THE KAUFMANN–CLOTE QUESTION ON END EXTENSIONS OF MODELS OF ARITHMETIC AND THE WEAK REGULARITY PRINCIPLE

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Abstract. We investigate the end extendibility of models of arithmetic with restricted elementarity. By utilizing the restricted ultrapower construction in the second-order context, for each $n \in \mathbb{N}$ and any countable model of $\mathrm{B}\Sigma_{n+2}$, we construct a proper Σ_{n+2}-elementary end extension satisfying $\mathrm{B}\Sigma_{n+1}$, which answers a question by Clote positively. We also give a characterization of the countable models of $\mathrm{I}\Sigma_{n+2}$ in terms of their end extendibility, similar to the case of $\mathrm{B}\Sigma_{n+2}$. Along the proof, we introduce a new type of regularity principle in arithmetic called the weak regularity principle, which serves as a bridge between the model's end extendibility and the amount of induction or collection it satisfies.

§1. Introduction. End extensions play a fundamental role in the model theory of arithmetic and have been studied intensively. The classical MacDowell–Specker theorem [13] showed that every model of PA admits a proper elementary end extension. Around two decades later, Paris and Kirby [16] studied the hierarchical version of the MacDowell–Specker theorem for fragments of PA. In fact, they showed that for countable models, end extendibility with elementarity characterizes the collection strength of the ground model.

THEOREM 1.1 (Paris–Kirby). *Let M be a countable model of $\mathrm{I}\Delta_0$. For each $n \in \mathbb{N}$, M satisfies $\mathrm{B}\Sigma_{n+2}$ if and only if M has a proper Σ_{n+2}-elementary end extension K.*

For the left-to-right direction, the above theorem does not explicitly specify what theory the end extension K can satisfy. The amount of elementarity stated in the theorem already implies $K \models \mathrm{I}\Sigma_n$. Paris–Kirby's proof actually indicates that K cannot always satisfy $\mathrm{I}\Sigma_{n+1}$, since this would imply $M \models \mathrm{B}\Sigma_{n+3}$. Moreover, for each $n \in \mathbb{N}$, Cornaros and Dimitracopoulos [3] constructed a countable model of $\mathrm{B}\Sigma_{n+2}$ which does not Σ_{n+1}-elementarily end extend to any model of $\mathrm{I}\Sigma_{n+1}$. So with Σ_{n+2}-elementarity, the theory that the end extension K can always satisfy lies between $\mathrm{I}\Sigma_n$ and $\mathrm{I}\Sigma_{n+1}$, and the following question arises naturally.

QUESTION 1.2 (Kaufmann–Clote). *For $n \in \mathbb{N}$, does every countable model $M \models \mathrm{B}\Sigma_{n+2}$ have a proper Σ_{n+2}-elementary end extension $K \models \mathrm{B}\Sigma_{n+1}$?*

The question was included in the list of open problems in [1, p. 12, Problem 33] edited by Clote and Krajíček. It was first raised by Clote in [2], where he noted that

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the same question in the context of models of set theory had previously been posed by Kaufmann in [8]. In the same paper, Clote [2, Proposition 7] showed that every countable model $M \models \mathrm{I}\Sigma_{n+2}$ admits a Σ_{n+2}-elementary proper end extension to some $K \models M\text{-}\mathrm{B}\Sigma_{n+1}$, which is defined as follows: $M\text{-}\mathrm{B}\Sigma_{n+1}$ is the class of formulas of the form

$$\forall x < a\ \exists y\ \varphi(x, y) \rightarrow \exists b\ \forall x < a\ \exists y < b\ \varphi(x, y),$$

where $a \in M$ and $\varphi(x, y) \in \Sigma_{n+1}$ with parameters in K. Cornaros and Dimitracopoulos [3] showed that every countable model $M \models \mathrm{B}\Sigma_{n+2}$ has a Σ_{n+2}-elementary proper end extension $K \models \mathrm{B}\Sigma^-_{n+1}$ (the parameter-free Σ_{n+1}-collection). In this paper, we give an affirmative answer to Question 1.2.

The original proof of Theorem 1.1 is based on a first-order restricted ultrapower construction. The ultrapower is generated by a single element when viewed from the ground model. One can show that, by relativizing the proof of the fact that pointwise Σ_{n+1}-definable models do not satisfy $\mathrm{B}\Sigma_{n+1}$ (e.g.,[7, Lemma IV.1.41]), such ultrapowers always fail to satisfy $\mathrm{B}\Sigma_{n+1}$ as required in Question 1.2. To address this issue, we expand our model to a second-order structure that satisfies WKL_0; the end extension K will then be a second-order restricted ultrapower with respect to this second-order structure.

For us, one of the motivations for studying this question is to find a model-theoretic characterization of (countable) models of $\mathrm{I}\Sigma_{n+2}$ analogous to Theorem 1.1. Despite the fact that the end extension in Question 1.2 is insufficient for characterization, a slight generalization of it will suffice. We will show that for any countable model $M \models \mathrm{I}\Delta_0 + \exp$, $M \models \mathrm{I}\Sigma_{n+2}$ if and only if M admits a proper Σ_{n+2}-elementary end extension $K \models M\text{-}\mathrm{I}\Sigma_{n+1}$, whose definition is similar to that of $M\text{-}\mathrm{B}\Sigma_{n+1}$.

The regularity-type principles are the keys to connecting end extensions and the arithmetic theories satisfied by the ground model. Through the end extension, we can employ a "nonstandard analysis" style argument to prove certain types of regularity principles in the ground model. An example of such an argument related to Question 1.2 is provided below.

Proposition 1.3. *For each $n \in \mathbb{N}$, let $M \models \mathrm{I}\Delta_0$. If M admits a proper Σ_{n+2}-elementary end extension $K \models \mathrm{B}\Sigma_{n+1}$, then M satisfies the following principle*:

$$\forall x\ \exists y < a\ \varphi(x, y) \rightarrow \exists y < a\ \exists^{\mathrm{cf}} x\ \varphi(x, y),$$

where $a \in M$, $\varphi(x, y) \in \Pi_{n+1}(M)$ and $\exists^{\mathrm{cf}} x$ abbreviates $\forall b\ \exists x > b$.

Proof. Suppose $M \models \forall x\ \exists y < a\ \varphi(x, y)$. Because M and K both satisfy $\mathrm{B}\Sigma_{n+1}$, the formula $\forall x\ \exists y < a\ \varphi(x, y)$ is equivalent to some Π_{n+1} formula in both M and K. Then by Σ_{n+2}-elementarity, $K \models \forall x\ \exists y < a\ \varphi(x, y)$. Pick some arbitrary $d > M$ in K and let $c < a$ such that $K \models \varphi(d, c)$. Now for each $b \in M$, $K \models \exists x > b\ \varphi(x, c)$ and it is witnessed by d. Transferring each of these formulas back to M by elementarity, we have $M \models \exists x > b\ \varphi(x, c)$ for any $b \in M$, which means $M \models \exists^{\mathrm{cf}} x\ \varphi(x, c)$. ⊣

We call the principle in Proposition 1.3 the *weak regularity principle*, and denote it by $\mathrm{WR}\varphi$. The above proposition, together with the affirmative answer to Question 1.2, implies that $\mathrm{B}\Sigma_{n+2} \vdash \mathrm{WR}\Pi_{n+1}$ for each $n \in \mathbb{N}$.

Similarly to the argument above, we will show that if the end extension K is a model of M-$\mathrm{I}\Sigma_{n+1}$, then the ground model M will satisfy some stronger form of the weak regularity principle that implies $\mathrm{I}\Sigma_{n+2}$.

The paper is organized as follows. In Section 2, we present the necessary notations and fundamental facts regarding models of arithmetic. In Section 3, we review the definition of the second-order restricted ultrapower, and state some basic properties. In Section 4, we provide an affirmative answer to Question 1.2 in Theorem 4.3, and present the construction of an end extension that characterizes countable models of $\mathrm{I}\Sigma_{n+2}$ as mentioned above. In Section 5, we formally introduce the weak regularity principle $\mathrm{WR}\varphi$, and calibrate its strength within the I-B hierarchy. Finally, combining the results of Section 4 and Section 5, we establish a model-theoretic characterization of countable models of $\mathrm{I}\Sigma_{n+2}$ analogous to Theorem 1.1.

§2. Preliminaries. We assume the reader is familiar with some basic concepts and facts in the model theory of first- and second-order arithmetic [7, 9]. We reserve both the symbols $\mathbb{N}$ and ω for the set of standard natural numbers. For each $n \in \mathbb{N}$, let Σ_n and Π_n be the usual classes of formulas in the arithmetic hierarchy of first-order arithmetic. Given a model of first-order arithmetic M, $\Sigma_n(M)$ is the class of Σ_n formulas, potentially including parameters from M that are not explicitly shown. $\Pi_n(M)$ and other formula classes are defined similarly. A formula is Δ_n over M if it is equivalent to both a Σ_n and a Π_n formula in M; if M is clear from context, we simply write Δ_n for short. $\Sigma_n \wedge \Pi_n$ is the class of formulas which is the conjunction of a Σ_n and a Π_n formula, and $\Sigma_n \vee \Pi_n$ is defined similarly. $\Sigma_0(\Sigma_n)$ is the closure of Σ_n formulas under Boolean operations and bounded quantification. Σ^0_n, Π^0_n, Δ^0_n and $\Sigma_0(\Sigma^0_n)$ are their second-order variants, respectively. Given a model of second-order arithmetic $(M, \mathcal{X})$, $\Sigma^0_n(M, \mathcal{X})$ is defined similarly to the definition of $\Sigma_n(M)$, where the implicit parameters include both first- and second-order parameters in $(M, \mathcal{X})$. $\Pi^0_n(M, \mathcal{X})$ and $\Delta^0_n(M, \mathcal{X})$ are defined similarly. Finally, $\exists^{\mathrm{cf}} x \dots$ is the abbreviation of $\forall b\, \exists x > b \dots$.

