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# The vanishing levels of a tree

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*Abstract.* We initiate the study of the spectrum of sets that can be realized as the vanishing levels $V(\mathbf{T})$ of a normal $\kappa$-tree $\mathbf{T}$. This is an invariant in the sense that if $\mathbf{T}$ and $\mathbf{T}'$ are club-isomorphic, then $V(\mathbf{T}) \triangle V(\mathbf{T}')$ is nonstationary. Additional features of this invariant imply that the spectrum is closed under finite unions and intersections. The set $V(\mathbf{T})$ must be stationary for a homogeneous normal $\kappa$-Aronszajn tree $\mathbf{T}$, and if there exists a special $\kappa$-Aronszajn tree, then there exists one $\mathbf{T}$ that is homogeneous and satisfies that $V(\mathbf{T})$ covers a club in $\kappa$. It is consistent (from large cardinals) that there is an $\aleph_2$-Souslin tree, and yet $V(\mathbf{T})$ is co-stationary for every $\aleph_2$-tree $\mathbf{T}$. Both $V(\mathbf{T}) = \emptyset$ and $V(\mathbf{T}) = \kappa$ (modulo nonstationary) are shown to be feasible using $\kappa$-Souslin trees, even at some large cardinal close to a weakly compact. It is also possible to have a family of $2^\kappa$ many $\kappa$-Souslin trees for which the corresponding family of vanishing levels forms an antichain in the Boolean algebra of the powerset of $\kappa$, modulo the nonstationary ideal.

## 1 Introduction

By a classical theorem of Cantor, every countable dense linear ordering with no endpoints is order-isomorphic to the rational numbers. Consequently, every separable, dense linear order with no endpoints in which every nonempty bounded set has a least upper bound is order-isomorphic to the real line. In a problem list published in 1920 [Sou20], Souslin asked whether the characterization remains valid once replacing separability by the property *ccc* asserting that every pairwise disjoint family of open intervals is countable. In the early 1930s, this problem led Kurepa to the discovery of set-theoretic trees. Most notably, Kurepa proved that Souslin's proposed characterization of the real line is equivalent to the following purely Ramsey-theoretic assertion: *every uncountable set-theoretic tree must admit an uncountable chain or an uncountable antichain*. Soon after learning about the latter, Aronszajn was able to construct an uncountable set-theoretic tree all of whose levels are countable, and yet admitting no uncountable chains. Kurepa who possibly did not fully appreciate this partial result, named this object *an Aronszajn tree*, insisting that the main question is whether a *Souslin tree* can be constructed. It then took three more decades until it was proven that, unlike Aronszajn trees, the existence of Souslin trees is independent of

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the usual axioms of set theory ZFC and even of ZFC + GCH (see the surveys [Rud69, Alv99, Kan11]).

Famously, long before Souslin trees were shown to consistently exist, Rudin [Rud55] boldly used them to construct a *Dowker space* [Dow51], i.e., a normal topological space whose product with the unit interval is not normal.[1] The Dowker space problem has its own rich history, which we will not elaborate on in here (but see [KS98]). For our purpose, it suffices to mention that a few years ago, Rinot and Shalev [RS23] found a new proof of Rudin's theorem by introducing a combinatorial guessing principle $\clubsuit_{\mathrm{AD}}$ and proving the following two implications:

$$\exists \text{ Souslin tree} \implies \clubsuit_{\mathrm{AD}} \text{ holds} \implies \exists \text{ Dowker space}.$$

Their proof of the first implication goes through an analysis of the *vanishing levels* of set-theoretic trees, highlighting its importance and raising a few fundamental questions. In order to formulate them, let us provide a couple of basic definitions.

***Definition 1.1*** (Set-theoretic trees) For an infinite cardinal $\kappa$, a partially ordered set $\mathbf{T} = (T, <_T)$ is a $\kappa$-*tree* iff the following two requirements hold:

(1) For every node $x \in T$, the set $x_\downarrow := \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$. Hereafter, write $\mathrm{ht}(x) := \mathrm{otp}(x_\downarrow, <_T)$ for the *height* of $x$;
(2) For every ordinal $\alpha < \kappa$, the $\alpha^{\mathrm{th}}$-level of the tree, $T_\alpha := \{x \in T \mid \mathrm{ht}(x) = \alpha\}$, is nonempty and has size less than $\kappa$. The level $T_\kappa$ is empty.

$\mathbf{T}$ is *normal* iff each $x \in T$ admits an extension to every level $\alpha < \kappa$.

For an ordinal $\alpha$, a subset $B \subseteq T$ is an $\alpha$-*branch* iff $(B, <_T)$ is linearly ordered and $\{\mathrm{ht}(x) \mid x \in B\} = \alpha$; it is said to be *vanishing* iff it has no upper bound in $\mathbf{T}$. With this terminology, Kőnig's *infinity lemma* [Kon27] is nothing but the assertion that every $\aleph_0$-tree has an $\aleph_0$-branch, and Kurepa's theorem [Kur35] is that Souslin's question admits an affirmative answer iff every $\aleph_1$-tree with no uncountable antichains has an $\aleph_1$-branch. Curiously, his proof of the contrapositive of the forward implication goes through showing that the collection of vanishing branches of a counterexample tree admits a lexicographic-like ordering that makes it into a ccc nonseparable linear order (see the proof of [Kun80, Theorem II.5.13]).

As said before, Kurepa named these objects after people, thus, a $\kappa$-*Aronszajn tree* is a $\kappa$-tree with no $\kappa$-branches, and a $\kappa$-*Souslin tree* is a $\kappa$-Aronszajn tree with no $\kappa$-sized antichains. Our next key definition reads as follows.

***Definition 1.2*** (Vanishing levels) For a $\kappa$-tree $\mathbf{T} = (T, <_T)$, let $V(\mathbf{T})$ denote the set of all nonzero (limit) ordinals $\alpha < \kappa$ such that for any node $x \in T$ of height less than $\alpha$ there exists a vanishing $\alpha$-branch containing $x$.

The general case of the first implication from [RS23] presented earlier asserts that if $\mathbf{T}$ is a $\kappa$-Souslin tree, then $\clubsuit_{\mathrm{AD}}(S)$ holds over some subset $S \subseteq \kappa$ that is equal to the intersection of $V(\mathbf{T})$ with some closed and unbounded subset of $\kappa$ (a *club*). In particular, if $V(\mathbf{T})$ is "large" (e.g., *stationary*), then a nontrivial instance of $\clubsuit_{\mathrm{AD}}$ holds true, which in turn has important applications in set-theoretic topology.

[1] The Sorgenfrey line is a standard example of a normal space whose square is not normal. What's striking about Dowker spaces is that their product with the compact metric space $[0, 1]$ is not normal.

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For $\mathbf{T}$ a normal $\aleph_1$-Aronszajn tree, $V(\mathbf{T})$ is readily large and must in fact cover a club, but complications arise naturally once dealing with higher trees. For instance, a $\kappa$-tree $\mathbf{T}$ is *$\sigma$-complete* iff any increasing sequence of nodes in $\mathbf{T}$, and of length less than $\sigma$, has an upper bound in $\mathbf{T}$. For such a tree, $V(\mathbf{T})$ cannot contain points of cofinality smaller than $\sigma$. Dually, if a normal splitting $\kappa$-tree $\mathbf{T}$ is *regressive*,[2] then $V(\mathbf{T})$ must contain all points of cofinality $\aleph_0$. But completeness and regressivity are too coarse and what's missing is a sincere understanding of the spectrum of sets that can be realized as the vanishing levels of normal $\kappa$-trees. After all, what we are facing here is an invariant of trees in the sense that if two normal $\kappa$-trees $\mathbf{T}, \mathbf{T}'$ are isomorphic on a club, then $V(\mathbf{T})$ is equal to $V(\mathbf{T}')$ modulo a nonstationary set, and it possesses various algebraic features, such as $V(\mathbf{T} \otimes \mathbf{T}') = V(\mathbf{T}) \cup V(\mathbf{T}')$ and $V(\mathbf{T} + \mathbf{T}') = V(\mathbf{T}) \cap V(\mathbf{T}')$ for any two normal $\kappa$-trees $\mathbf{T}, \mathbf{T}'$. Moving forward, Definition 1.2 opens the door to formulating deep questions about higher trees: for instance, Krueger [Kru18] proved it is consistent that every two $\aleph_1$-complete $\aleph_2$-Aronszajn trees are club-isomorphic, and so the next natural step is seeking models in which, for some fixed subset $X \subseteq \aleph_2$, any two $\aleph_2$-trees whose set of vanishing levels coincide with $X$ modulo some nonstationary set are moreover club-isomorphic.[3]

As made clear earlier, in view of applications, the primary problem in this vein is whether $V(\mathbf{T})$ of a $\kappa$-tree $\mathbf{T}$ must be large. Our first main result shows that this is not the case. This is best demonstrated in Gödel's constructible universe, $\mathsf{L}$, where we obtain the following characterization.

***Theorem A*** *In $\mathsf{L}$, for every regular uncountable cardinal $\kappa$ that is not weakly compact, the following are equivalent:*

- *there exists a $\kappa$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \emptyset$;*
- *there exists a normal and splitting $\kappa$-tree $\mathbf{T}$ such that $V(\mathbf{T}) = \emptyset$;*
- *$\kappa$ is not the successor of a cardinal of countable cofinality.*

Our second main result deals with the other extreme and continues our discussion on completeness and regressivity: is it possible to have a $\kappa$-Souslin tree whose set of vanishing levels is as large as possible? Here, again, we obtain a complete characterization for Gödel's universe.

***Theorem B*** *In $\mathsf{L}$, for every regular uncountable cardinal $\kappa$ that is not weakly compact, the following are equivalent:*

- *there exists a $\kappa$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \operatorname{acc}(\kappa)$;*
- *there exists a $\kappa$-tree $\mathbf{T}$ such that $V(\mathbf{T}) = \operatorname{acc}(\kappa)$;*
- *$\kappa$ is not subtle.*

An interesting feature of the proof of Theorem B is that it goes through a pump-up theorem generating $\kappa$-Souslin trees from other input trees with weaker properties. Before we turn to describe some of our pump-up theorems, let us introduce another piece of notation. For a $\kappa$-tree $\mathbf{T}$, we write $V^-(\mathbf{T})$ for the set of all $\alpha$'s such that there exists a vanishing $\alpha$-branch. If $\mathbf{T}$ is homogeneous, then $V^-(\mathbf{T})$ coincides with $V(\mathbf{T})$,

[2] See Definition 2.8.
[3] See Example 5.12 for a complementary scenario.

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but in contrast with Theorem A, for every normal $\kappa$-Aronszajn tree $\mathbf{T}$, the set $V^-(\mathbf{T})$ is necessarily stationary.[4]

Our first pump-up theorem asserts that the existence of a special $\kappa$-Aronszajn tree $\mathbf{T}$ is equivalent to the existence of one with $V(\mathbf{T}) = \mathrm{acc}(\kappa)$. Our second pump-up theorem asserts that for every $\kappa$-tree $\mathbf{K}$ there exists a $\kappa$-tree $\mathbf{T}$ such that $V^-(\mathbf{K})\setminus V(\mathbf{T})$ is nonstationary. Our third pump-up theorem asserts that assuming an instance of the proxy principle $\mathrm{P}(\dots)$ from [BR17a],[5] the corresponding tree $\mathbf{T}$ may moreover be made to be $\kappa$-Souslin.

***Theorem C*** *Suppose that $\mathrm{P}(\kappa, 2, \sqsubseteq^*, 1)$ holds. Then:*

(1) *For every $\kappa$-tree $\mathbf{K}$, there exists a $\kappa$-Souslin tree $\mathbf{T}$ such that $V^-(\mathbf{K})\setminus V(\mathbf{T})$ is nonstationary. In particular:*
(2) *There exists a $\kappa$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T})$ is stationary.*

The preceding addresses the problem of ensuring $V(\mathbf{T})$ to cover some stationary set $S$. The next theorem addresses its dual. Along the way, it provides a cheap way to obtain a family of $2^\kappa$-many $\kappa$-Souslin trees that are not pairwise club-isomorphic.[6]

***Theorem D*** *If $\diamondsuit(S)$ holds for some nonreflecting stationary subset $S$ of a strongly inaccessible cardinal $\kappa$, then there is a family $\mathcal{S}$ of $2^\kappa$-many stationary subsets of $S$ such that:*

- *for every $S' \in \mathcal{S}$, there is a $\kappa$-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = S'$;*
- *for all $S' \neq S''$ in $\mathcal{S}$, $|S' \cap S''| < \kappa$.*

In the last two sections of this article, we come back to the motivating problem of getting instances of $\clubsuit_{\mathrm{AD}}$. By [RS23, Theorem 2.30], if $\kappa$ is weakly compact, then $\clubsuit_{\mathrm{AD}}(S)$ fails for every $S$ with $\mathrm{Reg}(\kappa) \subseteq S \subseteq \kappa$. This raises the question as to whether $\clubsuit_{\mathrm{AD}}(S)$ may hold over a large subset $S$ of a cardinal $\kappa$ that is close to being weakly compact. We answer this question in the affirmative.

***Theorem E*** *Assuming the consistency of a weakly compact cardinal, it is consistent that for some strongly inaccessible cardinal $\kappa$ satisfying $\chi(\kappa) = \omega$,[7] there is a $\kappa$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \mathrm{acc}(\kappa)$.*

Our final theorem sheds a new light on the classical problem of getting Dowker spaces and can be read independently of the rest of the article. In [dC77], de Caux constructed a Dowker space of size $\aleph_1$ assuming the combinatorial principle $\clubsuit$. Addressing higher cardinals, Good [Goo95] constructed a Dowker space of size $\lambda^+$ assuming $\clubsuit(S)$ for some nonreflecting stationary subset $S$ of $E^{\lambda^+}_\omega$.[8] Thanks to advances in [RS23] and the insights gathered through the study of vanishing levels of trees, we

[4] Note that any $\kappa$-Souslin must be normal on a tail end.
[5] See Definition 3.3 and Conventions 3.4 and 3.5 below. An aggregated list of sufficient conditions for it to hold is given in [BRY25, Section 3].
[6] The most general condition for the existence of such a family is given in [BRY25, Section 5], but it has nothing to do with the set of vanishing levels.
[7] $\chi(\kappa)$ can be understood as measuring how far $\kappa$ is from being weakly compact (see Definition 6.5 below).
[8] Strictly speaking, the hypothesis in [Goo95] is $\clubsuit_{\lambda^+}(S, 2)$, but [BR21, Lemma 3.5] shows that this is no stronger than the vanilla $\clubsuit(S)$.

