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Quantization dimensions of negative order

BY MARC KESSEBÖHMER AND ALJOSCHA NIEMANN
Institute for Dynamical Systems, FB 3 *– Mathematics and Computer Science, University of Bremen, Bibliothekstr.* 5, 28359 *Bremen, Germany.*
e-mails: mhk@uni-bremen.de, niemann1@uni-bremen.de

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Abstract

We investigate the possibility of defining meaningful upper and lower quantization dimensions for a compactly supported Borel probability measure of order r, including negative values of r. To this end, we employ the concept of partition functions, which generalises the notion of the L^q-spectrum, thus extending the authors' earlier work with Sanguo Zhu in a natural way. In particular, we derive inherent fractal-geometric bounds and easily verifiable necessary conditions for the existence of quantization dimensions. We state the exact asymptotics of the quantization error of negative order for absolutely continuous measures, thereby providing an affirmative answer to an open question regarding the geometric mean error posed by Graf and Luschgy in this journal in 2004.

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1. *Introduction and statement of main results*

The quantization problem for probability measures dates back to the 1980s (cf. [22]), where it was first established in the context of information theory, and has recently received renewed attention, as appropriate quantization of continuous data is fundamental to many machine learning applications [17, 20].

The aim is to examine the asymptotic behavior of the errors in the convergence of a sequence of approximations of a given random variable with a quantised version of that random variable (i.e., with a random variable that takes at most $n \in \mathbb{N}$ different values) in terms of rth power mean with $r \geq 0$. The quantization dimension (of order r) is then defined as the exponential rate of this convergence as n tends to infinity. These ideas have been extensively explored in the mathematical literature by numerous authors, including [3–6, 12, 14–16, 18, 19, 21, 29–32, 34–36]. The objective of this study is to demonstrate that this concept can be naturally extended to encompass also negative values of r. In turn, we gain new insights into the asymptotics of the geometric mean error for absolutely continuous measures, for which only the upper bound has been established in [6]. In this paper, Graf and Luschgy asked whether the upper bound they established already represents the exact asymptotic for any

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absolute continuous measure, a question we can now answer in the affirmative under very mild conditions (see Theorem 1·3 and the following remark).

1·1. *Basic set-up and first observations*

Let X be a bounded random variable with values in the normed vector space $\left(\mathbb{R}^d, \|\cdot\|\right)$, $d \in \mathbb{N}$, defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and we let $\nu := \mathbb{P} \circ X^{-1}$ denote its (compactly supported) distribution. For a given $n \in \mathbb{N}$, let $\mathcal{F}_n$ denote the set of all Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}^d$ which take at most n different values, i.e. with $\operatorname{card}\left(f\left(\mathbb{R}^d\right)\right) \leq n$, and call an element of $\mathcal{F}_n$ an *n-quantiser*. Our aim is to approximate X with a quantised version of X, i.e. X will be approximated by $f \circ X$ with $f \in \mathcal{F}_n$ where we quantify this approximation with respect to the r-quasi-norm. More precisely, we are interested in the *nth quantization error* of ν *of order* $r \in (0, +\infty]$ given by

$$\mathfrak{e}_{n,r}(\nu) := \inf_{f \in \mathcal{F}_n} \|X - f \circ X\|_{L^r_{\mathbb{P}}} = \inf_{f \in \mathcal{F}_n} \|\mathrm{id} - f\|_{L^r_\nu},$$

where $\|g\|_{L^r_{\mathbb{P}}} := \left(\int \|g\|^r \, \mathrm{d}\mathbb{P}\right)^{1/r}$ for $0 < r < \infty$ and $\|g\|_{L^\infty_{\mathbb{P}}} := \inf\{c \geq 0 : \|g\| \leq c \ \mathbb{P}\text{-a.s.}\}$ and $\mathrm{id} : x \mapsto x$ denotes the identity map on $\mathbb{R}^d$.

The problem can also be expressed using only the distribution ν, which we assume is compactly supported throughout the paper. For every $n \in \mathbb{N}$, we write for the set of non-empty sets with at most n elements $\mathcal{A}_n := \left\{A \subset \mathbb{R}^d : 1 \leq \operatorname{card}(A) \leq n\right\}$. Then due to [4, lemma 3·1] an equivalent formulation of the nth quantization error of ν of order $r \in [0, +\infty)$ is given by

$$\mathfrak{e}_{n,r}(\nu) = \begin{cases} \inf_{A \in \mathcal{A}_n} \|d(\cdot, A)\|_{L^r_\nu}, & r > 0, \\ \inf_{A \in \mathcal{A}_n} \exp \int \log d(x, A) \, \mathrm{d}\nu(x), & r = 0, \end{cases}$$

with $d(x, A) := \min_{y \in A} \|x - y\|$. By [4, lemma 6·1] it follows that $\mathfrak{e}_{n,r}(\nu) \to 0$ or more precisely, $\mathfrak{e}_{n,r}(\nu) = O(n^{-1/d})$ and, if ν is singular with respect to the Lebesgue measure, then $\mathfrak{e}_{n,r}(\nu) = o\left(n^{-1/d}\right)$ (see [4, theorem 6·2]). We define the *upper* and *lower quantization dimension for* ν *of order* r by

$$\overline{D}_r(\nu) := \limsup_{n \to \infty} \frac{\log n}{-\log \mathfrak{e}_{n,r}(\nu)}, \quad \underline{D}_r(\nu) := \liminf_{n \to \infty} \frac{\log n}{-\log \mathfrak{e}_{n,r}(\nu)},$$

with the natural convention that $1/\log(0) = 0$. In particular, when $\mathfrak{e}_{n,r}(\nu) = 0$ for all n large, then the dimension vanishes. If $\overline{D}_r(\nu) = \underline{D}_r(\nu)$, we call the common value the *quantization dimension for* ν *of order* r and denote it by $D_r(\nu)$.

We point out that for $r \geq 1$ the relevant r-quasi-norm is indeed a norm and only for $r \in (0, 1)$ a quasi-norm, which means that only a generalised triangular inequality holds. In this note, we will investigate the possibility and significance of also considering negative values for r in the definition of the quantization error, i.e. we extend the definition of $\mathfrak{e}_{n,r}(\nu)$ to values $r < 0$, by setting

$$\mathfrak{e}_{n,r}(\nu) := \inf_{A \in \mathcal{A}_n} \left(\int d(x, A)^r \, \mathrm{d}\nu(x) \right)^{1/r}.$$

Our first observation is that if ν has an atom, that is there exists $x \in \mathbb{R}^d$ with $\nu(\{x\}) > 0$, then for all $r < 0$ we have that the integrant in the definition of $\mathfrak{e}_{n,r}(\nu)$ is equal to $+\infty$ on a

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set of positive measure whenever $x \in A$. Therefore, in this situation $\mathfrak{e}_{n,r}(\nu) = 0$ for all $n \in \mathbb{N}$ and hence $D_r(\nu) = 0$ for $r < 0$. This means that if the measure has a too high concentration of mass, the quantization dimension will vanish for negative values of r. This idea can be taken further: we will assume without loss of generality that the support of ν is contained in the half-open unit cube $\mathcal{Q} := (0, 1]^d$, see [12, Introduction]. Although it is not strictly necessary to assume that ν is normalised, it will always be assumed to be finite throughout the paper. Let us define the ∞-dimension of ν by

$$\dim_\infty(\nu) := \liminf_{n\to\infty} \frac{\max_{Q\in\mathcal{D}_n} \log \nu(Q)}{\log 2^{-n}} = \liminf_{r\to 0} \frac{\sup_{x\in\mathbb{R}^d} \log \nu\left(B_r(x)\right)}{\log r},$$

where $B_r(x)$ denotes the ball in $\left(\mathbb{R}^d, \|\cdot\|\right)$ of radius $r > 0$ and center x, and $\mathcal{D}_n$ denotes the partition of $\mathcal{Q}$ by half-open cubes of the form $\prod_{i=1}^d \left(k_i 2^{-n}, (k_i + 1)\, 2^{-n}\right]$ with $(k_1, \ldots, k_d) \in \mathbb{Z}^d$. We set $\mathcal{D} := \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$, which defines –after adding the empty set– a semiring of sets. Note that $\dim_\infty(\nu) \le d$. The following observation provides us with the relevant range of meaningful values for the order of quantization.

PROPOSITION 1·1. *For a probability measure ν on $\mathbb{R}^d$ and $r < -\dim_\infty(\nu)$, we have*

$$\mathfrak{e}_{n,r}(\nu) = 0, \text{ for all } n \in \mathbb{N} \tag{1·1}$$

and, in particular, $D_r(\nu) = 0$. Furthermore, if the measure ν has an absolutely continuous part with respect to the Lebesgue measure, then (1·1) holds for $r = -d$.

Remark 1·2. Note that for absolutely continuous measure the second claim in the above proposition is only meaningful if $d = \dim_\infty(\nu)$. In instances where the quantity $\dim_\infty(\nu)$ is strictly smaller than d and ν is absolutely continuous, we refer to Examples 2·11 and 2·12.

Also note that Proposition 1·1 is consistent with our first observation, since $\dim_\infty(\nu) = 0$ whenever ν has atoms. Hence the interesting range of values for r in the formulation of the quantization problem is the interval $[-\dim_\infty(\nu), +\infty]$ which we will focus on in this paper. This observation also shows that the quantization dimension with negative order is particularly sensitive to regions of high concentration of the underlying measure — a fact that could prove particularly useful for applications.

Proof. Fix $r < -\dim_\infty(\nu)$ and $t \in (r, -\dim_\infty(\nu))$. By definition of $\dim_\infty(\nu)$, we find sequences $(s_\ell) \in (\mathbb{R}_{>0})^{\mathbb{N}}$ and $(x_\ell) \in \left(\mathbb{R}^d\right)^{\mathbb{N}}$ such that $s_\ell \searrow 0$ and $\nu\left(B_{s_\ell}(x_\ell)\right) \ge s_\ell^{-t}$. Setting $V_{n,r}(\nu) := \mathfrak{e}_{n,r}(\nu)^r$, this gives, for $\ell \to \infty$,

$$\begin{aligned} V_{n,r}(\nu) &\ge V_{1,r}(\nu) \ge \sup_{A\in\mathcal{A}_1} \int d(x, A)^r \,\mathrm{d}\nu(x) \\ &\ge \int_{B_{s_\ell}(x_\ell)} d(x, \{x_\ell\})^r \,\mathrm{d}\nu(x) \ge \nu\left(B_{s_\ell}(x_\ell)\right) s_\ell^r \ge s_\ell^{r-t} \to \infty. \end{aligned}$$

This proves the first claim.

To see that $\mathfrak{e}_{n,-d}(\nu) = 0$ for each $n \in \mathbb{N}$, by Lebesgue's differentiation theorem we find a point $x_0 \in \mathbb{R}^d$ such that $\lim_{k\to\infty} \nu\left(B_{2^{-k}}(x_0)\right)/2^{-kd} = c > 0$. Consequently, for $k \in \mathbb{N}$ large enough (say $k \ge k_0$) we have for $C_k := B_{2^{-k}}(x_0) \setminus B_{2^{-k-1}}(x_0)$ that $\nu(C_k) > (c/2)\, 2^{-d(k+1)}$.

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This gives for each $n \in \mathbb{N}$

$$V_{n,-d}(\nu) \geq \sum_{k \geq k_0} \int_{C_k} d\left(x, \{x_0\}\right)^{-d} \, \mathrm{d}\nu(x) \geq \sum_{k \geq k_0} \frac{c}{2} 2^{-d(k+1)} 2^{dk} = \infty.$$

This shows the second claim.

1·2. *Absolutely continuous measures – exact asymptotics*

We are able to extend the classical result in quantization theory for absolutely continuous probability measures with respect to the Lebesgue measure, which goes back to Zador [**22**] and was generalised by Bucklew and Wise in [**2**]; for a rigorous proof for $r \geq 1$, which also works for $r > 0$, see [**4**, theorem 6·2].

