator

$$J(x; \tau, \sigma, f) = h(x)\mathbb{E}_x^h\left[\mathbb{E}_{X_\tau}^h\left[\frac{g(X_U)}{h(X_U)}\mathbf{1}(U < \sigma) + \frac{f(X_\sigma) + K_2}{h(X_\sigma)}\mathbf{1}(U > \sigma)\right] - \frac{K_1}{h(X_\tau)}\right]. \tag{3.5}$$

Suppose there is a sequence of functions $f_n \in \hat{\mathbf{B}}$ such that $f_n \to f$ with respect to the norm $\|\cdot\|_{\mathbf{B}}$. We need to prove that $(\Lambda f)(x) \geq f(x)$. To do that, we first show that $(\Lambda f)(x) = \lim_{n\to\infty} (\Lambda f_n)(x)$.

Let τ, σ be any stopping times such that $\tau \leq \sigma$. For a given x, there exists n_ε for each $\varepsilon > 0$ such that for each $n > n_\varepsilon$, the functions satisfy

$$\left|\frac{f_n(x)}{h(x)} - \frac{f(x)}{h(x)}\right| < \frac{\varepsilon}{h(x)}.$$

Hence

$$\begin{aligned}
&|J(x; \tau, \sigma, f_n) - J(x; \tau, \sigma, f)| \\
&\quad = h(x)\left|\mathbb{E}_x^h\left[\mathbb{E}_{X_\tau}^h\left[\frac{g(X_U)}{h(X_U)}\mathbf{1}(U < \sigma) + \left(\frac{f_n(X_\sigma) + K_2}{h(X_\sigma)}\right)\mathbf{1}(U \geq \sigma)\right] - \frac{K_1}{h(X_\tau)}\right]\right. \\
&\qquad \left. - \mathbb{E}_x^h\left[\mathbb{E}_{X_\tau}^h\left[\frac{g(X_U)}{h(X_U)}\mathbf{1}(U < \sigma) + \left(\frac{f(X_\sigma) + K_2}{h(X_\sigma)}\right)\mathbf{1}(U \geq \sigma)\right] - \frac{K_1}{h(X_\tau)}\right]\right| \\
&\quad = h(x)\left|\mathbb{E}_x^h\left[\mathbb{E}_{X_\tau}^h\left[\left(\frac{f_n(X_\sigma) - f(X_\sigma)}{h(X_\sigma)}\right)\mathbf{1}(U \geq \sigma)\right]\right]\right| \\
&\quad \leq h(x)\mathbb{E}_x^h\left[\mathbb{E}_{X_\tau}^h\left[\left|\left(\frac{f_n(X_\sigma) - f(X_\sigma)}{h(X_\sigma)}\right)\mathbf{1}(U \geq \sigma)\right|\right]\right] \\
&\quad \leq h(x)\mathbb{E}_x^h\left[\mathbb{E}_{X_\tau}^h\left[\frac{\varepsilon}{h(X_\sigma)}\right]\right] \\
&\quad \leq \varepsilon,
\end{aligned}$$

which implies that

$$|\Lambda f_n(x) - \Lambda f(x)| = |\sup_{\tau,\sigma} J(x; \tau, \sigma, f_n) - \sup_{\tau,\sigma} J(x; \tau, \sigma, f)| \leq \varepsilon.$$

Then $\Lambda f(x) = \lim_{n\to\infty} \Lambda f_n(x) \geq \lim_{n\to\infty} f_n(x) = f(x)$, and thus $\hat{\mathbf{B}}$ is a complete metric space.

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Next we show that $\Lambda f \in \hat{\mathbf{B}}$. We start by proving that the function $\Lambda f/h$ is bounded. First, by definition of Λ, we have $\Lambda f \geq 0$. Moreover, since h is r-harmonic, it is strictly positive inside the state space. Now, since $\tau_f^* \leq \sigma_f^*$ and $K_2 < K_1$, we find by monotonicity and Jensen's inequality that

$$\begin{aligned}
\frac{\Lambda f(x)}{h(x)} &= \left(\frac{\Lambda f(x)}{h(x)}\right)^+ \\
&= \mathbb{E}_x^h\left[\mathbb{E}^h_{X_{\tau_f^*}}\left[\frac{g(X_U)}{h(X_U)}\mathbf{1}\big(U < \sigma_f^*\big) + \left(\frac{f\big(X_{\sigma_f^*}\big) + K_2}{h\big(X_{\sigma_f^*}\big)}\right)\mathbf{1}\big(U \geq \sigma_f^*\big)\right] - \frac{K_1}{h\big(X_{\tau_f^*}\big)}\right]^+ \\
&\leq \left(\|g\|_{\mathbf{B}} + \|f\|_{\mathbf{B}} + \mathbb{E}_x^h\left[\mathbb{E}^h_{X_{\tau_f^*}}\left[\frac{K_2}{h\big(X_{\sigma_f^*}\big)}\right] - \frac{K_1}{h\big(X_{\tau_f^*}\big)}\right]\right)^+ \\
&\leq (\|g\|_{\mathbf{B}} + \|f\|_{\mathbf{B}})^+ + \left(\frac{1}{h(x)}\big(K_2\mathbb{E}_x\big[\mathrm{e}^{-r\sigma_f^*}\big] - K_1\mathbb{E}_x\big[\mathrm{e}^{-r\tau_f^*}\big]\big)\right)^+ \\
&= \|g\|_{\mathbf{B}} + \|f\|_{\mathbf{B}}
\end{aligned}$$

for all x. Thus $\|\Lambda f\|_{\mathbf{B}} < \infty$. We also find that

$$\begin{aligned}
&(\Lambda^2 f)(x) \\
&\quad = \sup\nolimits_\tau \mathbb{E}_x\big[\mathrm{e}^{-r\tau}\big(\sup\nolimits_\sigma \mathbb{E}_x\big[\mathrm{e}^{-rU}g(X_U)\mathbf{1}(U < \sigma) + ((\Lambda f)(X_\sigma) + K_2)\mathbf{1}(U > \sigma)\big] - K_1\big)\big] \\
&\quad \geq \sup\nolimits_\tau \mathbb{E}_x\big[\mathrm{e}^{-r\tau}\big(\sup\nolimits_\sigma \mathbb{E}_x\big[\mathrm{e}^{-rU}g(X_U)\mathbf{1}(U < \sigma) + (f(X_\sigma) + K_2)\mathbf{1}(U > \sigma)\big] - K_1\big)\big] \\
&\quad = (\Lambda f)(x).
\end{aligned}$$

The function Λf is continuous by Assumption 3.2. Finally, since $\Lambda f \in \mathbf{B}$, the function $\Lambda^2 f$ is also continuous by Assumption 3.2. Thus Λ is a function from $\hat{\mathbf{B}}$ to $\hat{\mathbf{B}}$. □

Lemma 3.2. *The mapping* $\Lambda : \hat{\mathbf{B}} \to \hat{\mathbf{B}}$ *is a contraction.*

Proof. Let $f_1, f_2 \in \hat{\mathbf{B}}$. Then the distance

$$\begin{aligned}
|(\Lambda f_1)(x) - (\Lambda f_2)(x)| &= ((\Lambda f_1)(x) - (\Lambda f_2)(x))\mathbf{1}((\Lambda f_1)(x) > (\Lambda f_2)(x)) \\
&\quad + ((\Lambda f_2)(x) - (\Lambda f_1)(x))\mathbf{1}((\Lambda f_2)(x) > (\Lambda f_1)(x)) \\
&= \big(J\big(x; \tau_{f_1}^*, \sigma_{f_1}^*, f_1\big) - J\big(x; \tau_{f_2}^*, \sigma_{f_2}^*, f_2\big)\big)\mathbf{1}((\Lambda f_1)(x) > (\Lambda f_2)(x)) \\
&\quad + \big(J\big(x; \tau_{f_2}^*, \sigma_{f_2}^*, f_2\big) - J\big(x; \tau_{f_1}^*, \sigma_{f_1}^*, f_1\big)\big)\mathbf{1}((\Lambda f_2)(x) > (\Lambda f_1)(x)) \\
&\leq \big(J\big(x; \tau_{f_1}^*, \sigma_{f_1}^*, f_1\big) - J\big(x; \tau_{f_1}^*, \sigma_{f_1}^*, f_2\big)\big)\mathbf{1}((\Lambda f_1)(x) > (\Lambda f_2)(x)) \\
&\quad + \big(J\big(x; \tau_{f_2}^*, \sigma_{f_2}^*, f_2\big) - J\big(x; \tau_{f_2}^*, \sigma_{f_2}^*, f_1\big)\big)\mathbf{1}((\Lambda f_2)(x) > (\Lambda f_1)(x)).
\end{aligned}$$

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By the definition of J, we find that

$$
\begin{aligned}
&J\big(x; \tau^*_{f_1}, \sigma^*_{f_1}, f_1\big) - J\big(x; \tau^*_{f_1}, \sigma^*_{f_1}, f_2\big) \\
&\quad = \mathbb{E}_x^{\psi_r}\Bigg[\mathbb{E}^{\psi_r}_{X_{\tau^*_{f_1}}}\Bigg[\Bigg(\frac{f_1\big(X_{\sigma^*_{f_1}}\big)}{\psi_r\big(X_{\sigma^*_{f_1}}\big)} - \frac{f_2\big(X_{\sigma^*_{f_1}}\big)}{\psi_r\big(X_{\sigma^*_{f_1}}\big)}\Bigg)\mathbf{1}\big(U > \sigma^*_{f_1}\big)\Bigg]\Bigg] \\
&\quad \le \| f_1 - f_2\|_{\mathbf{B}} \mathbb{E}_x^{\psi_r}\Big[\mathbb{E}^{\psi_r}_{X_{\tau^*_{f_1}}}\big[\mathbf{1}\big(U > \sigma^*_{f_1}\big)\big]\Big].
\end{aligned}
$$

Define

$$
\gamma = \sup_x\Big(\max\Big\{\mathbb{E}_x^{\psi_r}\Big[\mathbb{E}^{\psi_r}_{X_{\tau^*_{f_1}}}\big[\mathbf{1}\big(U > \sigma^*_{f_1}\big)\big]\Big], \mathbb{E}_x^{\psi_r}\Big[\mathbb{E}^{\psi_r}_{X_{\tau^*_{f_2}}}\big[\mathbf{1}\big(U > \sigma^*_{f_2}\big)\big]\Big]\Big\}\Big).
$$

Since $(\Lambda f) \ge f$, we must have $S^a_f \cap S^i_f = \emptyset$ for all $f \in \hat{\mathbf{B}}$ and consequently $\gamma < 1$.

These yield the desired result:

$$
\| \Lambda f_1 - \Lambda f_2\|_{\mathbf{B}} \le \gamma \| f_1 - f_2\|_{\mathbf{B}},
$$

where $\gamma < 1$. □

Using the operator Λ, we can rewrite the value functions V_i^k as follows.