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found the way to waive Good's need for guessing using ladders of order-type $\omega$. Thus, we found a new sufficient condition for the existence of a Dowker space of size $\kappa$.

**Theorem F** *If $\clubsuit(S)$ holds over a nonreflecting stationary $S \subseteq \kappa$, then there exists a Dowker space of size $\kappa$.*

## 1.1 Organization of this article

Throughout this article, $\kappa$ denotes a regular uncountable cardinal.

In Section 2, we develop the basic theory of vanishing levels of $\kappa$-trees. It is proved that if $\kappa$ is not a strong limit, then $V^-(\mathbf{T})$ is stationary for every normal and splitting $\kappa$-tree $\mathbf{T}$. It is proved that for every $\kappa$-tree $\mathbf{K}$, there exists a $\kappa$-tree $\mathbf{T}$ such that $V^-(\mathbf{K})\setminus V(\mathbf{T})$ is nonstationary, and that the existence of a special $\kappa$-Aronszajn tree $\mathbf{T}$ is equivalent to the existence of a homogeneous one with $V(\mathbf{T}) = \mathrm{acc}(\kappa)$.

In Section 3, we prove Theorem C and some variations of it. As a corollary, we get Theorem B and infer that if $\square_\lambda + \diamondsuit(\lambda^+)$ holds for an infinite cardinal $\lambda$, or if $\square(\lambda^+)$ + GCH holds for a regular uncountable $\lambda$, then there exists a $\lambda^+$-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = \mathrm{acc}(\lambda^+)$.

In Section 4, we address the problem of realizing a given nonreflecting stationary subset of $\kappa$ as $V(\mathbf{T})$ for some $\kappa$-Souslin tree $\mathbf{T}$. The proof of Theorem D will be found there.

In Section 5, we address the problem of constructing a homogeneous $\kappa$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) \in x\}$ for a prescribed finite nonempty set $x \subseteq \mathrm{Reg}(\kappa)$. In particular, this is shown to be feasible in $\mathsf{L}$ whenever $\kappa$ is a limit cardinal or the successor of a cardinal of cofinality at least $\max(x)$. The proof of Theorem A will be found there.

In Section 6, we deal with Souslin trees admitting an ascent path. It is proved that for every uncountable cardinal $\lambda$, $\square_\lambda$ + GCH entails that for every $\mu \in \mathrm{Reg}(\mathrm{cf}(\lambda))$, there exists a $\lambda^+$-Souslin tree $\mathbf{T}$ with a $\mu$-ascent path such that $V(\mathbf{T}) = \mathrm{acc}(\lambda^+)$. The proof of Theorem E will be found there.

In Section 7, we improve [RS23, Lemma 2.10] from which we obtain the proof of Theorem F. As said, this section can be read independently of the rest of the article.

## 1.2 Notation and conventions

$H_\kappa$ denotes the collection of all sets of hereditary cardinality less than $\kappa$. $\mathrm{Reg}(\kappa)$ denotes the set of all infinite regular cardinals $< \kappa$. For $\chi \in \mathrm{Reg}(\kappa)$, $E^\kappa_\chi$ denotes the set $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \chi\}$, and $E^\kappa_{\geq\chi}$, $E^\kappa_{>\chi}$, $E^\kappa_{\leq\chi}$, $E^\kappa_{<\chi}$, $E^\kappa_{\neq\chi}$, are defined analogously.

For a set of ordinals $C$, we write $\mathrm{ssup}(C) := \sup\{\alpha + 1 \mid \alpha \in C\}$, $\mathrm{acc}^+(C) := \{\alpha < \mathrm{ssup}(C) \mid \sup(C \cap \alpha) = \alpha > 0\}$, $\mathrm{acc}(C) := C \cap \mathrm{acc}^+(C)$, and $\mathrm{nacc}(C) := C \setminus \mathrm{acc}(C)$. For a set $S$, we write $[S]^\chi$ for $\{A \subseteq S \mid |A| = \chi\}$, and $[S]^{<\chi}$ is defined analogously. For a set of ordinals $S$, we identify $[S]^2$ with $\{(\alpha, \beta) \mid \alpha, \beta \in S, \alpha < \beta\}$, and we let $\mathrm{Tr}(S) := \{\beta < \mathrm{ssup}(S) \mid \mathrm{cf}(\beta) > \omega \;\&\; S \cap \beta \text{ is stationary in } \beta\}$.

We define four binary relations over sets of ordinals, as follows:

- $D \sqsubseteq C$ iff there exists some ordinal $\beta$ such that $D = C \cap \beta$;
- $D \sqsubseteq^* C$ iff $D \setminus \varepsilon \sqsubseteq C \setminus \varepsilon$ for some $\varepsilon < \sup(D)$;

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- $D \ {}^{S}\!\!\sqsubseteq C$ iff $D \sqsubseteq C$ and $\sup(D) \notin S$;
- $D \ {}_{\chi}\!\!\sqsubseteq C$ iff $D \sqsubseteq C$ or $\mathrm{cf}(\sup(D)) < \chi$.

A *list* over a set of ordinals $S$ is a sequence $\vec{A} = \langle A_\alpha \mid \alpha \in S\rangle$ such that, for each $\alpha \in S$, $A_\alpha$ is a subset of $\alpha$. It is said to be *thin* if $|\{A_\alpha \cap \varepsilon \mid \alpha \in S\}| < \mathrm{ssup}(S)$ for every $\varepsilon < \mathrm{ssup}(S)$. It is said to be *$\xi$-bounded* if $\mathrm{otp}(A_\alpha) \leq \xi$ for all $\alpha \in S$. A *ladder system* over $S$ is a list $\vec{A} = \langle A_\alpha \mid \alpha \in S\rangle$ such that $\mathrm{ssup}(A_\alpha) = \alpha$ for every $\alpha \in S$.[9] It is said to be *almost disjoint* if $\sup(A_\alpha \cap A_{\alpha'}) < \alpha$ for every pair $\alpha < \alpha'$ of ordinals in $S$. A *$C$-sequence* over $S$ is a ladder system $\vec{C} = \langle C_\alpha \mid \alpha \in S\rangle$ such that each $C_\alpha$ is a closed subset of $\alpha$. Finally, a (resp., $\xi$-bounded) *$\mathcal{C}$-sequence* over $S$ is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \in S\rangle$ of nonempty sets such that every element of $\prod_{\alpha\in S}\mathcal{C}_\alpha$ is a (resp., $\xi$-bounded) $C$-sequence.

## 2 The basic theory of vanishing levels

***Definition 2.1*** A $\kappa$-tree $\mathbf{T} = (T, <_T)$ is said to be:

- *Hausdorff* iff for every limit ordinal $\alpha$ and all $x, y \in T_\alpha$, if $x_\downarrow = y_\downarrow$, then $x = y$;
- *normal* iff for all $\alpha < \beta < \kappa$ and $x \in T_\alpha$ there exists $y \in T_\beta$ with $x <_T y$;
- *$\chi$-complete* iff any $<_T$-increasing sequence of elements of $\mathbf{T}$, and of length $< \chi$, has an upper bound in $\mathbf{T}$;
- *$\varsigma$-splitting* iff every node of $\mathbf{T}$ admits at least $\varsigma$-many immediate successors, that is, for every $x \in T$, $|\{y \in T \mid x <_T y, \mathrm{ht}(y) = \mathrm{ht}(x)+1\}| \geq \varsigma$. By *splitting*, we mean 2-splitting;
- *Aronszajn* iff $\mathbf{T}$ has no $\kappa$-branches;
- *Souslin* iff $\mathbf{T}$ has no cofinal branches nor antichains of size $\kappa$;
- *special* iff there exists a map $\rho : T \to T$ satisfying the following:
  - for every non-minimal $x \in T$, $\rho(x) <_T x$;
  - for every $y \in T$, $\rho^{-1}\{y\}$ is covered by less than $\kappa$ many antichains.

***Remark 2.2*** All the $\kappa$-Souslin trees constructed in this article will be Hausdorff, normal, and splitting.

***Definition 2.3*** For a $\kappa$-tree $\mathbf{T} = (T, <_T)$ and an ordinal $\alpha$:

(1) a subset $B \subseteq T$ is an *$\alpha$-branch* iff $(B, <_T)$ is linearly ordered and $\{\mathrm{ht}(x) \mid x \in B\} = \alpha$; it is said to be *vanishing* iff it has no upper bound in $\mathbf{T}$;
(2) $V^-(\mathbf{T})$ denotes the set of all $\alpha \in \mathrm{acc}(\kappa)$ such that there exists a vanishing $\alpha$-branch;
(3) $V(\mathbf{T})$ denotes the set of all $\alpha \in \mathrm{acc}(\kappa)$ such that for every $x \in T$ with $\mathrm{ht}(x) < \alpha$ there exists a vanishing $\alpha$-branch containing $x$;
(4) For $A \subseteq \kappa$, we write $T \restriction A := \{x \in T \mid \mathrm{ht}(x) \in A\}$.

***Remark 2.4*** $V(\mathbf{T}) \subseteq V^-(\mathbf{T}) \subseteq \mathrm{acc}(\kappa)$, and if $V(\mathbf{T})$ is cofinal in $\kappa$, then $\mathbf{T}$ is normal.

***Lemma 2.5*** *Suppose that $\mathbf{T} = (T, <_T)$ is a $\kappa$-tree such that $V^-(\mathbf{T})$ (resp., $V(\mathbf{T})$) covers a club in $\kappa$. Then, there exists a subset $T' \subseteq T$ such that the tree $\mathbf{T}' := (T', <_T)$ satisfies that $V^-(\mathbf{T}')$ (resp., $V(\mathbf{T}')$) is equal to* $\mathrm{acc}(\kappa)$.

[9] It follows that for every successor ordinal $\beta + 1$ in $S$, $\max(A_{\beta+1}) = \beta$.

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**Proof** Let $D$ be a subclub of $V^-(\mathbf{T})$ (resp., $V(\mathbf{T})$), and consider the tree $\mathbf{T}' := (T \upharpoonright D, <_T)$. We claim that $V^-(\mathbf{T}') = \mathrm{acc}(\kappa)$ (resp., $V(\mathbf{T}') = \mathrm{acc}(\kappa)$). To this end, pick $\alpha \in \mathrm{acc}(\kappa)$. Let $\delta < \kappa$ be least ordinal to satisfy $\mathrm{otp}(D \cap \delta) = \alpha$. Then, $\delta \in \mathrm{acc}(D) \subseteq D$ and hence $\delta \in V^-(\mathbf{T})$ (resp., $\delta \in V(\mathbf{T})$). As every vanishing $\delta$-branch in $\mathbf{T}$ induces a vanishing $\alpha$-branch in $\mathbf{T}'$, we infer that $\alpha \in V^-(\mathbf{T}')$ (resp., $\alpha \in V(\mathbf{T}')$). ■

***Proposition 2.6*** *For a $\kappa$-tree $\mathbf{T} = (T, <_T)$:*

(1) *If $\mathbf{T}$ is a normal $\kappa$-Aronszajn tree, then $V^-(\mathbf{T})$ is stationary;*
(2) *If $\mathbf{T}$ is homogeneous,*[10] *then $V^-(\mathbf{T}) = V(\mathbf{T})$.*

**Proof** (1) Suppose not, and fix a club $D \subseteq \kappa$ disjoint from $V^-(\mathbf{T})$. We shall construct a $<_T$-increasing sequence $\langle t_\alpha \mid \alpha \in D\rangle$ in such a way that $t_\alpha \in T_\alpha$ for all $\alpha \in D$, contradicting the fact that $\mathbf{T}$ is $\kappa$-Aronszajn. We start by letting $t_{\min(D)}$ be an arbitrary element of $T_{\min(D)}$. Next, for every $\alpha \in D$ such that $t_\alpha$ has already been successfully defined, we set $\beta := \min(D \setminus (\alpha+1))$, and use the normality of $\mathbf{T}$ to pick $t_\beta$ in $T_\beta$ extending $t_\alpha$. For every $\alpha \in \mathrm{acc}(D)$ such that $\langle t_\varepsilon \mid \varepsilon \in D \cap \alpha\rangle$ has already been defined, the latter clearly induces an $\alpha$-branch, so the fact that $\alpha \notin V^-(\mathbf{T})$ implies that there exists some $t_\alpha \in T_\alpha$ such that $t_\varepsilon <_T t_\alpha$ for all $\varepsilon \in D \cap \alpha$. This completes the description of the recursion.

(2) Suppose that $\mathbf{T}$ is homogeneous. Let $\alpha \in V^-(\mathbf{T})$, and fix a vanishing $\alpha$-branch $b$. Now, given a node $x$ of $\mathbf{T}$ of height less than $\alpha$, let $y$ be the unique element of $b$ to have the same height as $x$. Since $\mathbf{T}$ is homogeneous, there exists an automorphism $\pi$ of $\mathbf{T}$ sending $y$ to $x$, and it is clearly the case that $\pi[b]$ is a vanishing $\alpha$-branch through $x$. ■

***Proposition 2.7*** *If $\square(\kappa)$ holds, then there exists a $\kappa$-Aronszajn tree $\mathbf{T}$ such that $V(\mathbf{T}) = E^\kappa_\omega$.*

**Proof** By [Kön03, Theorem 3.9], $\square(\kappa)$ yields a sequence of functions $\langle f_\beta : \beta \to \beta \mid \beta \in \mathrm{acc}(\kappa)\rangle$ such that:

- for every $(\beta, \gamma) \in [\mathrm{acc}(\kappa)]^2$, $\{\alpha < \beta \mid f_\beta(\alpha) \neq f_\gamma(\alpha)\}$ is finite;
- there is no cofinal $B \subseteq \mathrm{acc}(\kappa)$ such that $\{f_\beta \mid \beta \in B\}$ is linearly ordered by $\subseteq$.