If for $\kappa > 0$ the limit

$$\mathfrak{C}_{r,\kappa}(\nu) := \lim_{n \to \infty} n^{1/\kappa} \mathfrak{e}_{n,r}(\nu)$$

exists in $[0, +\infty]$, we refer to its value as the *κ-dimensional quantization coefficient of order r of ν*. In the literature, the lower and upper quantization coefficients defined via limes inferior and superior have also frequently been considered [**5**, **6**, **16**, **26**, **33**]. If this number (or more generally, its upper and lower version) is positive and finite, the quantization dimension of order r also exists and is equal to κ. Note that $\mathfrak{C}_{r,\kappa}(\nu) = 0$ is equivalent to saying that $\mathfrak{e}_{n,r}(\nu) = o\left(n^{-1/\kappa}\right)$.

In Section 2·5, we will see that, for the Lebesgue measure restricted to $\mathcal{Q}$ and for which we write Λ, the coefficient $\mathfrak{C}_{r,d}(\Lambda)$ also exists and is finite for negative r, and positive only for $r > -d$ (cf. Lemma 2·10). In general, for an absolutely continuous probability $\nu := h\Lambda$ with density h we set $s_h := \sup\{s > 0 \colon \|h\|_s < \infty\}$, where for abbreviation we write $\|\cdot\|_s$ instead of $\|\cdot\|_{L^s_\Lambda}$. By Lemma 2·6 and 2·7 we have that $s_h \leq d/\left(d - \dim_\infty(\nu)\right)$. In particular, $-d \leq -\dim_\infty(\nu) \leq d/s_h - d$ and these inequalities might be strict as Example 2·11 demonstrates. For a measurable function h we set

$$\Phi_r(h) := \begin{cases} \|h\|_{d/(d+r)}^{1/r}, & r \in (-d, \infty) \setminus \{0\}, \\ \exp\left(-(1/d) \int h \log(h) \, \mathrm{d}\Lambda\right), & r = 0, \\ 0, & r \leq -d, \end{cases}$$

where $\|h\|_{d/(d+r)}^{1/r} = 0$ whenever $r < 0$ and $h^{d/(d+r)}$ not integrable.

THEOREM 1·3. *Let $\nu = h\Lambda$ be an absolutely continuous Borel probability measure on $\mathcal{Q}$ with density h and s_h given as above. For $r \neq d/s_h - d$, and also for $r = d/s_h - d$ provided $\|h\|_{s_h} = \infty$, the d-dimensional quantization coefficient of order r of ν exists and its finite value equals*

$$\mathfrak{C}_{r,d}(\nu) = \mathfrak{C}_{r,d}(\Lambda)\Phi_r(h).$$

Moreover, we have $\mathfrak{C}_{r,d}(\nu) > 0$ for $r > d/s_h - d$, and $\mathfrak{C}_{r,d}(\nu) = 0$ for $r < d/s_h - d$ or, if $\|h\|_{s_h} = \infty$, also for $r = d/s_h - d$.

The proof of this first theorem is outlined at the end of our paper in Section 2·5. This section covers the necessary prerequisites and explains the main steps of the proof.

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Remark 1·4. The special value $r=0$, for which $\mathfrak{e}_{n,0}(\nu)$ coincides with the geometric mean error, has been previously examined in several papers (see [**6**, **23**–**25**, **27**–**29**, **35**, **36**]). The present approach, which now takes negative orders into account, provides a straightforward answer to a question posed by Graf and Luschgy over 20 years ago. In [**6**] they showed that

$$\limsup_{n\to\infty} n^{1/d}\mathfrak{e}_{n,0}(\nu) \leq \mathfrak{C}_{0,d}(\Lambda)\Phi_0(h), \tag{1·2}$$

and pointed out that it "*remains an open question whether one has a genuine limit in* (1·2) *for absolutely continuous probabilities* $\mathbb{P}$ *different from uniform distributions on cubes*." Whenever $s_h > 1$, our first main result covers the geometric mean error, i.e. $r=0$, and hence answers the question in the affirmative.

It should also be noted that explicit formulae for $\mathfrak{C}_{r,d}(\Lambda)$ are often difficult to obtain and that its value is dependent on the underlying norm on $\mathbb{R}^d$. Nevertheless, the quantization dimension is independent of the underlying norm. For the sake of simplicity, both the euclidean and the maximum norm will be used in the subsequent proofs.

In Example 2·12 we provide an example with $\|h\|_{s_h} < \infty$ such that the d-dimensional quantization coefficient of order $d/s_h - d$ exists and is positive. This critical situation $\|h\|_{s_h} < \infty$ will be revisited in Remark 1·9.

The fact that $\|h\|_{d/(d+r)} = \infty$ for a certain negative r indicates a strong concentration of mass and can be considered analogous to singular measures and positive r. In both cases, the convergence rate of the quantization error is $o\left(n^{-1/d}\right)$.

1·3. *Partition functions and related concepts*

Building upon the ideas developed in the context of spectral problems for Kreĭn–Feller operators in [**9**, **10**] and for approximation orders of Kolmogorov, Gel'fand, and linear widths in [**8**, **13**], we will address the quantization problem as initiated in [**12**], where we considered positive order and determine the exact value of the upper quantization dimension with the help of the L^q-spectrum. In this instance, however, we will consider negative order, for which novel concepts and strategies are required, particularly the more general concept of the $\mathfrak{J}$-partition function. Interestingly, the underlying methods elaborated in [**11**] and initially utilised in the context of spectral problems prove to be precisely the appropriate tools in this context. The main difference to the results obtained for the positive order case is that, this time, our formalism provides the exact value of the lower quantization dimension, as well as an upper bound for the upper quantization dimension (see Theorem 1·5). Additionally, we provide the exact value for the upper and lower quantization dimension in the regular cases (see Section 1·6).

With $\mathcal{D}_n(Q) := \left\{Q' \in \mathcal{D}_n : Q' \subset Q\right\}$, $n\in\mathbb{N}$, we let $\mathcal{D}(Q) := \bigcup_{n\in\mathbb{N}} \mathcal{D}_n(Q)$. For $r > -\dim_\infty(\nu)$, we then set

$$\mathfrak{J} := \mathfrak{J}_{\nu,r} : \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad Q \mapsto \max_{Q'\in\mathcal{D}(Q)} \nu\left(Q'\right)\Lambda\left(Q'\right)^{r/d}$$

and define the $\mathfrak{J}$-*partition function* $\tau_{\mathfrak{J}}$, for $q\in\mathbb{R}_{\geq 0}$, by

$$\tau_{\mathfrak{J}}(q) := \limsup_{n\to\infty} \tau_{\mathfrak{J},n}(q), \quad \text{with} \quad \tau_{\mathfrak{J},n}(q) := \frac{\log\left(\sum_{Q\in\mathcal{D}_n,\mathfrak{J}(Q)>0} \mathfrak{J}(Q)^q\right)}{\log\left(2^n\right)}.$$

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Define the critical value

$$q_r := \inf\left\{q > 0 : \tau_{\mathfrak{J}_{\nu,r}}(q) < 0\right\}.$$

Note that for $r \geq 0$ we have $\mathfrak{J}_{\nu,r}(Q) = \nu(Q)\Lambda(Q)^{r/d}$ and consequently for non-negative values of r, $\tau_{\mathfrak{J}}(q) = \beta_\nu(q) - qr$, where

$$\beta_\nu : q \mapsto \limsup_{n\to\infty} \beta_{\nu,n}(q) \quad \text{with} \quad \beta_{\nu,n}(q) := \frac{\log\left(\sum_{Q\in\mathcal{D}_n,\nu(Q)>0} \nu(Q)^q\right)}{\log\left(2^n\right)}$$

denotes the *L^q-spectrum of* ν, which was the central object in [**12**]. In Example 1·12 (see also Figure 1 on page 10) we provide a measure for which $\tau_{\mathfrak{J}_{\nu,r}}(q) = \beta_\nu(q) - qr$ holds also for negative values of r. We always have $\tau_{\mathfrak{J}_{\nu,r}}(q) \geq \beta_\nu(q) - qr$ by the definition $\tau_{\mathfrak{J}}$. It is easy to construct purely atomic measures such that $q_r = 0$ for all $r > 0$ and $\beta_\nu(0) > 0$, see [**9**]. In this case it turns out that the upper quantization dimension is also 0.

For the following fundamental inequalities for q_r we assume $\dim_\infty(\nu) > 0$. Then $\beta_\nu(0) = \tau_{\mathfrak{J}_{\nu,r}}(0) = \overline{\dim}_M(\nu) \geq \dim_\infty(\nu) > 0$ and for $-\dim_\infty(\nu) < r < 0$ and $q \geq 0$, we have $\tau_{\mathfrak{J}_{\nu,r}}(q) \geq \beta_\nu(q) - qr$. For $q \geq 1$, the convexity of $\tau_{\mathfrak{J}_{\nu,r}}$ gives

$$\tau_{\mathfrak{J}_{\nu,r}}(q) \geq \left(\tau_{\mathfrak{J}_{\nu,r}}(1) - \tau_{\mathfrak{J}_{\nu,r}}(0)\right) q + \tau_{\mathfrak{J}_{\nu,r}}(0) \geq \left(-r - \beta_\nu(0)\right) q + \beta_\nu(0).$$

On the other hand, again the convexity of $\tau_{\mathfrak{J}_{\nu,r}}$ gives for $q \geq 1$

$$\tau_{\mathfrak{J}_{\nu,r}}(q) \leq \left(-\dim_\infty(\nu) - r\right) q + \dim_\infty(\nu) + r + \tau_{\mathfrak{J}_{\nu,r}}(1).$$

Combining both inequalities and using the definition of q_r proves for $r \in (-\dim_\infty(\nu), 0)$

$$1 < \frac{\overline{\dim}_M(\nu)}{\overline{\dim}_M(\nu) + r} \leq q_r \leq 1 + \frac{\tau_{\mathfrak{J}_{\nu,r}}(1)}{\dim_\infty(\nu) + r} \leq 1 + \frac{\overline{\dim}_M(\nu) - \dim_\infty(\nu)}{\dim_\infty(\nu) + r}. \tag{1·3}$$

Moreover, for $r > 0$, due the convexity of β_ν and the fact that $\beta_\nu(q) = \tau_{\mathfrak{J}_{\nu,r}}(q) + rq \leq (1-q)\,\overline{\dim}_M(\nu)$ for $q \in [0, 1]$, we conclude

$$0 < q_r \leq \frac{\overline{\dim}_M(\nu)}{\overline{\dim}_M(\nu) + r} < 1. \tag{1·4}$$

Further, we will need some ideas from entropy theory: Let Π denote the set of finite partitions of $\mathcal{Q}$ by cubes from $\mathcal{D}$. We define

$$\mathcal{M}_{\nu,r}(x) := \inf\left\{\mathrm{card}(P) : P \in \Pi,\ \max_{Q\in P} \mathfrak{J}_{\nu,r}(Q) < 1/x\right\}$$

and

$$\overline{h}_{\nu,r} := \limsup_{x\to\infty} \frac{\log \mathcal{M}_{\nu,r}(x)}{\log x}, \quad \underline{h}_{\nu,r} := \liminf_{x\to\infty} \frac{\log \mathcal{M}_{\nu,r}(x)}{\log x}$$

will be called the *upper,* resp. *lower,* (ν, r)*-partition entropy.* We write $\overline{h}_r := \overline{h}_{\nu,r}$ and $\underline{h}_r := \lim_{\varepsilon\downarrow 0} \underline{h}_{\nu,r-\varepsilon}$.

For all $n \in \mathbb{N}$ and $\alpha > 0$, we define

$$\mathcal{N}_{\nu,\alpha,r}(n) := \mathrm{card} N_{\nu,\alpha,r}(n), \quad N_{\nu,\alpha,r}(n) := \left\{Q \in \mathcal{D}_n : \mathfrak{J}_{\nu,r}(Q) \geq 2^{-\alpha n}\right\},$$

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and set

$$\overline{F}_{\nu,r}(\alpha) := \limsup_n \frac{\log^+\left(\mathcal{N}_{\nu,\alpha,r}(n)\right)}{\log 2^n} \text{ and } \underline{F}_{\nu,r}(\alpha) := \liminf_n \frac{\log^+\left(\mathcal{N}_{\nu,\alpha,r}(n)\right)}{\log 2^n},$$

with $\log^+(x) := \max\{0, \log(x)\}$, $x \geq 0$. Following [10], we refer to the quantities

$$\overline{F}_r := \overline{F}_{\nu,r} := \sup_{\alpha>0} \frac{\overline{F}_{\nu,r}(\alpha)}{\alpha} \text{ and } \underline{F}_r := \underline{F}_{\nu,r} := \sup_{\alpha>0} \frac{\underline{F}_{\nu,r}(\alpha)}{\alpha}$$

as the (ν, r)-*upper*, resp. *lower, optimised coarse multifractal dimension*. We know from general consideration obtained in [11] that we always have $\overline{F}_r = q_r = \overline{h}_r \geq \underline{h}_r \geq \underline{F}_r$, see Proposition 2·1.