For each $n \in \mathbb{N}$, let $\mathrm{B}\Sigma_n$ and $\mathrm{I}\Sigma_n$ be the collection scheme and the induction scheme for Σ_n formulas respectively. We assume that all the $\mathrm{B}\Sigma_n$ include $\mathrm{I}\Delta_0$, and all the theories considered include PA^-, which is the theory of the non-negative parts of discretely ordered rings. Paris and Kirby [16, Theorem A] proved that $\mathrm{I}\Sigma_{n+1} \vdash \mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$ and none of the converses holds for each $n \in \mathbb{N}$. The hierarchy of theories containing all the $\mathrm{I}\Sigma_n$ and $\mathrm{B}\Sigma_n$ is referred to as the *I-B hierarchy*. $\mathrm{I}\Sigma^0_n$ and $\mathrm{B}\Sigma^0_n$ are their second-order counterparts, respectively.

We adopt the standard pairing function, where the code of an ordered pair (a, b) under this pairing function is denoted by $\langle a, b\rangle$. For any element c in some model $M \models \mathrm{I}\Delta_0 + \exp$, we identify c with a subset of M by defining $x \in c$ to mean the xth digit in the binary expansion of c is 1. Fixing any proper cut I of M, we say that a set $A \subseteq I$ is *coded* in M if there is some $c \in M$ such that $A = \{x \in I \mid M \models x \in c\}$. We define

$$\mathrm{SSy}_I(M) := \{A \subseteq I \mid A \text{ is coded in } M\}.$$

RCA_0 is the subsystem of second-order arithmetic consisting of $\mathrm{I}\Sigma^0_1$ and Δ^0_1-comprehension. The system WKL_0 consists of RCA_0 and a statement asserting that every infinite binary tree has an infinite path. For each $n \in \mathbb{N}$, every countable model of $\mathrm{B}\Sigma^0_{n+2}$ admits a countable ω-extension (i.e., an extension only adding second-order objects) to some model satisfying $\mathrm{WKL}_0 + \mathrm{B}\Sigma^0_{n+2}$ [6].

Considering extensions of models of first-order arithmetic, we say that an extension $M \subseteq K$ of models of first-order arithmetic is Σ_n-*elementary*, if all the $\Sigma_n(M)$ formulas are absolute between M and K, and we write $M \preccurlyeq_{\Sigma_n} K$ to denote this. For any extension $M \subseteq K$ and $d \in K$, we write $d > M$ if $K \models d > c$ for any $c \in M$. We say that an extension $M \subseteq K$ is an *end extension*, if for every $d \in K \setminus M$, we have $d > M$. This is denoted by $M \subseteq_{\mathrm{e}} K$, or $M \preccurlyeq_{\mathrm{e},\Sigma_n} K$ if we also have $M \preccurlyeq_{\Sigma_n} K$. We say an extension $M \subseteq K$ is *proper* if $M \neq K$.

Formally, for a model of second-order arithmetic $(M, \mathcal{X})$, we view $(M, \mathcal{X})$ as a two-sorted first-order structure with number sort M and set sort $\mathcal{X}$, where elements in $\mathcal{X}$ are treated as syntactic objects and are not necessarily subsets of M. Under this convention, we write $(M, \mathcal{X}) \subseteq (K, \mathcal{Y})$ if it is an extension of the corresponding two-sorted structure, i.e., $M \subseteq K$, $\mathcal{X} \subseteq \mathcal{Y}$, and for any $x \in M$ and $A \in \mathcal{X}$,

$$(M, \mathcal{X}) \models x \in A \iff (K, \mathcal{Y}) \models x \in A.$$

Here, on the left-hand side of the equivalence, the second-order object A is interpreted by a subset of M; while it is interpreted by a subset of K on the right-hand side. In this paper, the second-order parts of our extensions usually remain the same, i.e., $\mathcal{X} = \mathcal{Y}$. We write $(M, \mathcal{X}) \preccurlyeq_{\Sigma^0_n} (K, \mathcal{Y})$ if all the $\Sigma^0_n(M, \mathcal{X})$ formulas are absolute between the two structures. We say that an extension of second-order structures is an end extension if its first-order part is an end extension. We denote this by $(M, \mathcal{X}) \subseteq_{\mathrm{e}} (K, \mathcal{Y})$, or $(M, \mathcal{X}) \preccurlyeq_{\mathrm{e},\Sigma^0_n} (K, \mathcal{Y})$ if we also have $(M, \mathcal{X}) \preccurlyeq_{\Sigma^0_n} (K, \mathcal{Y})$.

§3. Second-order restricted ultrapowers. The second-order restricted ultrapower resembles the usual ultrapower construction in model theory. Usually, in the model theory of arithmetic, we take the index set for the ultrapower construction to be the model itself, but instead of working on the class of all subsets of the model and all functions from the model to itself, we only consider a restricted class of subsets and functions. For completeness, we review the definition and some basic facts. All the results in this section appear in [10] except Lemma 3.6, Corollary 3.7, and Theorem 3.9. Throughout this section, we fix some arbitrary second-order structure $(M, \mathcal{X}) \models \mathrm{RCA}_0$.

DEFINITION 3.1 (Second-order restricted ultrapower). The second-order part $\mathcal{X}$ of $(M, \mathcal{X})$ forms a Boolean algebra under inclusion and Boolean operations. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathcal{X}$ whose elements are all cofinal in M, and $\mathcal{F}$ be the class of all the total functions from M to M in $\mathcal{X}$. Define an equivalence relation $\sim$ on $\mathcal{F}$ by

$$f \sim g \iff \{i \in M \mid f(i) = g(i)\} \in \mathcal{U},$$

where $f, g \in \mathcal{F}$. Let $\mathcal{F}/\mathcal{U}$ be the set of equivalence classes $[f]$ for $f \in \mathcal{F}$ modulo $\sim$. The interpretations of symbols in the language of first-order arithmetic in $\mathcal{F}/\mathcal{U}$ are defined by

$$[f] + [g] = [f + g],$$
$$[f] \times [g] = [f \times g],$$
$$[f] < [g] \iff \{i \in M \mid f(i) < g(i)\} \in \mathcal{U}.$$

Here $f + g$ and $f \times g$ are the pointwise addition and multiplication of f and g as functions. M naturally embeds into $\mathcal{F}/\mathcal{U}$ by identifying elements of M with constant functions. Moreover, $\mathcal{F}/\mathcal{U}$ is a proper extension of M by considering the equivalence class of the identity function on M.

$\mathcal{F}/\mathcal{U}$ admits a natural second-order expansion inherited from $\mathcal{X}$, namely for $A \in \mathcal{X}$ and $[f] \in \mathcal{F}/\mathcal{U}$, we define

$$[f] \in A \iff \{i \in M \mid f(i) \in A\} \in \mathcal{U}.$$

We denote the expanded structure of the ultrapower by $(\mathcal{F}/\mathcal{U}, \mathcal{X})$. It is easy to show that for $i \in M$ and $A \in \mathcal{X}$,

$$(M, \mathcal{X}) \models i \in A \iff (\mathcal{F}/\mathcal{U}, \mathcal{X}) \models i \in A.$$

So we may view $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ as an extension of $(M, \mathcal{X})$, where the second-order part remains the same. We call $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ the *second-order restricted ultrapower* of $(M, \mathcal{X})$ (with respect to $\mathcal{U}$).

The first-order restricted ultrapower is defined similarly, but with $\mathcal{F}$ and $\mathcal{U}$ replaced by the corresponding first-order definable classes. For example, in the construction of an Δ_1-ultrapower, $\mathcal{U}$ is an ultrafilter on the class of Δ_1-definable subsets, and $\mathcal{F}$ is the class of Δ_1-definable total functions.

From now on, we also fix an ultrafilter $\mathcal{U}$ on $\mathcal{X}$ whose elements are all cofinal in M.

Generally, Łoś's theorem does not hold for restricted ultrapowers, but a restricted version of it does hold:

Theorem 3.2 (Restricted Łoś's theorem). *Let $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ be the second-order restricted ultrapower of $(M, \mathcal{X})$. Then the following holds:*

(1) *If $\varphi(\overline{x})$ is a $\Sigma^0_1(M, \mathcal{X})$ formula, then*

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \varphi(\overline{[f]}) \iff \exists A \in \mathcal{U}, A \subseteq \{i \in M \mid (M, \mathcal{X}) \models \varphi(\overline{f(i)})\}.$$

(2) *If $\varphi(\overline{x})$ is a $\Delta^0_1(M, \mathcal{X})$ formula over $(M, \mathcal{X})$, then*

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \varphi(\overline{[f]}) \iff \{i \in M \mid (M, \mathcal{X}) \models \varphi(\overline{f(i)})\} \in \mathcal{U}.$$

Here the right-hand side makes sense by the Δ^0_1-comprehension of $(M, \mathcal{X})$.

Corollary 3.3. $(M, \mathcal{X}) \preccurlyeq_{\Sigma^0_2} (\mathcal{F}/\mathcal{U}, \mathcal{X})$.