Set $T := \{f \in {}^\alpha\alpha \mid \alpha < \kappa, f \text{ disagrees with } f_\alpha \text{ on a finite set}\}$. Then, $\mathbf{T} = (T, \subsetneq)$ is a uniformly coherent $\kappa$-Aronszajn tree. By [RS23, Remark 2.20], then, $V(\mathbf{T}) = E^\kappa_\omega$. ■

***Definition 2.8*** For a $\kappa$-tree $\mathbf{T} = (T, <_T)$ and a subset $S \subseteq \kappa$, we say that $\mathbf{T}$ is *$S$-regressive* iff there exists a map $\rho : T \upharpoonright S \to T$ satisfying the following:

- for every $x \in T \upharpoonright S$, $\rho(x) <_T x$;
- for all $\alpha \in S$ and $x, y \in T_\alpha$, if $\rho(x) <_T y$ and $\rho(y) <_T x$, then $x = y$.

We say that $\mathbf{T}$ is *regressive* if it is $\mathrm{acc}(\kappa)$-regressive.

***Remark 2.9*** If $\rho$ is as above, then every map $\rho : T \upharpoonright S \to T$ satisfying $\rho(x) \leq_T \rho(x) <_T x$ for all $x \in T \upharpoonright S$ is as well a witness to $\mathbf{T}$ being $S$-regressive.

The next lemma generalizes [RS23, Lemmas 2.19 and 2.21].

[10] That is, for all $\alpha < \kappa$ and $s, t \in T_\alpha$, there is an automorphism of $\mathbf{T}$ sending $s$ to $t$.

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***Lemma 2.10*** *Suppose that:*

- $\mathbf{T}$ *is a normal, $\varsigma$-splitting $\kappa$-tree, for some fixed cardinal $\varsigma < \kappa$;*
- $S \subseteq E^\kappa_\chi$ *is stationary for some fixed regular cardinal $\chi < \kappa$;*
- *Any of the following:*
  (1) $\varsigma^\chi \geq \kappa$;
  (2) $\mathbf{T}$ *is $S$-regressive and $\varsigma^{<\chi} < \varsigma^\chi$;*
  (3) $\mathbf{T}$ *is $S$-regressive, $\chi = \varsigma$ and there exists a weak $\chi$-Kurepa tree.*[11]

*Then, for every $\alpha \in S$, either $\alpha \in V(\mathbf{T})$ or $(\mathrm{cf}(\alpha) > \omega$ and) $V^-(\mathbf{T}) \cap \alpha$ is stationary in $\alpha$. In particular, $V^-(\mathbf{T}) \cap E^\kappa_{\leq\chi}$ is stationary.*

**Proof** Write $\mathbf{T} = (T, <_T)$. Toward a contradiction, suppose that $\alpha \in S$ is a counterexample. As $\alpha \notin V(\mathbf{T})$, we may fix $x \in T$ with $\mathrm{ht}(x) < \alpha$ such that every $\alpha$-branch $B$ with $x \in B$ has an upper bound in $\mathbf{T}$. Since either $\mathrm{cf}(\alpha) \leq \omega$ or $V^-(\mathbf{T}) \cap \alpha$ is nonstationary in $\alpha$, we may fix a club $C$ in $\alpha$ of order-type $\chi$ such that $\min(C) = \mathrm{ht}(x)$ and such that $\mathrm{acc}(C) \cap V^-(\mathbf{T}) = \emptyset$.

Let $\langle \alpha_i \mid i < \chi \rangle$ denote the increasing enumeration of $C$. We shall recursively construct an array of nodes $\langle t_s \mid s \in {}^{<\chi}\varsigma \rangle$ in such a way that $t_s \in T_{\alpha_{\mathrm{dom}(s)}}$. Set $t_\emptyset := x$. For every $i < \chi$ and every $s : i \to \varsigma$ such that $t_s$ has already been defined, since $T$ is normal and $\varsigma$-splitting, we may find an injective sequence $\langle t_{s^\frown\langle j \rangle} \mid j < \varsigma \rangle$ of nodes of $T_{\alpha_{i+1}}$ all extending $t_s$. For every $i \in \mathrm{acc}(\chi)$ such that $\langle t_s \mid s \in {}^{<i}\varsigma \rangle$ has already been defined, for every $s : i \to \varsigma$, since $\{t_{s \restriction \iota} \mid \iota < i\}$ induces an $\alpha_i$-branch, the fact that $\alpha_i \notin V^-(\mathbf{T})$ implies that we may find $t_s \in T_{\alpha_i}$ that is a limit of that $\alpha_i$-branch. This completes the recursive construction of our array.

For every $s \in {}^\chi\varsigma$, $B_s := \{t \in T \mid \exists i < \chi\,(t <_T t_{s\restriction i})\}$ is an $\alpha$-branch containing $x$, and hence there must be some $b_s \in T_\alpha$ extending all elements of $B_s$. Our construction also ensures that $B_s \neq B_{s'}$ whenever $s \neq s'$. We now consider a few options:

(1) Suppose that $\varsigma^\chi \geq \kappa$. Then, $|T_\alpha| \geq |\{b_s \mid s \in {}^\chi\varsigma\}| = \varsigma^\chi \geq \kappa$. This is a contradiction.

(2) Suppose that $\mathbf{T}$ is $S$-regressive, as witnessed by $\rho : T \restriction S \to T$. For every $s \in {}^\chi\varsigma$, $\rho(b_s)$ belongs to $B_s$, but by Remark 2.9, we may assume that $\rho(b_s) = t_{s\restriction i}$ for some $i < \chi$.
   - If $\varsigma^{<\chi} < \varsigma^\chi$, then we may now find $s \neq s'$ in ${}^\chi\varsigma$ such that $\rho(b_s) = \rho(b_{s'})$. Then, $\rho(b_{s'}) <_T t_s$ and $\rho(b_s) <_T t_{s'}$, contradicting the fact that $b_s \neq b_{s'}$.
   - If $\chi = \varsigma$ and there exists a weak $\chi$-Kurepa tree, then this may be witnessed by a tree of the form $(K, \subsetneq)$ for some $K \subseteq {}^{<\chi}\varsigma$. Let $\langle s_\beta \mid \beta < \chi^+ \rangle$ be an injective enumeration of branches through $(K, \subsetneq)$. Since $|K| \leq \chi$, there must exist $\beta \neq \beta'$ such that $\rho(b_{s_\beta}) = \rho(b_{s_{\beta'}})$, which yields a contradiction as in the previous case.

■

***Corollary 2.11*** *If $\kappa$ is not a strong limit, then for every normal and splitting $\kappa$-tree $\mathbf{T}$, $V^-(\mathbf{T})$ is stationary.*

**Proof** Suppose that $\kappa$ is not a strong limit. It is not hard to see that there exists some infinite cardinal $\varsigma < \kappa$ for which there exists a regular cardinal $\chi < \kappa$ such that $\varsigma^\chi \geq \kappa$. Now, given a normal and splitting $\kappa$-tree $\mathbf{T} = (T, <_T)$, as shown in the proof of

[11] That is, a tree of height and size $\chi$ admitting at least $\chi^+$-many branches. We also allow the case $\chi = \omega$, though this is already covered by Clause (2) above.

[RS23, Proposition 2.16], the club $D := \{\alpha < \kappa \mid \alpha = \varsigma^\alpha\}$ satisfies that $\mathbf{T}' = (T \restriction D, <_T)$ is normal and $\varsigma$-splitting. By Lemma 2.10, $V^-(\mathbf{T}')$ is stationary. As $D$ is a club in $\kappa$, this means that $V^-(\mathbf{T})$ is stationary, as well. ∎

***Corollary 2.12*** *If $\kappa = \lambda^+$ is a successor cardinal and $\lambda^{\aleph_0} \geq \kappa$, then for every normal and splitting $\kappa$-tree $\mathbf{T}$, $E^\kappa_\omega \setminus V(\mathbf{T})$ is nonstationary.*

**Proof** Suppose that $\kappa$ and $\lambda$ are as above. Now, given a normal and splitting $\kappa$-tree $\mathbf{T} = (T, <_T)$, the club $D := \{\alpha < \kappa \mid \alpha = \lambda^\alpha\}$ satisfies that $\mathbf{T}' = (T \restriction D, <_T)$ is normal and $\lambda$-splitting. By Lemma 2.10, $V(\mathbf{T}') \supseteq E^\kappa_\omega$. As $D$ is a club in $\kappa$, this means that $E^\kappa_\omega \setminus V(\mathbf{T})$ is nonstationary. ∎

***Definition 2.13*** [BR21] A *streamlined $\kappa$-tree* is a subset $T \subseteq {}^{<\kappa}H_\kappa$ such that the following two conditions are satisfied:

(1) $T$ is downward-closed, i.e., for every $t \in T$, $\{t \restriction \alpha \mid \alpha < \kappa\} \subseteq T$;
(2) for every $\alpha < \kappa$, the set $T \cap {}^\alpha H_\kappa$ is nonempty and has size $< \kappa$.

For every $\alpha \leq \kappa$, we denote $\mathcal{B}(T \restriction \alpha) := \{f \in {}^\alpha H_\kappa \mid \forall \beta < \alpha\, (f \restriction \beta \in T)\}$.

***Remark 2.14*** We identify a streamlined $\kappa$-tree $T$ with the poset $\mathbf{T} = (T, \subsetneq)$ which is a Hausdorff $\kappa$-tree in the sense of Definition 2.1 satisfying that $\mathrm{ht}(x) = \mathrm{dom}(x)$ for every $x \in T$. In particular, $T_\alpha = T \cap {}^\alpha H_\kappa$ for every $\alpha < \kappa$. Furthermore, for $\alpha \in \mathrm{acc}(\kappa)$, a subset $B \subseteq T$ is a vanishing $\alpha$-branch iff there exists an $f \in \mathcal{B}(T \restriction \alpha) \setminus T_\alpha$ such that $B = \{f \restriction \beta \mid \beta < \alpha\}$.

We now extend Lemma 2.5 to streamlined trees.

***Lemma 2.15*** *Suppose that $T$ is a streamlined $\kappa$-tree such that $V^-(T)$ (resp., $V(T)$) covers a club in $\kappa$. Then, there exists a streamlined $\kappa$-tree $X$ that is club-isomorphic to $T$, and $V^-(X)$ (resp., $V(X)$) is equal to* $\mathrm{acc}(\kappa)$.

**Proof** Let $D$ be an arbitrary club in $\kappa$. Let $\pi : \kappa \leftrightarrow D$ be the unique order-preserving map. For every $\gamma < \kappa$, for every $t \in T_{\pi(\gamma)}$, define a corresponding $x_t : \gamma \to T$ via

$$x_t(\alpha) := t \restriction \pi(\alpha).$$

Consider $X := \{x_t \mid t \in T \restriction D\}$, which is again a subset of ${}^{<\kappa}H_\kappa$.

***Claim 2.15.1*** *$X$ is downward-closed.* ∎

**Proof** Let $\beta < \gamma < \kappa$ and $x \in X \cap {}^\gamma H_\kappa$. Pick $t \in T_{\pi(\gamma)}$ such that $x = x_t$. As $T$ is streamlined, $t \restriction \pi(\beta)$ is in $T$, so that $x_{t \restriction \pi(\beta)}$ is in $X$. In addition, for every $\alpha < \beta$,

$$x_{t\restriction\pi(\beta)}(\alpha) = (t \restriction \pi(\beta)) \restriction \pi(\alpha) = t \restriction \pi(\alpha) = x_t(\alpha) = x(\alpha),$$

and hence $x \restriction \beta$ is in $X$. ∎

It follows that for every $\gamma < \kappa$, $X_\gamma = X \cap {}^\gamma H_\kappa = \{x_t \mid t \in T_{\pi(\gamma)}\}$ and in particular, $0 < |X_\gamma| < \kappa$. That is, $X$ is a streamlined $\kappa$-tree.

Next, consider the club of fixed-points $E := \{\gamma \in \mathrm{acc}(\kappa) \mid \pi(\gamma) = \gamma\}$.

***Claim 2.15.2*** *$t \mapsto x_t$ forms an isomorphism from $(T \restriction E, \subsetneq)$ to $(X \restriction E, \subsetneq)$.*

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**Proof** It is clear that for every pair $t \subsetneq t'$ of nodes in $T \restriction D$, $x_t \subsetneq x_{t'}$. For every $\gamma \in E$, the map $t \mapsto x_t$ sends $T_\gamma$ onto $X_\gamma$. To verify it is injective, let $\gamma \in E$ and $t \neq t'$ in $T_\gamma$. Pick $\alpha < \gamma$ such that $t \restriction \alpha \neq t' \restriction \alpha$, in particular, $t \restriction \pi(\alpha) \neq t' \restriction \pi(\alpha)$, and hence $x_t(\alpha) \neq x_{t'}(\alpha)$, so that $x_t \neq x_{t'}$. ∎

***Claim 2.15.3*** *If $D \subseteq V(T)$, then $V(X) = \mathrm{acc}(\kappa)$.*

**Proof** Let $\gamma \in \mathrm{acc}(\kappa)$ and $x \in X \restriction \gamma$. Fix a $t \in T_{\pi(\mathrm{dom}(x))}$ such that $x = x_t$. If $D \subseteq V(T)$, then $\pi(\gamma) \in V(T)$ and $t \in T \restriction \pi(\gamma)$, so we may find some vanishing $\pi(\gamma)$-branch $B$ through $T$ containing $t$. Evidently, $\{x_s \mid s \in B \cap (T \restriction D)\}$ is a $\gamma$-branch containing $x$. If it is not vanishing, then $\bigcup\{x_s \mid s \in B \cap (T \restriction D)\}$ belongs to $X$, so that it must equal $x_{t^*}$ for $t^* := \bigcup(B \cap (T \restriction D))$, and the latter must belong to $T \restriction D$. However, $\mathrm{otp}(B \cap (T \restriction D), \subseteq) = \gamma \in \mathrm{acc}(\kappa)$ and hence $t^* = \bigcup B$, whereas $\bigcup B$ is not in $T$. Thus, the said $\gamma$-branch is indeed vanishing. ∎

A similar proof shows that if $D \subseteq V^-(T)$, then $V^-(X) = \mathrm{acc}(\kappa)$.