1·4. *Main results*

In our main theorem we combine the main result in [12] for positive r with our new results on negative r. Note that our assumption in [12], that $\sup_{x\in(0,1)} \beta_\nu(x) > 0$, is implied by our stronger assumption $\dim_\infty(\nu) > 0$. It is important to note that under this stronger assumption $r > 0$ implies $0 < q_r < 1$, just as $-\dim_\infty(\nu) < r < 0$ implies $q_r > 1$ (see (1·4) and (1·3)). Finally, we set

$$\dim_H(\nu) := \inf\{\dim_H(A) : \nu(A) > 0\}$$

with $\dim_H(A)$ referring to the *Hausdorff dimension* of $A \subset \mathbb{R}^d$. The proof of our main theorem is postponed to the last section.

THEOREM 1·5. *For a compactly supported probability measure ν on $\mathbb{R}^d$ we have*:

(i) *for $r \in (0, +\infty)$,*

$$\frac{r\underline{F}_r}{1-\underline{F}_r} \leq \underline{D}_r(\nu) \leq \frac{r\underline{h}_r}{1-\underline{h}_r} \leq \frac{r\overline{h}_r}{1-\overline{h}_r} = \overline{D}_r(\nu) = \frac{rq_r}{1-q_r} = \frac{r\overline{F}_r}{1-\overline{F}_r};$$

(ii) *for $r \in (-\dim_\infty(\nu), 0)$,*

(*a*) $\dfrac{r\overline{F}_r}{1-\overline{F}_r} = \dfrac{rq_r}{1-q_r} = \underline{D}_r(\nu) = \dfrac{r\overline{h}_r}{1-\overline{h}_r}$,

(*b*) $\dfrac{r\overline{h}_r}{1-\overline{h}_r} \leq \dfrac{r\underline{h}_r}{1-\underline{h}_r} \leq \overline{D}_r(\nu) \leq \dfrac{r\underline{F}_r}{1-\underline{F}_r}$ *under the assumption* $\underline{F}_r > 1$;

(iii) *for $r = 0$, and under the assumption that* $\dim_\infty(\nu) > 0$ *and $r \mapsto q_r$ is differentiable at 0, we have β_ν is differentiable at 1 and*

$$D_0(\nu) = -\beta'_\nu(1) = \dim_H(\nu).$$

Remark 1·6. It should be noted that in [12], the notion of the *(upper) generalised Rényi dimension of* ν [1] was employed. In our context, it is necessary to replace the concept of the L^q-spectrum with the more general concept of the $\tau_{\mathfrak{J}_{\nu,r}}$-partition function, where for $r > -\dim_\infty(\nu)$, we set

$$\overline{\mathfrak{R}}_\nu(q) := \begin{cases} \tau_{\mathfrak{J}_{\nu,r}}(q) / (1-q), & \text{for } q \in \mathbb{R}_{\geq 0} \setminus \{1\}, \\ \limsup_n \left(\sum_{C\in\mathcal{D}_n} \nu(C)\log\nu(C)\right) / \log\left(2^{-n}\right), & \text{for } q = 1. \end{cases}$$

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In this manner, we derive for both positive and negative values of r the remarkable identity

$$\overline{\mathfrak{R}}_{\nu}\left(q_r\right)=\frac{rq_r}{1-q_r},$$

which can be applied to Theorem 1·5 (i) and (ii), providing an alternative expression for $\overline{D}_r(\nu)$ whenever r is positive and for $\underline{D}_r(\nu)$ whenever r is negative.

The connection between the derivatives of $r\mapsto q_r$ and β_ν becomes apparent in the proof at the beginning of Section 2, where we show that β_ν' is differentiable at 1 with $\beta_\nu'(1)=1/\partial_r q_r|_{r=0}$. This fact is contingent on the monotonicity of $r\mapsto\underline{D}_r(\nu)$ on $(-\dim_\infty(\nu),0]$, which we establish in Lemma 1·7.

1·5. *Fractal-geometric bounds and basic properties*

For a compactly supported probability measure ν on $\mathbb{R}^d$, we let $\overline{\dim}_M(\nu):=\overline{\dim}_M(\operatorname{supp}(\nu))$, where $\overline{\dim}_M(A)$, denotes the *upper Minkowski dimension* of the bounded set $A\subset\mathbb{R}^d$.

The fact that $\tau_{\mathfrak{J}}$ is proper, convex with $\beta_\nu(0)=\tau_{\mathfrak{J}}(0)=\overline{\dim}_M(\nu)$, $\tau_{\mathfrak{J}}(q)\geq\beta_\nu(q)-rq$ for $q\geq 0$, and that its asymptotic slope $\lim_{q\to\infty}\tau_{\mathfrak{J}}(q)/q$ is equal to $-r-\dim_\infty(\nu)$ plays a crucial role in many fundamental properties of the quantization dimension. To begin this investigation, we first observe a simple fact that is well known for positive r.

LEMMA 1·7. *If* $\dim_\infty(\nu)>0$, *then the functions* $r\mapsto\underline{D}_r(\nu)$ *and* $r\mapsto\overline{D}_r(\nu)$ *are both monotonically increasing on the interval* $(-\dim_\infty(\nu),0]$.

Proof. Note that for $-\dim_\infty(\nu)<s<r<0$ and $A\in\mathcal{A}_n$ we have by Hölder's inequality (assuming without loss of generality that ν is normalised)

$$\int d(x,A)^r\,\mathrm{d}\nu\leq\left(\int d(x,A)^s\,\mathrm{d}\nu\right)^{r/s}\leq\left(V_{n,s}(\nu)\right)^{r/s}\Longrightarrow\left(\int d(x,A)^r\,\mathrm{d}\nu\right)^{1/r}\geq\mathfrak{e}_{n,s}(\nu)$$

and by Jensen's inequality, to cover the case $r=0$,

$$\int\log d(x,A)^r\,\mathrm{d}\nu\leq\log\int d(x,A)^r\,\mathrm{d}\nu$$
$$\Longrightarrow\exp\int\log d(x,A)\,\mathrm{d}\nu\geq\left(\int d(x,A)^r\,\mathrm{d}\nu\right)^{1/r}\geq\mathfrak{e}_{n,r}(\nu).$$

Taking in both cases the infimum over all $A\in\mathcal{A}_n$, gives

$$\mathfrak{e}_{n,s}(\nu)\leq\mathfrak{e}_{n,r}(\nu)\leq\mathfrak{e}_{n,0}(\nu)$$

and the claim follows.

Let us now discuss the behaviour of the quantization dimension at the relevant boundary points $-\dim_\infty(\nu)$ and $+\infty$. The case $+\infty$ follows from [12] and by observing

$$\lim_{r\nearrow+\infty}\overline{D}_r(\nu)=\lim_{q\searrow 0}\beta_\nu(q)\leq\beta_\nu(0)=\overline{\dim}_M(\nu)=\overline{D}_\infty(\nu),\tag{1·5}$$

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where the last equality can be found in [**4**, theorem 11·7, Proposition 11·9]. To discuss the boundary point $-\dim_\infty(\nu)$ we need the following auxiliary quantity

$$a_\nu := \sup\{q_r : r \in (-\dim_\infty(\nu), 0)\}.$$

LEMMA 1·8. *For a compactly supported probability measure* ν *on* $\mathbb{R}^d$ *with* $\dim_\infty(\nu) > 0$,

$$\dim_\infty(\nu) \le \lim_{r \searrow -\dim_\infty(\nu)} \underline{D}_r(\nu) = \frac{a_\nu}{a_\nu - 1}\dim_\infty(\nu), \tag{1·6}$$

where we set $a_\nu/(a_\nu - 1) = 1$ *whenever* $a_\nu = +\infty$.

The proofs will be given at the end of Section 2·3, where we also show that a_ν is strictly greater than 1. See also [**12**, proposition 1·7] for the corresponding upper bounds for the upper quantization dimension of order $r > 0$.

Remark 1·9. It is easy to construct an absolutely continuous measure with a density that has one singularity, such that β_ν is piecewise linear, which gives rise to the above described behavior with $a_\nu < \infty$, see Example 2·12.

The above observation and lemma demonstrate that the boundary behavior is only partially encoded by $\tau_{\mathfrak{J}}$. Equation (1·5) shows that continuity of $r \mapsto \overline{D}_r(\nu)$ in $r = +\infty$ if and only if the L^q-spectrum β_ν is continuous from the right in 0 or, equivalently, the same holds for $\tau_{\mathfrak{J}_{\nu,r}}$ for all $r > 0$. In general, continuity of $r \mapsto \underline{D}_r(\nu)$ in $r = -\dim_\infty(\nu)$ from the right cannot be derived from $\tau_{\mathfrak{J}}$ alone. In point of fact, Proposition 1·1 together with Theorem 1·5 covers all values of r except the critical value $r = -\dim_\infty(\nu)$. We would like to point out that the behaviour at this critical value is not easily accessible and depends very much on the measure under consideration:

For instance, as we have already observed in Theorem 1·3, for the uniform distribution Λ we have $\dim_\infty(\Lambda) = d$, while

$$D_r(\Lambda) = \begin{cases} d, & r \in (-\dim_\infty(\Lambda), \infty), \\ 0, & r \le -\dim_\infty(\Lambda). \end{cases}$$

That is to say, $r \mapsto D_r(\Lambda)$ is discontinuous and continuous from the left in $r = -\dim_\infty(\Lambda)$.

On the other hand, in Example 2·12 we construct an absolutely continuous measure ν with $d > \dim_\infty(\nu) > 0$ such that

$$\lim_{r \searrow -\dim_\infty(\nu)} \underline{D}_r(\nu) = \underline{D}_{-\dim_\infty(\nu)}(\nu) = d,$$

that is, $r \mapsto D_r(\nu)$ is this time discontinuous and continuous from the right in $r = -\dim_\infty(\nu)$.

1·6. *Regularity results*

As a second main result we find necessary conditions for the upper and lower quantization dimension to coincide, which are easy to check in many situations.

Definition 1·10. We define two notions of regularity for a compactly supported probability measure ν on $\mathbb{R}^d$ such that $\dim_\infty(\nu) > 0$.

(1) The measure ν is called *multifractal-regular at r (r-MF-regular)* if $\underline{F}_{\nu,r} = \overline{F}_{\nu,r}$.

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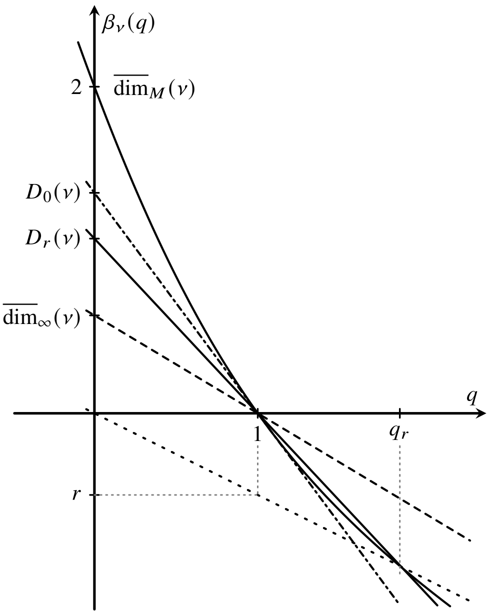

Figure 1. For the L^q-spectrum of the self-similar measure ν supported on a *dyadic Menger sponge* in $\mathbb{R}^3$ with four contractions and with probability vector $(0.66, 0.2, 0.08, 0.06)$ we have $\beta_\nu(q) = \tau_{\mathfrak{J}_{\nu,r}}(q) + qr$, $\beta_\nu(0) = 2$ and $\dim_\infty(\nu) = \log 0.66/\log 2 < -0.599$. For $r = -0.5 > -\dim_\infty(\nu)$ the intersection of the graph of β_ν and the dashed line determines q_r. The (solid) line through the points $(q_r, \beta_\nu(q_r))$ and $(1, 0)$ intersects the vertical axis in $D_r(\nu)$. The (dash-dotted) tangent to β_ν in 1 intersects the vertical axis in $D_0(\nu)$. Also the lower bound $D_r(\nu) \geq \dim_\infty(\nu)$ becomes obvious.