Proof. Let $(M, \mathcal{X}) \models \forall x \, \exists y \, \psi(x, y)$ for some $\psi(x, y) \in \Delta^0_0(M, \mathcal{X})$. By choosing the least witness y of $\psi(x, y)$, we may assume $\psi(x, y)$ defines the graph of a total function $f \in \mathcal{F}$, so $(M, \mathcal{X}) \models \psi(x, f(x))$ for all $x \in M$. In particular, for each $g \in \mathcal{F}$ and $x \in M$,

$$(M, \mathcal{X}) \models \psi(g(x), f \circ g(x)).$$

Here $f \circ g$ is the composition of f and g. By Theorem 3.2,

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \psi([g], [f \circ g])$$

for each $[g] \in \mathcal{F}/\mathcal{U}$. So $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall x\, \exists y\, \psi(x, y)$. ⊣

We ensure that the ultrapower is an end extension of the ground model by the following definition and lemma.

DEFINITION 3.4. We say that $\mathcal{U}$ is *additive*, if whenever $f \in \mathcal{F}$ is bounded, then there is a $c \in M$ such that $\{i \in M \mid f(i) = c\} \in \mathcal{U}$.

LEMMA 3.5. *If $\mathcal{U}$ is additive, then $\mathcal{F}/\mathcal{U}$ is an end extension of M.*

PROOF. If $\mathcal{F}/\mathcal{U} \models [f] < b$ for some $b \in M$, then we may define

$$g(i) = \begin{cases} 0, & \text{if } f(i) \geqslant b, \\ f(i), & \text{if } f(i) < b. \end{cases}$$

Then g is a bounded function in $\mathcal{F}$ and $[g] = [f]$. The additiveness of $\mathcal{U}$ implies that there is some $c \in M$ such that $\{i \in M \mid g(i) = c\} \in \mathcal{U}$, that is, $\mathcal{F}/\mathcal{U} \models [g] = [f] = c$. ⊣

The following lemma and corollary enable us to transfer the comprehension in $(M, \mathcal{X})$ into the ultrapower via Σ^0_2-elementarity, and reduce the case of $\mathrm{B}\Sigma_{n+2}$ to $\mathrm{B}\Sigma^0_2$ uniformly in the construction of our main result.

LEMMA 3.6. *For each $n \geqslant 1$, if $(M, \mathcal{X})$ satisfies Σ_n-comprehension, then each instance of Σ_n- and Π_n-comprehension in $(M, \mathcal{X})$ is transferred to $(\mathcal{F}/\mathcal{U}, \mathcal{X})$. Formally, for any first-order formula $\varphi(x)$ in $\Sigma_n(M)$ or $\Pi_n(M)$, if there is some $A \in \mathcal{X}$ such that*

$$(M, \mathcal{X}) \models \forall x\, (x \in A \leftrightarrow \varphi(x)),$$

then $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall x\, (x \in A \leftrightarrow \varphi(x))$ as well.

PROOF. Fix some $(M, \mathcal{X})$ that satisfies Σ_n-comprehension. We prove the statement for all the $\varphi(x)$ in $\Sigma_k(M)$ and $\Pi_k(M)$ simultaneously by induction on k.

For $k = 1$, let $\varphi(x)$ be any formula in $\Sigma_1(M)$ or $\Pi_1(M)$, $A \in \mathcal{X}$ and $(M, \mathcal{X}) \models \forall x\, (x \in A \leftrightarrow \varphi(x))$. Since $\forall x\, (x \in A \leftrightarrow \varphi(x))$ is a $\Pi^0_2(M, \mathcal{X})$ formula, $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall x\, (x \in A \leftrightarrow \varphi(x))$ by Corollary 3.3.

For the induction step, suppose $k < n$ and the statement holds for all the formulas in $\Sigma_k(M)$ and $\Pi_k(M)$. Take any $\varphi(x) := \exists y\, \psi(x, y) \in \Sigma_{k+1}(M)$ where $\psi(x, y) \in \Pi_k(M)$. Suppose

$$(M, \mathcal{X}) \models \forall x\, (x \in A \leftrightarrow \exists y\, \psi(x, y)) \tag{3.6.1}$$

for some $A \in \mathcal{X}$. By Σ_n-comprehension in $(M, \mathcal{X})$, there exists some $B \in \mathcal{X}$ such that $(M, \mathcal{X})$ satisfies

$$\forall \langle x, y\rangle\, (\langle x, y\rangle \in B \leftrightarrow \psi(x, y)). \tag{3.6.2}$$

(3.6.1) and (3.6.2) in $(M, \mathcal{X})$ imply that $(M, \mathcal{X})$ also satisfies

$$\forall x\, (x \in A \leftrightarrow \exists y\, \langle x, y\rangle \in B). \tag{3.6.3}$$

By the induction hypothesis and Corollary 3.3, (3.6.2) and (3.6.3) are transferred to $(\mathcal{F}/\mathcal{U}, \mathcal{X})$. Combining (3.6.2) and (3.6.3) in $(\mathcal{F}/\mathcal{U}, \mathcal{X})$, we have

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall x\ (x \in A \leftrightarrow \exists y\ \psi(x, y)).$$

The case for $\varphi(x) \in \Pi_{k+1}(M)$ is exactly the same. This completes the induction. ⊣

Corollary 3.7. *If* $(M, \mathcal{X})$ *satisfies* Σ_n*-comprehension for some* $n \in \mathbb{N}$, *then* $M \preccurlyeq_{\Sigma_{n+2}} \mathcal{F}/\mathcal{U}$ *when viewed as an extension of models of first-order arithmetic.*

Proof. Let $M \models \forall x\ \exists y\ \psi(x, y)$ for some $\psi(x, y) \in \Pi_n(M)$. By Σ_n-comprehension in $(M, \mathcal{X})$, there exists some $A \in \mathcal{X}$ such that $(M, \mathcal{X})$ satisfies

$$\forall \langle x, y\rangle\ (\langle x, y\rangle \in A \leftrightarrow \psi(x, y)).$$

Therefore, $(M, \mathcal{X})$ also satisfies

$$\forall x\ \exists y\ \langle x, y\rangle \in A.$$

By Lemma 3.6 and Corollary 3.3, both formulas hold in $(\mathcal{F}/\mathcal{U}, \mathcal{X})$, which implies $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall x\ \exists y\ \psi(x, y)$. ⊣

For technical reasons, we need another kind of ultrapower construction in the model theory of arithmetic called the *coded ultrapower*, which appears in [15, Theorem 9] and [11, Theorem 3]. We fix some $M \models \mathrm{I}\Sigma_1$ and a proper end extension $M \subseteq_{\mathrm{e}} L$ such that $L \models \mathrm{I}\Delta_0 + \exp$.

Definition 3.8. Let $\mathcal{U}$ be an ultrafilter on $\mathrm{SSy}_M(L)$ whose elements are all cofinal in M, and let $\mathcal{G}$ be the class of all total functions mapping from M to L that are coded in L. Then $\mathcal{G}/\mathcal{U}$ is defined in the same way as $\mathcal{F}/\mathcal{U}$ in Definition 3.1, and we call it the *M-coded ultrapower* of L with respect to $\mathcal{U}$.

For our purpose, only Łoś's Theorem for Δ_0 formulas in $\mathcal{G}/\mathcal{U}$ and the fact that $\mathcal{G}/\mathcal{U} \models \mathrm{I}\Delta_0$ are needed. Their proofs essentially appear in [15, Theorem 11], but using a slightly different definition of the coded ultrapowers.

Theorem 3.9. *Let* $\mathcal{G}/\mathcal{U}$ *be a M-coded ultrapower of L. Then the following hold*:

(1) *Let* $\varphi(\overline{x})$ *be a* $\Delta_0(L)$ *formula, then*

$$\mathcal{G}/\mathcal{U} \models \varphi(\overline{[f]}) \iff \{i \in M \mid L \models \varphi(\overline{f(i)})\} \in \mathcal{U}.$$

(2) $\mathcal{G}/\mathcal{U} \models \mathrm{I}\Delta_0$.

§4. Constructions of end extensions.

In this section, we present the constructions of end extensions by the second-order restricted ultrapower construction. We first answer Question 1.2 affirmatively. In view of Corollary 3.7, it suffices to deal with the case in which $(M, \mathcal{X}) \models \mathrm{B}\Sigma^0_2$.

Theorem 4.1. *For any countable model* $(M, \mathcal{X}) \models \mathrm{B}\Sigma^0_2 + \mathrm{WKL}_0$, *there is a proper end extension* $(M, \mathcal{X}) \preccurlyeq_{\mathrm{e},\Sigma^0_2} (K, \mathcal{X}) \models \mathrm{B}\Sigma^0_1$.

We will give two proofs. In both of our constructions, the end extension is given by a second-order restricted ultrapower construction. The first proof, presented below, was suggested by Tin Lok Wong. We ensure that the ultrapower of $(M, \mathcal{X})$ satisfies $\mathrm{B}\Sigma^0_1$ by properly embedding it into a coded ultrapower as an initial segment. The second proof, presented on Page 11, guarantees that the ultrapower satisfies $\mathrm{B}\Sigma^0_1$ directly by the construction of the ultrafilter. WKL_0 plays a central role in both constructions.

THE FIRST PROOF OF THEOREM 4.1. First by [5, Theorem 4.6], there is a countable end extension $M \subseteq_{\mathrm{e}} L \models \mathrm{I}\Delta_0$ such that $\mathcal{X} = \mathrm{SSy}_M(L)$.