***Definition 2.16*** For two elements $s, t$ of $H_\kappa$, we define $s * t$ to be the empty set, unless $s, t \in {}^{<\kappa}H_\kappa$ with $\mathrm{dom}(s) \leq \mathrm{dom}(t)$, in which case $s * t : \mathrm{dom}(t) \to H_\kappa$ is defined by stipulating:

$$(s * t)(\beta) := \begin{cases} s(\beta), & \text{if } \beta \in \mathrm{dom}(s); \\ t(\beta), & \text{otherwise.} \end{cases}$$

***Remark 2.17*** The above operation is associative in the sense that $(r * s) * t = r * (s * t)$ whenever $\mathrm{dom}(r) \leq \mathrm{dom}(s) \leq \mathrm{dom}(t)$.

***Definition 2.18*** A streamlined $\kappa$-tree $T$ is *uniformly homogeneous* iff for all $\alpha < \beta < \kappa$, $s \in T_\alpha$ and $t \in T_\beta$, $s * t$ is in $T$.

The following fact must be folklore.

***Proposition 2.19*** *Suppose that $T$ is a streamlined $\kappa$-tree that is uniformly homogeneous. Then, $T$ is indeed homogeneous.*

**Proof** Let $s \neq s'$ be two nodes in $T_\beta$ for some $\beta < \kappa$. For every $t \in T$, consider

$$\begin{aligned} \alpha(t) &:= \min(\{\varepsilon < \beta \mid s(\varepsilon) \neq t(\varepsilon)\} \cup \{\beta, \mathrm{dom}(t)\}), \text{ and} \\ \alpha'(t) &:= \min(\{\varepsilon < \beta \mid s'(\varepsilon) \neq t(\varepsilon)\} \cup \{\beta, \mathrm{dom}(t)\}). \end{aligned}$$

For the next definition, we take the convention that for a function $f$, whenever we write $f(x)$, then we implicitly express that $x$ is in $\mathrm{dom}(f)$. Now, define a map $\pi : T \to T$ via:

$$\pi(t) := \begin{cases} (s' \restriction \alpha(t)) * t, & \text{if } t \subseteq s \text{ or } s \subseteq t; \\ (s' \restriction \alpha(t)) * t, & \text{if } \alpha'(t) < \alpha(t) \text{ and } t(\alpha(t)) \neq s'(\alpha(t)); \\ ((s' \restriction \alpha(t)) * (s \restriction (\alpha(t)+1))) * t, & \text{if } \alpha'(t) < \alpha(t) \text{ and } t(\alpha(t)) = s'(\alpha(t)); \\ (s \restriction \alpha'(t)) * t, & \text{if } t \subseteq s' \text{ or } s' \subseteq t; \\ (s \restriction \alpha'(t)) * t, & \text{if } \alpha(t) < \alpha'(t) \text{ and } t(\alpha'(t)) \neq s(\alpha'(t)) \\ ((s \restriction \alpha'(t)) * (s' \restriction (\alpha'(t)+1)) * t, & \text{if } \alpha(t) < \alpha'(t) \text{ and } t(\alpha'(t)) = s(\alpha'(t)); \\ t, & \text{otherwise.} \end{cases}$$

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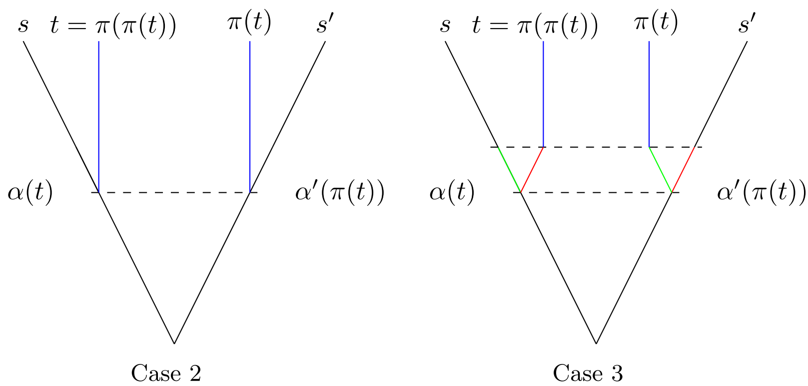


*Figure 1*: Two main cases from the proof of Claim 2.19.1.

It is not hard to see that $\pi$ is well-defined and satisfies $\pi(s) = s'$.

***Claim 2.19.1*** *Let $t \in T$. Then, $\pi(\pi(t)) = t$.* ■

**Proof** Write $\gamma := \mathrm{dom}(t)$. Let $\delta < \beta$ be the least such that $s(\delta) \neq s'(\delta)$. Note that if $\alpha(t) \neq \alpha'(t)$, then $\min\{\alpha(t), \alpha'(t)\} = \delta$. Now, by symmetry, it suffices to analyze the first three cases in the definition of $\pi$. See Figure 1 for an illustration.

Case 1: If $t \subseteq s$, then $\pi(t) = s' \restriction \gamma$ and hence $\pi(\pi(t)) = s \restriction \gamma = t$. Likewise, if $s \subseteq t$, then $\pi(t) = s' * t$ and hence $\pi(\pi(t)) = s * t = t$.

Case 2: If $\delta = \alpha'(t) < \alpha(t) < \min\{\beta, \gamma\}$ and $t(\alpha(t)) \neq s'(\alpha(t))$, then $\pi(t) = (s' \restriction \alpha(t)) * t$ and hence $\alpha'(\pi(t)) = \alpha(t)$ and $\alpha(\pi(t)) = \delta = \alpha'(t)$. So $\alpha(\pi(t)) < \alpha'(\pi(t))$. In addition, $\pi(t)(\alpha(t)) = t(\alpha(t))$, so that $\pi(t)(\alpha(t)) \neq s(\alpha(t))$. Altogether,

$$\pi(\pi(t)) = (s \restriction \alpha'(\pi(t))) * \pi(t) = (s \restriction \alpha(t)) * t = t.$$

Case 3: If $\delta = \alpha'(t) < \alpha(t) < \min\{\beta, \gamma\}$ and $t(\alpha(t)) = s'(\alpha(t))$, then $\pi(t) = ((s' \restriction \alpha(t)) * (s \restriction (\alpha(t) + 1))) * t$. So $\alpha(\pi(t)) = \delta = \alpha'(t)$ and $\alpha'(\pi(t)) \geq \alpha(t)$. If $\alpha'(\pi(t)) > \alpha(t)$, then $t(\alpha(t)) = s'(\alpha(t)) = \pi(t)(\alpha(t)) = s(\alpha(t))$, contradicting the definition of $\alpha(t)$. So $\alpha'(\pi(t)) = \alpha(t)$. Therefore, $\pi(t)(\alpha'(\pi(t))) = \pi(t)(\alpha(t)) = s(\alpha(t)) = s(\alpha'(\pi(t)))$. Altogether,

$$\begin{aligned}\pi(\pi(t)) &= (s \restriction \alpha'(\pi(t))) * (s' \restriction (\alpha'(\pi(t)) + 1)) * \pi(t)\\ &= (s \restriction \alpha(t)) * (s' \restriction (\alpha(t) + 1)) * t\\ &= (t \restriction \alpha(t)) * (s' \restriction (\alpha(t) + 1)) * t\\ &= (t \restriction \alpha(t))^\frown\langle t(\alpha(t) + 1)\rangle * t = t.\end{aligned}$$

For the reader's convenience, the above proof is illustrated in Figure 1.

■

At this point, to prove that $\pi$ is an automorphism, it suffices to show that it is order-preserving.

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***Claim 2.19.2*** *Let $t_0 \subseteq t_1$ be a pair of nodes in $T$. Then, $\pi(t_0) \subseteq \pi(t_1)$.*

**Proof** We may assume that $t_0 \subsetneq t_1$.

▸ If $\alpha(t_0) < \alpha(t_1)$, then $t_0 \subseteq s$, so that $\pi(t_0) = s' \restriction \alpha(t_0)$.

▸▸ If $\alpha'(t_1) < \alpha(t_1)$, then $s' \restriction \alpha(t_0) \subseteq s' \restriction \alpha(t_1) = \pi(t_1) \restriction \alpha(t_1)$.

▸▸ If $\alpha'(t_1) \geq \alpha(t_1)$, then $t_1 \restriction \alpha(t_0) = s \restriction \alpha(t_0) = s' \restriction \alpha(t_0)$ and hence $s' \restriction \alpha(t_0) \subseteq \pi(t_1)$.

▸ If $\alpha'(t_0) < \alpha'(t_1)$, then $t_0 \subseteq s'$, so that $\pi(t_0) = s \restriction \alpha'(t_0)$.

▸▸ If $\alpha(t_1) < \alpha'(t_1)$, then $s \restriction \alpha'(t_0) \subseteq \pi(t_1) \restriction \alpha'(t_1)$.

▸▸ If $\alpha(t_1) \geq \alpha'(t_1)$, then $t_1 \restriction \alpha'(t_0) = s' \restriction \alpha'(t_0) = s \restriction \alpha'(t_0)$ and hence $s \restriction \alpha'(t_0) \subseteq \pi(t_1)$.

▸ If $\alpha(t_0) = \alpha(t_1)$ and $\alpha'(t_0) = \alpha'(t_1)$, then surely $\pi(t_0) = \pi(t_1)$. ∎

This completes the proof.

The implication (4) $\implies$ (3) of the next lemma is what was dubbed in the article's Introduction as the *second pump-up theorem*.

***Lemma 2.20*** *For a stationary $S \subseteq \kappa$, the following are equivalent:*

(1) *There exist a club $D \subseteq \kappa$ and a thin almost disjoint ladder system over $S \cap D$;*
(2) *There exist a club $D \subseteq \kappa$ and a thin ladder system $\langle A_\alpha \mid \alpha \in S \cap D\rangle$ such that, for every $(\alpha, \beta) \in [S \cap D]^2$, $A_\alpha \neq A_\beta \cap \alpha$;*
(3) *There exist a club $D \subseteq \kappa$ and a uniformly homogeneous streamlined $\kappa$-tree $T$ such that $V(T) \supseteq S \cap D$;*
(4) *There exist a club $D \subseteq \kappa$ and a $\kappa$-tree $\mathbf{T}$ such that $V^-(\mathbf{T}) \supseteq S \cap D$.*

**Proof** (1) $\implies$ (2): This is immediate.

(2) $\implies$ (3): Suppose that $D$ and $\langle A_\alpha \mid \alpha \in S \cap D\rangle$ are as in (2). Let $\langle x_i \mid i < \kappa\rangle$ be an injective enumeration of $\langle A_\alpha \cap \varepsilon \mid \varepsilon < \alpha, \alpha \in S \cap D\rangle$. For each $\alpha \in S \cap D$, let $k_\alpha : \alpha \to \kappa$ be the unique function to satisfy for all $\varepsilon < \alpha$:

$$A_\alpha \cap \varepsilon = x_{k_\alpha(\varepsilon)}.$$

Define first an auxiliary collection $K$ by letting

$$K := \{k_\beta \restriction \alpha \mid \alpha < \beta, \beta \in S \cap D\}.$$

Note that $\{\mathrm{dom}(y) \mid y \in K\} = \kappa$ and that $K$ is closed under taking initial segments. So $K$ is a streamlined $\kappa$-tree because otherwise there must exist some $\varepsilon < \kappa$ such that $\{k_\beta \restriction \varepsilon \mid \beta \in S \cap D\}$ has size $\kappa$, contradicting the fact that $\langle A_\beta \mid \beta \in S \cap D\rangle$ is thin. We shall use $K$ to construct a uniformly homogeneous streamlined $\kappa$-tree $T$ by defining its levels $T_\alpha$ by recursion on $\alpha < \kappa$.

Start by letting $T_0 := K_0$. Clearly, $T_0 = \{\emptyset\}$, so that $|T_0| < \kappa$. Next, for every nonzero $\alpha < \kappa$ such that $T \restriction \alpha$ has already been defined and has size less than $\kappa$, let

$$T_\alpha := \{x * y \mid x \in T \restriction \alpha,\ y \in K_\alpha\}$$

and note that $|T_\alpha| < \kappa$. Altogether, $T$ is a streamlined $\kappa$-tree.

***Claim 2.20.1*** *$T$ is uniformly homogeneous.* ∎

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**Proof** We prove that $x * y \in T$ for all $x, y \in T$ with $\mathrm{dom}(x) < \mathrm{dom}(y)$. The proof is by induction on $\mathrm{dom}(y)$. So suppose that $\alpha < \kappa$ is such that for all $x, y \in T$ with $\mathrm{dom}(x) < \mathrm{dom}(y) < \alpha$, it is the case that $x * y \in T$, and let $x, y \in T$ with $\mathrm{dom}(x) < \mathrm{dom}(y) = \alpha$. Recalling the definition of $T_\alpha$, pick $x' \in T \restriction \alpha$ and $y' \in K_\alpha$ such that $y = x' * y'$.

▸ If $\mathrm{dom}(x) < \mathrm{dom}(x')$, then $x * y = x * (x' * y') = (x * x') * y'$. As $\mathrm{dom}(x) < \mathrm{dom}(x') < \alpha$, the induction hypothesis implies that $x * x' \in T \restriction \alpha$, and then the definition of $T_\alpha$ implies that $(x * x') * y'$ is in $T$.

▸ If $\mathrm{dom}(x) \geq \mathrm{dom}(x')$, then $x * y = x * (x' * y') = x * y'$, and then the definition of $T_\alpha$ implies that $x * y'$ is in $T$. ■

By the preceding claim together with Proposition 2.6, it now suffices to prove that $V^-(T) \supseteq S \cap D \cap \mathrm{acc}(\kappa)$. To this end, let $\alpha \in S \cap D \cap \mathrm{acc}(\kappa)$. Clearly, $b := \{k_\alpha \restriction \varepsilon \mid \varepsilon < \alpha\}$ is an $\alpha$-branch in $K$ and hence in $T$. If $b$ is not vanishing in $T$, then we may find $x \in T \restriction \alpha$ and $y \in K_\alpha$ such that $x * y = k_\alpha$. Recalling the definition of $K_\alpha$, we may pick $\beta \in S \cap D$ above $\alpha$ such that $y = k_\beta \restriction \alpha$. As $\alpha < \beta$, it is the case that $A_\alpha \neq A_\beta \cap \alpha$, so we may pick $\delta \in A_\alpha \Delta (A_\beta \cap \alpha)$. Then, $\varepsilon := \max\{\delta, \mathrm{dom}(x)\} + 1$ is smaller than $\alpha$ and satisfies $k_\alpha(\varepsilon) \neq k_\beta(\varepsilon)$, contradicting the fact that $k_\alpha(\varepsilon) = (x * y)(\varepsilon) = y(\varepsilon) = k_\beta(\varepsilon)$.