(2) The measure ν is called *partition function regular at r (r-PF-regular)*

a. $\tau_{\mathfrak{J}}(q) = \liminf_n \tau_{\mathfrak{J},n}(q) \in \mathbb{R}$ for all $q \in (q_r - \varepsilon, q_r)$, for some $\varepsilon > 0$, or
b. $\tau_{\mathfrak{J}}(q_r) = \liminf_n \tau_{\mathfrak{J},n}(q_r)$ and $\tau_{\mathfrak{J}}$ is differentiable at q_r.

The following theorem, which is a direct consequence of [11, theorem 1·12], shows that the spectral partition function is a valuable auxiliary concept to determine the quantization dimension for a given measure ν.

THEOREM 1·11. *The following regularity implications hold for* $r \in (-\dim_\infty(\nu), +\infty) \setminus \{0\}$*:*

$$\nu \text{ is } r\text{-PF-regular} \implies \nu \text{ is } r\text{-MF-regular} \implies \underline{D}_r(\nu) = \overline{D}_r(\nu) = \frac{rq_r}{1-q_r}.$$

We would like to point out that all examples discussed in [12] for which the quantization dimension exists (*self-similar measure, inhomogeneous self-similar measure, Gibbs measures with possible overlap*) remain literally valid for $r \in (-\dim_\infty(\nu), 0)$ as well.

Example 1·12. We briefly discuss one particularly regular example. We consider the self-similar measure ν supported on a *dyadic Menger sponge* in $\mathbb{R}^3$ with the four defining contractions

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$$\{\varphi_k : z \mapsto 1/2 \cdot z + k : k \in \{(0,0,0), (1/2,0,0), (0,1/2,0), (0,0,1/2)\}\}$$

and probability vector $(0.66, 0.2, 0.08, 0.06)$. In this example, we have that $\tau_{\mathfrak{J}_{\nu,r}}(q) = \beta_\nu(q) - qr$ even for $r < 0$ and the L^q-spectrum exists as a limit and is differentiable on $\mathbb{R}_{>0}$. Furthermore, we have $\beta_\nu(0) = \overline{\dim}_M(\nu) = 2$. Therefore, our second main result (see Theorem 1·11) on regularity applies and we can read off the value of q_r and, as a consequence of Theorem 1·5 also of $D_r(\nu)$, directly from β_ν for all values $r > -\dim_\infty(\nu) = \log_2 0.66 \approx -0.599$ as demonstrated in Figure 1 on page 10.

2. *Proof of main theorems*

In this section, we only give a proof for the upper and lower bounds of the quantization dimension of order $r \in (-\dim_\infty(\nu), 0)$ as stated in Theorem 1·5 (ii) and the proof of part (iii). This is sufficient for proving Theorem 1·5, since part (i) is fully covered by [**12**]. We conclude this section by outlining the proof of the theorem on absolutely continuous measures, Theorem 1·3.

2·1. *Optimal partitions and partition entropy*

We make use of some general observations from [**11**] which are valid for arbitrary set functions $\mathfrak{J} : \mathcal{D} \to \mathbb{R}_{\geq 0}$ on the dyadic cubes $\mathcal{D}$, which are monotone, $\dim_\infty(\mathfrak{J}) > 0$ (in particular uniformly vanishing) and locally non-vanishing with $\mathfrak{J}(\mathcal{Q}) > 0$ and such that $\liminf_n \tau_{\mathfrak{J},n}(q) \in \mathbb{R}$ for some $q > 0$. Here, *uniformly vanishing* means $\lim_{k\to\infty} \sup_{Q \in \bigcup_{n\geq k} \mathcal{D}_n} \mathfrak{J}(Q) = 0$ and *locally non-vanishing* means $\mathfrak{J}(Q) > 0$ implies that there exists $Q' \in \mathcal{D}(Q) \setminus \{Q\}$ with $\mathfrak{J}(Q') > 0$. It is important to note that all these conditions on the set function are fulfilled for our particular choice $\mathfrak{J} = \mathfrak{J}_{\nu,r}$ whenever $r > -\dim_\infty(\nu)$ since $\dim_\infty(\mathfrak{J}_{\nu,r}) = \dim_\infty(\nu) + r$.

We also recall the closely connected *dual problem,* where we consider

$$\gamma_{\mathfrak{J},n} := \inf_{P \in \Pi, \mathrm{card}(P) \leq n} \max_{Q \in P} \mathfrak{J}(Q)$$

and convergence rates

$$\overline{\alpha}_{\mathfrak{J}} := \limsup_{n\to\infty} \frac{\log(\gamma_{\mathfrak{J},n})}{\log(n)} \text{ and } \underline{\alpha}_{\mathfrak{J}} := \liminf_{n\to\infty} \frac{\log(\gamma_{\mathfrak{J},n})}{\log(n)}.$$

With this at hand, we can state the crucial results used in the proofs of our main theorem as follows.

PROPOSITION 2·1 ([**11**, theorems 1·4, 1·8 and section 1·3]). *We have*

$$\overline{F}_{\mathfrak{J}} = \overline{h}_{\mathfrak{J}} = \frac{-1}{\overline{\alpha}_{\mathfrak{J}}} = q_{\mathfrak{J}} \quad \textit{and} \quad \underline{F}_{\mathfrak{J}} \leq \underline{h}_{\mathfrak{J}} = \frac{-1}{\underline{\alpha}_{\mathfrak{J}}}.$$

For the proof of the main theorem, we divide the problem into upper and lower bounds.

2·2. *Upper bounds*

We first prove the upper bounds, which are somewhat less demanding than the lower bounds discussed thereafter. In this section, the underlying norm on $\mathbb{R}^d$ is assumed to be euclidean.

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PROPOSITION 2·2. *For $r \in (-\dim_\infty(\nu), 0)$, we have*

$$\underline{D}_r(\nu) \le \frac{r\overline{F}_r}{1-\overline{F}_r} = \frac{r\overline{h}_r}{1-\overline{h}_r} = \frac{rq_r}{1-q_r}$$

and, if $\underline{F}_r > 1$,

$$\overline{D}_r(\nu) \le \frac{r\underline{F}_r}{1-\underline{F}_r}.$$

Proof. The second and third equalities follow from Proposition 2·1. For fixed $\alpha > 0$ define $c_{\alpha,n} := \operatorname{card}\left(\mathcal{N}_{\nu,\alpha,r}(n)\right)$. We begin with the upper bound for the lower quantization dimension. For each $Q \in \mathcal{N}_{\nu,\alpha,r}(n)$ we consider $Q' \in \mathcal{D}(Q)$ such that

$$\nu\left(Q'\right)\Lambda\left(Q'\right)^{r/d} = \max_{C\in\mathcal{D}(Q)} \nu\left(C\right)\Lambda\left(C\right)^{r/d} = \mathfrak{J}_{\nu,r}(Q)$$

and let A denote the set of all centers of the elements Q' for $Q \in \mathcal{N}_{\nu,\alpha,r}(n)$. Then,

$$\sup_{x\in Q'} d(x,A) \le \sqrt{d}\Lambda\left(Q'\right)^{1/d}$$

keeping in mind that r is negative,

$$\begin{aligned}\mathfrak{e}_{c_{\alpha,n},r}(\nu) &\le \left(\int d(x,A)^r \,\mathrm{d}\nu(x)\right)^{1/r} \le \left(\sum_{Q\in\mathcal{N}_{\nu,\alpha,r}(n)} \int_{Q'} d(x,A)^r \,\mathrm{d}\nu(x)\right)^{1/r} \\ &\le \sqrt{d}\left(\sum_{Q\in\mathcal{N}_{\nu,\alpha,r}(n)} \Lambda\left(Q'\right)^{r/d}\nu\left(Q'\right)\right)^{1/r} = \sqrt{d}\left(\sum_{Q\in\mathcal{N}_{\nu,\alpha,r}(n)} \mathfrak{J}_{\nu,r}(Q)\right)^{1/r} \\ &\le \sqrt{d}c_{\alpha,n}^{1/r}2^{-\alpha n/r}.\end{aligned}$$

Since $\overline{F}_r = q_r > 1$ we consider only α in $\overline{\mathscr{A}} := \left\{\alpha > 0 : \overline{F}_r(\alpha)/\alpha > 1\right\} \neq \varnothing$. For such $\alpha \in \overline{\mathscr{A}}$, take a subsequence (n_k) such that $\lim_k \log c_{\alpha,n_k}/n_k = \overline{F}_r(\alpha) > \alpha > 0$ and $c_{\alpha,n_k}^{1/r}2^{-\alpha n_k/r} < 1$. Then

$$\frac{\log c_{\alpha,n_k}}{-\log \mathfrak{e}_{c_{\alpha,n_k},r}(\nu)} \le \frac{r\log c_{\alpha,n_k}}{-r/2\log d - \log c_{\alpha,n_k} + \alpha n_k} \le \frac{r\log c_{\alpha,n_k}/\left(\alpha n_k\right)}{1 - r\log\left(d\right)/\left(2\alpha n_k\right) - \log c_{\alpha,n_k}/\left(\alpha n_k\right)}$$

and therefore

$$\underline{D}_r(\nu) = \liminf_{n\to\infty} \frac{\log n}{-\log \mathfrak{e}_{n,r}(\nu)} \le \lim_{k\to\infty} \frac{\log c_{\alpha,n_k}}{-\log \mathfrak{e}_{c_{\alpha,n_k},r}(\nu)} \le \frac{r\overline{F}_r(\alpha)/\alpha}{1-\overline{F}_r(\alpha)/\alpha}.$$

Taking the infimum over $\alpha \in \overline{\mathscr{A}}$ and, keeping in mind that $r < 0$, yields

$$\underline{D}_r(\nu) \le \inf_{\alpha\in\overline{\mathscr{A}}} \frac{r\overline{F}_r(\alpha)/\alpha}{1-\overline{F}_r(\alpha)/\alpha} = \frac{r\sup_{\alpha>0}\overline{F}_r(\alpha)/\alpha}{1-\sup_{\alpha>0}\overline{F}_r(\alpha)/\alpha} = \frac{r\overline{F}_r}{1-\overline{F}_r} = \frac{rq_r}{1-q_r}.$$

Finally, we show the upper bound for the upper quantization dimension under the assumption $\underline{F}_r > 1$. Let $Q \in E_{\alpha,n}$, $Q' \in \mathcal{D}(Q)$ and A be given as above. Then for $\alpha > 0$ such that $\underline{F}_r(\alpha) > 0$ and every $\varepsilon \in \left(0, \underline{F}_r(\alpha)\right)$ and n large, we have $c_{\alpha,n} \ge 2^{n\left(\underline{F}_r(\alpha)-\varepsilon\right)}$. With

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$n_k := \lfloor \log_2(k) / (\underline{F}_r(\alpha) - \varepsilon) \rfloor$, $k \in \mathbb{N}$, we find

$$\mathfrak{e}_{k,r}(\nu) \leq \mathfrak{e}_{c_{\alpha,n_k},r}(\nu) \leq \sqrt{d} c_{\alpha,n_k}^{1/r} 2^{-\alpha n_k / r}.$$

Since $\underline{F}_r > 1$ we have $\alpha \in \underline{\mathscr{A}} := \{\alpha > 0 : \underline{F}_r(\alpha)/\alpha > 1\} \neq \varnothing$ and $\sqrt{d} c_{\alpha,n_k}^{1/r} 2^{-\alpha n_k / r} < 1$ for k large. Hence, we find

$$\begin{aligned}\frac{\log k}{-\log \mathfrak{e}_{k,r}(\nu)} &\leq \frac{r \log k}{-r/2 \log d - \log c_{\alpha,n_k} + \alpha n_k \log(2)} \\ &\leq \frac{r \log k}{-r/2 \log d - \log k + \alpha \log(k) / (\underline{F}_r(\alpha) - \varepsilon)}.\end{aligned}$$

Taking the upper limit and since $\underline{F}_r(\alpha)/\alpha > 1$ we find

$$\overline{D}_r(\nu) \leq \frac{r\underline{F}_r(\alpha)/\alpha}{1 - \underline{F}_r(\alpha)/\alpha}.$$

Taking the infimum over $\alpha \in \underline{\mathscr{A}}$ yields

$$\overline{D}_r(\nu) \leq \inf_{\alpha \in \underline{\mathscr{A}}} \frac{r\underline{F}_r(\alpha)/\alpha}{1 - \underline{F}_r(\alpha)/\alpha} = \frac{r \sup_{\alpha > 0} \underline{F}_r(\alpha)/\alpha}{1 - \sup_{\alpha > 0} \underline{F}_r(\alpha)/\alpha} = \frac{r\underline{F}_r}{1 - \underline{F}_r}.$$

2·3. *Lower bounds*

In this section we give the proof of the following lower bounds.