Lemma 3.3. *The value functions V_i^k in the problem (2.2) can be written as*

$$
V_i^n(x) = \Lambda^{n+1}(\lambda(R_{r+\lambda}g)(x) - K_2).
$$

Proof. First we study the function V_i^0. We find that

$$
\begin{aligned}
V_i^0(x) &= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}\mathbb{E}_{X_\tau}[\mathrm{e}^{-rU}g(X_U)]] \\
&= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}\lambda(R_{r+\lambda}g)(X_\tau)] \\
&= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}(\lambda(R_{r+\lambda}g)(X_\tau) + \sup_\sigma \mathbb{E}_{X_\tau}[\mathrm{e}^{-(r+\lambda)\sigma}\cdot 0])] \\
&= \sup_\tau \mathbb{E}_x[\mathrm{e}^{-r\tau}(\lambda(R_{r+\lambda}g)(X_\tau) \\
&\qquad + \sup_\sigma \mathbb{E}_{X_\tau}[\mathrm{e}^{-(r+\lambda)\sigma}(\lambda(R_{r+\lambda}g)(X_\sigma) - K_2 - \lambda(R_{r+\lambda}g)(X_\sigma) + K_2)])] \\
&= \Lambda(\lambda(R_{r+\lambda}g)(x) - K_2).
\end{aligned}
$$

The claim then follows inductively from the fact that $V_i^{n+1} = \Lambda V_i^n$. □

Now we can write a similar expression for the value function of the problem with infinitely many reversals as

$$
V_i(x) = \lim_{n\to\infty} V_i^n(x) = \lim_{n\to\infty} \Lambda^{n+1}(\lambda(R_{r+\lambda})g(x) - K_2) = \Lambda^{\infty}(\lambda(R_{r+\lambda}g)(x) - K_2).
$$

Next we show that $\lambda(R_{r+\lambda}g)(x) - K_2 \in \mathbf{B}$. The first condition is true by Assumption 3.4; next we will prove a lemma to guarantee the second one.

Lemma 3.4. *If f/h is a bounded function, for some function f and some r-harmonic function h, and then $R_{r+\lambda}f/h$ is bounded for all $\lambda > 0$.*

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Proof. Let M be a constant such that $f(x)/h(x) \leq M$, and let $r, \lambda > 0$. Now, for all x, we find that

$$\begin{aligned}\frac{(R_{r+\lambda}f)(x)}{h(x)} &= \frac{\mathbb{E}_x\big[\int_0^\infty \mathrm{e}^{-(r+\lambda)t} f(X_t)\,\mathrm{d}t\big]}{h(x)}\\ &= \frac{\mathbb{E}_x\big[\int_0^\infty \mathrm{e}^{-(r+\lambda)t} \frac{f(X_t)}{h(X_t)} h(X_t)\,\mathrm{d}t\big]}{h(x)}\\ &\leq \frac{M\mathbb{E}_x\big[\int_0^\infty \mathrm{e}^{-(r+\lambda)t} h(X_t)\,\mathrm{d}t\big]}{h(x)}\\ &= \frac{M(R_{r+\lambda}h)(x)}{h(x)}\\ &= \frac{\frac{M}{\lambda}h(x)}{h(x)}\\ &= \frac{M}{\lambda},\end{aligned}$$

and thus $(R_{r+\lambda}f)(x)/h(x)$ is bounded by definition. □

To show that the value function V_i is the unique fixed point in $\hat{\mathbf{B}}$, we still have to show that $V_i \in \hat{\mathbf{B}}$. First we see that since the resolvent $\lambda(R_{r+\lambda}g)(x)$ is continuous, and by Lemma 3.4 the ratio $\lambda(R_{r+\lambda}g)(x)/h(x)$ is bounded, and those conditions apply for constants and sums of functions fulfilling the conditions, it follows that $\lambda(R_{r+\lambda}g) - K_2 \in \mathbf{B}$.

By recalling that $V_i^0 = \Lambda(\lambda(R_{r+\lambda}g) - K_2)$, we see that V_i^0 is continuous by Assumptions 3.2 and 3.3, and the boundedness of V_i^0/h is inherited from $\lambda(R_{r+\lambda}g) - K_2$ since the operator Λ is a contraction with respect to the norm $\|\cdot\|_{\mathbf{B}}$ by Lemma 3.2. Then, since $\Lambda V_i^0 = V_i^1$, it follows by the monotonous order of V_i^k that $\Lambda V_i^0 \geq V_i^0$. Finally, by Assumptions 3.2 and 3.3, it follows that Λf is continuous, and hence $V_i^0 \in \hat{\mathbf{B}}$.

Since the operator Λ is closed in space $\hat{\mathbf{B}}$, it follows inductively that $V_i^k \in \hat{\mathbf{B}}$ for all k. Finally, since $\hat{\mathbf{B}}$ is a complete metric space, it follows that $V_i = \lim_{k\to\infty} V_i^k \in \hat{\mathbf{B}}$.

Using that expression and the previous lemma, we can prove the main result of this section.

Theorem 3.1. (Bellman principle.) *The value function V_i is the unique fixed point of the operator Λ in $\hat{\mathbf{B}}$.*

Proof. Since by Lemma 3.2 the operator Λ is a contraction, for every $\varepsilon > 0$, there exists n_ε such that

$$\begin{aligned}|V_i(x) - V_i^n(x)| &= |(\Lambda^\infty \xi)(x) - (\Lambda^n \xi)(x)|\\ &\leq |(\Lambda^{n+i}\xi)(x) - (\Lambda^n \xi)(x)| + \frac{\varepsilon}{2}\\ &\leq \gamma^n |(\Lambda^i \xi)(x) - \xi(x)| + \frac{\varepsilon}{2}\\ &\leq \gamma^n |(\Lambda^\infty \xi)(x) - \xi(x)| + \frac{\varepsilon}{2}\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon\end{aligned}$$

for all i and all $n > n_\varepsilon$, where $\xi(x) = \lambda(R_{r+\lambda}g)(x) - K_2$. Consequently we have

$$\begin{aligned} |V_i - \Lambda V_i| &\leq |V_i - \Lambda V_i^n| + |\Lambda V_i^n - \Lambda V_i| \\ &\leq |V_i - V_i^{n+1}| + \gamma |V_i^n - V_i| \\ &\to 0 \end{aligned}$$

when $n \to \infty$. Thus the value function V_i is a fixed point of Λ. By Banach's fixed point theorem, the fixed point is unique. □

4. A class of solvable problems: Linear diffusion dynamics

4.1. The problem specification

We assume that the process X follows a regular linear diffusion on the positive real line $\mathbb{R}_+$. Furthermore, we assume that the boundaries of the state space are natural. Now, the evolution of X is completely determined by its *scale function* S and *speed measure* m inside $\mathbb{R}^+$; see [7, pp. 13–14]. Furthermore, we assume that the function S and the measure m are both absolutely continuous with respect to the Lebesgue measure, have smooth derivatives, and that S is twice continuously differentiable. Under these assumptions, we know that the infinitesimal generator $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \to C_b(\mathbb{R}_+)$ of X can be expressed as

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu(x)\frac{\mathrm{d}}{\mathrm{d}x},$$

where the functions σ and μ are related to S and m via the formulæ

$$m'(x) = \frac{2}{\sigma^2(x)}\,\mathrm{e}^{B(x)} \quad \text{and} \quad S'(x) = \mathrm{e}^{-B(x)} \quad \text{for all } x \in \mathbb{R}_+,$$

where

$$B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}\,\mathrm{d}y,$$

see [7, p. 17]. From these definitions we find that

$$\sigma^2(x) = \frac{2}{S'(x)m'(x)} \quad \text{and} \quad \mu(x) = -\frac{S''(x)}{S'^2(x)m'(x)} \quad \text{for all } x \in \mathbb{R}^+.$$

In what follows, we assume that the functions μ and σ^2 are continuous. The assumption that the state space is $\mathbb{R}^+$ is done for convenience. In fact we could assume that the state space is any interval $\mathcal{I}$ in $\mathbb{R}$ and the subsequent analysis would hold with obvious modifications.

Denote the hitting time for set S as τ_S and the hitting time for point y as τ_y. Then we call a state in $E = \mathbb{R}^+$ regular if $\mathbb{P}(\tau_{(0,x)} = 0) = \mathbb{P}(\tau_{(x,\infty)} = 0) = 1$. Under our assumptions, the process X is a regular linear diffusion and the speed measure m is absolutely continuous with respect to the Lebesgue measure. Thus we see that by [7, p. 13], all states in $\mathbb{R}_+$ are regular.

Then we let ψ_r and φ_r, respectively, denote the increasing and the decreasing solution of the second-order linear ordinary differential equation $\mathcal{A}u = ru$, where $r > 0$, defined on the domain of the characteristic operator of X. The functions ψ_r and φ_r can be identified as the minimal

r-excessive functions ψ_r and φ_r of X; see [7, pp. 18–20]. Finally, it is well known (see [7, p. 19]) that for a given $f \in L_1^r$ the resolvent $R_r f$ can be expressed as

$$(R_r f)(x) = B_r^{-1}\varphi_r(x)\int_0^x \psi_r(y)f(y)m'(y)\,\mathrm{d}y + B_r^{-1}\psi_r(x)\int_x^\infty \varphi_r(y)f(y)m'(y)\,\mathrm{d}y \tag{4.1}$$

for all $x \in \mathbb{R}_+$, where

$$B_r = \frac{\psi_r'(x)}{S'(x)}\varphi_r(x) - \frac{\varphi_r'(x)}{S'(x)}\psi_r(x)$$

denotes the Wronskian determinant.

Next, we propose the class of payoff functions for which we study the problem (2.3). In what follows, we use the notation

$$g_l(x) = g(x) - K_l\frac{r+\lambda}{\lambda}, \quad l = 1, 2, \tag{4.2}$$

for brevity.

Assumption 4.1. For $l = 1, 2$,

(1) the payoff $g_l \in L_1^r$ is bounded from below and continuous,

(2) there exists a unique $0 \le x_l^0 < \infty$ such that $g_l(x) \gtreqqless 0$, when $x \gtreqqless x_l^0$,

(3) there is a unique state x_l^* which maximizes the function

$$x \mapsto \frac{g_l(x)}{\psi_r(x)}$$

and that this function is non-decreasing on $(0, x_l^*)$ and non-increasing on (x_l^*, ∞). Further, the limiting conditions

$$\lim_{x\to 0+}\frac{g_l(x)}{\psi_r(x)} \le 0 \le \lim_{x\to\infty}\frac{g_l(x)}{\psi_r(x)} < \infty$$

hold,

(4) the function

$$x \mapsto \frac{g_l(x)}{\varphi_r(x)}$$

is non-decreasing.

Assumption 4.1 is fairly weak and easy to verify; similar assumptions appear frequently in optimal stopping. Roughly speaking, the assumption means that the payoff g should be continuous and non-decreasing, and satisfy suitable limiting conditions at the boundaries. Furthermore, the payoff can be unbounded but the rate of growth is constrained by item (3). The assumptions are similar to the irreversible problem with exercise lag studied in [17].

We show that that the problem specification of this section satisfies Assumptions 3.2 and 3.3; Assumption 3.1 is assumed to hold. To this end, we recall the definition of the fine topology. A set A is finely open with respect to a process X if for each $x \in A$ there exists a nearly Borel set $B \subset A$ such that $\mathbb{P}_x(\tau_{B^c} > 0) = 1$. We prove the following lemma.