(3) $\implies$ (4): This is immediate.

(4) $\implies$ (1) Every $\kappa$-tree is order-isomorphic to an ordinal-based tree (see, e.g., [RS23, Proposition 2.16]), so we may assume that we are given a tree $\mathbf{T}$ of the form $(\kappa, <_T)$ and a club $D \subseteq \kappa$ such that $V^-(\mathbf{T}) \supseteq S \cap D$. By possibly shrinking $D$, we may also assume that $D \subseteq \mathrm{acc}\{\beta < \kappa \mid T \restriction \beta = \beta\}$. It follows that for every $\alpha \in D$, every $\alpha$-branch is a cofinal subset of $\alpha$. For every $\alpha \in S \cap D$, let $A_\alpha$ be a vanishing $\alpha$-branch. As $\mathbf{T}$ is a $\kappa$-tree, the ladder system $\langle A_\alpha \mid \alpha \in S \cap D\rangle$ is thin. In addition, for every $(\alpha, \beta) \in [S \cap D]^2$, if it were the case that $\sup(A_\beta \cap A_\alpha) = \alpha$, then $\min(A_\beta \setminus A_\alpha)$ is a node extending all elements of $A_\alpha$, contradicting the fact that $A_\alpha$ is vanishing. So, $\sup(A_\beta \cap A_\alpha) < \alpha$.

When $S = \kappa$, the preceding is related to the subtle tree property.

***Definition 2.21*** (Weiß, [Wei10]) $\kappa$ has the *subtle tree property* ($\kappa$-STP for short) iff for every thin list $\langle A_\alpha \mid \alpha \in D\rangle$ over a club $D \subseteq \kappa$, there exists a pair $(\alpha, \beta) \in [D]^2$ such that $A_\alpha = A_\beta \cap \alpha$.

Note that for every thin list $\langle A_\alpha \mid \alpha \in D\rangle$, if $\{\alpha \in D \mid \sup(A_\alpha) < \alpha\}$ is stationary, then by Fodor's lemma, there exists a pair $(\alpha, \beta) \in [D]^2$ such that $A_\alpha = A_\beta \cap \alpha$. Thus, $\kappa$-STP is really about thin ladder systems.

***Corollary 2.22*** *All of the following are equivalent:*

- $\kappa$-STP *fails;*
- *there is a $\kappa$-tree $\mathbf{T}$ with $V^-(\mathbf{T}) = \mathrm{acc}(\kappa)$;*
- *there is a homogeneous streamlined $\kappa$-tree $T$ with $V(T) = \mathrm{acc}(\kappa)$;*
- *there is a uniformly homogeneous streamlined $\kappa$-tree $T$ such that $V(T)$ covers a club in $\kappa$.*

**Proof** By Lemmas 2.20 and 2.15. ■

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***Remark 2.23*** By [Wei10, Theorem 3.2.5], PFA implies that $\aleph_2$-STP holds. By [HS20, Theorem 1.2], if $\lambda$ is the singular limit of supercompact cardinals then $\lambda^+$-STP fails.[12]

***Corollary 2.24*** *Assuming the consistency of a subtle cardinal,[13] it is consistent that the conjunction of the following holds true:*

- *there exists an $\aleph_2$-Souslin tree;*
- *for every normal and splitting $\aleph_2$-tree $\mathbf{T}$, $E^{\aleph_2}_{\aleph_1}\setminus V(\mathbf{T})$ is stationary.*

**Proof** Work in $\mathsf{L}$, and fix a subtle cardinal $\kappa$ that is not weakly compact. Now, pass to a generic extension $\mathsf{L}[G]$ by Mitchell's forcing of length $\kappa$. By [Wei10, Theorem 2.3.1], $\aleph_2$-STP holds, and hence $V(\mathbf{T})$ cannot contain a club for every $\aleph_2$-tree $\mathbf{T}$. Since $\kappa$ is not weakly compact in $\mathsf{L}$, $\square(\aleph_2)$ holds. In addition, this is a model in which $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and hence Corollary 2.12 implies that $E^{\aleph_2}_{\aleph_0}\setminus V(\mathbf{T})$ is nonstationary for every normal and splitting $\aleph_2$-tree $\mathbf{T}$. Therefore, $E^{\aleph_2}_{\aleph_1}\setminus V(\mathbf{T})$ is stationary for every normal and splitting $\aleph_2$-tree $\mathbf{T}$.

Since the model $\mathsf{L}[G]$ is a forcing extension by a projection of $\mathrm{Add}(\omega,\kappa)\times\mathbb{R}$ for some countably-closed forcing $\mathbb{R}$, all of our reals live in $\mathsf{L}^{\mathrm{Add}(\omega,\kappa)}$. Recalling that already $\mathrm{Add}(\omega,\omega_1)$ adds a Luzin set, we altogether infer that $\mathfrak{b} = \aleph_1$. Finally, by [Rin22, Theorem A], whenever $\mathfrak{b} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ and $\square(\aleph_2)$ all hold, there indeed exist an $\aleph_2$-Souslin tree. ∎

***Corollary 2.25*** *Suppose that $S$ is a stationary subset of a strongly inaccessible $\kappa$. Then, there exists a $\kappa$-tree $\mathbf{T}$ such that $V(\mathbf{T})\cap S$ is stationary.*

**Proof** By Lemma 2.20, it suffices to find a stationary $S^-\subseteq S$ that carries a thin almost disjoint $C$-sequence. We consider two cases:

▸ If $S\cap E^\kappa_\omega$ is stationary, then set $S^- := S\cap E^\kappa_\omega$, and let $\langle C_\alpha \mid \alpha\in S^-\rangle$ be some $\omega$-bounded $C$-sequence over $S^-$.

▸ Otherwise, let $S^- := S\setminus(E^\kappa_\omega\cup\mathrm{Tr}(S))$. Then, $S^-$ is stationary, and for every $\alpha\in S^-$, we may pick a club $C_\alpha$ in $\alpha$ that is disjoint from $S$. Evidently, $\sup(C_\alpha\cap C_{\alpha'})<\alpha$ for every pair $\alpha<\alpha'$ of ordinals in $S^-$. ∎

***Lemma 2.26*** *If $\theta\in\mathrm{Reg}(\kappa)$ is such that $\lambda^{<\theta}<\kappa$ for all $\lambda<\kappa$, then there exists an almost disjoint thin $C$-sequence over $E^\kappa_\theta$.*

**Proof** Just take a $\theta$-bounded $C$-sequence over $E^\kappa_\theta$. ∎

Building on the work of Todorčević [Tod07] and Krueger [Kru13], we obtain the following pump-up theorem for special $\kappa$-Aronszajn trees.

***Theorem 2.27*** *The following are equivalent:*

(i) *There exists a special $\kappa$-Aronszajn tree;*
(ii) *There exists a streamlined $\kappa$-Aronszajn tree $K$, a club $D\subseteq\mathrm{acc}(\kappa)$, and a function $f: K\restriction D\to\kappa$ such that all of the following hold:*
   - *$V^-(K)\supseteq D$;*
   - *$f(x)<\mathrm{dom}(x)$ for all $x\in K\restriction D$;*

[12] The statement of the theorem in [HS20] is limited to countable cofinality, but the proof works unconditionally.

[13] See Definition 3.9 below.

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– *$f(x) \neq f(y)$ for every pair $x \subsetneq y$ of nodes from $K \restriction D$;*
– *for all $x, y \in K$ and $\varepsilon \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$, if $x(\varepsilon) = y(\varepsilon)$, then $x \restriction \varepsilon = y \restriction \varepsilon$. [(iv)]*

(iii) *There exists a streamlined uniformly homogeneous special $\kappa$-Aronszajn tree $T$ for which $V(T)$ covers a club in $\kappa$;*
(iv) *There exists a streamlined homogeneous special $\kappa$-Aronszajn tree $T$ with $V(T) = \mathrm{acc}(\kappa)$.*

**Proof** $(i) \implies (ii)$ Assuming that there exists a special $\kappa$-Aronszajn tree, by [Kru13, Lemma 1.2 and Theorem 2.5], we may fix a $C$-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ and a club $C \subseteq \mathrm{acc}(\kappa)$ satisfying the following:
(1) for every $\beta \in C$, $\min(C_\beta) > \mathrm{otp}(C_\beta)$;
(2) for every $\beta \in \mathrm{acc}(\kappa) \setminus C$, $\min(C_\beta) > \sup(C \cap \beta)$;
(3) for every $\varepsilon < \kappa$, $|\{C_\beta \cap \varepsilon \mid \beta < \kappa\}| < \kappa$.
Consider the following additional requirement:
(4) $\min(C_\beta) = \mathrm{otp}(C_\beta) + 1$ for every $\beta \in C$.

***Claim 2.27.1*** *We may moreover assume that Clause (4) holds.* ■

**Proof** For every $\beta \in C$, let $C^\bullet_\beta := C_\beta \cup \{\mathrm{otp}(C_\beta) + 1\}$, and for every $\beta \in \kappa \setminus C$, let $C^\bullet_\beta := C_\beta$. We just need to verify that $|\{C^\bullet_\beta \cap \varepsilon \mid \beta < \kappa\}| < \kappa$ for every $\varepsilon < \kappa$. Toward a contradiction, suppose that $\varepsilon$ is a counterexample. From (3), it follows that we may fix $B \in [C]^\kappa$ on which the map $\beta \mapsto C^\bullet_\beta \cap \varepsilon$ is injective. We may moreover assume that $\beta \mapsto C_\beta \cap \varepsilon$ is constant over $B$. By possibly removing one element of $B$, we may assume that $C^\bullet_\beta \cap \varepsilon$ is nonempty for all $\beta \in B$. So, we may moreover assume the existence of $\tau < \varepsilon$ such that $\min(C^\bullet_\beta) = \tau$ for every $\beta \in B$. But then $C^\bullet_\beta \cap \varepsilon = (C_\beta \cap \varepsilon) \cup \{\tau\}$ for every $\beta \in B$. This is a contradiction. ■

Now, let $\rho_0$ be the characteristic function from [Tod07, Section 6] obtained by walking along $\vec{C}$ satisfying (1)–(4), and consider the following streamlined $\kappa$-tree:

$$T(\rho_0) := \{\rho_{0\beta} \restriction \alpha \mid \alpha \leq \beta < \kappa\}.$$

Using (1)–(3), the proof of [Kru13, Theorem 4.4] provides a club $D \subseteq C$ and a function $g : T(\rho_0) \restriction D \to \kappa$ satisfying the following two:

- $g(t) < \mathrm{dom}(t)$ for all $t \in T(\rho_0) \restriction D$;
- for every pair $s \subsetneq t$ of nodes from $T(\rho_0) \restriction D$, $g(s) \neq g(t)$.

Next, consider the following subfamily of $T(\rho_0)$:

$$T := \{\rho_{0\beta} \restriction \alpha \mid \alpha < \beta < \kappa\}.$$

Clearly, $T$ is downward-closed and $\{\mathrm{dom}(y) \mid y \in T\} = \kappa$, so that $T$ is a streamlined $\kappa$-Aronszajn subtree of $T(\rho_0)$.

***Claim 2.27.2*** *$T \cap \{\rho_{0\alpha} \mid \alpha \in C\} = \emptyset$. In particular, $V^-(T) \supseteq C \supseteq D$.*

**Proof** The "in particular" part will follow from the fact that $\{\rho_{0\alpha} \restriction \varepsilon \mid \varepsilon < \alpha\}$ is an $\alpha$-branch of $T$ for every $\alpha < \kappa$. Thus, let $\alpha \in C$ and we shall prove that $\rho_{0\alpha} \notin T$. Suppose not, and pick some $\beta > \alpha$ such that $\rho_{0\alpha} = \rho_{0\beta} \restriction \alpha$. Recall that for every $\gamma < \kappa$,

$$C_\gamma = \{\xi < \gamma \mid \rho_{0\gamma}(\xi) \text{ is a sequence of length } 1\}.$$

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In particular, $\min(C_\alpha) = \min(C_\beta)$. As $\sup(C \cap \beta) \geq \alpha > \min(C_\alpha)$, it follows from Clause (2) that $\beta \in C$. So, by Clause (4), $\mathrm{otp}(C_\alpha) = \mathrm{otp}(C_\beta)$. It follows that may fix some $\delta \in C_\alpha \setminus C_\beta$. But then $\rho_{0\alpha}(\delta)$ is a sequence of length 1, whereas $\rho_{0\beta}(\delta)$ is a longer sequence. This is a contradiction. ∎

For every $t \in T \restriction \mathrm{acc}(\kappa)$, define a function $k_t : \mathrm{dom}(t) \to T$ via

$$k_t(\varepsilon) := t \restriction \varepsilon.$$

Let $K$ be the following downward-closed subfamily of ${}^{<\kappa}H_\kappa$:

$$K := \{k_t \restriction \alpha \mid \alpha \leq \mathrm{dom}(t), t \in T \restriction \mathrm{acc}(\kappa)\}.$$

Evidently, for all $x, y \in K$ and $\varepsilon \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$, if $x(\varepsilon) = y(\varepsilon)$, then $x \restriction \varepsilon = y \restriction \varepsilon$. In addition, $t \mapsto k_t$ constitutes an isomorphism between $(T \restriction \mathrm{acc}(\kappa), \subsetneq)$ and $(K \restriction \mathrm{acc}(\kappa), \subsetneq)$, and hence $K$ is a streamlined $\kappa$-Aronszajn tree with $V^-(K) \supseteq D$. The fact that the above map is an isomorphism also implies that a function $f : K \restriction D \to \kappa$ defined via $f(k_t) := g(t)$ satisfies that $f(x) < \mathrm{dom}(x)$ for all $x \in K \restriction D$, and that $f(x) \neq f(y)$ for every pair $x \subsetneq y$ of nodes from $K \restriction D$.