PROPOSITION 2·3. *If $r \in (-\dim_\infty(\nu), 0)$, then $q_r > 1$ and we have*

$$\frac{rq_r}{1 - q_r} \leq \underline{D}_r(\nu) \text{ and } \lim_{\varepsilon \downarrow 0} \frac{r\underline{h}_{r-\varepsilon}}{1 - \underline{h}_{r-\varepsilon}} \leq \overline{D}_r(\nu).$$

For convenience in this section we choose the maximum norm on $\mathbb{R}^d$. For any $Q \in \mathcal{D}$, we let $|Q|$ denote the side length of Q. Before we proceed, we need the following two elementary lemmas. For $\widetilde{Q} \in \mathcal{D}$ we let $B_n(\widetilde{Q}) := \bigcup \left\{ Q \in \mathcal{D}_{n-1} : \overline{\widetilde{Q}} \cap \overline{Q} \neq \varnothing \right\}$ denote the 2^{-n+1}-parallel set of $\widetilde{Q}$.

LEMMA 2·4. *For $\widetilde{Q} \in \mathcal{D}$, a finite set $A \subset \mathbb{R}^d$, and an integer $n > |\log_2 |\widetilde{Q}||$, we have*

$$\begin{aligned}\mathit{card}\, \{Q \in \mathcal{D}_n(\widetilde{Q}) : d(Q, A) \geq |Q| \,\&\forall Q' \in \mathcal{D}_{n-1}(\widetilde{Q}) : Q' \supset Q, d(Q', A) < |Q'|\} \\ \leq 6^d \mathit{card}\, (A \cap B_n(\widetilde{Q})).\end{aligned}$$

Proof. First we show that for any $a \in \mathbb{R}^d$ we have

$$\begin{aligned}\text{card}\, \{Q \in \mathcal{D}_n(\widetilde{Q}) : d(Q, a) \geq |Q| \,\&\forall Q' \in \mathcal{D}_{n-1}(\widetilde{Q}) : Q' \supset Q, d(Q', a) < |Q'|\} \\ \leq 6^d \mathbb{1}_{B_n(\widetilde{Q})}(a).\end{aligned}$$

The case $a \notin B_n(\widetilde{Q})$ follows by observing that $n > |\log_2 |\widetilde{Q}||$ implies that for every $Q \in \mathcal{D}_n(\widetilde{Q})$, there exists $Q' \in \mathcal{D}_{n-1}(\widetilde{Q})$ with $Q' \supset Q$ and $d(Q', a) \geq d(\widetilde{Q}, a) \geq 2^{-n+1} = |Q'|$ and the relevant set is therefore empty.

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For the case $a \in B_n\left(\widetilde{Q}\right)$, observe that we have at most 3^d cubes $Q' \in \mathcal{D}_{n-1}\left(\widetilde{Q}\right)$ such that $d\left(Q', a\right) < \left|Q'\right|$ and in each such cube there are at most 2^d subcubes from $\mathcal{D}_n$. This gives the upper bound for the cardinality in question of $2^d \cdot 3^d$.

Now the claim follows from

$$\begin{aligned}
&\operatorname{card}\left\{Q \in \mathcal{D}_n(\widetilde{Q}) : d\left(Q, A\right) \geq |Q| \;\&\forall Q' \in \mathcal{D}_{n-1}(\widetilde{Q}) : Q' \supset Q, d\left(Q', A\right) < \left|Q'\right|\right\} \\
&\leq \sum_{a\in A} \operatorname{card}\left\{Q \in \mathcal{D}_n(\widetilde{Q}) : d\left(Q, a\right) \geq |Q| \;\&\forall Q' \in \mathcal{D}_{n-1}(\widetilde{Q}) : Q' \supset Q, d\left(Q', a\right) < \left|Q'\right|\right\}.
\end{aligned}$$

LEMMA 2·5. *For $A \subset \mathbb{R}^d$ and $P \subset \bigcup_{k=1}^{n-1} \mathcal{D}_k$, $n \in \mathbb{N}$ a finite disjoint family of sets we have*

$$\sum_{\widetilde{Q}\in P} \operatorname{card}\left(A \cap B_n\left(\widetilde{Q}\right)\right) \leq 3^d \operatorname{card}(A).$$

Proof. Since for all $\widetilde{Q} \in P$ we have $\left|\widetilde{Q}\right| \geq 2^{-n+1}$ we find

$$\begin{aligned}
\sum_{\widetilde{Q}\in P} \operatorname{card}\left(A \cap B_n\left(\widetilde{Q}\right)\right) &= \sum_{\widetilde{Q}\in P} \sum_{Q\in\mathcal{D}_{n-1}, \bar{Q}\cap\bar{\widetilde{Q}}\neq\varnothing} \operatorname{card}\left(A \cap Q\right) \\
&= \sum_{Q\in\mathcal{D}_{n-1}} \sum_{\widetilde{Q}\in P, \bar{Q}\cap\bar{\widetilde{Q}}\neq\varnothing} \operatorname{card}\left(A \cap Q\right) \\
&\leq \sum_{Q\in\mathcal{D}_{n-1}} \sum_{\widetilde{Q}\in \mathcal{D}_{n-1}, \bar{Q}\cap\bar{\widetilde{Q}}\neq\varnothing} \operatorname{card}\left(A \cap Q\right) \\
&\leq \sum_{Q\in\mathcal{D}_{n-1}} 3^d \operatorname{card}\left(A \cap Q\right) = 3^d \operatorname{card}(A).
\end{aligned}$$

Proof of Proposition 2·3. For $k \in \mathbb{N}$ choose $A \in \mathcal{A}_k$, and $\varepsilon > 0$ such that $r - \varepsilon > -\dim_\infty(\nu)$. Let $t_k := \gamma_{\mathfrak{J}_{\nu,r-\varepsilon},k}$ be as in Proposition 2·1 and $P \in \Pi$ an optimal partition realising $\gamma_{\mathfrak{J}_{\nu,r-\varepsilon},k}$ that is $\operatorname{card}(P) \leq k$ and $\mathfrak{J}_{\nu,r-\varepsilon}\left(\widetilde{Q}\right) \leq t_k$ for all $\widetilde{Q} \in P$ (see [**11**]). Let us define

$$P_1 := \left\{\widetilde{Q} \in P, d\left(\widetilde{Q}, A\right) \geq \left|\widetilde{Q}\right|\right\} \text{ and } P_2 := P \setminus P_1.$$

This allows us to estimate

$$\begin{aligned}
\int d(x, A)^r \,\mathrm{d}\nu(x) &= \sum_{\widetilde{Q}\in P_1} \int_{\widetilde{Q}} d(x, A)^r \,\mathrm{d}\nu(x) + \sum_{\widetilde{Q}\in P_2} \int_{\widetilde{Q}} d(x, A)^r \,\mathrm{d}\nu(x) \\
&\leq \sum_{\widetilde{Q}\in P_1} \underbrace{\Lambda\left(\widetilde{Q}\right)^{r/d} \nu\left(\widetilde{Q}\right)}_{\leq t_k} + \sum_{\widetilde{Q}\in P_2} \int_{\widetilde{Q}} d(x, A)^r \,\mathrm{d}\nu(x) \\
&\leq t_k \operatorname{card}(P) + \sum_{\widetilde{Q}\in P_2} \int_{\widetilde{Q}} d(x, A)^r \,\mathrm{d}\nu(x).
\end{aligned}$$

Further, for $n \in \mathbb{N}$, and $\widetilde{Q} \in P_2$ we set

$$E_n\left(\widetilde{Q}\right) := \left\{Q \in \mathcal{D}_n(\widetilde{Q}) : d\left(Q, A\right) \geq |Q| \;\&\forall Q' \in \mathcal{D}_{n-1}(\widetilde{Q}) : Q' \supset Q, d\left(Q', A\right) < \left|Q'\right|\right\}.$$

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Note that for $2^{-n} > |\widetilde{Q}|$ we have $\mathcal{D}_n(\widetilde{Q}) = \varnothing$ and therefore $E_n(\widetilde{Q}) = \varnothing$. For $2^{-n} < |\widetilde{Q}|$ and $Q \in \mathcal{D}_n(\widetilde{Q})$ there is exactly one $Q' \in \mathcal{D}_{n-1}(\widetilde{Q})$ with $Q' \supset Q$. Finally, if $d(\widetilde{Q}, A) < |\widetilde{Q}| = 2^{-n}$ we again have $E_n(\widetilde{Q}) = \varnothing$. Since ν has no atoms,

$$\bigcup_{n\in\mathbb{N}} \bigcup_{\widetilde{Q}\in P_2} E_n(\widetilde{Q}) = \bigcup_{\widetilde{Q}\in P_2} \widetilde{Q} \setminus A,$$

i.e. $(E_n(\widetilde{Q}))_{n\in\mathbb{N},\widetilde{Q}\in P_2}$ is a countable ν-almost sure infinite partition of $\bigcup_{\widetilde{Q}\in P_2} \widetilde{Q}$. With this at hand, we find for the second summand in the above estimate

$$\begin{aligned}
\sum_{\widetilde{Q}\in P_2} \int d(x,A)^r \, \mathrm{d}\nu(x) &\le \sum_{n=0}^{\infty} \sum_{\widetilde{Q}\in P_2} \sum_{Q\in E_n(\widetilde{Q})} \int_Q d(x,A)^r \, \mathrm{d}\nu(x) \\
&\le \sum_{n=0}^{\infty} \sum_{\widetilde{Q}\in P_2} \sum_{Q\in E_n(\widetilde{Q})} \underbrace{\Lambda(Q)^{(r/d-\varepsilon/d)}\nu(Q)}_{\le t_k} \underbrace{\Lambda(Q)^{\varepsilon/d}}_{=2^{-n\varepsilon}} \\
&\le t_k \sum_{n=0}^{\infty} 2^{-\varepsilon n} \sum_{\widetilde{Q}\in P_2, |\widetilde{Q}|>2^{-n}} 6^d \operatorname{card}\left(A \cap B_n(\widetilde{Q})\right) \\
&\le t_k \sum_{n=0}^{\infty} 2^{-\varepsilon n} 6^d \sum_{\widetilde{Q}\in P_2, |\widetilde{Q}|>2^{-n}} \operatorname{card}\left(A \cap B_n(\widetilde{Q})\right) \\
&\le \frac{18^d}{1-2^{-\varepsilon}} t_k \operatorname{card}(A),
\end{aligned} \tag{2·1}$$

where for the third inequality we used Lemma 2·4 and for the fifth inequality we used Lemma 2·5. Combining the above then gives

$$\int d(x,A)^r \, \mathrm{d}\nu(x) \le \left(1 + \frac{18^d}{1-2^{-\varepsilon}}\right) t_k k.$$

With (2·1) this gives

$$\mathfrak{e}_{r,k}(\nu) \ge \left(\left(1 + \frac{18^d}{1-2^{-\varepsilon}}\right) t_k k\right)^{1/r}.$$

Now, using $q_{r-\varepsilon} + \varepsilon \ge \log(n)/(-\log(t_n))$ for all n large and Proposition 2·1, gives

$$\underline{D}_r(\nu) \ge \liminf_{k\to\infty} \frac{r\log k}{-\log k - \log t_k} = \frac{r \limsup_{k\to\infty} \log(k)/\log(1/t_k)}{1 - \limsup_{k\to\infty} \log(k)/\log(1/t_k)} \ge \frac{r(q_{r-\varepsilon}+\varepsilon)}{1-(q_{r-\varepsilon}+\varepsilon)}.$$

Letting ε tend to zero and by the continuity of $a \mapsto q_a$, $a \in (-\dim_\infty(\nu), 0)$ the first inequality follows.