Let

$$\mathcal{G} := \{g \mid g \text{ is a total function from } M \text{ to } L \text{ coded in } L\},$$
$$\mathcal{F} := \{f \in \mathcal{X} \mid f \text{ is a total function from } M \text{ to } M\}.$$

For any ultrafilter $\mathcal{U}$ on $\mathcal{X}$ whose elements are all cofinal in M, let $\mathcal{G}/\mathcal{U}$ and $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ be the coded ultrapower (see Definition 3.8) and the second-order restricted ultrapower with respect to $\mathcal{U}$ respectively. Since $\mathcal{X} = \mathrm{SSy}_M(L)$, we may regard $\mathcal{F}$ as a subset of $\mathcal{G}$, and thus $\mathcal{F}/\mathcal{U}$ naturally embeds into $\mathcal{G}/\mathcal{U}$.

We want to construct a sufficiently generic ultrafilter $\mathcal{U}$, such that both M and $\mathcal{F}/\mathcal{U}$ are proper initial segments of $\mathcal{G}/\mathcal{U}$. We construct $\mathcal{U}$ in ω many stages. For each $k \in \mathbb{N}$, we construct some $A_k \in \mathcal{X}$ that is cofinal in M and $A_k \supseteq A_{k+1}$. We enumerate all the pairs $\langle f, g\rangle$ such that $f \in \mathcal{F}$ and $g \in \mathcal{G}$ as $\{\langle f_k, g_k\rangle\}_{k\in\mathbb{N}}$, and all the bounded functions in $\mathcal{F}$ as $\{h_k\}_{k\in\mathbb{N}}$. Here we identify each element $g \in \mathcal{G}$ with some element in L that codes g.

Stage 0: Set $A_0 = M$.

Stage $2k+1$ ($\mathcal{F}/\mathcal{U} \subseteq_{\mathrm{e}} \mathcal{G}/\mathcal{U}$): Consider

$$A := A_{2k} \cap \{x \in M \mid L \models g_k(x) < f_k(x)\}.$$

Since $L \models \mathrm{I}\Delta_0$, $A \in \mathrm{SSy}_M(L) = \mathcal{X}$. If A is cofinal in M, then let $A_{2k+1} = A$. Otherwise let $A_{2k+1} = A_{2k}$ and proceed to the next stage.

Stage $2k+2$ ($M \subseteq_{\mathrm{e}} \mathcal{F}/\mathcal{U}$): Assume h_k is bounded by $b \in M$. Then

$$(M, \mathcal{X}) \models \exists^{\mathrm{cf}} x\, \exists y < b\, (x \in A_{2k+1} \wedge h_k(x) = y).$$

Since $(M, \mathcal{X}) \models \mathrm{B}\Sigma^0_2$, there is some $c < b$ such that the set

$$A_c := \{x \in M \mid (M, \mathcal{X}) \models x \in A_{2k+1} \wedge h_k(x) = c\}$$

is cofinal in M. $A_c \in \mathcal{X}$ by Δ^0_1-comprehension in $(M, \mathcal{X})$. Let $A_{2k+2} = A_c$ and proceed to the next stage.

Finally, let $\mathcal{U} := \{A \in \mathcal{X} \mid \exists k \in \mathbb{N}\ A_k \subseteq A\}$. It is not hard to see that $\mathcal{U}$ is an ultrafilter and each element of $\mathcal{U}$ is cofinal in M. This completes the construction of $\mathcal{U}$.

Verification: First we verify that $\mathcal{F}/\mathcal{U} \subseteq_{\mathrm{e}} \mathcal{G}/\mathcal{U}$ and it is proper. We show that for any $g \in \mathcal{G}$ and $f \in \mathcal{F}$, either $\mathcal{G}/\mathcal{U} \models [g] \geqslant [f]$ or there exists some $\hat{g} \in \mathcal{F}$ such that $\mathcal{G}/\mathcal{U} \models [g] = [\hat{g}]$. Take $k \in \mathbb{N}$ such that $\langle f_k, g_k\rangle = \langle f, g\rangle$. At Stage $2k$, if

$$A = A_{2k} \cap \{x \in M \mid L \models g(x) < f(x)\}$$

is cofinal in M, then $\{x \in M \mid L \models g(x) < f(x)\} \in \mathcal{U}$. Let $\hat{g} \in \mathcal{G}$ be defined by

$$\hat{g}(x) = \begin{cases} g(x), & \text{if } g(x) < f(x), \\ 0, & \text{if } g(x) \geqslant f(x). \end{cases}$$

Then $[\hat{g}] \in \mathcal{F}/\mathcal{U}$ and $\mathcal{G}/\mathcal{U} \models [g] = [\hat{g}]$. Otherwise, if A is bounded in M, then we are forced to have $\{x \in M \mid L \models g(x) \geqslant f(x)\} \in \mathcal{U}$. By Theorem 3.9, $\mathcal{G}/\mathcal{U} \models [g] \geqslant [f]$. The fact that $\mathcal{F}/\mathcal{U} \subseteq_{\mathrm{e}} \mathcal{G}/\mathcal{U}$ is proper follows from considering the constant function $g(x) \equiv c$ for any $c > M$ in L.

Next we verify that $M \subseteq_{\mathrm{e}} \mathcal{F}/\mathcal{U}$. By Lemma 3.5, it suffices to show that $\mathcal{U}$ is additive with respect to $\mathcal{F}$. Suppose $h \in \mathcal{F}$ is bounded and $h = h_k$ for some $k \in \mathbb{N}$. At Stage $2k+2$, the choice of A_{2k+2} forces

$$\{x \in M \mid (M, \mathcal{X}) \models h(x) = c\} \in \mathcal{U}$$

for some $c \in M$, so $\mathcal{U}$ is additive with respect to $\mathcal{F}$.

The fact that $(M, \mathcal{X}) \preccurlyeq_{\Sigma_2^0} (\mathcal{F}/\mathcal{U}, \mathcal{X})$ follows from Corollary 3.3.

Finally we verify that $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \mathrm{B}\Sigma_1^0$. By Theorem 3.9, $\mathcal{G}/\mathcal{U} \models \mathrm{I}\Delta_0$. Since $\mathcal{G}/\mathcal{U}$ is a proper end extension of $\mathcal{F}/\mathcal{U}$, $(\mathcal{F}/\mathcal{U}, \mathrm{SSy}_{\mathcal{F}/\mathcal{U}}(\mathcal{G}/\mathcal{U}))$ satisfies $\mathrm{B}\Sigma_1^0$. It suffices to show that $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ embeds into $(\mathcal{F}/\mathcal{U}, \mathrm{SSy}_{\mathcal{F}/\mathcal{U}}(\mathcal{G}/\mathcal{U}))$. For each $A \in \mathcal{X} = \mathrm{SSy}_M(L)$, let $a \in L$ be the element that codes $A \subseteq M$. By Theorem 3.9 and Definition 3.1, it is not hard to prove that for each $f \in \mathcal{F}$,

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models [f] \in A \iff (\mathcal{G}/\mathcal{U}, \mathcal{X}) \models [f] \in a.$$

So we may embed the second-order part $\mathcal{X}$ of $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ into $\mathrm{SSy}_{\mathcal{F}/\mathcal{U}}(\mathcal{G}/\mathcal{U})$ by sending A to the subset of $\mathcal{F}/\mathcal{U}$ coded by a. Since $(\mathcal{F}/\mathcal{U}, \mathrm{SSy}_{\mathcal{F}/\mathcal{U}}(\mathcal{G}/\mathcal{U})) \models \mathrm{B}\Sigma_1^0$, we have $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \mathrm{B}\Sigma_1^0$. □

Even though it is simple, this construction does not reveal a syntactical proof of the fact that $\mathrm{B}\Sigma_{n+2} \vdash \mathrm{WR}\Pi_{n+1}$. We ensure that $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \mathrm{B}\Sigma_1^0$ by embedding it in a larger ultrapower $\mathcal{G}/\mathcal{U}$ as an initial segment, and the core argument is wrapped within the construction of $\mathcal{G}/\mathcal{U}$.

Our second construction directly guarantees each instance of $\mathrm{B}\Sigma_1^0$ in the ultrapower, and therefore provides more insights. It relies on a simple yet powerful lemma, which states that second-order universes of models of WKL_0 are closed under the operation that produces choice functions for ranges of Π_1^0-definable multi-valued bounded functions. The lemma also leads to a syntactical proof of the fact that $\mathrm{B}\Sigma_{n+2} \vdash \mathrm{WR}\Pi_{n+1}$ (see Lemma 5.2).

Lemma 4.2. *Fix a model* $(M, \mathcal{X}) \models \mathrm{WKL}_0$. *Let* $\theta(x, y, z) \in \Delta_0^0(M, \mathcal{X})$. *If* $(M, \mathcal{X}) \models \forall x\ \exists y < f(x)\ \forall z\ \theta(x, y, z)$ *for some total function* $f \in \mathcal{X}$, *then there is a total function* $P \in \mathcal{X}$ *such that*

$$(M, \mathcal{X}) \models \forall x\ (P(x) < f(x) \wedge \forall z\ \theta(x, P(x), z)).$$

Proof. Consider the following tree T which is Δ_1^0-definable in $(M, \mathcal{X})$:

$$\sigma \in T \iff \forall x, z < \operatorname{len}\sigma\ (\sigma(x) < f(x) \wedge \theta(x, \sigma(x), z)).$$

Obviously T is *bounded* by the total function $f \in \mathcal{X}$, which means that for any $\sigma \in T$ and $x < \operatorname{len}\sigma$, we have $\sigma(x) < f(x)$. We show that T is infinite. Let $F(x) = \max_{x'<x} f(x')$. For any $x \in M$, by $\mathrm{I}\Sigma^0_1$, let $\sigma_x \in M$ be a coded sequence of length x such that for any $x' < x$ and $y' < F(x)$,

$$\sigma_x(x') = y' \iff (M, \mathcal{X}) \models \forall z\, \theta(x', y', z) \wedge \forall w < y'\, \neg\forall z\, \theta(x', w, z).$$

Then we have

$$(M, \mathcal{X}) \models \forall x' < x\, \forall z\, (\sigma_x(x') < f(x) \wedge \theta(x', \sigma_x(x'), z)).$$

This means σ_x is an element of T of length x, so T is infinite. It is provable in WKL_0 that every infinite bounded tree has an infinite path (see [17, Lemma IV.1.4]). Take any such infinite path $P \in \mathcal{X}$ of T. Clearly P satisfies the requirement as in the statement. ⊣

THE SECOND PROOF OF THEOREM 4.1. We construct an ultrafilter $\mathcal{U}$ on $\mathcal{X}$ in ω many stages. Along the construction, we gradually guarantee that the ultrapower $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \mathrm{B}\Sigma^0_1$ and that $\mathcal{U}$ is additive.