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Lemma 4.1. *Let X be a strong Markov process in $\mathbb{R}^+$ with almost surely continuous paths, such that each $x \in E = I$ is a regular state. Then a set A is finely open if and only if it is open.*

Proof. First, let A be an open set. Thus, for each $x \in A$, there exists a ball $B(x, \delta) \subset A$ for some $\delta > 0$. Now, since the paths of X are almost surely continuous, it follows that for each $\omega \in \Omega$, barring a zero-measured set of exceptions, there exists $t_\delta(\omega)$ such that $X_t(\omega) \in B(x, \delta/2)$ for all $0 \le t \le t_\delta(\omega)$. Consequently, $\tau_{B(x,\delta)^c}(\omega) \ge \tau_{B(x,\delta/2)^c}(\omega) \ge t_\delta(\omega) > 0$, yielding

$$\mathbb{P}_x(\tau_{b(x,\delta)^c} > 0) = 1.$$

Since this is true for all $x \in A$, the set A is finely open by definition.

Next, assume that A is not open. Then there exists a sequence x_i of points in A^c that converges to some $x \in A$. Either an infinite number of those points are on the left side of x, or an infinite number of them are on the right side of x. By symmetry, let us assume the left side of x has infinite number of points x_i. Then those points form a subsequence that also converges to x.

Now, since x is a regular point, the stopping time $\tau_{(0,x)} = 0$ almost surely. Thus, if $t > 0$, for each ω, barring a zero-measured set of exceptions, there exists a time index $u(\omega) < t$ such that $X_{u(\omega)}(\omega) < x$. There exists a member of x_i such that $X_{u(\omega)}(\omega) < x_{i(\omega)} < x$, implying that

$$\tau_{B^c} \le \tau_{A^c} \le \tau_{x_i} \le u < t$$

for chosen ω, where the inequality $\tau_{x_i} \le u$ follows from the path of X_t being almost surely continuous (the exceptions to this can also be disregarded as a null set). Since the claim is true for any nearly Borel $B \subset A$ and for any $t > 0$, it follows that $\tau_{B^c} = 0$ almost surely for any nearly Borel $B \subset A$, implying that A is not finely open, and completing the proof. □

As an immediate consequence we get the following.

Corollary 4.1. *Let the process X satisfy the assumptions of Lemma 4.1. Then a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is finely continuous with respect to X.*

We have shown that fine continuity and continuity are equivalent in one-dimensional Markov processes satisfying the assumptions of Lemma 4.1. Thus the next result follows by [28, Theorem 5, p. 135].

Corollary 4.2. *Let the process X satisfy the assumptions of Lemma 4.1 and let g be a continuous function. Then Assumptions 3.2 and 3.3 hold.*

The process X does satisfy the assumptions of Lemma 4.1, so we only need to show that Assumption 3.4 holds. By [22, Lemma 8.1.4, p. 143] the function $u(t, x) = \mathbb{E}_x[f(X_t)]$ is continuous for all bounded and continuous functions f. Then, if $\varepsilon > 0$ and M is an upper bound for $|g(x)|$, there exists a number $T > 0$ such that

$$\int_T^\infty \mathrm{e}^{-rt}\,\mathrm{d}t = \frac{1}{r}\,\mathrm{e}^{-rT} < \frac{\varepsilon}{4M}.$$

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Then, if $\delta > 0$ is such that for all $y \in B(x, \delta)$ we have $|g(x) - g(y)| < r\varepsilon/2$, we get

$$\begin{aligned}
|(R_r g)(x) - (R_r g)(y)| &\le \int_0^T \mathrm{e}^{-rt}|u(t, x) - u(t, y)|\,\mathrm{d}t \\
&\quad + \left|\mathbb{E}_x\left[\int_T^\infty \mathrm{e}^{-rt} g(X_t)\,\mathrm{d}t\right] - \mathbb{E}_y\left[\int_T^\infty \mathrm{e}^{-rt} g(X_t)\,\mathrm{d}t\right]\right| \\
&\le \int_0^T \mathrm{e}^{-rt}\frac{r\varepsilon}{2}\,\mathrm{d}t + 2\int_T^\infty M\,\mathrm{e}^{-rt}\,\mathrm{d}t \\
&= \frac{\varepsilon(1-\mathrm{e}^{-rT})}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Thus the resolvent $(R_r g)(x)$ is also continuous when g is bounded, as also claimed by [22, Lemma 8.1.3, p. 143]. Now, if g is continuous and the ratio g/h is bounded for some r-harmonic function h, we get

$$(R_{r+\lambda} g)(x) = \int_0^\infty \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)t} g(X_t)]\,\mathrm{d}t = \int_0^\infty \mathbb{E}_x^h\left[\mathrm{e}^{-(\lambda)t}\frac{g(X_t)}{h(X_t)}\right]\mathrm{d}t = R_\lambda^h\left(\frac{g}{h}\right)(x)$$

in other words, the resolvent can be represented in terms of another resolvent in the h-transformed space, where the argument is g/h and the rate is λ. That resolvent is continuous, since its argument is bounded and continuous. Thus we have shown the following proposition.

Proposition 4.1. *Assumption 3.4 holds for the process X and the gain function g satisfying Assumption 4.1.*

To close the subsection, we present the following lemma without a proof, as it follows from the representation (4.1) and [16, Lemma 2.1] by means of differentiation.

Lemma 4.2. *Let $f \in L_1^r$. Then*

$$\psi_r'(x)\varphi_{r+\lambda}(x) - \psi_r(x)\varphi_{r+\lambda}'(x) = \lambda S'(x)(\Phi\psi_r)(x),$$

$$\lambda(R_{r+\lambda}f)'(x)\varphi_{r+\lambda}(x) - \lambda(R_{r+\lambda}f)(x)\varphi_{r+\lambda}'(x) = \lambda S'(x)(\Phi f)(x),$$

$$\lambda(R_{r+\lambda}f)'(x)\psi_r(x) - \lambda(R_r f)(x)\psi_r'(x) = \frac{\lambda^2 S'(x)}{B_{r+\lambda}}((\Phi f)(x)(\Psi\psi_r)(x) - (\Psi f)(x)(\Phi\psi_r)(x)),$$

where

$$(\Phi f)(x) = \int_x^\infty \varphi_{r+\lambda}(y) f(y) m'(y)\,\mathrm{d}y, \quad (\Psi f)(x) = \int_0^x \psi_{r+\lambda}(y) f(y) m'(y)\,\mathrm{d}y.$$

4.2. The solution

We start by first deriving a candidate solution to our main problem, then show that this function is in the function space $\hat{\mathbf{B}}$ (for the r-harmonic function ψ_r), and finally show that it satisfies the Bellman principle of Proposition 3.1.

4.2.1. *Deriving the candidate.* Since we are dealing with a time-homogeneous problem, we start with the working assumption that the optimal policy is of the following type: consider thresholds $y_1 > y_2$ and the rule that

- an inactive investor should engage the investment once the state variable X is above the threshold y_1,
- an active investor should disengage the investment once the state variable X is below the threshold y_2.

Denote the candidate solution for the inactive investor as G_i. Then the rule described above can be expressed as

$$G_i(x) = \psi_r(x)\mathbb{E}_x^{\psi_r}\left[\left(\mathbb{E}_{X_{\tau_{y_1}}}^{\psi_r}\left[\frac{g(X_U)}{\psi_r(X_U)}\mathbf{1}(U < \sigma_{y_2}) + \frac{G_i(X_{\sigma_{y_2}}) + K_2}{\psi_r(X_{\sigma_{y_2}})}\mathbf{1}(U > \sigma_{y_2})\right] - \frac{K_1}{\psi_r(X_{\tau_{y_1}})}\right)\right]. \tag{4.3}$$

This condition is expressed in terms of the ψ_r-transform; we have already used this way of writing in the previous section. The reason for this is the same as above: it lends itself well to fixed point arguments. The following lemma tells us that it is reasonable to work with the condition (4.3) in the first place.

Lemma 4.3. *There is a unique continuous function G_i satisfying the condition (4.3) such that G_i/ψ_r is bounded.*

Proof. Recall the function space $\mathbf{B}$. For $y_1 > y_2$, define the operator $\Theta\colon \mathbf{B} \to \mathbf{B}$ as $(\Theta f) = J(\,\cdot\,;\tau_{y_1}, \sigma_{y_2}, f)$; see the definition of the operator Λ in (3.5). Take $f_1, f_2 \in \mathbf{B}$. Then

$$\begin{aligned}\|\Theta f_1 - \Theta f_2\|_{\mathbf{B}} &= \sup_x\left|\mathbb{E}_x^{\psi_r}\left[\mathbb{E}_{y_1}^{\psi_r}\left[\left(\frac{f_1(y_2)}{\psi_r(y_2)} - \frac{f_2(y_2)}{\psi_r(y_2)}\right)\mathbf{1}(U > \sigma_{y_2})\right]\right]\right| \\ &\le \eta\|f_1 - f_2\|_{\mathbf{B}},\end{aligned}$$

where

$$\eta = \sup_x \mathbb{E}_x^{\psi_r}\left[\mathbb{E}_{y_1}^{\psi_r}[\mathbf{1}(U > \sigma_{y_2})]\right] < 1.$$

Thus there is a unique fixed point G_i to the operator Θ. □

By reversing the ψ_r-transform, we rewrite the condition (4.3) as

$$G_i(x) = \mathbb{E}_x\left[\mathrm{e}^{-r\tau_{y_1}}(G_a(X_{\tau_{y_1}}) - K_1)\right], \tag{4.4}$$

where

$$G_a(x) = \mathbb{E}_x\left[\mathrm{e}^{-rU}g(X_U)\mathbf{1}(U < \sigma_{y_2}) + \mathrm{e}^{-r\sigma_{y_2}}(G_i(X_{\sigma_{y_2}}) + K_2)\mathbf{1}(U > \sigma_{y_2})\right].$$

Let $x < y_1$ and stopping time $\tau < \tau_{y_1}$. Then the condition (4.4) and the strong Markov property yield

$$G_i(x) = \mathbb{E}_x[\mathrm{e}^{-r\tau}G_i(X_\tau)].$$

In other words, the function G_i is r-harmonic for $x < y_1$, so we can write $G_i(x) = A\psi_r(x) + A'\varphi_r(x)$, where A and A' are constants, for $x < y_1$. Since G_i/ψ_r is bounded, the constant $A' = 0$. Moreover, since G_i is continuous, we can write

$$G_i(x) = \begin{cases} G_a(x) - K_1, & x \ge y_1, \\ A\psi_r(x), & x < y_1. \end{cases}$$

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Let $x \geq y_1$. Then

$$
\begin{aligned}
G_a(x) &= \mathbb{E}_x\big[\mathrm{e}^{-rU} g(X_U)\mathbf{1}(U < \sigma_{y_2}) + \mathrm{e}^{-r\sigma_{y_2}}(G_i(X_{\sigma_{y_2}}) + K_2)\mathbf{1}(U > \sigma_{y_2})\big] \\
&= \lambda(R_{r+\lambda}g)(x) + \mathbb{E}_x\big[\mathrm{e}^{-(r+\lambda)\sigma_{y_2}}(G_i(y_2) + K_2 - \lambda(R_{r+\lambda}g)(y_2))\big] \\
&= \lambda(R_{r+\lambda}g)(x) + (A\psi_r(y_2) + K_2 - \lambda(R_{r+\lambda}g)(y_2))\frac{\varphi_{r+\lambda}(x)}{\varphi_{r+\lambda}(y_2)}.
\end{aligned}
$$