For the second inequality, recall that by our assumption and Proposition 2·1 we have $1 < \underline{F}_{\mathfrak{J}} \le \underline{h}_{\mathfrak{J}}$. For the second inequality we consider $P \in \Pi$ with $\operatorname{card}(P) = \mathcal{M}_{\nu,r-\varepsilon}(1/t)$. This time (2·1) gives

$$\mathfrak{e}_{r,\mathcal{M}_{\nu,r-\varepsilon}(1/t)}(\nu) \ge \left(\frac{18^d}{1-2^{-\varepsilon}} t \mathcal{M}_{\nu,r-\varepsilon}(1/t)\right)^{1/r}$$

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and hence using $\underline{h}_{r-\varepsilon}+\varepsilon \geq \log\left(\mathcal{M}_{\nu,r-\varepsilon}\left(1/t_n\right)\right)/\log\left(1/t_n\right)$ on a subsequence $t_n \searrow 0$,

$$\begin{aligned}\overline{D}_r(\nu) &\geq \limsup_{t\to 0}\frac{r\log\mathcal{M}_{\nu,r-\varepsilon}\left(1/t\right)}{-\log\mathcal{M}_{\nu,r-\varepsilon}\left(1/t\right)+\log\left(1/t\right)}=\frac{r\liminf\log\left(n\right)/\log\left(1/t_n\right)}{1-\liminf\log\left(n\right)/\log\left(1/t_n\right)}\\ &\geq \frac{r\left(\underline{h}_{r-\varepsilon}+\varepsilon\right)}{1-\left(\underline{h}_{r-\varepsilon}+\varepsilon\right)}.\end{aligned}$$

Proof of Lemma 1·8. First note that $a_\nu := \sup\left\{q_r : r\in\left(-\dim_\infty\left(\nu\right),0\right)\right\}>1$, due to the fact that $q_0=1$ and $r\mapsto q_r$ is strictly decreasing on $\left(-\dim_\infty\left(\nu\right),0\right]$. Since $r\mapsto q_r$ is decreasing we have

$$\lim_{r\searrow-\dim_\infty(\nu)} q_r=\sup\left\{q_r : r\in\left(-\dim_\infty\left(\nu\right),0\right)\right\}=a_\nu,$$

and therefore

$$\lim_{r\searrow-\dim_\infty(\nu)}\underline{D}_r(\nu)=\begin{cases}\lim\limits_{r\searrow-\dim_\infty(\nu)} r\dfrac{q_r}{1-q_r}=\dfrac{a_\nu}{a_\nu-1}\dim_\infty\left(\nu\right), & a_\nu<\infty,\\ \lim\limits_{r\searrow-\dim_\infty(\nu)}\dfrac{r}{1/q_r-1}=\dim_\infty\left(\nu\right), & a_\nu=\infty.\end{cases}$$

2·4. *Quantization via geometric mean error*

In this subsection we settle the quantziation problem for the geometric mean error as stated in Theorem 1·5 (iii).

Proof of Theorem 1·5 (iii). Part (iii) follows by combining Lemma 1·7 with [**7**, **25**] and our results from part (ii) by noting, on the one hand,

$$\frac{-1}{\partial_r q_r|_{r=0}}=\lim_{r\uparrow 0}\frac{r}{1-q_r}q_r\leq\lim_{r\uparrow 0}\frac{rq_r}{1-q_r}=\lim_{r\uparrow 0}\underline{D}_r(\nu)\leq\underline{D}_0(\nu), \tag{2·2}$$

where we used $\lim_{r\uparrow 0} q_r=1$. Using $\beta_\nu\left(q_r\right)\leq\tau_{\nu,r}\left(q_r\right)+rq_r=rq_r$, this also shows

$$\frac{-1}{\partial_r q_r|_{r=0}}\leq\lim_{r\uparrow 0}\frac{\beta_\nu\left(q_r\right)}{1-q_r}=-\beta_\nu'\left(1+\right). \tag{2·3}$$

On the other hand, using $\beta_\nu\left(q_r\right)=rq_r$ for $r>0$, as established in [**12**] and the monotonicity of $r\mapsto\overline{D}_r(\nu)$ on $r\geq 0$ obtained e. g. in [**6**, lemma 3·5], we have that

$$\begin{aligned}\overline{D}_0(\nu)\leq\lim_{r\downarrow 0}\overline{D}_r(\nu)&=\lim_{r\downarrow 0}\frac{\beta_\nu(q_r)-\beta_\nu(1)}{1-q_r}=-\beta_\nu'\left(1-\right)\\ &=\lim_{r\downarrow 0}\frac{r}{1-q_r}q_r=\frac{-1}{\partial_r q_r|_{r=0}}.\end{aligned} \tag{2·4}$$

Combining $-\beta_\nu'\left(1+\right)\leq-\beta_\nu'\left(1-\right)$, (2·3), and (2·4) shows that β_ν is differentiable at 1 with $\beta_\nu'(1)=1/\partial_r q_r|_{r=0}$. Combining (2·2) and (2·4) $-1/\partial_r q_r|_{r=0}\leq\underline{D}_0(\nu)\leq\overline{D}_0(\nu)\leq -1/\partial_r q_r|_{r=0}$. This proves the claim.

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2·5. *Absolutely continuous case*

We start with the following lemma, which has been stated in a similar context in [10, lemma 3·15].

LEMMA 2·6. *Let ν be a non-zero absolutely continuous measure with Lebesgue density $f \in L^s_\Lambda$ for some $s \geq 1$. Then, for all $q \in [0, s]$, the L^q-spectrum is linear with $\liminf_{n\to\infty} \beta_{\nu,n}(q) = \beta_\nu(q) = d(1-q)$ and $\tau_{\mathfrak{J}_{\nu,r}}(q) = \beta_\nu(q) - rq$ for $r \geq d/s - d$.*

Proof. First, we remark that, $\beta_\nu(1) = 0$ and $\beta_\nu(0) = d$. Hence, the convexity of β_ν implies $\beta_\nu(q) \leq d(1-q)$ for all $q \in [0, 1]$ and $\beta_\nu(q) \geq d(1-q)$ for $q > 1$. Moreover, by Jensen's inequality, for all $q \in [0, 1]$ and n large, we have

$$\sum_{Q\in\mathcal{D}_n} \nu(Q)^q = \sum_{Q\in\mathcal{D}_n} \left(\frac{\int_Q f \,\mathrm{d}\Lambda}{\Lambda(Q)}\right)^q \Lambda(Q)^q \geq \sum_{Q\in\mathcal{D}_n} \Lambda(Q)^{q-1} \int_Q f^q \,\mathrm{d}\Lambda \geq \Lambda(Q)^{q-1} \int_Q f^q \,\mathrm{d}\Lambda,$$

implying $\liminf_{n\to\infty} \beta_{\nu,n}(q) \geq d(1-q)$. Further, Jensen's inequality, for all $q \in [1, s]$, yields

$$\sum_{Q\in\mathcal{D}_n} \nu(Q)^q = \sum_{Q\in\mathcal{D}_n} \left(\frac{\int_Q f \,\mathrm{d}\Lambda}{\Lambda(Q)}\right)^q \Lambda(Q)^q \leq \Lambda(Q)^{q-1} \sum_{Q\in\mathcal{D}_n} \int_Q f^q \,\mathrm{d}\Lambda \leq \Lambda(Q)^{q-1} \int_Q f^q \,\mathrm{d}\Lambda.$$

Hence, we obtain $\limsup_{n\to\infty} \beta_{\nu,n}(q) \leq d(1-q)$. To prove the last equality we again use Jensen's inequality, for $Q \in \mathcal{D}_n$, $r \geq d/s - d$ and $q = s$:

$$\nu(Q)^q = \left(\int_Q f \Lambda(Q)^{-1} \,\mathrm{d}\Lambda\right)^q \Lambda(Q)^q \leq \left(\int_Q f^q \,\mathrm{d}\Lambda\right) \Lambda(Q)^{q-1},$$

implying $\nu(Q)^q \Lambda(Q)^{qr/d} \leq \left(\int_Q f^q \,\mathrm{d}\Lambda\right) \Lambda(Q)^{q(1+r/d)-1}$. Note that

$$q(1 + r/d) - 1 = \left(\frac{d+r}{d}\right) q - 1 \geq \frac{q}{s} - 1 = 0$$

and since have that $Q \mapsto \left(\int_Q f^q \,\mathrm{d}\Lambda\right) \Lambda(Q)^{q(1+r/d)-1}$ is monotonic. Therefore, we get the following upper bound:

$$\begin{aligned}\sum_{Q\in\mathcal{D}_n} \max_{Q'\in\mathcal{D}(Q)} \nu\left(Q'\right) \Lambda\left(Q'\right)^r &\leq \sum_{Q\in\mathcal{D}_n} \left(\int_Q f^q \,\mathrm{d}\Lambda\right) \Lambda(Q)^{q(1+r/d)-1} \\ &\leq \|f\|^q_{L^q_\Lambda(\mathcal{Q})} 2^{-q(d+r)+d}\end{aligned}$$

showing

$$\tau_{\mathfrak{J}_{\nu,r}}(q) \leq d(1-q) - rq.$$

Further, $\tau_{\mathfrak{J}_{\nu,r}}(0) \leq d$ combined with the convexity of $\tau_{\mathfrak{J}_{\nu,r}}$, we conclude that for all $q \in [0, s]$

$$\tau_{\mathfrak{J}_{\nu,r}}(q) \leq d(1-q) - rq.$$

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The lower bound follows from

$$d(1-q)-rq \le \liminf_{n\to\infty} \beta_{\nu,n}(q)-rq \le \liminf_{n\to\infty} \tau_{\mathfrak{J}_{\nu,r,n}}(q) \le \tau_{\mathfrak{J}_{\nu,r}}(q).$$

We also need the following easy observation.

LEMMA 2·7. *For any compactly supported probability measure* ν *we have for* $q \ge 0$,

$$\beta_\nu(q) \ge -\dim_\infty(\nu)\cdot q.$$

Proof. By the definition of $\dim_\infty(\nu)$ we have that for every $\varepsilon > 0$ there exists infinitely many $n \in \mathbb{N}$ with $\max_{Q\in\mathcal{D}_n} \nu(Q) \ge 2^{(-\dim_\infty(\nu)-\varepsilon)n}$ and therefore

$$\limsup_{n\to\infty} \frac{1}{\log 2^n} \log \sum_{Q\in\mathcal{D}_n} \nu(Q)^q \ge (-\dim_\infty(\nu)-\varepsilon)\, q.$$

Combining the previous two lemmas gives $s_h \le d/\left(d-\dim_\infty(h\Lambda)\right)$, as claimed at the beginning of Section 1·2.

For what follows, we take advantage of the fact that many basic inequalities for positive r can simply be reversed to hold for negative r. To see this, recall from the introduction, that for negative r,

$$V_{n,r}(\nu) = \mathfrak{e}_{n,r}(\nu)^r = \begin{cases} \inf_{A\in\mathcal{A}_n} \int d(x,A)^r \, \mathrm{d}\nu(x), & r>0,\\ \sup_{A\in\mathcal{A}_n} \int d(x,A)^r \, \mathrm{d}\nu(x), & r<0.\end{cases}$$

For the reversed inequalities of the following lemma for the case $r > 0$ we refer to [**4**, lemma 4·14].