Enumerate all the triples $\{\langle \exists z\ \ \theta_k(x, y, z), f_k, g_k\rangle\}_{k\in\mathbb{N}}$, where $\theta_k(x, y, z) \in \Delta^0_0(M, \mathcal{X})$ and f_k, g_k are total functions in $\mathcal{X}$. Enumerate all the bounded total functions in $\mathcal{X}$ as $\{h_k\}_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$, at Stage k we construct a cofinal set $A_k \in \mathcal{X}$ such that $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$, and the resulting ultrafilter $\mathcal{U} := \{A \in \mathcal{X} \mid \exists k \in \mathbb{N}\ A \supseteq A_k\}$.

Stage 0: Set $A_0 = M \in \mathcal{X}$.

Stage $2k+1$ ($(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \mathrm{B}\Sigma^0_1$): At these stages we want to guarantee the following instances of $\mathrm{B}\Sigma^0_1$ in $(\mathcal{F}/\mathcal{U}, \mathcal{X})$:

$$\forall y < [g_k]\, \exists z\, \theta_k([f_k], y, z) \to \exists b\, \forall y < [g_k]\, \exists z < b\, \theta_k([f_k], y, z).$$

The general idea is that we first try to "force" the consequent of the implication above to be true in $(\mathcal{F}/\mathcal{U}, \mathcal{X})$. If we succeed, then the entire instance is true. Otherwise, we apply Lemma 4.2 to argue that the antecedent is already guaranteed to be false in the ultrapower.

Consider the Σ^0_1-definable set

$$A := A_{2k} \cap \{x \in M \mid \exists b\, \forall y < g_k(x)\, \exists z < b\, \theta_k(f_k(x), y, z)\}.$$

It is provable in RCA_0 that every cofinal Σ^0_1-definable set has a cofinal subset in the second-order universe (see [7, Theorem I.3.22]). If A is cofinal in M, then let $A_{2k+1} \in \mathcal{X}$ be such a cofinal subset of A and proceed to Stage $2k+2$. If A is not cofinal in M, then we let $A_{2k+1} = A_{2k}$ and proceed directly to Stage $2k+2$.

Stage $2k+2$ (Additiveness of $\mathcal{U}$): This part is exactly the same as the construction of Stage $2k+2$ in the first proof of Theorem 4.1.

Finally, let $\mathcal{U} := \{A \in \mathcal{X} \mid \exists k \in\ \mathbb{N}\ A \supseteq A_k\}$. This completes the construction of $\mathcal{U}$.

Verification: Let $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ be the corresponding second-order restricted ultrapower. The fact that $(M, \mathcal{X}) \preccurlyeq_{\mathrm{e},\Sigma^0_2} (\mathcal{F}/\mathcal{U}, \mathcal{X})$ follows from the exact same reasoning as in the first proof of Theorem 4.1. To show that $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \mathrm{B}\Sigma^0_1$, consider arbitrary instance of $B\Sigma^0_1$ in $(\mathcal{F}/\mathcal{U}, \mathcal{X})$:

$$\forall y < [g]\, \exists z\, \theta([f], y, z) \to \exists b\, \forall y < [g]\, \exists z < b\, \theta([f], y, z),$$

where $\theta \in \Delta^0_0(M, \mathcal{X})$ and $[f], [g] \in \mathcal{F}/\mathcal{U}$. Here without loss of generality, we assume that $[f]$ is the only first-order parameter in θ. Assume that at Stage $2k + 1$, we enumerate $\langle \exists z\, \theta_k(x, y, z), f_k, g_k\rangle = \langle \exists z\, \theta(x, y, z), f, g\rangle$, and $A_{2k} \in \mathcal{X}$ is the cofinal subset of M we obtained from the previous stage. Suppose we are in the first case of the construction at this stage, i.e.,

$$A = A_{2k} \cap \{x \in M \mid \exists b\, \forall y < g(x)\, \exists z < b\, \theta(f(x), y, z)\}$$

is cofinal in M, then by the construction there exists some cofinal subset of A in $\mathcal{U}$. By Theorem 3.2, $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \exists b\ \forall y < [g]\ \exists z < b\ \theta([f], y, z)$, so the instance of $B\Sigma^0_1$ is true. If we are in the second case, assuming that A is bounded by some $d \in M$, then

$$(M, \mathcal{X}) \models \forall x > d\ (x \in A_{2k} \to \forall b\, \exists y < g(x)\, \forall z < b\, \neg\theta(f(x), y, z)).$$

By $B\Sigma^0_1$ in $(M, \mathcal{X})$, this is equivalent to:

$$(M, \mathcal{X}) \models \forall x > d\ \exists y < g(x)\ (x \in A_{2k} \to \forall z\, \neg\theta(f(x), y, z)).$$

By Lemma 4.2, there is a total function $P \in \mathcal{X}$ such that

(1) $(M, \mathcal{X}) \models \forall x > d\ P(x) < g(x)$.
(2) $(M, \mathcal{X}) \models \forall x > d\ (x \in A_{2k} \to \forall z\, \neg\theta(f(x), P(x), z))$.

Since P is a total function, $[P] \in \mathcal{F}/\mathcal{U}$. By Theorem 3.2, (1) implies $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models [P] < [g]$.

We claim that $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall z\, \neg\theta([f], [P], z)$. Suppose not, then by Theorem 4.1, there is some $A' \in \mathcal{U}$ such that $A' \subseteq \{x \in M \mid \exists z\, \theta(f(x), P(x), z)\}$. But by (2), $A' \cap A_{2k}$ is bounded by d, which contradicts the fact that $A' \cap A_{2k} \in \mathcal{U}$. So $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall z\, \neg\theta([f], [P], z)$, and the instance of $B\Sigma^0_1$ we considered is vacuously true. □

THEOREM 4.3. *For each $n \in \mathbb{N}$ and any countable model $M \models B\Sigma_{n+2}$, there is a Σ_{n+2}-elementary proper end extension $M \preccurlyeq_{e,\Sigma_{n+2}} K \models B\Sigma_{n+1}$.*

PROOF. We first expand M to a second-order structure satisfying $B\Sigma^0_2$ by adding all the Σ_n-definable sets into the second-order universe, then we further ω-extend it to some countable $(M, \mathcal{X}) \models B\Sigma^0_2 + WKL_0$. By Theorem 4.1, there is an ultrapower extension $(M, \mathcal{X}) \preccurlyeq_{e,\Sigma^0_2} (\mathcal{F}/\mathcal{U}, \mathcal{X})$ that satisfies $B\Sigma^0_1$. Since all Σ_n-definable subsets of M are in $\mathcal{X}$, $(M, \mathcal{X})$ satisfies Σ_n-comprehension and $M \preccurlyeq_{e,\Sigma_{n+2}} \mathcal{F}/\mathcal{U}$ by Corollary 3.7.

To show that $\mathcal{F}/\mathcal{U} \models B\Sigma_{n+1}$, suppose that $\mathcal{F}/\mathcal{U} \models \forall x < [g]\ \exists y\ \theta(x, y, [f])$ for some $[g] \in \mathcal{F}/\mathcal{U}$ and $\theta \in \Pi_n$, where $[f] \in \mathcal{F}/\mathcal{U}$ is the only parameter in θ. By Π_n-comprehension in M, let $A \in \mathcal{X}$ be so that $(M, \mathcal{X})$ satisfies

$$\forall \langle x, y, z\rangle\ (\langle x, y, z\rangle \in A \leftrightarrow \theta(x, y, z)).$$

The structure $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ satisfies the same formula by Lemma 3.6, and thus

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall x < [g]\ \exists y\ \langle x, y, [f]\rangle \in A.$$

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By $\mathrm{B}\Sigma_1^0$ in $(\mathcal{F}/\mathcal{U}, \mathcal{X})$,

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \exists b\, \forall x < [g]\, \exists y < b\, \langle x, y, [f] \rangle \in A,$$

which means $\mathcal{F}/\mathcal{U} \models \exists b\, \forall x < [g]\, \exists y < b\, \theta(x, y)$, so $\mathcal{F}/\mathcal{U} \models \mathrm{B}\Sigma_{n+1}$. ⊣

We now proceed with the construction of end extensions in order to characterize countable models of $\mathrm{I}\Sigma_{n+2}$. We first define M-$\mathrm{I}\Sigma_{n+1}$ for an end extension $M \subseteq_{\mathrm{e}} K$, and introduce some equivalent definitions of it.

Definition 4.4. For each $n \in \mathbb{N}$, let M, K be models of $\mathrm{I}\Delta_0 + \exp$ and $M \subseteq_{\mathrm{e}} K$. We say $K \models M$-$\mathrm{I}\Sigma_{n+1}$ if for any $\varphi(x) \in \Sigma_{n+1}(K)$ and $a \in M$,

$$K \models \varphi(0) \wedge \forall x < a\, (\varphi(x) \to \varphi(x+1)) \to \forall x < a\, \varphi(x).$$

Notice that we allow parameters from K in φ, while the bound a must be in M.