By continuity, this yields

$$
A\psi_r(y_1) = \lambda(R_{r+\lambda}g)(y_1) + (A\psi_r(y_2) + K_2 - \lambda(R_{r+\lambda}g)(y_2))\frac{\varphi_{r+\lambda}(y_1)}{\varphi_{r+\lambda}(y_2)} - K_1.
$$

Solving for A, we find that

$$
A = A(y_1, y_2) = \frac{(K_2 - \lambda(R_{r+\lambda}g)(y_2))\varphi_{r+\lambda}(y_1) - (K_1 - \lambda(R_{r+\lambda}g)(y_1))\varphi_{r+\lambda}(y_2)}{\psi_r(y_1)\varphi_{r+\lambda}(y_2) - \psi_r(y_2)\varphi_{r+\lambda}(y_1)}.
$$

Summarizing, we can write the candidate solutions as

$$
G_i(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + C(y_1, y_2)\varphi_{r+\lambda}(x) - K_1, & x \geq y_1, \\ A(y_1, y_2)\psi_r(x), & x > y_1, \end{cases} \tag{4.5}
$$

and

$$
G_a(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + C(y_1, y_2)\varphi_{r+\lambda}(x), & x \geq y_2, \\ A(y_1, y_2)\psi_r(x) + K_2, & x > y_2, \end{cases} \tag{4.6}
$$

where

$$
\begin{aligned}
C(y_1, y_2) &= \frac{1}{\varphi_{r+\lambda}(y_2)}(A(y_1, y_2)\psi_r(y_2) + K_2 - \lambda(R_{r+\lambda}g)(y_2)) \\
&= \frac{(K_2 - \lambda(R_{r+\lambda}g)(y_2))\psi_r(y_1) - (K_1 - \lambda(R_{r+\lambda}g)(y_1))\psi_r(y_2)}{\psi_r(y_1)\varphi_{r+\lambda}(y_2) - \psi_r(y_2)\varphi_{r+\lambda}(y_1)}.
\end{aligned}
$$

To find optimal thresholds, we impose the smooth-pasting condition: we assume that the candidate functions are continuously differentiable over their respective boundaries. This leads to the conditions

$$
\begin{aligned}
\lambda(R_{r+\lambda}g)'(y_1) + C(y_1, y_2)\varphi'_{r+\lambda}(y_1) &= A(y_1, y_2)\psi'_r(y_1), \\
\lambda(R_{r+\lambda}g)'(y_2) + C(y_1, y_2)\varphi'_{r+\lambda}(y_2) &= A(y_1, y_2)\psi'_r(y_2).
\end{aligned} \tag{4.7}
$$

A simple computation yields

$$
\lambda(R_{r+\lambda}g)(x) - K_l = \lambda(R_{r+\lambda}g_l)(x);
$$

recall expression (4.2). Using this notation, we readily verify that the necessary conditions

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(4.7) can be rewritten as

$$\begin{aligned}\lambda(R_{r+\lambda}g_2)(y_2)G(y_1)+F_2(y_1)\varphi_{r+\lambda}(y_2)&=F_1(y_1)\psi_r(y_2),\\ \lambda(R_{r+\lambda}g_1)(y_1)I(y_2)+J_2(y_2)\varphi_{r+\lambda}(y_1)&=J_1(y_2)\psi_r(y_1),\end{aligned}\tag{4.8}$$

where

$$\begin{aligned}F_1(x)&=\lambda(R_{r+\lambda}g_1)(x)\varphi'_{r+\lambda}(x)-\lambda(R_{r+\lambda}g_1)'(x)\varphi_{r+\lambda}(x),\\ F_2(x)&=\lambda(R_{r+\lambda}g_1)(x)\psi'_r(x)-\lambda(R_{r+\lambda}g_1)'(x)\psi_r(x),\\ G(x)&=-I(x)=\psi'_r(x)\varphi_{r+\lambda}(x)-\psi_r(x)\varphi'_{r+\lambda}(x),\\ J_1(x)&=\lambda(R_{r+\lambda}g_2)'(x)\varphi_{r+\lambda}(x)-\lambda(R_{r+\lambda}g_2)(x)\varphi'_{r+\lambda}(x),\\ J_2(x)&=\lambda(R_{r+\lambda}g_2)'(x)\psi_r(x)-\lambda(R_{r+\lambda}g_2)(x)\psi'_r(x).\end{aligned}$$

We observe that the conditions (4.8) can be further expressed as

$$\begin{aligned}\lambda(R_{r+\lambda}g)(y_2)+\frac{F_2(y_1)}{G(y_1)}\varphi_{r+\lambda}(y_2)&=\frac{F_1(y_1)}{G(y_1)}\psi_r(y_2)+K_2,\\ \lambda(R_{r+\lambda}g)(y_1)+\frac{J_2(y_2)}{I(y_2)}\varphi_{r+\lambda}(y_1)-K_1&=\frac{J_1(y_2)}{I(y_2)}\psi_r(y_1).\end{aligned}$$

By coupling these with expressions (4.5) and (4.6), we obtain the conditions

$$\begin{aligned}\frac{F_2(y_1)}{G(y_1)}&=\frac{J_2(y_2)}{I(y_2)},\\ \frac{F_1(y_1)}{G(y_1)}&=\frac{J_1(y_2)}{I(y_2)}.\end{aligned}$$

Using Lemma 4.2, these can be further expressed as

$$\begin{aligned}H_1(y_1)&=H_2(y_2),\\ R_1(y_1)&=R_2(y_2),\end{aligned}\tag{4.9}$$

where

$$\begin{aligned}H_l(x)&=\frac{(\Phi g_l)(y_l)}{(\Phi\psi_r)(y_l)},\\ R_l(x)&=(\Phi g_l)(y_l)\frac{(\Psi\psi_r)(y_l)}{(\Phi\psi_r)(y_l)}-(\Psi g_l)(y_l),\end{aligned}$$

for $l=1,2$.

To analyse the solvability of the pair (4.9), we prove some auxiliary results. The next lemma shows that the fractions

$$\frac{(\Phi g_l)(x)}{(\Phi\psi_r)(x)}\quad\text{and}\quad\frac{(\Psi g_l)(x)}{(\Psi\psi_r)(x)}$$

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have properties very similar to those of

$$\frac{g_l(x)}{\varphi_r(x)} \quad \text{and} \quad \frac{g_l(x)}{\psi_r(x)}$$

in Assumption 4.1.

Lemma 4.4. *There exist unique states $\hat{x}_l < x_l^*$ and $\check{x}_l > x_l^*$ such that*

$$\hat{x}_l = \operatorname{argmax}\left\{\frac{\Phi g_l}{\Phi \psi_r}\right\} \quad \textit{and} \quad \check{x}_l = \operatorname{argmax}\left\{\frac{\Psi g_l}{\Psi \psi_r}\right\}$$

and the functions

$$x \mapsto \frac{(\Phi g_l)(x)}{(\Phi \psi_r)(x)} \quad \textit{and} \quad x \mapsto \frac{(\Psi g_l)(x)}{(\Psi \psi_r)(x)}$$

are non-decreasing on $(0, \hat{x}_l)$ and $(0, \check{x}_l)$, and non-increasing on $(\hat{x}_l, \infty)$ and $(\check{x}_l, \infty)$. Furthermore, $\hat{x}_2 < \hat{x}_1$ and $\check{x}_2 < \check{x}_1$.

Proof. The main claim is [17, Lemma 3.4]. Thus we show that $\hat{x}_2 < \hat{x}_1$, and then $\check{x}_2 < \check{x}_1$ follows similarly.

Let $\hat{x}_1$ be the unique maximum of the function $(\Phi g_1)/(\Phi \psi_r)$, so that it satisfies the equation

$$\psi_r(\hat{x}_1)(\Phi g_1)(\hat{x}_1) = g_1(\hat{x}_1)(\Phi \psi_r)(\hat{x}_1).$$

Then, because $g_2 = g_1 + \Omega$, where $\Omega = (K_1 - K_2)(r + \lambda)/\lambda$, we find that

$$\begin{aligned}
&\psi_r(\hat{x}_1)(\Phi g_2)(\hat{x}_1) - g_2(\hat{x}_1)(\Phi \psi_r)(\hat{x}_1) \\
&\quad < \psi_r(\hat{x}_1)(\Phi g_1)(\hat{x}_1) - g_1(\hat{x}_1)(\Phi \psi_r)(\hat{x}_1) + \Omega(\psi_r(\hat{x}_1)(\Phi 1)(\hat{x}_1) - (\Phi \psi_r)(\hat{x}_1)) \\
&\quad < 0,
\end{aligned}$$

where the last inequality follows because ψ_r is increasing. Hence the point $\hat{x}_1$ is on the part where

$$x \mapsto \frac{(\Phi g_2)(x)}{(\Phi \psi_r)(x)}$$

is decreasing, which implies that $\hat{x}_2 < \hat{x}_1$. □

The following lemma summarizes the needed properties of the functions H_l and R_l, $l = 1, 2$.

Lemma 4.5. *Let $\hat{x}_1$ and $\hat{x}_2$ be as in Lemma 4.4. Then*

(1) $H_l'(x) = -\dfrac{\varphi_{r+\lambda}(x) m'(x)}{(\Phi \psi_r)(x)^2}(\psi_r(x)(\Phi g_l)(x) - (\Phi \psi_r)(x) g_l(x)),$

(2) $R_l'(x) = \dfrac{B_{r+\lambda} \psi_r(x) m'(x)}{\lambda (\Phi \psi_r)(x)^2}(\psi_r(x)(\Phi g_l)(x) - (\Phi \psi_r)(x) g_l(x)),$

(3) $H_2(\hat{x}_2) < H_1(\hat{x}_1),$

(4) $R_2(\hat{x}_2) < R_1(\hat{x}_1),$

(5) $\lim_{x \to 0} H_2(x) > 0$ *and* $\lim_{x \to 0} R_2(x) < 0.$

Proof. Parts (1) and (2) follow by straightforward differentiation and the formula (4.1).

For parts (3) and (4), since $g_1(x) < g_2(x)$ we find that $H_2(x) < H_1(x)$ for all x. By Lemma 4.4 we have $\hat{x}_2 < \hat{x}_1$ and thus $H_2(\hat{x}_2) < H_2(\hat{x}_1)$. We find that

$$H_2(\hat{x}_2) < H_2(\hat{x}_1) < H_1(\hat{x}_1).$$

The proof of part (4) is analogous.

For part (5), let $x < x_0$. Then we find that

$$(\Phi g_2)(x) = \int_x^{x_0} g_2(z)\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z + \int_{x^0}^{\infty} g_2(z)\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z.$$

The last integral is finite since $g_2 \in L_1^r$, and for the first we find by the mean value theorem that

$$\int_x^{x_0} g_2(z)\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z = g_2(\xi)\int_x^{x_0} \varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z = g_2(\xi)\left(\frac{\varphi'_{r+\lambda}(x_0)}{S'(x_0)} - \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\right),$$

where $\xi \in [x, x_0]$. As $g_2(\xi) < 0$ and

$$\lim_{x\to 0} -\frac{\varphi'_{r+\lambda}(x)}{S'(x)} = \infty$$

(the lower boundary is natural), we find that $\lim_{x\to 0}(\Phi g_2)(x) = -\infty$. As ψ_r is increasing, we find by similar calculations that $\lim_{x\to 0}(\Phi\psi_r)(x) = \infty$. Thus L'Hôpital's rule with part (4) of Assumption 4.1 yields

$$\lim_{x\to 0} H_2(x) = \lim_{x\to 0} -\frac{g_2(x)\varphi_{r+\lambda}(x)m'(x)}{\psi_r(x)\varphi_{r+\lambda}(x)m'(x)} = -\lim_{x\to 0}\frac{g_2(x)}{\psi_r(x)} \geq 0.$$

The proof for the limit of $R_1(x)$ is analogous. □

We are now in a position to prove that the pair (4.9) has a unique solution under our assumptions. With the next result, we can continue our analysis with a unique candidate function G_i.