$(ii) \implies (iii)$: Suppose that $K$ and $f : K \restriction D \to \kappa$ are as in Clause (ii). By possibly shrinking $D$, we may assume that for all $\beta \in D$ and $\alpha < \beta$, it is the case that $\omega \cdot \alpha < \beta$.

Using Remark 2.17, we may define a family $T$ to be the collection of all elements of the form $x_0 * \cdots * x_n$, where[14]

(a) $n < \omega$,
(b) $x_i \in K$ for all $i \leq n$, and
(c) $\mathrm{dom}(x_i) < \mathrm{dom}(x_{i+1})$ for all $i < n$.

It is clear that $t \restriction \alpha \in T$ for all $t \in T$ and $\alpha < \kappa$. Thus, recalling the proof of Claim 2.20.1, to establish that $T$ is a uniformly homogeneous streamlined $\kappa$-tree, it suffices to prove the following claim.[15]

***Claim 2.27.3*** *$T_0 = \{\emptyset\}$ and $T_\alpha = \{x * y \mid x \in T \restriction \alpha, y \in K_\alpha\}$ for every nonzero $\alpha < \kappa$.*

**Proof** Suppose that $\alpha$ is a nonzero ordinal such that $T_\varepsilon = \{x * y \mid x \in T \restriction \alpha, y \in K_\varepsilon\}$ for every $\varepsilon < \alpha$. Let $t \in T_\alpha$. Pick a sequence $(x_0, \ldots, x_n)$ satisfying (a)–(c) for which $t = x_0 * \cdots * x_n$.

- If $n = 0$, then $t = \emptyset * x_0$ with $\emptyset \in T \restriction \alpha$ and $x_0 \in K_\alpha$.
- If $n = m + 1$ for some $m < \omega$, then $t = x * y$ with $x := x_0 * \cdots * x_m$ in $T \restriction \alpha$ and $y := x_{m+1}$ in $K_\alpha$. ∎

For each node $t \in T$, we define $n(t)$ and $x(t)$ by first letting $n(t)$ denote the least $n$ for which there exists a sequence $(x_0, \ldots, x_n)$ satisfying (a)–(c) for which $t = x_0 * \cdots * x_n$, and then letting $x(t)$ be such an $x_n$. Note that $\mathrm{dom}(x(t)) = \mathrm{dom}(t)$, and that $K = \{t \in T \mid n(t) = 0\}$.

[14] To clarify, in the special case that $n = 0$, $x_0 * \cdots * x_n$ stands for $x_0$.

[15] Altogether, this will show that the recursive procedure described in the proof of Lemma 2.20 of constructing a uniformly homogeneous tree $T$ out of a given streamlined $\kappa$-tree $K$ coincides with the closed-form construction given here. Both yield the minimal uniformly homogeneous streamlined $\kappa$-tree $T$ that includes a given streamlined $\kappa$-tree $K$.

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Define a function $g : T \restriction D \to \kappa$ via

$$g(t) := (\omega \cdot f(x(t))) + n(t).$$

***Claim 2.27.4***

(1) $g(t) < \mathrm{dom}(t)$ *for all* $t \in T \restriction D$;
(2) *Let* $s \subsetneq t$ *be a pair of nodes from* $T \restriction D$. *Then,* $g(s) \neq g(t)$.

**Proof** (1) Since $\omega \cdot \alpha < \beta$ for all $\beta \in D$ and $\alpha < \beta$.

(2) Suppose not. Let $\tau < \kappa$ and $n < \omega$ be such that $f(x(s)) = \tau = f(x(t))$ and $n(s) = n = n(t)$. By the choice of $f$, it follows that $x(s) \nsubseteq x(t)$, so since $s \subsetneq t$, it must be the case that $n = m + 1$ for some $m < \omega$. Fix a sequence $(x_0, \ldots, x_m, x_{m+1})$ of nodes from $K$ such that $s = x_0 * \cdots * x_m * x_{m+1}$ and $x_{m+1} = x(s)$. Likewise, fix a sequence $(y_0, \ldots, y_m, y_{m+1})$ of nodes from $K$ such that $t = y_0 * \cdots * y_m * y_{m+1}$ and $y_{m+1} = x(t)$.

▸ As $x_{m+1} \nsubseteq y_{m+1}$, we may fix $\delta \in \mathrm{dom}(x_{m+1})$ such that $x_{m+1}(\delta) \neq y_{m+1}(\delta)$.

▸ As $s \subseteq t = y_0 * \cdots * y_m * y_{m+1}$ and $n(s) > m$, it must be the case that $\mathrm{dom}(y_m) < \mathrm{dom}(s)$.

Altogether, $\varepsilon := \max\{\delta + 1, \mathrm{dom}(x_m), \mathrm{dom}(y_m)\}$ is an ordinal less than $\mathrm{dom}(s)$, satisfying $x_{m+1}(\varepsilon) = s(\varepsilon) = t(\varepsilon) = y_{m+1}(\varepsilon)$, but then $x_{m+1} \restriction \varepsilon = y_{m+1} \restriction \varepsilon$, contradicting the fact that $\delta < \varepsilon$. ■

It is easy to see that the two features of $g$ together imply that $T$ admits no $\kappa$-branch. The beginning of the proof of [Kru13, Theorem 4.4] shows furthermore that $T$ must be a special $\kappa$-Aronszajn tree.

***Claim 2.27.5*** $V(T) \supseteq D$.

**Proof** Let $\alpha \in D$. As $D \subseteq V^-(K)$, we may fix a function $t : \alpha \to H_\kappa$ such that $\{t \restriction \varepsilon \mid \varepsilon < \alpha\} \subseteq K$, but $t \notin K$. As $K \subseteq T$, it thus suffices to prove that $t \notin T$. Toward a contradiction, suppose that $t \in T$. In particular, $n(t) > 0$. Fix $m < \omega$ and a sequence $(x_0, \ldots, x_m, x_{m+1})$ of nodes from $K$ such that $t = x_0 * \cdots * x_m * x_{m+1}$. As $x_{m+1} \neq t$, we may fix some $\delta < \alpha$ such that $t(\delta) \neq x_{m+1}(\delta)$. Pick $\varepsilon < \alpha$ above $\max\{\delta, \mathrm{dom}(x_m)\}$. Then, $t(\varepsilon) = x_{m+1}(\varepsilon)$. But $t \restriction (\varepsilon + 1)$ and $x_{m+1} \restriction (\varepsilon + 1)$ are two nodes in $K$ that agree on $\varepsilon$ and hence $t \restriction (\varepsilon + 1) = x_{m+1} \restriction (\varepsilon + 1)$, contradicting the fact that $\delta < \varepsilon$. ■

The implication $(iii) \implies (iv)$ follows from Lemma 2.15, and the implication $(iv) \implies (i)$ is trivial.

***Definition 2.28*** (Products) For a sequence of $\kappa$-trees $\langle \mathbf{T}^i \mid i < \tau \rangle$ with $\mathbf{T}^i = (T^i, <_{T^i})$ for each $i < \tau$, the product $\bigotimes_{i<\tau} \mathbf{T}^i$ is defined to be the tree $\mathbf{T} = (T, <_T)$, where:

- $T = \bigcup\{\prod_{i<\tau} T^i_\alpha \mid \alpha < \kappa\}$;
- $\vec{s} <_T \vec{t}$ iff $\vec{s}(i) <_{T^i} \vec{t}(i)$ for every $i < \tau$.

***Remark 2.29*** The product of streamlined trees may be realized as a streamlined tree (see Definition 5.4 below).

***Proposition 2.30*** *For a sequence* $\langle \mathbf{T}^i \mid i < \tau \rangle$ *of normal* $\kappa$*-trees, if* $\lambda^\tau < \kappa$ *for all* $\lambda < \kappa$, *then:*

(1) $\bigotimes_{i<\tau}\mathbf{T}^i$ *is a normal* $\kappa$*-tree;*
(2) $V(\bigotimes_{i<\tau}\mathbf{T}^i) = \bigcup\{V(\mathbf{T}^i) \mid i<\tau\}$;
(3) $V^-(\bigotimes_{i<\tau}\mathbf{T}^i) = \bigcup\{V^-(\mathbf{T}^i) \mid i<\tau\}$.

**Proof** Left to the reader. ∎

***Definition 2.31*** (Sums) The *disjoint sum* $\sum\mathcal{P}$ of a family of posets $\mathcal{P}$ is the poset $(A,<_A)$ defined as follows:

- $A := \{((P,<_P),x) \mid (P,<_P)\in\mathcal{P}, x\in P\}$;
- $((P,<_P),x) <_A ((Q,<_Q),y)$ iff $(P,<_P) = (Q,<_Q)$ and $x <_P y$.

In the special case of doubleton, we write $\mathbf{T}+\mathbf{S}$ instead of $\sum\{\mathbf{T},\mathbf{S}\}$.

***Proposition 2.32*** *Suppose that $\mathcal{T}$ is a family of less than $\kappa$ many (resp., normal) $\kappa$-trees. Then:*

(1) $\sum\mathcal{T}$ *is a (resp., normal)* $\kappa$*-tree;*
(2) $V(\sum\mathcal{T}) = \bigcap\{V(\mathbf{T}) \mid \mathbf{T}\in\mathcal{T}\}$;
(3) $V^-(\sum\mathcal{T}) = \bigcup\{V^-(\mathbf{T}) \mid \mathbf{T}\in\mathcal{T}\}$.

**Proof** Left to the reader. ∎

***Remark 2.33*** The disjoint sum of two Hausdorff trees need not be Hausdorff for the mere reason it does not have a unique root, but this is inessential. Furthermore, there is a natural operation of disjoint sum for streamlined trees (as in the proof of Claim 6.3.1) whose outcome is a streamlined tree (hence Hausdorff) maintaining the features of Proposition 2.32.

It follows from Propositions 2.30 and 2.32 that the spectrum of sets that arise as the vanishing levels of normal $\kappa$-trees is closed under finite unions and intersections.

***Corollary 2.34*** *Suppose $\chi\in\mathrm{Reg}(\kappa)$ is such that $\lambda^{<\chi}<\kappa$ for all $\lambda<\kappa$. Then, there exists a $\kappa$-tree $\mathbf{T}$ with $V^-(\mathbf{T})\supseteq\mathrm{acc}(\kappa)\cap E^\kappa_{\le\chi}$.*

**Proof** Denote $\Theta := \mathrm{Reg}(\chi+1)$. By Lemmas 2.26 and 2.20, for every $\theta\in\Theta$, we may pick a $\kappa$-tree $\mathbf{T}^\theta$ such that $E^\kappa_\theta\setminus V^-(\mathbf{T}^\theta)$ is nonstationary. In fact, the proof of (2) $\implies$ (3) of Lemma 2.20 shows that we may secure $V^-(\mathbf{T}^\theta)\supseteq E^\kappa_\theta$. Let $\mathbf{T} := \sum\{\mathbf{T}^\theta \mid \theta\in\Theta\}$ be the disjoint sum of these trees. By Proposition 2.32, $V^-(\mathbf{T}) = \bigcup_{\theta\in\Theta}V^-(\mathbf{T}^\theta)\supseteq\bigcup_{\theta\in\Theta}E^\kappa_\theta = \mathrm{acc}(\kappa)\cap E^\kappa_{\le\chi}$. ∎

***Remark 2.35*** In Section 5, we provide sufficient conditions for getting a homogeneous $\kappa$-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = \bigcup_{\chi\in x}E^\kappa_\chi$ for a prescribed finite and nonempty $x\subseteq\mathrm{Reg}(\kappa)$.

***Question 2.36*** *Is it consistent that for some regular uncountable cardinal $\kappa$, there are $\kappa$-Souslin trees, but $V(\mathbf{T})$ is nonstationary for every $\kappa$-Souslin tree $\mathbf{T}$?*

By Proposition 2.6, Corollary 2.11, and [BR17b, Lemma 2.4], in such a model there cannot be a homogeneous $\kappa$-Souslin tree. A model with an $\aleph_1$-Souslin tree but no homogeneous one was constructed by Abraham and Shelah in [AS93].

## 3 Consulting another tree

In this section, we present a method for constructing a $\kappa$-Souslin tree $\mathbf{T}$ while consulting another input tree $\mathbf{K}$ in order to ensure $V(\mathbf{T}) \supseteq V^-(\mathbf{K})$. This is how we will be proving Theorems B and C. The main result of this section is Theorem 3.7 below. A sample corollary of it reads as follows.

***Corollary 3.1*** *Suppose that $\kappa = \lambda^+$ for an infinite cardinal $\lambda$.*

(1) *If $\square_\lambda + \diamondsuit(\kappa)$ holds, then there exists a $\kappa$-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = \operatorname{acc}(\kappa)$;*
(2) *If $\square(\kappa)$ holds and $\aleph_0 < \lambda^{<\lambda} < \lambda^+ = 2^\lambda$, then there exists a $\kappa$-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = \operatorname{acc}(\kappa)$;*
(3) *If $\mathrm{P}_\lambda(\kappa, \kappa, \sqsubseteq, 1)$ holds, then there exists a $\kappa$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) \supseteq E^\kappa_{>\omega}$.*

**Proof** (1) Suppose that $\square_\lambda + \diamondsuit(\kappa)$ holds. By Lemma 2.5, it suffices to find a $\kappa$-Souslin tree $\mathbf{T}$ for which $V(\mathbf{T})$ covers a club in $\kappa$.

- For $\lambda = \aleph_0$, $\diamondsuit(\aleph_1)$ implies the existence of a normal and splitting $\aleph_1$-Souslin tree $\mathbf{T}$, and by Corollary 2.12, $V(\mathbf{T})$ covers a club in $\aleph_1$.
- For $\lambda \geq \aleph_1$, by [BR17a, Corollary 3.9], $\square_\lambda + \mathrm{CH}_\lambda$ is equivalent to $\mathrm{P}_\lambda(\kappa, 2, \sqsubseteq, 1)$. In addition, by a theorem of Jensen, $\square_\lambda$ gives rise to a special $\lambda^+$-Aronszajn tree. Thus, we infer from Theorem 2.27 the existence of a streamlined $\kappa$-tree $K$ for which $V(K)$ covers a club in $\kappa$. It thus follows from Theorem 3.7(1) below that there exists a $\kappa$-Souslin tree $\mathbf{T}$ for which $V(\mathbf{T})$ is a club in $\kappa$.