Lemma 4.5. *For each* $n \in \mathbb{N}$, *let* M *be a model of* $\mathrm{I}\Delta_0 + \exp$, $K \models \mathrm{I}\Sigma_n$ *and* $M \subseteq_{\mathrm{e}} K$. *Then the following are equivalent*:

(i) $K \models M$-$\mathrm{I}\Sigma_{n+1}$.

(ii) *For any* $\varphi(x) \in \Sigma_{n+1}(K)$ *and* $a \in M$,

$$K \models \exists c\, \forall x < a\, (\varphi(x) \leftrightarrow x \in c).$$

(iii) *For any* $\theta(x, y) \in \Pi_n(K)$ *and* $a \in M$,

$$K \models \exists b\, \forall x < a\, (\exists y\, \theta(x, y) \leftrightarrow \exists y < b\, \theta(x, y)).$$

Proof. We show (i) ⇔ (ii) and (ii) ⇔ (iii). To show (i) ⇒ (ii), first by modifying a standard argument, one can show that (i) implies the least number principle for $\Pi_{n+1}(K)$ formulas that are satisfied by some element of M. Then we can pick the least $c < 2^a \in M$ such that

$$K \models \forall x < a\, (\varphi(x) \to x \in c).$$

Such c will code $\varphi(x)$ for $x < a$ by the minimality of c. To show (ii) ⇒ (i), take some $c \in M$ that codes $\{x < a \mid K \models \varphi(x)\}$. Then, one can prove the instance of M-$\mathrm{I}\Sigma_{n+1}$ for $\varphi(x)$ by replacing $\varphi(x)$ with $x \in c$ and applying $K \models \mathrm{I}\Delta_0$.

To show (ii) ⇒ (iii), take some $c \in M$ that codes $\{x < a \mid K \models \exists y\, \theta(x, y)\}$ by (ii). Consider the following Σ_{n+1} formula (over $\mathrm{I}\Sigma_n$):

$$\Phi(v) := \exists b\, \forall x < v\, (x \in c \leftrightarrow \exists y < b\, \theta(x, y)).$$

It is not hard to show that $K \models \Phi(0) \wedge \forall v\, (\Phi(v) \to \Phi(v+1))$. By M-$\mathrm{I}\Sigma_{n+1}$ (from (ii) ⇒ (i)) we have $K \models \Phi(a)$, which implies (iii). Finally, to show (iii) ⇒ (ii), let $\varphi(x) := \exists y\, \theta(x, y)$ for some $\theta \in \Pi_n(K)$. By (iii), there is some $b \in K$ such that

$$K \models \forall x < a\, (\exists y\, \theta(x, y) \leftrightarrow \exists y < b\, \theta(x, y)).$$

By $\mathrm{I}\Sigma_n$ in K, there is some $c \in K$ that codes $\{x < a \mid K \models \exists y < b\; \theta(x, y)\}$. This element c will serve as a witness for (ii). ⊣

The condition (ii) in Lemma 4.5 was first studied in [12] by Kossak. An extension $M \subseteq K$ is called an $(n+1)$-*conservative* extension if $M \preccurlyeq_{\Sigma_{n+2}} K$ and it satisfies the condition (ii) in Lemma 4.5. In the same paper, Kossak showed that every countable model of $I\Sigma_{n+2}$ has a proper $(n+1)$-conservative extension. The construction is also a mild generalization of [2, Proposition 7]. For the sake of completeness, we include a proof of this result here, employing the framework of the second-order restricted ultrapowers.

THEOREM 4.6. *For any countable model* $(M, \mathcal{X}) \models I\Sigma^0_2$, *there is a proper end extension* $(M, \mathcal{X}) \preccurlyeq_{e,\Sigma^0_2} (K, \mathcal{X})$ *such that for any* $\Sigma^0_1(K, \mathcal{X})$ *formula* $\varphi(z)$ *and* $a \in M$, *the set* $\{z < a \mid (K, \mathcal{X}) \models \varphi(z)\}$ *is coded in K.*

PROOF. We ensure that the ultrapower $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ satisfies the coding requirement by maximizing each Σ^0_1-definable subset of $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ that is bounded by some element of M.

Enumerate all the pairs $\{\langle \exists y\, \theta_k(x, y, z), a_k\rangle\}_{k\in\mathbb{N}}$ such that $\theta_k(x, y, z) \in \Delta^0_0(M, \mathcal{X})$ and $a_k \in M$. Enumerate all the bounded total functions in $\mathcal{X}$ as $\{h_k\}_{k\in\mathbb{N}}$. At each stage k we construct a cofinal set $A_k \in \mathcal{X}$ such that $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$, and we define the ultrafilter $\mathcal{U} := \{A \in \mathcal{X} \mid \exists k \in \mathbb{N}\, A \supseteq A_k\}$.

Stage 0: Set $A_0 = M$.

Stage 2*k* + 1 (Coding Σ^0_1 sets): Consider the following Π^0_2 formula where $c < 2^{a_k}$:

$$\Phi(c) := \exists^{\mathrm{cf}} x\ (x \in A_{2k} \wedge \forall z \in\ c\ \exists y\ \theta_k(x, y, z)).$$

Since A_{2k} is cofinal in M, $(M, \mathcal{X}) \models \Phi(0)$. By $I\Sigma^0_2$ in $(M, \mathcal{X})$, there exists a maximal $c_0 < 2^{a_k}$ satisfying the formula above. Similar to the second proof of Theorem 4.1, let $A_{2k+1} \in \mathcal{X}$ be a cofinal subset of the following Σ^0_1-definable subset of M:

$$\{x \in M \mid (M, \mathcal{X}) \models x \in A_{2k} \wedge \forall z \in\ c_0\ \exists y\ \theta_k(x, y, z)\}.$$

Stage 2*k* + 2 (Additiveness of $\mathcal{U}$): This part is exactly the same as Stage $2k+2$ in the proof of Theorem 4.1.

Finally, let $\mathcal{U} := \{A \in \mathcal{X} \mid \exists k \in\ \mathbb{N}\ A \supseteq A_k\}$. This completes the construction.

Verification: Let $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ be the corresponding second-order restricted ultrapower. The fact that $(M, \mathcal{X}) \preccurlyeq_{e,\Sigma^0_2} (\mathcal{F}/\mathcal{U}, \mathcal{X})$ follows in exactly the same way as in the second proof of Theorem 4.1.

To show the coding requirement of $(\mathcal{F}/\mathcal{U}, \mathcal{X})$, consider any Σ^0_1 formula $\exists y\ \theta([f], y, z)$ where $\theta \in \Delta^0_0(\mathcal{F}/\mathcal{U}, \mathcal{X})$ and $a \in M$. Without loss of generality, we may assume that $[f] \in \mathcal{F}/\mathcal{U}$ is the only first-order parameter of θ.

Let $k \in \mathbb{N}$ be such that the pair $\langle \exists y\ \theta(f(x), y, z), a\rangle$ was considered at Stage $2k+1$ of the construction. We claim that the maximal $c_0 \in M$ we obtained in the construction codes $\{z < a \mid (\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \exists y\ \theta([f], y, z)\}$.

On the one hand, for each $z < a$ such that $z \in c_0$, since $A_{2k+1} \in \mathcal{U}$ is a subset of $\{x \in M \mid (M, \mathcal{X}) \models \exists y\, \theta(f(x), y, z)\}$, $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \exists y\, \theta([f], y, z)$ by Theorem 3.2. On the other hand, for each $z' < a$ such that $z' \notin c_0$, if $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \exists y\, \theta([f], y, z')$, then by Theorem 3.2 again there is some $B \in \mathcal{U}$ such that

$$B \subseteq \{x \in M \mid (M, \mathcal{X}) \models \exists y\ \theta(f(x), y, z')\}.$$

Since $B \cap A_{2k+1} \in \mathcal{U}$, $B \cap A_{2k+1}$ is cofinal in M. Then we have

$$(M, \mathcal{X}) \models \exists^{\mathrm{cf}} x (x \in A_{2k} \wedge \forall z \in\ c_0 \cup \{z'\}\ \exists y\ \theta(f(x), y, z)),$$

which contradicts the maximality of c_0 in the construction. So $(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \neg\exists y\, \theta([f], y, z')$ for $z' \notin c_0$. ⊣

THEOREM 4.7 (Kossak [12]). *For each $n \in \mathbb{N}$ and any countable model $M \models \mathrm{I}\Sigma_{n+2}$, there is a Σ_{n+2}-elementary proper end extension $M \preccurlyeq_{\mathrm{e},\Sigma_{n+2}} K \models M\text{-}\mathrm{I}\Sigma_{n+1}$.*

PROOF. The proof is mostly the same as that of Theorem 4.3. We expand M to a second-order structure $(M, \mathcal{X})$ satisfying $\mathrm{I}\Sigma^0_2 + \mathrm{RCA}_0$ by adding all the Δ_{n+1}-definable subsets of M into the second-order universe. Such $(M, \mathcal{X})$ satisfies Σ_n-comprehension. By Theorem 4.6, there exists an ultrapower extension $(M, \mathcal{X}) \preccurlyeq_{\mathrm{e},\Sigma^0_2} (\mathcal{F}/\mathcal{U}, \mathcal{X})$ that codes all the Σ^0_1-definable subsets bounded by some element of M. Since all the Σ_n-definable sets of M are in $\mathcal{X}$, $(M, \mathcal{X})$ satisfies Σ_n-comprehension and $M \preccurlyeq_{\mathrm{e},\Sigma_{n+2}} \mathcal{F}/\mathcal{U}$ by Corollary 3.7.