LEMMA 2·8. *For* $r < 0$ *and a linear combination* $\nu := \sum s_i\nu_i$, $s_i \ge 0$, *of finite measures* ν_i *and* $n \in \mathbb{N}$, *we have*

$$V_{n,r}(\nu) \le \sum s_i V_{n,r}(\nu_i).$$

Further, for $n \ge \sum n_i$, $n_i \in \mathbb{N}$, *we have*

$$V_{n,r}(\nu) \ge \sum s_i V_{n_i,r}(\nu_i).$$

Proof. For $A \in \mathcal{A}_n$,

$$\int d(x,A)^r \, \mathrm{d}\nu(x) = \sum s_i \int d(x,A)^r \, \mathrm{d}\nu_i(x) \le \sum s_i V_{n,r}(\nu_i).$$

Taking the supremum over $A \in \mathcal{A}_n$ gives the first inequality.

For the second inequality, assume $n \ge \sum n_i$ and note that for $A_i \in \mathcal{A}_{n_i}$ we have $A := \bigcup_i A_i \in \mathcal{A}_n$. Hence,

$$V_{n,r}(\nu) \ge \int d(x,A)^r \, \mathrm{d}\nu(x) = \sum s_i \int d(x,A)^r \, \mathrm{d}\nu_i(x) \ge \sum s_i \int d(x,A_i)^r \, \mathrm{d}\nu_i(x).$$

Now taking the supremum over all $A_i \in \mathcal{A}_{n_i}$ gives the desired second inequality.

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LEMMA 2·9. *For $r \in (-d, 0)$ we let $C_{r,d} := 2^{-r} + 18^d / \left(2^r - 2^{-d}\right)$. Then for all $m \in \mathbb{N}$ and $A \in \mathcal{A}_m$ we have*

$$\int d\,(x, A)^r \;\mathrm{d}\Lambda(x) \le C_{r,d} m^{-r/d}.$$

Proof. Let us first consider the case $m = 2^{nd}$ for $n \in \mathbb{N}$. We follow the estimates in Section 2·3 for the lower bound of the quantization dimension with $\nu = \Lambda$. Observe that for $n \in \mathbb{N}$ the optimal partition P of cardinality 2^{nd} is in this situation given by $\mathcal{D}_n$. As in the proof Proposition 2·3 we partition P into P_1, P_2 and obtain

$$\begin{aligned}\int d(x, A)^r \,\mathrm{d}\Lambda(x) &= \sum_{\widetilde{Q} \in P_1} \int_{\widetilde{Q}} d(x, A)^r \,\mathrm{d}\Lambda(x) + \sum_{\widetilde{Q} \in P_2} \int_{\widetilde{Q}} d(x, A)^r \,\mathrm{d}\Lambda(x) \\ &\le \mathrm{card}(P) \cdot \underbrace{\Lambda\left(\widetilde{Q}\right)^{r/d} \Lambda\left(\widetilde{Q}\right)}_{=2^{-n(d+r)}} + \sum_{\widetilde{Q} \in P_2} \int_{\widetilde{Q}} d(x, A)^r \,\mathrm{d}\Lambda(x) \\ &\le m^{-r/d} + \sum_{\widetilde{Q} \in P_2} \int_{\widetilde{Q}} d(x, A)^r \,\mathrm{d}\Lambda(x).\end{aligned}$$

Using Lemma 2·4 and Lemma 2·5 we estimate the second summand as follows:

$$\begin{aligned}\sum_{\widetilde{Q} \in P_2} \int d(x, A)^r \,\mathrm{d}\Lambda(x) &\le \sum_{k=0}^{\infty} \sum_{\widetilde{Q} \in P_2} \sum_{Q \in E_k(\widetilde{Q})} \int_Q d\,(x, A)^r \;\mathrm{d}\Lambda(x) \\ &\le \sum_{k=0}^{\infty} \sum_{\widetilde{Q} \in P_2} \sum_{Q \in E_k(\widetilde{Q})} \underbrace{\Lambda(Q)^{r/d} \Lambda(Q)}_{=2^{-k(d+r)}} \\ &\le \sum_{k=n+1}^{\infty} \sum_{\widetilde{Q} \in P_2, |\widetilde{Q}| > 2^{-k}} 6^d \mathrm{card}\left(A \cap B_k\left(\widetilde{Q}\right)\right) \cdot 2^{-k(d+r)} \\ &\le 18^d \mathrm{card}(A) \sum_{k=n}^{\infty} 2^{-k(d+r)} = \frac{18^d}{1 - 2^{-(d+r)}} m^{-r/d}.\end{aligned}$$

Combining the above, we obtain with $C := \left(1 + 18^d / (1 - 2^{-(d+r)})\right)$

$$\int d(x, A)^r \,\mathrm{d}\Lambda(x) \le \left(1 + \frac{18^d}{1 - 2^{-(d+r)}}\right) m^{-r/d} = C m^{-r/d}.$$

Now for $m \in \mathbb{N}$ and $A \in \mathcal{A}_m$ arbitrary, there find $n \in \mathbb{N}$ with $2^{nd} < m \le 2^{(n+1)d}$. Using our result on the special subsequence, we get

$$\int d\,(x, A)^r \;\mathrm{d}\Lambda(x) \le C 2^{-(n+1)r} = \left(C 2^{-r}\right) m^{-r/d},$$

which proves our claim by setting $C_{r,d} := 2^{-r} C$.

LEMMA 2·10. *For the uniform distribution Λ on $\mathcal{Q}$, the d-dimensional quantization coefficient $\mathfrak{C}_{r,d}(\Lambda)$ of order $r \in \mathbb{R}$ exists, is finite, and positive only for $r \in (-d, +\infty)$; otherwise, it is equal to zero.*

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Proof. This statement is well known for $r \geq 1$ (see e. g. [4]; the proof there also works for $r \in (0, 1]$) and for $r = 0$ see [6, theorem 3·2]. Now for $r \in (-d, 0)$, note that by an observation from the introduction of [12] we know that for any subcube $Q \subset \mathcal{Q}$ with side length $a \in (0, 1)$ and Λ_Q denoting the normalised restriction of Λ to Q, we have

$$V_{n,r}\left(\Lambda_Q\right) = a^r V_{n,r}(\Lambda)$$

Now, for $k \in \mathbb{N}$, let us divide $\mathcal{Q}$ evenly into k^d axis-parallel subcubes $\left\{Q_{i,k} : i = 1, \ldots, k^d\right\}$ with side length $1/k$. Then we have $\Lambda = \sum k^{-d} \Lambda_{Q_{i,k}}$ and hence by part (ii) of Lemma 2·8, for $m \in \mathbb{N}$ and $n := k^d m$,

$$V_{n,r}(\Lambda) \geq \sum_{i=1}^{k^d} k^{-d} V_{m,r}\left(\Lambda_{Q_{i,k}}\right) = k^{-r} V_{m,r}(\Lambda).$$

Therefore,

$$n^{r/d} V_{n,r}(\Lambda) = k^r m^{r/d} V_{n,r}(\Lambda) \geq m^{r/d} V_{m,r}(\Lambda).$$

This gives for each $m \in \mathbb{N}$

$$\limsup_{n\to\infty} n^{1/d} \mathfrak{e}_{n,r}(\Lambda) \leq m^{1/d} \mathfrak{e}_{m,r}(\Lambda).$$

Hence, in the chain of inequalities

$$\limsup_{n\to\infty} n^{1/d} \mathfrak{e}_{n,r}(\Lambda) \leq \inf_m m^{1/d} \mathfrak{e}_{m,r}(\Lambda) \leq \liminf_{n\to\infty} n^{1/d} \mathfrak{e}_{n,r}(\Lambda)$$

in fact equality holds and the claimed limit exists and is smaller than $V_{1,r}(\Lambda)^{1/r} < \infty$. The limit is also positive by Lemma 2·9. The fact that $\mathfrak{C}_{r,d}(\Lambda) = 0$ for $r \leq -d$ follows from Proposition 1·1.

Proof of Theorem 1·3. With $\nu := h\Lambda$, the statement for positive r follows by [4, theorem 6·2] and $r < -\dim_\infty(\nu)$ is also clear from Proposition 1·1. For $r \in (-d, 0)$, we first follow almost literally the proof of [4, theorem 6·2]:

First, let us consider only densities h that are constant on cubes from $\mathcal{D}_n$. By using Lemma 2·8 and Lemma 2·10 at the appropriate places and exchanging all relevant inequalities with their inverses and limes superior with limes inferior we see that the theorem holds for such densities. For example, for r negative [4, lemma 6·8] provides a unique maximiser (instead of minimiser for positive r) as required in the proof for the upper bound of $\limsup_n n^{r/d} V_{n,r}(h\Lambda)$. Namely, at the appropriate place we need the following general observation, which is an immediate consequence of Hölder's inequality: For $m \in \mathbb{N}$ and numbers $s_i > 0$, let $B = \left\{(v_1, \ldots, v_m) \in (0, \infty)^m : \sum_{i=1}^m v_i \leq 1\right\}$ and

$$t_i = \frac{s_i^{d/(d+r)}}{\sum_{j=1}^m s_j^{d/(d+r)}}, \quad 1 \leq i \leq m.$$

Then the function $F : B \to \mathbb{R}_+, \quad F(v_1, \ldots, v_m) = \sum_{i=1}^m s_i v_i^{-r/d}$ satisfies

$$F(t_1, \ldots, t_m) = \left(\sum_{i=1}^m s_i^{d/(d+r)}\right)^{(d+r)/d} = \max_{(v_1,\ldots,v_m)\in B} F(v_1, \ldots, v_m)$$

and $(t_1, \ldots, t_m)$ is the unique maximiser of F.

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For all $r \in (d/s_h - d, 0)$ we have $\|h\|_{\frac{d}{d+r}} < \infty$ and, by Jensen's inequality, we find that for the conditional expectation $h_k := \mathbb{E}(h \mid \mathcal{D}_k)$ we have $\int h_k^{d/(d+r)}\,\mathrm{d}\Lambda \le \int h^{d/(d+r)}\,\mathrm{d}\Lambda < \infty$. Since $\sigma(\mathcal{D}_k) \nearrow \sigma(\mathcal{D}) = \mathcal{B}$, by the martingale convergence theorem, we infer that the sequence (h_k) converges to h almost surely and, for all $q \le d/(d+r)$, also in L^q_Λ; in particular, we have $\|h_k\|_q \to \|h\|_q$.

For the final step in the proof we argue as follows. For $d/s_h - d < r' < r < 0$, $n \in \mathbb{N}$, and $A \in \mathcal{A}_n$ we have by Hölder's inequality

$$\begin{aligned}
\int d(\,\cdot\,, A)^r\, h\,\mathrm{d}\Lambda &= \int d(\,\cdot\,, A)^r\,(h - h_k + h_k)\,\mathrm{d}\Lambda \\
&\le V_{n,r}(h_k\Lambda) + \int d(\,\cdot\,, A)^r\,|h - h_k|\,\mathrm{d}\Lambda \\
&\le V_{n,r}(h_k\Lambda) + \left(\int d(\,\cdot\,, A)^{-dr/r'}\,\mathrm{d}\Lambda\right)^{-r'/d} \|h - h_k\|_{d/(d+r')} \\
&\le V_{n,r}(h_k\Lambda) + \left(V_{n,-rd/r'}(\Lambda)\right)^{-r'/d} \|h - h_k\|_{d/(d+r')}\,.
\end{aligned}$$

If we take the supremum over all $A \in \mathcal{A}_n$, multiply by $n^{r/d}$ and take the limes superior, we obtain,

$$\begin{aligned}
\limsup_{n\to\infty} n^{r/d} V_{n,r}(h\Lambda) &\le \limsup_{n\to\infty} n^{r/d} V_{n,r}(h_k\Lambda) \\
&\quad + \limsup_{n\to\infty} n^{r/d} V_{n,-rd/r'}(\Lambda)^{-r'/d} \|h - h_k\|_{d/(d+r')} \\
&= \left(\Phi_r(h_k)\,\mathfrak{C}_{r,d}(\Lambda)\right)^r + \mathfrak{C}_{rd/r',d}(\Lambda)^{-r'/d} \|h - h_k\|_{d/(d+r')} \\
&\to \left(\Phi_r(h_k)\,\mathfrak{C}_{r,d}(\Lambda)\right)^r \quad \text{for } k \to \infty.
\end{aligned}$$

Taking the rth root, this proves the lower bound on $\liminf_{n\to\infty} n^{1/d}\mathfrak{e}_{n,r}(h\Lambda)$.