Proposition 4.2. *Let Assumption 4.1 hold. Then there exists a unique solution to the pair of equations given by (4.9).*

Proof. Let $\hat{\cdot}$ and $\check{\cdot}$ denote the restrictions to the domains $(0, \hat{x}_2)$ and $(\hat{x}_1, \infty)$ respectively. Define the function $K\colon (0, \hat{x}_2) \to (0, \hat{x}_2)$ as

$$K(x) = \big(\hat{H}_2^{-1} \circ \check{H}_1 \circ \check{R}_1^{-1} \circ \hat{R}_2\big)(x).$$

We notice that y_2 is the fixed point of K if and only if the pair (y_2, y_1), where $y_1 = \big(\check{R}_1^{-1} \circ \hat{R}_2\big)(x)$, solves the pair of equations (4.9). By Lemma 4.5 the function K is well-defined. A direct differentiation and parts (1) and (2) of Lemma 4.5 yield

$$K'(x) = \big(\hat{H}_2^{-1'} \circ \check{H}_1 \circ \check{R}_1^{-1} \circ \hat{R}_2\big)(x) \cdot \big(\check{H}_1' \circ \check{R}_1^{-1} \circ \check{H}_2\big)(x) \cdot \big(\check{R}_1^{-1'} \circ \hat{R}_2\big)(x) \cdot \hat{R}_2'(x) > 0,$$

showing that K is increasing. Thus K is a monotonic mapping from an interval to its open subset and must have a fixed point. We denote this fixed point by y_2. Consequently, the pair (y_2, y_1), where $y_1 = (\check{R}_1^{-1} \circ \hat{R}_2)(x)$, gives a solution to the pair of equations (4.9).

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To prove uniqueness we use the fixed point property of y_2 and find by Lemma 4.5 parts (1) and (2) that

$$K'(y_2) = \frac{H_1'(y_1)}{R_1'(y_1)} \frac{R_2'(y_2)}{H_2'(y_2)} = \frac{\psi_r(y_2)}{\psi_r(y_1)} \frac{\varphi_{r+\lambda}(y_1)}{\varphi_{r+\lambda}(y_2)} < 1.$$

Thus, whenever K intersects the diagonal of $\mathbb{R}_+$, the intersection must be from above. □

4.2.2. *Candidate belongs to* $\hat{\mathbf{B}}$. To use Proposition 3.1 to show that the candidate value is the actual value of our main problem, we need to show that $G_i \in \hat{\mathbf{B}}$. To this end, we first prove the following lemma.

Now we are ready to prove the following proposition.

Proposition 4.3. *The candidate function G_i belongs to the space $\hat{\mathbf{B}}$.*

Proof. To complete the proof, we have to show that the following claims are true:

(1) G_i is continuous,

(2) G_i/h is bounded,

(3) $\Lambda G_i \geq G_i$.

Claim (1) is true since G_i is readily known to be continuous. Claim (3) is also trivially true, since it is known that $\Lambda G_i = G_i$. It is left to show that claim (2) holds.

First, let $x \geq y_1$. Then, for some constants a and b, we have

$$\begin{aligned}
\frac{G_i(x)}{h(x)} &= \frac{\lambda(R_{r+\lambda}g_1)(x) + C\varphi_{r+\lambda}(x)}{a\psi_r(x) + b\varphi_r(x)} \\
&\leq \frac{\lambda(R_{r+\lambda}g_1)(x)}{a\psi_r(x)} + \frac{C\varphi_{r+\lambda}(x)}{a\psi_r(x)} \\
&= \frac{\lambda(R_{r+\lambda}g_1)(x)}{a\psi_r(x)} + \frac{C\varphi_{r+\lambda}(y_1)}{a\psi_r(y_1)}.
\end{aligned}$$

Now, since $\lambda(R_{r+\lambda}g_1)/(a\psi_r)$ is bounded by Lemma 3.4, the rightmost function is also bounded. Thus $G_i(x)/h(x)$ is bounded for $x \geq y_1$.

Then let $x \leq y_1$. Now

$$\frac{G_i(x)}{h(x)} = \frac{A\psi_r(x)}{a\psi_r(x) + b\varphi_r(x)} \leq \frac{A\psi_r(x)}{a\psi_r(x)} = \frac{A}{a}$$

and thus $G_i(x)/h(x)$ is also bounded for $x \leq y_1$. This completes the proof. □

4.2.3. *Candidate satisfies the Bellman principle.* Knowing that $G_i \in \hat{\mathbf{B}}$, we show next that it satisfies the Bellman principle. We start by proving a series of lemmas.

Lemma 4.6. *Define the functions $f_\psi : \mathbb{R}_+ \to \mathbb{R}$ and $f_\varphi : \mathbb{R}_+ \to \mathbb{R}$ as*

$$f_\psi(x) = \frac{\lambda(R_{r+\lambda}g_1)(x) - C(y_1, y_2)\varphi_{r+\lambda}(x)}{\psi_r(x)}, \quad f_\varphi(x) = \frac{H_1(x) - C(y_1, y_2)\varphi_{r+\lambda}(x)}{\varphi_r(x)}.$$

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Then the following claims are true:

(i) f_ψ *is non-increasing at* $x \geq y_1$,

(ii) f_ψ *is non-decreasing at* x *for* $y_2 \leq x \leq y_1$,

(iii) f_φ *is non-decreasing at* x *for* $x \geq y_1$.

Proof. Define the function

$$\hat{C}(x) = \frac{\lambda(R_{r+\lambda}g_1)'(x)\psi_r(x) - \lambda(R_{r+\lambda}g_1)(x)\psi_r'(x)}{\varphi_{r+\lambda}'(x)\psi_r(x) - \varphi_{r+\lambda}(x)\psi_r'(x)}.$$

Then, differentiating and rearranging the terms, we find that the inequality $f_\psi'(x) \geq 0$ is equivalent to

$$\hat{C}(x) \leq \hat{C}(y_1). \tag{4.10}$$

This inequality holds if the function $\hat{C}$ is increasing on the interval $[x, y_1]$. In addition, if $\hat{C}(x)$ is also increasing in $[y_1, \infty)$, the converse of (4.10) holds for $x \geq y_1$, which would complete the proof of claim (i). Proceeding as in the derivation of (4.9), we can re-express $\hat{C}(x)$ as

$$\hat{C}(x) = -\frac{\lambda}{B_{r+\lambda}} R_1(x). \tag{4.11}$$

Thus, by Lemma 4.5 (2), we find that

$$\hat{C}'(x) \geq 0 \quad \text{if and only if} \quad \frac{g_1(x)}{\psi_r(x)} \geq \frac{(\Phi g_1)(x)}{(\Phi\psi_r)(x)}. \tag{4.12}$$

Since by the lemma (4.4) it holds that $\hat{x}_1 < y_1$, we find that inequality (4.12) holds when $x \in [\hat{x}_1, \infty)$. The same calculation also shows that $\hat{C}(x)$ is decreasing when $x \in [y_2, \hat{x}_1]$. This implies that (i) is proved, and for (ii) it remains to be shown that (4.10) is true at $x = y_2$.

Using the expression (4.11) for $\hat{C}(y_2)$ and the second equation in (4.9), we find that we need to show that

$$(\Psi g_1)(y_2) - \frac{(\Phi g_1)(y_2)(\Psi\psi_r)(y_2)}{(\Phi\psi_r)(y_2)} \leq (\Psi g_2)(y_2) - \frac{(\Psi g_2)(y_2)(\Phi\psi_r)(y_2)}{(\Phi\psi_r)(y_2)}.$$

Since the boundaries of the state space are natural, we have

$$(\Psi 1)(y_2) = \frac{\psi_{r+\lambda}'(y_2)}{S'(y_2)}, \quad (\Phi 1)(y_2) = -\frac{\varphi_{r+\lambda}'(y_2)}{S'(y_2)},$$

which implies, together with the application of Lemma 4.2 to function G defined in (4.8), that

$$\begin{aligned}
&(\Psi g_1)(y_2) - \frac{(\Phi g_1)(y_2)(\Psi\psi_r)(y_2)}{(\Phi\psi_r)(y_2)} - (\Psi g_2)(y_2) + \frac{(\Phi g_2)(y_2)(\Psi\psi_r)(y_2)}{(\Phi\psi_r)(y_2)} \\
&= -\Omega(\Psi 1)(y_2) + \frac{(\Psi\psi_r)(y_2)}{(\Phi\psi_r)(y_2)}\Omega(\Phi 1)(y_2) \\
&= \frac{\Omega}{S'(y_2)}\left(-\psi_{r+\lambda}'(y_2) - \frac{\psi_{r+\lambda}(y_2)\psi_r'(y_2) - \psi_{r+\lambda}'(y_2)\psi_r(y_2)}{\varphi_{r+\lambda}'(y_2)\psi_r(y_2) - \varphi_{r+\lambda}(y_2)\psi_r'(y_2)}\varphi_{r+\lambda}'(y_2)\right),
\end{aligned}$$

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where

$$\Omega = \frac{r+\lambda}{r}(K_1 + K_2) = g_2(x) - g_1(x).$$

Because $\varphi'_{r+\lambda}(y_2)\psi_r(y_2) - \varphi_{r+\lambda}(y_2)\psi'_r(y_2) < 0$, S' and Ω are always positive, we find that the above expression is negative if and only if

$$\begin{aligned} &- \psi'_{r+\lambda}(y_2)[\varphi'_{r+\lambda}(y_2)\psi_r(y_2) - \varphi_{r+\lambda}(y_2)\psi'_r(y_2)] \\ &\quad - \varphi'_{r+\lambda}(y_2)[\psi_{r+\lambda}(y_2)\psi'_r(y_2) - \psi'_{r+\lambda}(y_2)\psi_r(y_2)] > 0. \end{aligned}$$

Finally, cancelling the terms, we get the equivalent condition

$$\psi'_{r+\lambda}(y_2)\varphi_{r+\lambda}(y_2)\psi'_r(y_2) - \varphi'_{r+\lambda}(y_2)\psi_{r+\lambda}(y_2)\psi'_r(y_2) = B_{r+\lambda}\psi'_r(y_2) > 0,$$

which holds since ψ_r is increasing. This completes the proof for (ii).