To show that $\mathcal{F}/\mathcal{U} \models M\text{-}\mathrm{I}\Sigma_{n+1}$, we only need to show that $\mathcal{F}/\mathcal{U}$ satisfies condition (ii) in Lemma 4.5. Let $\varphi(x) := \exists y\, \theta(x, y, [f])$ be any Σ_{n+1} formula, where $\theta \in \Pi_n$ and $[f] \in \mathcal{F}/\mathcal{U}$ is the only parameter in θ. Since $(M, \mathcal{X})$ satisfies Σ_n-comprehension, there is some $A \in \mathcal{X}$ such that $(M, \mathcal{X})$ satisfies

$$\forall \langle x, y, z\rangle\, (\langle x, y, z\rangle \in A \leftrightarrow \theta(x, y, z)).$$

The same formula holds in $(\mathcal{F}/\mathcal{U}, \mathcal{X})$ by Lemma 3.6, and thus

$$(\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \forall x\, (\exists y\, \langle x, y, [f]\rangle \in A \leftrightarrow \varphi(x)).$$

For any $a \in M$, there is some $c \in K$ that codes

$$\{x < a \mid (\mathcal{F}/\mathcal{U}, \mathcal{X}) \models \exists y\, \langle x, y, [f]\rangle \in A\}.$$

Such c also codes $\{x < a \mid \mathcal{F}/\mathcal{U} \models \varphi(x)\}$. ⊣

In Section 5, we will prove the converse of Theorem 4.7 (see Theorem 5.9). We will relate the end extension constructed in Theorem 4.7 to the weak regularity principle, analogous to Proposition 1.3. Notably, the course of the proof will reveal some non-trivial syntactic consequences.

§5. The weak regularity principle.

In this final section, we introduce the weak regularity principle $\mathrm{WR}\varphi$, a variant of the regularity principle, and determine its strength within the I-B hierarchy.

Mills and Paris [14] introduced the regularity principle $\mathrm{R}\varphi$ to be the universal closure of the following formula:

$$\exists^{\mathrm{cf}} x\, \exists y < a\, \varphi(x, y) \to \exists y < a\, \exists^{\mathrm{cf}} x\, \varphi(x, y).$$

For any formula class Γ, let

$$\mathrm{R}\Gamma = \mathrm{I}\Delta_0 \cup \{\mathrm{R}\varphi \mid \varphi \in \Gamma\}.$$

It is shown in [14] that $\mathrm{R}\Pi_n \Leftrightarrow \mathrm{R}\Sigma_{n+1} \Leftrightarrow \mathrm{B}\Sigma_{n+2}$ for each $n \in \mathbb{N}$. The weak regularity principle is defined by replacing the $\exists^{\mathrm{cf}} x$ by $\forall x$ in the antecedent of implication in $\mathrm{R}\varphi$.

DEFINITION 5.1. Let $\varphi(x, y)$ be a formula in first-order arithmetic with possibly hidden variables. The *weak regularity principle* $\mathrm{WR}\varphi$ denotes the universal closure of the following formula:

$$\forall x\ \exists y < a\ \varphi(x, y) \to \exists y < a\ \exists^{\mathrm{cf}} x\ \varphi(x, y).$$

For any formula class Γ, define

$$\mathrm{WR}\Gamma = \mathrm{I}\Delta_0 \cup \{\mathrm{WR}\varphi \mid \varphi \in \Gamma\}.$$

The status of the strength of the weak regularity principle among the I-B hierarchy is more complex in comparison with that of the regularity principle. The principles for most natural classes of formulas are equivalent to the collection schemas, whereas induction schemas are only equivalent to the principle for a highly restricted subclass of $\Sigma_0(\Sigma_{n+1})$ formulas. We will show that for each $n \in \mathbb{N}$, $\mathrm{WR}\Sigma_0(\Sigma_n)$ and $\mathrm{WR}(\Sigma_{n+1} \vee \Pi_{n+1})$ are both equivalent to $\mathrm{B}\Sigma_{n+2}$, and $\mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1})$ is equivalent to $\mathrm{I}\Sigma_{n+2}$.

The weak regularity principle may also be viewed as an infinitary version of the *pigeonhole principle* (PHP), and similar phenomena arise with the strength of the pigeonhole principle in the I-B hierarchy. Dimitracopoulos and Paris proved in [4] that $\mathrm{PHP}\Sigma_{n+1}$ and $\mathrm{PHP}\Pi_{n+1}$ are equivalent to $\mathrm{B}\Sigma_{n+1}$, and that $\mathrm{PHP}(\Sigma_{n+1} \vee \Pi_{n+1})$ and $\mathrm{PHP}\Sigma_0(\Sigma_{n+1})$ are equivalent to $\mathrm{I}\Sigma_{n+1}$.

LEMMA 5.2. *For each* $n \in \mathbb{N}$, $\mathrm{B}\Sigma_{n+2} \vdash \mathrm{WR}\Pi_{n+1}$.

PROOF. We only show the case of $n = 0$ and the rest can be done by relativizing to the Σ_n-universal set.

Let $M \models \mathrm{B}\Sigma_2$. We first expand M to a second-order structure satisfying $\mathrm{B}\Sigma^0_2$ by adding all Δ_1-definable subsets of M, then further ω-extend it to $(M, \mathcal{X})$ satisfying $\mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2$.

Suppose $M \models \forall x\ \exists y < a\ \ \forall z\ \theta(x, y, z)$, where $\theta(x, y, z) \in \Delta_0(M)$. Applying Lemma 4.2 for θ and $f(x) \equiv a$ as a constant function, we obtain a total function $P \in \mathcal{X}$ such that

$$(M, \mathcal{X}) \models \forall x\ (P(x) < a \wedge \forall z\ \theta(x, P(x), z)).$$

By $\mathrm{B}\Sigma^0_2$, there is some $y_0 < a$ such that there are cofinally many x satisfying $P(x) = y_0$, which implies $M \models \exists y < a\ \exists^{\mathrm{cf}} x \forall z\ \theta(x, y, z)$. So $M \models \mathrm{WR}\Pi_1$. ⊣

COROLLARY 5.3. *For each* $n \in \mathbb{N}$, $\mathrm{B}\Sigma_{n+2} \vdash \mathrm{WR}(\Sigma_{n+1} \vee \Pi_{n+1})$.

PROOF. Fix $n \in \mathbb{N}$ and let $M \models \mathrm{B}\Sigma_{n+2}$, $\varphi(x, y) \in \Sigma_{n+1}(M)$ and $\psi(x, y) \in \Pi_{n+1}(M)$. Suppose $M \models \forall x\ \ \exists y <\ a\ \ (\varphi(x, y) \vee \psi(x, y))$ for some $a \in M$. If $M \models \forall x >\ b\ \ \exists y <\ a\ \ \psi(x, y)$ for some $b \in M$, then by Lemma 5.2, $M \models \exists y < a\ \exists^{\mathrm{cf}} x\ \psi(x, y)$ and the conclusion holds. Otherwise, $M \models \exists^{\mathrm{cf}} x\ \exists y < a\ \varphi(x, y)$, then by $\mathrm{R}\Sigma_{n+1}$, $M \models \exists y < a\ \exists^{\mathrm{cf}} x\ \varphi(x, y)$ and the conclusion holds again. ⊣

REMARK. There is also a direct model-theoretic proof similar to Proposition 1.3. One only needs to notice that over $\mathrm{B}\Sigma_{n+1}$, $\forall x\ \exists y < a\ (\varphi(x, y) \vee \psi(x, y))$ is equivalent to a Π_{n+2} formula.

LEMMA 5.4. *For each* $n \in \mathbb{N}$, $\mathrm{WR}(\Sigma_n \wedge \Pi_n) \vdash \mathrm{I}\Sigma_{n+1}$.

PROOF. We prove for each $k \leqslant n$, that $\mathrm{I}\Sigma_k + \mathrm{WR}(\Sigma_n \wedge \Pi_n) \vdash \mathrm{I}\Sigma_{k+1}$. Then the lemma follows by induction on k. Let $M \models \mathrm{I}\Sigma_k + \mathrm{WR}(\Sigma_n \wedge \Pi_n)$ and assume $M \models \neg\mathrm{I}\Sigma_{k+1}$. Then there is a proper cut $I \subseteq M$ defined by some formula $\varphi(y) := \exists x\ \theta(x, y)$, where $\theta(x, y) \in \Pi_k(M)$. Let

$$\mu(x, y) := \forall y' < y\ \exists x' < x\ \theta(x', y').$$

Then $\mu(x, y) \in \Pi_k(M)$ over $\mathrm{I}\Sigma_k$. Define

$$J = \{y \in M \mid M \models \exists x\ \mu(x, y)\}.$$

It is not hard to show that J is closed downward, closed under successors, and $J \subseteq I$ by its definition, that is, J is a proper cut of M contained in I.