(2) By [Rin17, Corollary 4.4], the hypothesis implies that $\mathrm{P}^-(\kappa, 2, \sqsubseteq, 1)$ holds. In addition, by a theorem of Specker, $\lambda = \lambda^{<\lambda}$ implies the existence of a special $\lambda^+$-Aronszajn tree. Now, continue as in the proof of Clause (1).

(3) Similar to the proof of Clause (1), using Theorem 3.7(2), instead. ∎

***Remark 3.2*** Sufficient conditions for $\mathrm{P}_\lambda(\kappa, \kappa, \sqsubseteq, 1)$ to hold are given by Corollaries 3.15 and 3.24 of [BR19c].

Before turning to the proofs of the main results of this section, we provide a few preliminaries.

***Definition 3.3*** (Proxy principle, [BR17a, BR21]) Suppose that $\mu, \theta \leq \kappa$ are cardinals, $\xi \leq \kappa$ is an ordinal, $\mathcal{R}$ is a binary relation over $[\kappa]^{<\kappa}$, and $\mathcal{S}$ is a collection of stationary subsets of $\kappa$. The principle $\mathrm{P}^-_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S})$ asserts the existence of a $\xi$-bounded $\mathcal{C}$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ such that:

- for every $\alpha < \kappa$, $|\mathcal{C}_\alpha| < \mu$;
- for all $\alpha < \kappa$, $C \in \mathcal{C}_\alpha$, and $\bar\alpha \in \operatorname{acc}(C)$, there exists some $D \in \mathcal{C}_{\bar\alpha}$ such that $D \mathrel{\mathcal{R}} C$;
- for every sequence $\langle B_i \mid i < \theta \rangle$ of cofinal subsets of $\kappa$, and every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that for all $C \in \mathcal{C}_\alpha$ and $i < \min\{\alpha, \theta\}$, $\sup(\operatorname{nacc}(C) \cap B_i) = \alpha$.

***Convention 3.4*** *We write $\mathrm{P}_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S})$ to assert that $\mathrm{P}^-_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S})$ and $\diamondsuit(\kappa)$ both hold.*

***Convention 3.5*** *If we omit $\xi$, then we mean $\xi := \kappa$. If we omit $\mathcal{S}$, then we mean $\mathcal{S} := \{\kappa\}$. In the case $\mu = 2$, we identify $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ with the unique element $\langle C_\alpha \mid \alpha < \kappa \rangle$ of $\prod_{\alpha<\kappa} \mathcal{C}_\alpha$.*

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***Fact 3.6*** [BR17a, Lemma 2.2] *The following are equivalent:*

(1) $\diamondsuit(\kappa)$, *i.e., there is a sequence* $\langle f_\beta \mid \beta < \kappa\rangle$ *such that for every function* $f : \kappa \to \kappa$, *the set* $\{\beta < \kappa \mid f \restriction \beta = f_\beta\}$ *is stationary in* $\kappa$.

(2) $\diamondsuit^-(H_\kappa)$, *i.e., there is a sequence* $\langle \Omega_\beta \mid \beta < \kappa\rangle$ *such that for all* $p \in H_{\kappa^+}$ *and* $\Omega \subseteq H_\kappa$, *there exists an elementary submodel* $\mathcal{M} \prec H_{\kappa^+}$ *such that:*
- $p \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa \in \kappa$;
- $\mathcal{M} \cap \Omega = \Omega_{\mathcal{M}\cap\kappa}$.

(3) $\diamondsuit(H_\kappa)$, *i.e., there are a partition* $\langle R_i \mid i < \kappa\rangle$ *of* $\kappa$ *and a sequence* $\langle \Omega_\beta \mid \beta < \kappa\rangle$ *such that for all* $p \in H_{\kappa^+}$, $\Omega \subseteq H_\kappa$, *and* $i < \kappa$, *there exists an elementary submodel* $\mathcal{M} \prec H_{\kappa^+}$ *such that:*
- $p \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa \in R_i$;
- $\mathcal{M} \cap \Omega = \Omega_{\mathcal{M}\cap\kappa}$.

***Theorem 3.7*** *Suppose that* $K$ *is some streamlined* $\kappa$*-tree.*

(1) *If* $\mathrm{P}(\kappa, 2, \sqsubseteq^*, 1)$ *holds, then there exists a normal and splitting streamlined* $\kappa$*-Souslin tree* $T$ *such that* $V(T) \supseteq V^-(K)$;

(2) *If* $\mathrm{P}(\kappa, \kappa, \sqsubseteq, 1)$ *holds, then there exists a normal and splitting streamlined* $\kappa$*-Souslin tree* $T$ *such that* $V(T) \supseteq V^-(K) \cap E^\kappa_{>\omega}$.

**Proof** Fix a well-ordering $\lhd$ of $H_\kappa$, and a sequence $\langle \Omega_\beta \mid \beta < \kappa\rangle$ witnessing $\diamondsuit^-(H_\kappa)$. If $\mathrm{P}^-(\kappa, \kappa, \sqsubseteq, 1)$ holds, then let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ be any $\mathrm{P}^-(\kappa, \kappa, \sqsubseteq, 1)$-sequence. If $\mathrm{P}^-(\kappa, 2, \sqsubseteq^*, 1)$ holds, then, by [BR21, Theorem 4.39], we may let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ be a $\mathrm{P}^-(\kappa, \kappa, \sqsubseteq, 1)$-sequence with the added feature that for every $\alpha \in \mathrm{acc}(\kappa)$ for all $C, D \in \mathcal{C}_\alpha$, $\sup(C \bigtriangleup D) < \alpha$.

Following the proof of [BR19b, Proposition 2.2], we shall recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ such that $T := \bigcup_{\alpha<\kappa} T_\alpha$ will constitute the tree of interest whose $\alpha^{\text{th}}$-level is $T_\alpha$. Note, however, that unlike the reference construction, here $T$ will not be a subset of ${}^{<\kappa}\kappa$, but of ${}^{<\kappa}H_\kappa$.

We start by letting $T_0 := \{\emptyset\}$, and once $T_\alpha$ has already been defined, we let

$$T_{\alpha+1} := \{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle, t^\frown\langle \eta\rangle \mid t \in T_\alpha, \eta \in K_\alpha\}.$$

Next, suppose that $\alpha \in \mathrm{acc}(\kappa)$ is such that $T \restriction \alpha$ has already been defined. For all $C \in \mathcal{C}_\alpha$ and $x \in T \restriction C$, we shall identify a set of potential nodes $\{\mathbf{b}^{C,\eta}_x \mid \eta \in \mathcal{B}(K \restriction \alpha)\}$ and then let

$$(\star) \qquad T_\alpha := \{\mathbf{b}^{C,\eta}_x \mid C \in \mathcal{C}_\alpha, \eta \in K_\alpha, x \in T \restriction C\}.$$

To this end, fix $C \in \mathcal{C}_\alpha$, $x \in T \restriction C$ and $\eta \in \mathcal{B}(K \restriction \alpha)$. The node $\mathbf{b}^{C,\eta}_x$ will be obtained as the limit $\bigcup \mathrm{Im}(b^{C,\eta}_x)$ of a sequence $b^{C,\eta}_x \in \prod_{\beta \in C\setminus \mathrm{dom}(x)} T_\beta$, as follows:

- Let $b^{C,\eta}_x(\mathrm{dom}(x)) := x$.
- For every $\beta \in \mathrm{nacc}(C)$ above $\mathrm{dom}(x)$ such that $b^{C,\eta}_x(\beta^-)$ has already been defined for $\beta^- := \sup(C \cap \beta)$, let

$$Q^{C,\eta}_x(\beta) := \{t \in T_\beta \mid \exists s \in \Omega_\beta[(s \cup (b^{C,\eta}_x(\beta^-)^\frown\langle \eta \restriction \beta^-\rangle)) \subseteq t]\}.$$

Now, consider the two possibilities:

– If $Q_x^{C,\eta}(\beta) \neq \emptyset$, then let $b_x^{C,\eta}(\beta)$ be its $\lhd$-least element;
– Otherwise, let $b_x^{C,\eta}(\beta)$ be the $\lhd$-least element of $T_\beta$ that extends $b_x^{C,\eta}(\beta^-)^\frown\langle\eta \restriction \beta^-\rangle$. Such an element must exist, as the level $T_\beta$ was constructed so as to preserve normality.

- For every $\beta \in \mathrm{acc}(C \setminus \mathrm{dom}(x))$ such that $b_x^{C,\eta} \restriction \beta$ has already been defined, let $b_x^{C,\eta}(\beta) := \bigcup \mathrm{Im}(b_x^{C,\eta} \restriction \beta)$.

For the last case, we need to argue that $b_x^{C,\eta}(\beta)$ is indeed an element of $T_\beta$. As $\vec{\mathcal{C}}$ is $\sqsubseteq$-coherent, the set $\bar{C} := C \cap \beta$ is in $\mathcal{C}_\beta$. Also, $K$ is a tree and hence $\bar{\eta} := \eta \restriction \beta$ is in $K_\beta$. So, since $\mathbf{b}_x^{\bar{C},\bar{\eta}} \in T_\beta$, to show that $b_x^{C,\eta}(\beta) \in T_\beta$, it suffices to prove the following.

***Claim 3.7.1*** $b_x^{C,\eta}(\beta) = \mathbf{b}_x^{\bar{C},\bar{\eta}}$. ■

**Proof** Clearly, $\mathrm{dom}(b_x^{C,\eta}(\beta)) = C \cap \beta \setminus \mathrm{dom}(x) = \bar{C} \setminus \mathrm{dom}(x) = \mathrm{dom}(b_x^{\bar{C},\bar{\eta}})$. So, we are left with showing that $b_x^{C,\eta}(\delta) = b_x^{\bar{C},\bar{\eta}}(\delta)$ for all $\delta \in \bar{C} \setminus \mathrm{dom}(x)$. The proof is by induction on $\delta \in \bar{C} \setminus \mathrm{dom}(x)$:

- For $\delta = \mathrm{dom}(x)$, we have that $b_x^{C,\eta}(\delta) = x = b_x^{\bar{C},\bar{\eta}}(\delta)$.
- Given $\delta \in \mathrm{nacc}(\bar{C})$ above $\mathrm{dom}(x)$ such that $b_x^{C,\eta}(\delta^-) = b_x^{\bar{C},\bar{\eta}}(\delta^-)$ for $\delta^- := \sup(\bar{C} \cap \delta)$, we argue as follows. Since
$$b_x^{C,\eta}(\delta^-)^\frown\langle\eta \restriction \delta^-\rangle = b_x^{\bar{C},\bar{\eta}}(\delta^-)^\frown\langle\bar{\eta} \restriction \delta^-\rangle,$$
the definitions of $b_x^{C,\eta}(\delta)$ and $b_x^{\bar{C},\bar{\eta}}(\delta)$ coincide.
- If $\delta \in \mathrm{acc}(\bar{C} \setminus \mathrm{dom}(x))$, then we take the limit of two identical sequences, and the unique limit is identical. ■

This completes the definition of $b_x^{C,\eta}$. For all $\eta \in \mathcal{B}(K \restriction \alpha)$, let $\mathbf{b}_x^{C,\eta} := \bigcup \mathrm{Im}(b_x^{C,\eta})$, and then we define $T_\alpha$ as promised in ($\star$).

Clearly, $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a normal and splitting $\kappa$-tree. The verification of Souslinness is standard (see [BR19b, Claims 2.2.2 and 2.2.3]).

***Claim 3.7.2*** *Suppose that $\alpha \in V^-(K)$ is such that $\sup(C \cap D) = \alpha$ for all $C, D \in \mathcal{C}_\alpha$. Then, $\alpha \in V(T)$.*

**Proof** As $\alpha \in V^-(K)$, we may fix $\eta \in \mathcal{B}(K \restriction \alpha) \setminus K_\alpha$. Let $x \in T \restriction \alpha$, and we shall find a vanishing $\alpha$-branch through $x$ in $T$. First fix $C \in \mathcal{C}_\alpha$. Using normality and by possibly extending $x$, we may assume that $x \in T \restriction C$. We have already established that $\{\mathbf{b}_x^{C,\eta} \restriction \varepsilon \mid \varepsilon < \alpha\}$ is an $\alpha$-branch through $x$. Toward a contradiction, suppose that it is not vanishing, so that $\bigcup \mathrm{Im}(b_x^{C,\eta})$ is in $T_\alpha$. It follows from ($\star$) that we may pick $D \in \mathcal{C}_\alpha$, $y \in T \restriction D$ and $\xi \in K_\alpha$ such that $\bigcup \mathrm{Im}(b_x^{C,\eta}) = \mathbf{b}_y^{D,\xi}$. Fix $\beta \in C \cap D$ large enough such that $\beta > \max\{\mathrm{dom}(x), \mathrm{dom}(y)\}$ and $\eta \restriction \beta \neq \xi \restriction \beta$. In particular, $\beta \in \mathrm{dom}(b_x^{C,\eta}) \cap \mathrm{dom}(b_y^{D,\xi})$. Consider $\beta^C := \min(C \setminus \beta + 1)$, the successor of $\beta$ in $C$ and $\beta^D := \min(D \setminus \beta + 1)$, the successor of $\beta$ in $D$. Then, the definition of the successor stage of $b_x^{C,\eta}$ ensures that $b_x^{C,\eta}(\beta^C)$ extends $b_x^{C,\eta}(\beta)^\frown\langle\eta \restriction \beta\rangle$, so that $b_x^{C,\eta}(\beta^C)(\beta) =$

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$\eta \restriction \beta$. Likewise, $b_y^{D,\xi}(\beta^D)(\beta) = \xi \restriction \beta$. From $\mathbf{b}_x^{C,\eta} = \mathbf{b}_y^{D,\xi}$, we infer that $b_x^{C,\eta}(\beta^C)(\beta) = \mathbf{b}_x^{C,\eta}(\beta) = \mathbf{b}_y^{D,\xi}(\beta) = b_y^{D,\xi}(\beta^D)(\beta)$, contradicting the fact that $\eta \restriction \beta \neq \xi \restriction \beta$. ■

This completes the proof.