By [26, Lemma 3.6], the mapping

$$x \mapsto \frac{\varphi_{r+\lambda}(x)}{\varphi_r(x)}$$

is decreasing, which implies that

$$\varphi'_{r+\lambda}(x)\varphi_r(x) - \varphi_{r+\lambda}(x)\varphi'_r(x) = \varphi_r^2(x)\left(\frac{\varphi_{r+\lambda}(x)}{\varphi_r(x)}\right)' < 0.$$

To show that $f_\varphi(x)$ is non-decreasing at $x \geq y_1$, we first find by differentiation that

$$f'_\varphi(x) = \frac{(H'_1(x)\varphi_r(x) - H_1(x)\varphi'_r(x)) - C(y_1, y_2)((\varphi'_{r+\lambda}(x)\varphi_r(x) - \varphi_{r+\lambda}(x)\varphi'_r(x)))}{\varphi_r^2(x)}.$$

Thus the claim is equivalent to

$$(H'_1(x)\varphi_r(x) - H_1(x)\varphi'_r(x)) - C(y_1, y_2)((\varphi'_{r+\lambda}(x)\varphi_r(x) - \varphi_{r+\lambda}(x)\varphi'_r(x))) \geq 0.$$

Noting that $C(y_1, y_2) = \hat{C}(y_1)$, we find after rearrangement that the above is equivalent to

$$\frac{\varphi'_r(x)H_1(x) - \varphi_r(x)H'_1(x)}{\varphi'_r(x)\varphi_{r+\lambda}(x) - \varphi_r(x)\varphi'_{r+\lambda}(x)} \leq \frac{\psi'_r(y_1)H_1(y_1) - \varphi_r(y_1)H'_1(y_1)}{\varphi'_r(y_1)\varphi_{r+\lambda}(y_1) - \varphi_r(y_1)\varphi'_{r+\lambda}(y_1)}.$$

Similar to (4.11), we can re-express the inequality as

$$\frac{(\Psi\varphi_r)(x)}{(\Phi\varphi_r)(x)}(\Phi g_1)(x) - (\Psi g_1)(x) \geq \frac{(\Psi\psi_r)(y_1)}{(\Phi\psi_r)(y_1)}(\Phi g_1)(y_1) - (\Psi g_1)(y_1). \tag{4.13}$$

We now show that (4.13) holds at $x = y_1$ and that the left-hand side is non-decreasing with respect to x. At $x = y_1$ it suffices to show that

$$\frac{(\Psi\varphi_r)(y_1)}{(\Phi\varphi_r)(y_1)} \geq \frac{(\Psi\psi_r)(y_1)}{(\Phi\psi_r)(y_1)}.$$

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Now, since φ_r is strictly decreasing and ψ_r strictly increasing, we have

$$\frac{\varphi(z)}{\varphi(y_1)} \geq 1 \geq \frac{\psi_r(z)}{\psi_r(y_1)}$$

whenever $z \leq y_1$ and

$$\frac{\varphi(z)}{\varphi(y_1)} \leq 1 \leq \frac{\psi_r(z)}{\psi_r(y_1)}$$

when $z \geq y_1$. Thus

$$\begin{aligned}
\frac{(\Psi\varphi_r)(y_1)}{(\Phi\varphi_r)(y_1)} &= \frac{\int_0^{y_1} \varphi_r(z)\psi_{r+\lambda}(z)m'(z)\,\mathrm{d}z}{\int_{y_1}^{\infty} \varphi_r(z)\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z} \\
&= \frac{\int_0^{y_1} \frac{\varphi(z)}{\varphi(y_1)}\psi_{r+\lambda}(z)m'(z)\,\mathrm{d}z}{\int_{y_1}^{\infty} \frac{\varphi(z)}{\varphi(y_1)}\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z} \\
&\geq \frac{\int_0^{y_1} \frac{\psi_r(z)}{\psi_r(y_1)}\psi_{r+\lambda}(z)m'(z)\,\mathrm{d}z}{\int_{y_1}^{\infty} \frac{\psi_r(z)}{\psi_r(y_1)}\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z} \\
&= \frac{(\Psi\psi_r)(y_1)}{(\Phi\psi_r)(y_1)}.
\end{aligned}$$

Denote the left-hand side of equation (4.13) by u, that is,

$$u(x) = \frac{(\Psi\varphi_r)(x)}{(\Phi\varphi_r)(x)}(\Phi g_1)(x) - (\Psi g_1)(x).$$

The derivative of u can be expressed as

$$\begin{aligned}
u'(x) = &\frac{(\Phi g_1)'(x)(\Psi\varphi_r)(x)(\Phi\varphi_r)(x) + (\Phi g_1)(x)(\Psi\varphi_r)'(x)(\Phi\varphi_r)(x)}{(\Phi\varphi_r(x))^2} \\
&- \frac{(\Phi g_1)(x)(\Psi\varphi_r)(x)(\Phi\varphi_r)'(x) + (\Psi g_1)'(x)(\Phi\varphi_r)^2(x)}{(\Phi\varphi_r(x))^2} \\
= &\frac{m'(x)}{(\Phi\varphi_r)^2(x)}[\varphi_{r+\lambda}(\Psi\varphi_r)(x) + \psi_{r+\lambda}(x)(\Phi\varphi_r)(x)]\,[\varphi_r(x)(\Phi g_1)(x) - g_1(x)(\Phi\varphi_r(x))(x)].
\end{aligned}$$

Since the first two factors are positive, it now suffices to prove

$$\frac{(\Phi g_1)(x)}{(\Phi\varphi_r(x))(x)} \geq \frac{g_1(x)}{\varphi_r(x)}.$$

Now, since $g_1(x)/\varphi_r(x)$ is non-decreasing (by item (4) of Assumption 4.1),

$$\begin{aligned}(\Phi g_1)(x) &= \int_x^\infty (g_1(z)\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z)\\ &= \int_x^\infty \left(\frac{g_1(z)}{\varphi_r(z)}\varphi_r(z)\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z\right)\\ &\geq \int_x^\infty \left(\frac{g_1(x)}{\varphi_r(x)}\varphi_r(z)\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z\right)\\ &= \frac{g_1(x)}{\varphi_r(x)}\int_x^\infty (\varphi_r(z)\varphi_{r+\lambda}(z)m'(z)\,\mathrm{d}z)\\ &= \frac{g_1(x)}{\varphi_r(x)}(\Phi\varphi_r(x))(x),\end{aligned}$$

which concludes the proof. □

The following lemma gives a useful ordering of the candidate functions G_i and G_a.

Lemma 4.7. *The candidate value functions G_i and G_a satisfy*

$$G_a(x) - K_2 \geq G_i(x) \geq G_a(x) - K_1$$

for all $x \in \mathbb{R}_+$.

Proof. For $x \in (y_1, \infty)$ and for $x \in (0, y_2)$, the claim holds by construction. Thus we assume that $x \in [y_2, y_1]$ for the rest of this proof.

By a direct calculation we find for $x \in [y_2, y_1]$ that

$$\begin{aligned}G_i(x) - G_a(x) + K_1 = {}&\frac{\psi_r(x)}{\psi_r(y_1)}(\lambda(R_{r+\lambda}g_1)(y_1) - C(y_1, y_2)\varphi_{r+\lambda}(y_1))\\ &- \lambda(R_{r+\lambda}g_1)(x) + C(y_1, y_2)\varphi_{r+\lambda}(x).\end{aligned}$$

Reorganizing the terms above, we find that the proposed inequality is equivalent to

$$f_\psi(x) \leq f_\psi(y_1),$$

where

$$f_\psi(x) = \frac{\lambda(R_{r+\lambda}g_1)(x) - C(y_1, y_2)\varphi_{r+\lambda}(x)}{\psi_r(x)}.$$

The claim then follows from part (ii) of Lemma 4.6. □

Next, we show that G_i is r-excessive and $G_a - \lambda(R_{r+\lambda}g_2)$ is $(r+\lambda)$-excessive, in order to use the inequalities

$$G_i(x) \geq \mathbb{E}_x[\mathrm{e}^{-r\tau}G_i(X_\tau)]$$

and

$$G_a(x) - \lambda(R_{r+\lambda}g_2)(x) \geq \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\tau}(G_a(X_\tau) - \lambda(R_{r+\lambda}g_2)(X_\tau))]$$

for all stopping times τ.

Lemma 4.8. *The function G_i is r-excessive.*

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Proof. Since $\lim_{x\to\infty} f_\psi(x) \ge 0$, we find that that $G_i(x) \ge 0$ for all $x \in \mathbb{R}_+$. We now show that

$$\mathbb{E}_x[\mathrm{e}^{-r\tau'} G_i(X_{\tau'})] \le G_i(x),$$

where $\tau' = \inf\{t \ge 0 \mid X_t \in [c, d]\}$, from which the r-excessivity follows by [7, p. 32]. When $x \in [c, d]$, this condition trivially holds, so we consider the remaining cases (1) $x < c$ and (2) $x > d$. We split these cases further depending on where y_1 is located. Let $x < c$. Then for $c < y_1$ we find by Lemma 4.6 that

$$\begin{aligned}
\mathbb{E}_x[\mathrm{e}^{-r\tau_c} G_i(X_{\tau_c})] &= \frac{\psi_r(x)}{\psi_r(c)} G_i(c) \\
&= \frac{\psi_r(x)}{\psi_r(c)} \psi_r(c) f_\psi(y_1) \\
&= \psi_r(x) f_\psi(y_1) \\
&= G_i(x),
\end{aligned}$$

and for $y_1 < c$,

$$\begin{aligned}
\mathbb{E}_x[\mathrm{e}^{-r\tau_c} G_i(X_{\tau_c})] &= \frac{\psi_r(x)}{\psi_r(c)} G_i(c) \\
&= \frac{\psi_r(x)}{\psi_r(c)} \psi_r(c) f_\psi(\max\{y_1, c\}) \\
&= \psi_r(x) f_\psi(c) \\
&\le \psi_r(x) f_\psi(\max\{x, y_1\}) \\
&= G_i(x).
\end{aligned}$$

The case $x > d$ is proved similarly by using part (iii) of Lemma 4.6. □

Lemma 4.9. *The function $G_a - \lambda(R_{r+\lambda} g)$ is $(r+\lambda)$-excessive.*

Proof. Recall that G_a is defined as

$$G_a(x) = \begin{cases} \lambda(R_{r+\lambda} g)(x) + C(y_1, y_2)\varphi_{r+\lambda}(x), & \text{if } x \ge y_2, \\ A\psi_r(x) + K_2, & \text{otherwise.} \end{cases}$$

Then differentiation and rearranging yields

$$(\mathcal{A} - (r+\lambda))(G_a(x) - \lambda R_{r+\lambda} g(x)) = \begin{cases} 0, & \text{if } x \ge y_2, \\ -\lambda A\psi_r(x) + \lambda^2 g_2(x), & \text{else.} \end{cases}$$

Thus we need to prove that $-\lambda A\psi_r(x) + \lambda^2 g_2(x) \le 0$ for all $x \le y_2$.