For any $x \in M$, if $M \models \mu(x, y)$ for all $y \in J$, then $y \in J$ is defined by $\mu(x, y)$ in M, which contradicts our assumption that $M \models \mathrm{I}\Sigma_k$. So for each $x \in M$, we may take the largest $y \in J$ satisfying $\mu(x, y)$ by $\mathrm{I}\Sigma_k$. Fixing some arbitrary $a > J$, we have

$$M \models \forall x\ \exists y < a\ (\mu(x, y) \wedge \neg\mu(x, y + 1)).$$

Applying $\mathrm{WR}(\Sigma_n \wedge \Pi_n)$, there is some $y_0 < a$ such that

$$M \models \exists^{\mathrm{cf}} x\ (\mu(x, y_0) \wedge \neg\mu(x, y_0 + 1)).$$

By the definition of $\mu(x, y)$, this implies $y_0 \in J$ and $y_0 + 1 \notin J$, which contradicts the fact that J is closed under successors. So $M \models \mathrm{I}\Sigma_{k+1}$. ⊣

LEMMA 5.5 (Independently by Leszek A. Kołodziejczyk). *For each* $n \in \mathbb{N}$, $\mathrm{WR}\Sigma_0(\Sigma_n) \vdash \mathrm{B}\Sigma_{n+2}$.

PROOF. Let $M \models \mathrm{WR}\Sigma_0(\Sigma_n)$. We show $M \models \mathrm{R}\Pi_n$, which is equivalent to $\mathrm{B}\Sigma_{n+2}$. By Lemma 5.4, $M \models \mathrm{I}\Sigma_{n+1}$. Suppose $M \models \exists^{\mathrm{cf}} x\ \exists y < a\ \varphi(x, y)$ for some $\varphi \in \Pi_n(M)$, and without loss of generality, we assume $M \models \exists y < a\ \varphi(0, y)$. For each $z \in M$, we find the largest $x < z$ such that $M \models \exists y < a\ \varphi(x, y)$, and associate z with all the witnesses $y < a$ such that $\varphi(x, y)$. Formally,

$$M \models \forall z\ \exists y < a\ \exists x < z\ (\varphi(x, y) \wedge \forall x' \in (x, z)\ \neg\exists y < a\ \varphi(x', y)),$$

where (x, z) refers to the open interval between x and z. By $\mathrm{WR}\Sigma_0(\Sigma_n)$,

$$M \models \exists y < a\ \exists^{\mathrm{cf}} z\ \exists x < z\ (\varphi(x, y) \wedge \forall x' \in (x, z)\ \neg\exists y < a\ \varphi(x', y)),$$

which implies $M \models \exists y < a\ \exists^{\mathrm{cf}} x\ \varphi(x, y)$. ⊣

THEOREM 5.6. *For each* $n \in \mathbb{N}$, $\mathrm{WR}\Sigma_0(\Sigma_n) \Leftrightarrow \mathrm{WR}(\Sigma_{n+1} \vee \Pi_{n+1}) \Leftrightarrow \mathrm{B}\Sigma_{n+2}$.

PROOF. The fact that $\mathrm{WR}\Sigma_0(\Sigma_n) \vdash \mathrm{B}\Sigma_{n+2}$ follows from Lemma 5.5. The fact that $\mathrm{B}\Sigma_{n+2} \vdash \mathrm{WR}(\Sigma_{n+1} \vee \Pi_{n+1})$ follows from Corollary 5.3. For $\mathrm{WR}(\Sigma_{n+1} \vee \Pi_{n+1}) \vdash \mathrm{WR}\Sigma_0(\Sigma_n)$, note that every $\Sigma_0(\Sigma_n)$ formula is equivalent to some Δ_{n+1} formula over $\mathrm{I}\Sigma_n$ (see [7, Lemma I.2.50]), and $\mathrm{WR}(\Sigma_{n+1} \vee \Pi_{n+1}) \vdash \mathrm{I}\Sigma_n$ by Lemma 5.4. ⊣

The following proposition is an analog of Proposition 1.3 for $M \preccurlyeq_{\mathrm{e},\Sigma_{n+2}} K \models M\text{-}\mathrm{I}\Sigma_{n+1}$ and $\mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1})$. It also leads to a model-theoretic proof of the fact that $\mathrm{I}\Sigma_{n+2} \vdash \mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1})$.

PROPOSITION 5.7. *Let* $M \models \mathrm{I}\Delta_0 + \exp$. *For each* $n \in \mathbb{N}$, *if there is a proper* Σ_{n+2}-*elementary end extension* $M \preccurlyeq_{\mathrm{e},\Sigma_{n+2}} K \models M\text{-}\mathrm{I}\Sigma_{n+1}$, *then* $M \models \mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1})$.

PROOF. Let $\theta(x, y, z) \in \Sigma_n(M)$, $\sigma(x, y, w) \in \Pi_n(M)$ and

$$\varphi(x, y) := \forall z\ \theta(x, y, z) \wedge \exists w\ \sigma(x, y, w).$$

Suppose $M \models \forall x\, \exists y < a\, \varphi(x, y)$ for some $a \in M$. Over $\mathrm{I}\Sigma_{n+1}$, this is equivalent to the following Π_{n+2} formula:

$$\forall x\, \forall b\, \exists y < a\, (\forall z < b\, \theta(x, y, z) \wedge \exists w\, \sigma(x, y, w)).$$

$M \models \mathrm{I}\Sigma_{n+1}$ by Theorem 1.1, so both M and K satisfy the above formula by elementarity. Pick some arbitrary $d > M$ in K, then

$$K \models \forall b\, \exists y < a\, (\forall z < b\, \theta(d, y, z) \wedge \exists w\, \sigma(d, y, w)).$$

By Lemma 4.5(iii), there is some $b \in K$ such that

$$K \models \forall y < a\, (\forall z\, \theta(d, y, z) \leftrightarrow \forall z < b\, \theta(d, y, z)),$$

which implies

$$K \models \exists y < a\, (\forall z\, \theta(d, y, z) \wedge \exists w\, \sigma(d, y, w)).$$

Pick a witness $c < a$ in M such that $K \models \forall z\ \ \theta(d, c, z) \wedge \exists w\ \ \sigma(d, c, w)$, i.e., $K \models \varphi(d, c)$. Now for each $b \in M$, $K \models \exists x > b\, \varphi(x, c)$, which is witnessed by d. Transferring each of these formulas to M, we have $M \models \exists x > b\, \varphi(x, c)$ for any $b \in M$. So $M \models \exists^{\mathrm{cf}} x\, \varphi(x, c)$. ⊣

THEOREM 5.8. *For each* $n \in \mathbb{N}$, $\mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1}) \Leftrightarrow \mathrm{I}\Sigma_{n+2}$.

PROOF. The fact that $\mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1}) \vdash \mathrm{I}\Sigma_{n+2}$ follows from Lemma 5.4. For the other direction, given any countable model $M \models \mathrm{I}\Sigma_{n+2}$, there is a proper end extension $M \preccurlyeq_{\mathrm{e},\Sigma_{n+2}} K \models M\text{-}\mathrm{I}\Sigma_{n+1}$ by Theorem 4.7, and then $M \models \mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1})$ by Proposition 5.7. ⊣

REMARK. In Hájek–Pudlák [7, Lemma I.2.49], it was shown that for each $n \in \mathbb{N}$, every $\Sigma_0(\Sigma_{n+1})$ formula is equivalent to the following normal form:

$$Q_1 u_1 < v_1\ \dots Q_k u_k < v_k\ \Psi(u_1 \dots u_k, v_1 \dots v_k, w_1 \dots w_l),$$

where $k, l \in \mathbb{N}$, each Q_i for $i \leq k$ is either $\forall$ or $\exists$. Ψ is a Boolean combination of Σ_{n+1} formulas and the variable sets $\{u_i\}_{i \leqslant k}$, $\{v_i\}_{i \leqslant k}$ and $\{w_i\}_{i \leqslant l}$ are pairwise disjoint.

Our proof of Proposition 5.7 can be refined to show that $\mathrm{I}\Sigma_{n+2} \vdash \mathrm{WR}\varphi$, where $\varphi(x, y) \in \Sigma_0(\Sigma_{n+1})$, and if written in the normal form above, x does not appear in $\{v_i\}_{i \leqslant k}$, i.e., x is not permitted to appear as the bound of any bounded quantifiers in front of a Boolean combination of Σ_{n+1} formulas. In contrast, the instance $\varphi(z, y)$ used to prove $\mathrm{WR}\Sigma_0(\Sigma_n) \vdash \mathrm{B}\Sigma_{n+2}$ in Lemma 5.5 starts with $\exists x < z$ explicitly.

Finally, we prove the converse of Theorem 4.7, and establish the characterization of countable models of $\mathrm{I}\Sigma_{n+2}$ as promised.

THEOREM 5.9. *Let M be a countable model of* $\mathrm{I}\Delta_0$. *For each* $n \in \mathbb{N}$, $M \models \mathrm{I}\Sigma_{n+2}$ *if and only if M admits a proper Σ_{n+2}-elementary end extension* $K \models M\text{-}\mathrm{I}\Sigma_{n+1}$.

PROOF. The direction from left to right follows by Theorem 4.7. For the other direction, if $M \preccurlyeq_{\mathrm{e},\Sigma_{n+2}} K \models M\text{-}\mathrm{I}\Sigma_{n+1}$, then $M \models \mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1})$ by Proposition 5.7, and thus $M \models \mathrm{I}\Sigma_{n+2}$ by Theorem 5.8. ⊣

The main remaining problem now is to find a purely syntactic proof of Theorem 5.8. We conjecture that a more refined tree construction similar to the approach in Lemma 4.2 would solve the problem.

Problem 5.10. *Give a direct proof of the fact that* $\mathrm{I}\Sigma_{n+2} \vdash \mathrm{WR}(\Sigma_{n+1} \wedge \Pi_{n+1})$ (*and also* $\mathrm{I}\Sigma_{n+2} \vdash \mathrm{WR}\varphi$ *where* $\varphi(x, y) \in \Sigma_0(\Sigma_{n+1})$ *as described in the remark after Theorem* 5.8) *without using end extensions.*

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