Similarly,

$$\begin{aligned}
V_{n,r}(h\Lambda) \ge \int d(\,\cdot\,, A)^r\, h\,\mathrm{d}\Lambda &= \int d(\,\cdot\,, A)^r\,(h - h_k + h_k)\,\mathrm{d}\Lambda \\
&\ge \int d(\,\cdot\,, A)^r\, h_k\,\mathrm{d}\Lambda - \int d(\,\cdot\,, A)^r\,|h - h_k|\,\mathrm{d}\Lambda \\
&\ge \int d(\,\cdot\,, A)^r\, h_k\,\mathrm{d}\Lambda - \left(\int d(\,\cdot\,, A)^{-dr/r'}\,\mathrm{d}\Lambda\right)^{-r'/d} \|h - h_k\|_{d/(d+r')} \\
&\ge \int d(\,\cdot\,, A)^r\, h_k\,\mathrm{d}\Lambda - \left(V_{n,-rd/r'}(\Lambda)\right)^{-r'/d} \|h - h_k\|_{d/(d+r')}\,.
\end{aligned}$$

Again, if we take the supremum over all $A \in \mathcal{A}_n$ —this time on the right-hand side—, multiply by $n^{r/d}$ and take the limes inferior, we obtain,

$$\begin{aligned}
\liminf_{n\to\infty} n^{r/d} V_{n,r}(h\Lambda) &\ge \liminf_{n\to\infty} n^{r/d} V_{n,r}(h_k\Lambda) \\
&\quad - \limsup_{n\to\infty} n^{r/d} V_{n,-rd/r'}(\Lambda)^{-r'/d} \|h - h_k\|_{d/(d+r')} \\
&= \left(\Phi_r(h_k)\,\mathfrak{C}_{r,d}(\Lambda)\right)^r - \mathfrak{C}_{rd/r',d}(\Lambda)^{-r/d'} \|h - h_k\|_{d/(d+r')} \\
&\to \left(\Phi_r(h)\mathfrak{C}_{r,d}(\Lambda)\right)^r \quad \text{for } k \to \infty.
\end{aligned}$$

This proves the upper bound on $\limsup_{n\to\infty} n^{1/d}\mathfrak{e}_{n,r}(h\Lambda)$.

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To cover the case $r = d/s_h - d$ with $\|h\|_{s_h} = \infty$, we consider the truncated version $h \wedge s\mathbb{1}$, for $s > 0$, which gives $\limsup n^{1/d} \mathfrak{e}_{n,r}(h\Lambda) \le \limsup n^{1/d} \mathfrak{e}_{n,r}((h \wedge s\mathbb{1})\Lambda) = \mathfrak{C}_{r,d}(\Lambda)\Phi_r(h \wedge s) \to 0$ for $s \nearrow \infty$.

For $r = 0$ the claimed upper bound is contained in [6, theorem 3·4]. For the lower bound we make use of our results for negative r together with the fact that $r \mapsto n^{1/d}(V_{n,r}(h\Lambda))^{1/r}$, $n \in \mathbb{N}$, is monotonically increasing on $r \in (-d, 0]$ and that

$$\begin{aligned}
\lim_{r\nearrow 0} \Phi_r(h) &= \lim_{r\nearrow 0}\left(\int h^{-r/(d+r)} h \,\mathrm{d}\Lambda\right)^{\frac{-(d+r)/r}{-d}}\\
&= \lim_{t\searrow 0} \exp\left(-1/d \cdot 1/t \cdot \log \int \exp(t\log(h))\, h \,\mathrm{d}\Lambda\right)\\
&= \exp \frac{\int \frac{d}{dt}\exp(t\log(h))\,|_{t=0} h\,\mathrm{d}\Lambda}{-d\int \exp(0\log(h))\, h\,\mathrm{d}\Lambda}\\
&= \exp - (1/d)\int \exp(0\log(h))\log(h) h\,\mathrm{d}\Lambda = \Phi_0(h).
\end{aligned}$$

In the third equality we used that $(\exp(t\log(h))\, h)/t < h^{1+\varepsilon} \in L^1_\Lambda$ for all $t \in (0, \varepsilon)$ and some $\varepsilon > 0$ in tandem with Lebesgue's dominated convergence theorem.

Example 2·11. We provide an example of an absolutely continuous measure ν on $\mathcal{Q} := [0, 1]$ such that $-d < -\dim_\infty(\nu) < d/s_h - d$. For this we consider a disjoint family $(I_{n,k} : n \in \mathbb{N}, 1 \le k \le 2^n)$ of pairwise disjoint subintervals of $[0, 1]$ such that for each $n \in \mathbb{N}$ and $k \in \{1, \ldots, 2^n\}$ we have $|I_{n,k}| = 2^{-3n+1}$. We define a measure by

$$\nu := \sum_{n\in\mathbb{N},k=1,\ldots,2^n} \Lambda(I_{n,k})^{-1/2}\,\Lambda|_{I_{n,k}}.$$

Since $\nu(I_{n,k}) = \Lambda(I_{n,k})^{1/2}$ and

$$\nu([0,1]) = \sum_{n\in\mathbb{N},k=1,\ldots,2^n} \Lambda(I_{n,k})^{1/2} = \sum_{n\in\mathbb{N}} 2^{(-3n+1)/2+n} < \infty$$

this measure is finite and absolutely continuous with density

$$h := \sum_{n\in\mathbb{N},k=1,\ldots,2^n} \Lambda(I_{n,k})^{-1/2}\,\mathbb{1}_{I_{n,k}}.$$

Then $\int_{\bigcup_k I_{n,k}} h^s \mathrm{d}\Lambda = 2^n 2^{s(n3/2-1/2)-3n+1} = 2^{n(s3/2-2)+1-s/2}$ and therefore

$$\int h^s\,\mathrm{d}\Lambda = \sum_{n\in\mathbb{N}} 2^{n(s3/2-2)+1-s/2} = 2^{-s/2+1}\sum_{n\in\mathbb{N}} 2^{(3/2s-2)n} \begin{cases} = \infty, & \text{for } s \ge 4/3,\\ < \infty, & \text{for } s < 4/3.\end{cases}$$

Hence, $s_h = 4/3$. On the other hand, $\dim_\infty(\nu) = 1/2$ and $d = 1$ giving

$$-d = -1 < -\dim_\infty(\nu) = -1/2 < -1/4 = d/s_h - d.$$

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Example 2·12. Our second example concerns an absolutely continuous measure ν on $\mathcal{Q} := [0,1]$ this time with density

$$h(x) = x^{-1/2}\left(\log\left(\frac{x}{100}\right)\right)^{-2},\ x \in (0,1]\,.$$

A straightforward calculation shows that $\dim_\infty(\nu) = 1/2$ and $s_h = 2$. This fact in combination with Lemma 2·6 and Lemma 2·7 gives

$$\beta_\nu(q) = \begin{cases} 1-q, & 0 \leq q \leq 2, \\ -(1/2)\,q, & q > 2, \end{cases}$$

and consequently, $\tau_{\mathfrak{J}_{\nu,r}}(q) = \beta_\nu(q) - rq$ for $q \in [0,2]$, $a_\nu = 2$, and $D_t(\nu) = 1$ for all $-1/2 < t < 0$. In particular, by right continuity (Lemma 1·8), monotonicity (Lemma 1·7) and regularity (Theorem 1·11) we have

$$\lim_{t \searrow -1/2} D_t(\nu) = 1 = \frac{a_\nu}{a_\nu - 1}\dim_\infty(\nu) \geq D_{-1/2}(\nu).$$

It is more involved to show that $D_{-1/2}(\nu) \geq 1$:

For $I(A) := \int_0^1 (d(x,A))^{-1/2}\, h(x)\,\mathrm{d}\Lambda(x)$ with $A = \{0 = a_1 < a_2 < \cdots < a_n = 1\} \subset [0,1]$, $n \geq 2$, our claim is that for a universal constant $C > 0$,

$$I(A) \leq C\sqrt{n}.$$

This implies that $\sup_{A \in \mathcal{A}_n} V_{n,-1/2}(\nu) \leq C\sqrt{n}$ and therefore $\underline{D}_{-1/2}(\nu) \geq 1$. Note that we have made use of the observation that the assumption $\{0,1\} \subset A$ does not result in a loss of generality as a consequence of $n \mapsto V_{n,-1/2}(\nu)$ being monotonically increasing. We proceed to prove this claim as follows:

Partitioning the interval. For each $1 \leq i \leq n-1$, set $J_i := \left[a_i, a_{i+1}\right)$, a partition of $[0,1)$, and define $\ell_i := a_{i+1} - a_i$. Then, $I(A) = \sum_{i=1}^{n-1} I_i$ with $I_i := \int_{J_i} (d(x,A))^{-1/2}\, h(x)\,\mathrm{d}\Lambda(x)$. We derive two useful upper bounds for this last integral: Since on each cell J_i, the density h is strictly positive and decreasing, we have

$$I_i = \int_{a_i}^{\frac{a_i+a_{i+1}}{2}} \frac{h(x)}{\sqrt{x-a_i}}\,\mathrm{d}\Lambda(x) + \int_{\frac{a_i+a_{i+1}}{2}}^{a_{i+1}} \frac{h(x)}{\sqrt{a_{i+1}-x}}\,\mathrm{d}\Lambda(x) \leq 2h(a_i)\sqrt{2}\ell_i^{1/2} \leq 3h\,(a_i)\,\ell_i^{1/2}. \tag{2·5}$$

The second bound considers the behavior for a_i close to 0, for which we get

$$I_i \leq 2\int_0^{\ell_i/2} \frac{1}{x(\log(x/100))^2}\,\mathrm{d}\Lambda(x) = \frac{2}{|\log(\ell_i/200)|}. \tag{2·6}$$

Partition of the indices. We now partition the set of indices into three disjoint cases according to the location and size of the intervals:

$$R := \left\{i : a_i < \mathrm{e}^{-\sqrt{n}}\right\}, \qquad S := \{i : \ell_i > a_i\} \setminus R, \qquad T := \{i : \ell_i \leq a_i\} \setminus R.$$

For each $P \in \{R,S,T\}$, define $I_P := \sum_{i \in P} I_i$. Thus, $I(A) = I_R + I_S + I_T$.

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Case 1: Small a_i ($i \in R$). Using bound (2·6) above, and that for $a_i < \mathrm{e}^{-\sqrt{n}}$, we have $\ell_i \le \mathrm{e}^{-\sqrt{n}}$ by construction, so $|\log(\ell_i/200)| \ge \sqrt{n}$ for large n. Therefore, $I_R \le n \cdot \max_{i\in R}(2/|\log(\ell_i/200)|) \le 2n/\sqrt{n} = 2\sqrt{n}$.

Case 2: Large intervals ($i \in S$). For intervals where $\ell_i > a_i$, using bound (2·6) again, $\int_{J_i} h(x)|x-a_i|^{-1/2}\,\mathrm{d}\Lambda(x) \le 1$, and the number of such intervals is controlled by $2^{\mathrm{card}(S)-1}\mathrm{e}^{-\sqrt{n}} \le 1$, so $\mathrm{card}\,(S) \le 3\sqrt{n}$, hence $I_S \le 3\sqrt{n}$.

Case 3: Small intervals away from 0 ($i \in T$). For these, applying Hölder's inequality, the bound (2·5), the fact that $h(a_i)^2/h(a_i+\ell_i)^2 \le 2$ for $i \in T$, and using the integral comparison criterion,

$$\begin{aligned} I_T &\le 3\sum_{i\in T} h\,(a_i)\,\sqrt{\ell_i} \le 3\sqrt{\mathrm{card}\,(T)}\left(\sum_{i\in T}(h\,(a_i))^2\,\ell_i\right)^{1/2} \\ &\le 3\sqrt{n}\left(\sum_{i\in T} 2\,(h\,(a_i+\ell_i))^2\,\ell_i\right)^{1/2} \le 6\sqrt{n}\,\underbrace{\left(\int h^2\,\mathrm{d}\Lambda\right)^{1/2}}_{=1/\sqrt{3}\cdot(\log(100))^{-3/2}} < \sqrt{n}. \end{aligned}$$

Combining the above, we can conclude that the constant C can be chosen as 6.

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