First, by Lemma 4.4 and the pair of equations (4.9), we have

$$\frac{g_2(x)}{\psi_r(x)} < \frac{(\Phi g_2)(x)}{(\Phi\psi_r)(x)} < \frac{(\Phi g_2)(y_2)}{(\Phi\psi_r)(y_2)} = \frac{(\Phi g_1)(y_1)}{(\Phi\psi_r)(y_1)}.$$

Then, similarly to the derivation of (4.9), we can rewrite the above as

$$\lambda \frac{g_2(x)}{\psi_r(x)} < f_\psi(y_1).$$

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Multiplying by $\psi_r(x)$ yields

$$\lambda g_2(x) < \frac{\psi_r(x)}{\psi_r(y_1)}(H_1(y_1) - C(y_1, y_2)\phi_{r+\lambda}(y_1)).$$

Consequently,

$$-\lambda A\psi_r(x) + \lambda^2 g_2(x) = -\lambda\frac{\psi_r(x)}{\psi_r(y_1)}(H_1(y_1) - C(y_1, y_2)\phi_{r+\lambda}(y_1)) + \lambda^2 g_2(x) < 0,$$

which concludes the proof. □

Having the auxiliary results at our disposal, we can now show that G_i satisfies the Bellman principle of Proposition 3.1. First, since the function G_i satisfies the condition (4.3), we have

$$G_i(x) = \mathbb{E}_x\big[\mathrm{e}^{-r\tau_{y_1}}(G_a(X_{\tau_{y_1}}) - K_1)\big],$$

$$G_a(x) = \lambda(R_{r+\lambda}g)(x) + \mathbb{E}_x\big[\mathrm{e}^{-(r+\lambda)\sigma_{y_2}}(G_i(X_{\sigma_{y_2}}) - \lambda(R_{r+\lambda}g)(X_{\sigma_{y_2}}) + K_2)\big].$$

On the other hand, by Lemmas 4.7–4.9 we find that

$$G_i(x) \geq \mathbb{E}_x[\mathrm{e}^{-r\tau}G_i(X_\tau)] \geq \mathbb{E}_x[\mathrm{e}^{-r\tau}(G_a(X_\tau) - K_1)] \tag{4.14}$$

and

$$\begin{aligned} G_a(x) - \lambda(R_{r+\lambda}g)(x) &\geq \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\tau}(G_a(X_\tau) - \lambda(R_{r+\lambda}g)(X_\tau))] \\ &\geq \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\tau}(G_i(X_\tau) - \lambda(R_{r+\lambda}g)(X_\tau) + K_2)], \end{aligned}$$

which implies

$$G_a(x) \geq \lambda(R_{r+\lambda}g)(x) + \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\tau}(G_a(X_\tau) - \lambda(R_{r+\lambda}g)(X_\tau))]. \tag{4.15}$$

Since inequalities (4.14) and (4.15) are true for all stopping times τ, it follows that

$$G_i(x) \geq \sup_\tau\, \mathbb{E}_x[\mathrm{e}^{-r\tau}(G_a(X_\tau) - K_1)],$$

$$G_a(x) \geq \sup_\sigma\, (\lambda R_{r+\lambda}g(x) + \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\sigma}(G_a(X_\sigma) - \lambda(R_{r+\lambda}g)(X_\sigma))]).$$

Then, since for stopping times $\tau^* = \tau_{y_1}$ and $\sigma^* = \sigma_{y_2}$ we have

$$G_i(x) = \mathbb{E}_x[\mathrm{e}^{-r\tau^*}(G_a(X_{\tau^*}) - K_1)],$$

$$G_a(x) = (\lambda(R_{r+\lambda}g)(x) + \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\sigma^*}(G_a(X_{\sigma^*}) - \lambda(R_{r+\lambda}g)(X_{\sigma^*}))]),$$

it follows that

$$G_i(x) = \sup_\tau\, \mathbb{E}_x[\mathrm{e}^{-r\tau}(G_a(X_\tau) - K_1)], \tag{4.16}$$
$$G_a(x) = \sup_\sigma \big(\lambda(R_{r+\lambda}g)(x) + \mathbb{E}_x[\mathrm{e}^{-(r+\lambda)\sigma}(G_a(X_\sigma) - \lambda(R_{r+\lambda}g)(X_\sigma))]\big). \tag{4.17}$$

Using these, we obtain that (i) for all pairs of stopping times (τ, σ), we can use equations (4.16) and (4.17) to conclude that

$$\begin{aligned} G_i(x) &\geq \mathbb{E}_x[\mathrm{e}^{-r\tau}(G_a(X_\tau) - K_1)] \\ &\geq \mathbb{E}_x[\mathrm{e}^{-r\tau}(\mathbb{E}_{X_\tau}[\mathrm{e}^{-rU}g(X_U)\mathbf{1}(U < \sigma) + \mathrm{e}^{-r\sigma}(G_i(X_\sigma) + K_2)\mathbf{1}(U > \sigma)] - K_1)]. \end{aligned}$$

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(ii) For the pair (τ^*, σ^*), we have

$$\begin{aligned} G_i(x) &= \mathbb{E}_x[\mathrm{e}^{-r\tau_{y_1}}(G_a(X_{\tau_{y_1}}) - K_1)] \\ &= \mathbb{E}_x[\mathrm{e}^{-r\tau_{y_1}}(\mathbb{E}_{X_{\tau_{y_1}}}[\mathrm{e}^{-rU}g(X_U)\mathbf{1}(U < \sigma_{y_2}) \\ &\quad + \mathrm{e}^{-r\sigma_{y_2}}(G_i(X_{\sigma_{y_2}}) + K_2)\mathbf{1}(U > \sigma_{y_2})] - K_1)]. \end{aligned}$$

Thus G_i is the solution to the problem (2.3). Summarizing, we have proved the following result.

Theorem 4.1. *Let Assumption 4.1 hold. Then*

$$V_i(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + C(y_1, y_2)\varphi_{r+\lambda}(x) - K_1, & x \geq y_1, \\ A(y_1, y_2)\psi_r(x), & x > y_1, \end{cases}$$

and

$$V_a(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + C(y_1, y_2)\varphi_{r+\lambda}(x), & x \geq y_2, \\ A(y_1, y_2)\psi_r(x) + K_2, & x > y_2, \end{cases}$$

where

$$A(y_1, y_2) = \frac{(K_2 - \lambda(R_{r+\lambda}g)(y_2))\varphi_{r+\lambda}(y_1) - (K_1 - \lambda(R_{r+\lambda}g)(y_1))\varphi_{r+\lambda}(y_2)}{\psi_r(y_1)\varphi_{r+\lambda}(y_2) - \psi_r(y_2)\varphi_{r+\lambda}(y_1)},$$

$$C(y_1, y_2) = \frac{(K_2 - \lambda(R_{r+\lambda}g)(y_2))\psi_r(y_1) - (K_1 - \lambda(R_{r+\lambda}g)(y_1))\psi_r(y_2)}{\psi_r(y_1)\varphi_{r+\lambda}(y_2) - \psi_r(y_2)\varphi_{r+\lambda}(y_1)}.$$

Here, the thresholds are uniquely given by the conditions

$$H_1(y_1) = H_2(y_2),$$

$$R_1(y_1) = R_2(y_2),$$

where

$$H_l(x) = \frac{(\Phi g_l)(y_l)}{(\Phi\psi_r)(y_l)},$$

$$R_l(x) = (\Phi g_l)(y_l)\frac{(\Psi\psi_r)(y_l)}{(\Phi\psi_r)(y_l)} - (\Psi g_l)(y_l)$$

for $l = 1, 2$.

We have shown that in the diffusion case the optimal rule is to activate the investment once revenue process X is above the threshold y_1 and abandon an active investment if the revenue process reaches the level y_2 before the project is completed. Figure 1 shows an example of a realization of using this kind of stopping strategy. The agent starts as inactive (the path for the inactive agent is plotted in black), and when the process hits the threshold y_1 the agent invests and his/her status changes to active (the path for the active agent is plotted in grey). When the status is changed to active a Poisson process with intensity λ is immediately started. This starting time is plotted as a dashed vertical line and marked as T_0. In this realization of the path the process hits the threshold y_2 before the first jump of the Poisson process (dashed vertical line T_1). Thus the agent abandons the project and goes back to inactive at y_2 so that she can wait for a better opportunity. Then the agent again invests and activates when the process hits y_1, but this time the Poisson process jumps before the process hits y_2, so the agent receives the payoff $g(X_{T_4})$.

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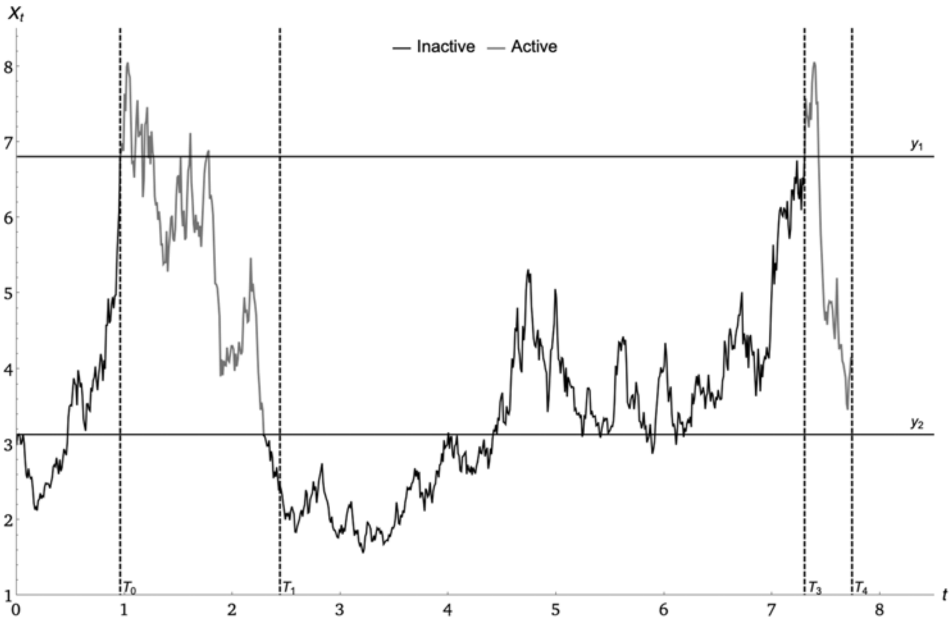

FIGURE 1. An illustration of a possible realization of the underlying process and the usage of the optimal policy given by Theorem 4.1.

4.3. An illustration

Let X be the diffusion with initial state $X_0 = x$ and the infinitesimal generator

$$\mathcal{A} = \mu x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\sigma^2 x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. This diffusion process is called a *geometric Brownian motion*. The state space of the process is $\mathbb{R}_+$ and the endpoints of the state space are natural. We further assume that $\mu < r$ and that $\mu - \frac{1}{2}\sigma^2 > 0$, so that $X_t \to \infty$ almost surely as $t \to \infty$. The scale density and the density of the speed measure read as

$$S'(x) = x^{-2\mu/\sigma^2}, \quad m'(x) = \frac{2}{\sigma^2} x^{2\mu/\sigma^2 - 2}.$$

We fix the constants r, $\lambda > 0$ and denote

$$\beta_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1,$$

$$\alpha_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0.$$

Then the minimal r-excessive functions for X are

$$\psi_r(x) = x^{\beta_0}, \quad \varphi_r(x) = x^{\alpha_0}.$$

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Further, we consider the linear payoff $g(x) = x^\theta - \eta$, where $\theta \in (0, 1]$ and $\eta > K_2(r+\lambda)/\lambda$. We notice now that our assumptions are satisfied.

By straightforward integration we find that

$$(\Phi g)(x) = \frac{2}{\sigma^2}\frac{x^{\theta-\beta_\lambda}}{\beta_\lambda-\theta} - \eta\frac{2}{\sigma^2}\frac{x^{-\beta_\lambda}}{\beta_\lambda}, \qquad (\Psi g)(x) = \frac{2}{\sigma^2}\frac{x^{\theta-\alpha_\lambda}}{\theta-\alpha_\lambda} + \eta\frac{2}{\sigma^2}\frac{x^{-\alpha_\lambda}}{\alpha_\lambda},$$
$$(\Phi\psi_r)(x) = \frac{2}{\sigma^2}\frac{x^{\beta_0-\beta_\lambda}}{\beta_\lambda-\beta_0}, \qquad (\Psi\psi_r)(x) = \frac{2}{\sigma^2}\frac{x^{\beta_0-\alpha_\lambda}}{\beta_0-\alpha_\lambda}.$$

Using the above calculations, we find using the representation (4.1) that

$$\lambda(R_{r+\lambda}g)(x) = \frac{2\lambda}{\sigma^2}\frac{x^\theta}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \frac{\eta\lambda}{r+\lambda}.$$

We note that analogous calculations also hold for g_1 and g_2 instead of g. Using these calculations, we first find that the solutions to the classical stopping problem $(V_c(x), x^*)$ and the problem with exercise lag but without reversibility (16) $(V_r(x), y^*)$ are given by

$$V_c(x) = \begin{cases} x^\theta - \eta - K_1, & x \geq x^*, \\ \dfrac{x^{*\theta} - \eta - K_1}{x^{*\beta_0}}x^{\beta_0}, & x < x^*, \end{cases}$$
$$V_r(x) = \begin{cases} \dfrac{2\lambda}{\sigma^2}\dfrac{x^\theta}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \dfrac{(\eta+K_1)\lambda}{r+\lambda}, & x \geq y^*, \\ \left(\dfrac{2\lambda}{\sigma^2}\dfrac{y^{*\theta-\beta_0}}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \dfrac{(\eta+K_1)\lambda y^{*-\beta_0}}{r+\lambda}\right)x^{\beta_0}, & x < y^*, \end{cases}$$

where x^* and y^* are given by

$$x^* = \left(\frac{(\eta+K_1)\beta_0}{\beta_0-\theta}\right)^{1/\theta},$$
$$y^* = \left(\frac{(\eta+K_1)\beta_0(\beta_\lambda-\theta)(\theta-\alpha_\lambda)}{\alpha_\lambda\beta_\lambda(\theta-\beta_0)}\right)^{1/\theta} = x^*\left(\frac{(\theta-\beta_\lambda)(\theta-\alpha_\lambda)}{\alpha_\lambda\beta_\lambda}\right)^{1/\theta}.$$

Similar calculations show that the solution to the reversible problem studied in previous sections is characterized by the value functions

$$V_i(x) = \begin{cases} \dfrac{2\lambda}{\sigma^2}\dfrac{x^\theta}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \dfrac{\eta\lambda}{r+\lambda} + C(y_1, y_2)x^{\alpha_\lambda} - K_1, & x \geq y_1, \\ A(y_1, y_2)x^{\beta_0}, & x \leq y_1, \end{cases} \tag{4.18}$$

and

$$V_a(x) = \begin{cases} \dfrac{2\lambda}{\sigma^2}\dfrac{x^\theta}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \dfrac{\eta\lambda}{r+\lambda} + C(y_1, y_2)x^{\alpha_\lambda}, & x \geq y_2, \\ A(y_1, y_2)x^{\beta_0} - K_2, & x \leq y_2, \end{cases} \tag{4.19}$$

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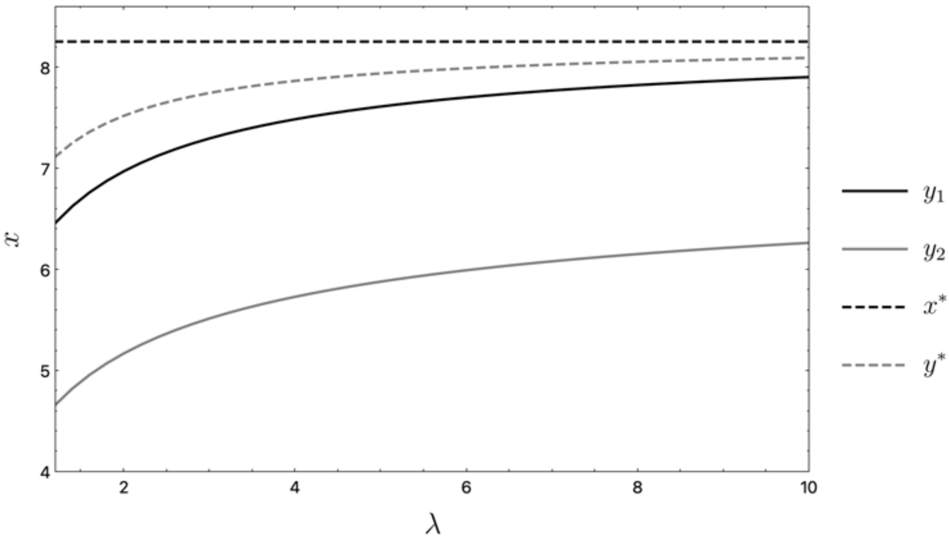

FIGURE 2. Optimal thresholds when the rate λ changes.

where

$$C(y_1, y_2) = \frac{2\lambda}{\sigma^2} \frac{\left(\frac{y_1^\theta}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \frac{\eta\lambda}{r+\lambda} - K_1\right)y_2^{\beta_0} - \left(\frac{y_2^\theta}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \frac{\eta\lambda}{r+\lambda} - K_2\right)y_1^{\beta_0}}{y_1^{\beta_0}y_2^{\alpha_\lambda} - y_1^{\alpha_\lambda}y_2^{\beta_0}},$$

$$A(y_1, y_2) = \frac{2\lambda}{\sigma^2} \frac{\left(\frac{y_1^\theta}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \frac{\eta\lambda}{r+\lambda} - K_1\right)y_2^{\alpha_\lambda} - \left(\frac{y_2^\theta}{(\beta_\lambda-\theta)(\theta-\alpha_\lambda)} - \frac{\eta\lambda}{r+\lambda} - K_2\right)y_1^{\alpha_\lambda}}{y_1^{\beta_0}y_2^{\alpha_\lambda} - y_1^{\alpha_\lambda}y_2^{\beta_0}}.$$

Further, the thresholds y_1 and y_2 in (4.18) and (4.19) are given as the unique solution to the pair of equations

$$y_1^{-\beta_0}\left(\frac{y_1^\theta}{\beta_\lambda-\theta} - \frac{\eta + K_1\frac{r+\lambda}{\lambda}}{\beta_\lambda}\right) = y_2^{-\beta_0}\left(\frac{y_2^\theta}{\beta_\lambda-\theta} - \frac{\eta + K_2\frac{r+\lambda}{\lambda}}{\beta_\lambda}\right),$$

$$y_1^{-\alpha_\lambda}\left[\frac{\beta_\lambda-\beta_0}{\beta_0-\alpha_\lambda}\left(\frac{y_1^\theta}{\beta_\lambda-\theta} - \frac{\eta + K_1\frac{r+\lambda}{\lambda}}{\beta_\lambda}\right) - \frac{y_1^\theta}{\theta-\alpha_\lambda} - \frac{\eta + K_1\frac{r+\lambda}{\lambda}}{\alpha_\lambda}\right]$$
$$= y_2^{-\alpha_\lambda}\left[\frac{\beta_\lambda-\beta_0}{\beta_0-\alpha_\lambda}\left(\frac{y_2^\theta}{\beta_\lambda-\theta} - \frac{\eta + K_2\frac{r+\lambda}{\lambda}}{\beta_\lambda}\right) - \frac{y_2^\theta}{\theta-\alpha_\lambda} - \frac{\eta + K_2\frac{r+\lambda}{\lambda}}{\alpha_\lambda}\right].$$

Since it seems that it is not possible to solve the pair of equations explicitly, we illustrate the results numerically. We select the parameters $\mu = 0.2$, $\sigma = 0.5$, $r = 0.25$, $\lambda = 1.0$, $\theta = 1.0$, $\eta = 1.0$, $K_1 = 0.05$, and $K_2 = 0.04$. Using these parameters we find that $y_2 \approx 4.46$ and $y_1 \approx 6.25$. If we let λ vary but keep the other parameters fixed, we find that the solution approaches a classical stopping problem, in the sense that $y_1 \to x^*$, as $\lambda \to \infty$; see Figure 2. This result is intuitive since in the limit $\lambda \to \infty$ there are no changes to reverse the investment and thus the payoff is immediately realized. Similarly, when $K_2 \to -\infty$ (so that it is never optimal to

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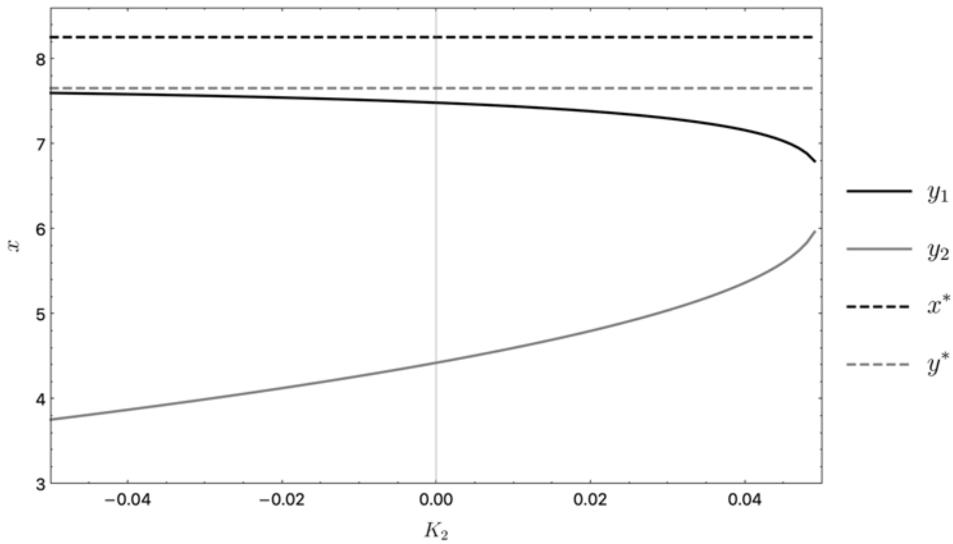

FIGURE 3. Optimal thresholds when the payoff/cost K_2 changes.

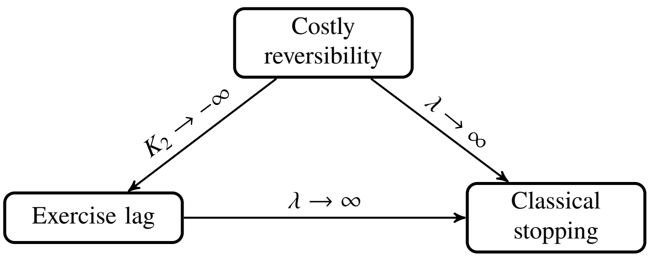

FIGURE 4. Limiting relations between the problems.

reverse the investment because the cost is too high), we find that $y_2 \to 0$ and $y_1 \to y^*$, and hence the problem reduces to the stopping problem with time-to-build considered in [17]; see Figure 3. These observations are also collected in Figure 4.

Lastly, in Figure 3 we also notice that $y_2 \to y_1$, when $K_2 \to K_1$. Interestingly, in this case it is reasonable to assume that the decision maker effectively follows a Poisson process and at each jump time makes the decision to either continue or stop and receive the payoff. Consequently, we conjecture that in this limiting case our considered problem could be represented as a Poisson stopping problem, as in [12, 16], for example. Unfortunately, the proper treatment of these considerations is beyond the scope of the present study and therefore left for future research.

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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