We now arrive at Theorem C.

***Corollary 3.8*** *Suppose that* $\mathrm{P}(\kappa, 2, \sqsubseteq^*, 1)$ *holds. Then:*

(1) *For every* $\chi \in \mathrm{Reg}(\kappa)$ *such that* $\lambda^{<\chi} < \kappa$ *for all* $\lambda < \kappa$, *and every* $\kappa$*-tree* $\mathbf{K}$, *there exists a* $\kappa$*-Souslin tree* $\mathbf{T}$ *such that* $(E^\kappa_{\leq\chi} \cup V^-(\mathbf{K})) \setminus V(\mathbf{T})$ *is nonstationary.*
(2) *There exists a* $\kappa$*-Souslin tree* $\mathbf{T}$ *such that* $V(\mathbf{T})$ *is stationary.*

**Proof** (1) Suppose $\chi$ and $\mathbf{K}$ are as above. By Corollary 2.34, we may fix a $\kappa$-tree $\mathbf{H}$ with $V^-(\mathbf{H}) \supseteq \mathrm{acc}(\kappa) \cap E^\kappa_{\leq\chi}$. By Proposition 2.32, $\mathbf{K} + \mathbf{H}$ is a $\kappa$-tree with $V^-(\mathbf{K} + \mathbf{H}) = V^-(\mathbf{K}) \cup V^-(\mathbf{H})$. By [BR21, Lemma 2.5], we may fix a streamlined $\kappa$-tree that $K$ that is club-isomorphic to $\mathbf{K} + \mathbf{H}$. Now, appeal to Theorem 3.7(1) with $K$.

(2) Appeal to Clause (1) with $\chi = \omega$. ■

***Definition 3.9*** (Jensen–Kunen, [JK69]) A cardinal $\kappa$ is *subtle* iff for every list $\langle A_\alpha \mid \alpha \in D\rangle$ over a club $D \subseteq \kappa$, there is a pair $(\alpha, \beta) \in [D]^2$ such that $A_\alpha = A_\beta \cap \alpha$.

We now arrive at Theorem B.

***Corollary 3.10*** *We have* $(1) \implies (2) \implies (3)$*:*

(1) *there exists a streamlined* $\kappa$*-Souslin tree* $T$ *such that* $V(T) = \mathrm{acc}(\kappa)$*;*
(2) *there exists a* $\kappa$*-tree* $\mathbf{T}$ *such that* $V^-(\mathbf{T})$ *covers a club in* $\kappa$*;*
(3) $\kappa$ *is not subtle.*

*In addition, in* $\mathsf{L}$*, for* $\kappa$ *not weakly compact,* $(3) \implies (1)$.

**Proof** $(1) \implies (2)$: This is immediate.

$(2) \implies (3)$: By Lemma 2.20.

Next, work in $\mathsf{L}$ and suppose that $\kappa$ is a regular uncountable cardinal that is not subtle and not weakly compact. If $\kappa$ is a successor cardinal, then by Corollary 3.1(1), Clause (1) holds, so assume that $\kappa$ is inaccessible. By GCH, $\kappa$ is moreover strongly inaccessible, and then Lemma 2.20 yields that Clause (3) holds. Since we work in $\mathsf{L}$ and $\kappa$ is not weakly compact, by [BR17a, Theorem 3.12], $\mathrm{P}(\kappa, 2, \sqsubseteq, 1)$ holds. So by Corollary 3.8(1), the hypothesis of Clause (3) yields a $\kappa$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T})$ covers a club in $\kappa$. Now, appeal to Lemma 2.15. ■

***Corollary 3.11*** *In* $\mathsf{L}$*, if* $\kappa$ *is not weakly compact, then for every stationary* $S \subseteq \kappa$*, there exists a* $\kappa$*-Souslin tree* $\mathbf{T}$ *for which* $V(\mathbf{T}) \cap S$ *is stationary.*

**Proof** By Corollary 3.1(1), we may assume that $\kappa$ is (strongly) inaccessible. By Corollary 2.25, we may fix a $\kappa$-tree $\mathbf{K}$ such that $V^-(\mathbf{K}) \cap S$ is stationary. By [BR17a, Theorem 3.12], $\mathrm{P}(\kappa, 2, \sqsubseteq, 1)$ holds. Finally, appeal to Corollary 3.8(1). ■

## 4 Realizing a nonreflecting stationary set

In this section, we provide conditions concerning a set $S \subseteq \kappa$ sufficient to ensure the existence of a $\kappa$-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) \supseteq S$ and possibly $V(\mathbf{T}) = S$. As a corollary, we obtain Theorem D.

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***Corollary 4.1*** *If $\diamondsuit(S)$ holds for some nonreflecting stationary subset $S$ of a strongly inaccessible cardinal $\kappa$, then there is an almost disjoint family $\mathcal{S}$ of $2^\kappa$ many stationary subsets of $S$ such that, for every $S' \in \mathcal{S}$, there is a $\kappa$-Souslin tree $\mathbf{T}$ with $V^-(\mathbf{T}) = V(\mathbf{T}) = S'$.*

**Proof** By Corollary 4.9 below, it suffices to prove that there exists a family $\mathcal{S}$ of $2^\kappa$ many stationary subsets of $S$ such that:

- for every $S' \in \mathcal{S}$, $\diamondsuit(S')$ holds;
- $|S' \cap S''| < \kappa$ for all $S' \neq S''$ from $\mathcal{S}$.

Now, as $\diamondsuit(S)$ holds, we may easily fix a sequence $\langle (A_\beta, B_\beta) \mid \beta \in S\rangle$ such that, for all $A, B \in \mathcal{P}(\kappa)$, the following set is stationary:

$$G_A(B) := \{\beta \in S \mid A \cap \beta = A_\beta \ \&\ B \cap \beta = B_\beta\}.$$

Set $\mathcal{S} := \{S_A \mid A \in \mathcal{P}(\kappa)\}$, where $S_A := \{\beta \in S \mid A \cap \beta = A_\beta\}$. Then, $\mathcal{S}$ is an almost disjoint family of $2^\kappa$ many stationary subsets of $S$, and for every $S' \in \mathcal{S}$, $\diamondsuit(S')$ holds, as witnessed by $\langle B_\beta \mid \beta \in S'\rangle$. ∎

***Definition 4.2*** [BR17a] A streamlined $\kappa$-tree $T \subseteq {}^{<\kappa}H_\kappa$ is *prolific* iff for all $\alpha < \kappa$ and $t \in T_\alpha$, $\{t^\frown\langle i\rangle \mid i < \max\{\omega, \alpha\}\} \subseteq T$.

A prolific tree is clearly splitting.

***Theorem 4.3*** *Suppose that $\mathrm{P}(\kappa, \kappa, {}^S\sqsubseteq, 1)$ holds for a given $S \subseteq \mathrm{acc}(\kappa)$. Then, there exists a normal, prolific, streamlined $\kappa$-Souslin tree $T$ such that $V(T) \supseteq S$.*

**Proof** Fix a well-ordering $\lhd$ of $H_\kappa$, a sequence $\langle \Omega_\beta \mid \beta < \kappa\rangle$ witnessing $\diamondsuit^-(H_\kappa)$, and a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ witnessing $\mathrm{P}^-(\kappa, \kappa, {}^S\sqsubseteq, 1)$. By ${}^S\sqsubseteq$-coherence, we may assume that for every $\alpha \in S$, $\mathcal{C}_\alpha$ is a singleton.

Following the proof of [BR19b, Proposition 2.2], we shall recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ such that $T := \bigcup_{\alpha<\kappa} T_\alpha$ will constitute a normal prolific streamlined $\kappa$-Souslin tree whose $\alpha^{\text{th}}$-level is $T_\alpha$.

Let $T_0 := \{\emptyset\}$, and for all $\alpha < \kappa$ let

$$T_{\alpha+1} := \{t^\frown\langle i\rangle \mid t \in T_\alpha, i < \max\{\omega, \alpha\}\}.$$

Next, suppose that $\alpha \in \mathrm{acc}(\kappa)$ is such that $T \restriction \alpha$ has already been defined. Constructing the level $T_\alpha$ involves deciding which $\alpha$-branches through $T \restriction \alpha$ will have their limits placed into our tree. For all $C \in \mathcal{C}_\alpha$ and $x \in T \restriction C$, we first define two elements $\mathbf{b}_x^{C,0}$ and $\mathbf{b}_x^{C,1}$ of $\mathcal{B}(T \restriction \alpha)$, ensuring that $\{\mathbf{b}_x^{C,0} \mid x \in T \restriction C\} \cap \{\mathbf{b}_x^{C,1} \mid x \in T \restriction C\} = \emptyset$, and then we shall let:

$$(\star) \qquad T_\alpha := \begin{cases} \{\mathbf{b}_x^{C,0} \mid C \in \mathcal{C}_\alpha, x \in T \restriction C\}, & \text{if } \alpha \in S;\\ \{\mathbf{b}_x^{C,0}, \mathbf{b}_x^{C,1} \mid C \in \mathcal{C}_\alpha, x \in T \restriction C\}, & \text{otherwise.}\end{cases}$$

For every $\alpha \in S$, since $|\mathcal{C}_\alpha| = 1$, this ensures that $\alpha \in V(T)$.

Let $C \in \mathcal{C}_\alpha$, $x \in T \restriction C$ and $i < 2$. $\mathbf{b}_x^{C,i}$ will be the limit $\bigcup \mathrm{Im}(b_x^{C,i})$ of a sequence $b_x^{C,i} \in \prod_{\beta \in C \setminus \mathrm{dom}(x)} T_\beta$ obtained by recursion, as follows. Set $b_x^{C,i}(\mathrm{dom}(x)) := x$. At successor step, for every $\beta \in C \setminus (\mathrm{dom}(x)+1)$ such that $b_x^{C,i}(\beta^-)$ has already been

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defined with $\beta^- := \sup(C\cap\beta)$, we consult the following set:

$$Q_x^{C,i}(\beta) := \{t\in T_\beta \mid \exists s\in\Omega_\beta[(s\cup(b_x^{C,i}(\beta^-)^\frown\langle i\rangle))\subseteq t]\}.$$

Now, consider the two possibilities:

- If $Q_x^{C,i}(\beta)\neq\emptyset$, then let $b_x^{C,i}(\beta)$ be its $\lhd$-least element;
- Otherwise, let $b_x^{C,i}(\beta)$ be the $\lhd$-least element of $T_\beta$ that extends $b_x^{C,i}(\beta^-)^\frown\langle i\rangle$. Such an element must exist, as the tree constructed so far is normal.

Finally, for every $\beta\in\operatorname{acc}(C\setminus\operatorname{dom}(x))$ such that $b_x^{C,i}\restriction\beta$ has already been defined, we let $b_x^{C,i}(\beta)=\bigcup\operatorname{Im}(b_x^C\restriction\beta)$. By $(\star)$, ${}^S\sqsubseteq$-coherence and the exact same proof of [BR19b, Claim 2.2.1], $b_x^{C,i}(\beta)$ is indeed in $T_\beta$.

***Claim 4.3.1*** *For every $C\in\mathcal{C}_\alpha$, $\{\mathbf{b}_x^{C,0}\mid x\in T\restriction C\}\cap\{\mathbf{b}_x^{C,1}\mid x\in T\restriction C\}=\emptyset$.* ∎

**Proof** Let $C\in\mathcal{C}_\alpha$ and $x,y\in T\restriction C$. Fix a large enough $\beta\in\operatorname{nacc}(C)$ for which $\beta^- := \sup(C\cap\beta)$ is bigger than $\max\{\operatorname{dom}(x),\operatorname{dom}(y)\}$. By the definitions of $b_x^{C,0}$ and $b_y^{C,1}$,

- $b_x^{C,0}(\beta)(\beta^-)=0$, and
- $b_y^{C,1}(\beta)(\beta^-)=1$.

In particular, $\mathbf{b}_x^{C,0}\neq\mathbf{b}_y^{C,1}$. ∎

This finishes the construction of $T_\alpha$. Finally, by [BR19b, Claims 2.2.2 and 2.2.3], $T := \bigcup_{\alpha<\kappa}T_\alpha$ is a $\kappa$-Souslin tree.

***Theorem 4.4*** *Suppose that $\chi$ is a cardinal such that $\lambda^{<\chi}<\kappa$ for all $\lambda<\kappa$, and that $\mathrm{P}(\kappa,\kappa,{}^S\sqsubseteq,1,\{S\cup E^\kappa_{\geq\chi}\})$ holds for a given $S\subseteq\operatorname{acc}(\kappa)\cap E^\kappa_{<\chi}$. Then, there exists a normal, prolific, streamlined $\kappa$-Souslin tree $T$ such that $V^-(T)\cap E^\kappa_{<\chi}=V(T)\cap E^\kappa_{<\chi}=S$.*

**Proof** The proof is almost identical to that of Theorem 4.3, where the only change is in that now, the definition of $T_\alpha$ for a limit $\alpha$ splits into three:

$$T_\alpha := \begin{cases}\{\mathbf{b}_x^{C,0} \mid C\in\mathcal{C}_\alpha, x\in T\restriction C\}, & \text{if } \alpha\in S;\\ \{\mathbf{b}_x^{C,0},\mathbf{b}_x^{C,1}\mid C\in\mathcal{C}_\alpha, x\in T\restriction C\}, & \text{if } \alpha\in E^\kappa_{\geq\chi};\\ \mathcal{B}(T\restriction\alpha), & \text{otherwise.}\end{cases}$$

The details are left to the reader. ∎

***Remark 4.5*** Sufficient conditions for the existence of $S\subseteq\kappa$ for which $\mathrm{P}(\kappa,\kappa,{}^S\sqsubseteq,1,\{S\})$ holds are given by [BR21, Corollary 4.22] and [BR21, Theorem 4.28]. In particular, for every (nonreflecting) stationary $E\subseteq\kappa$, if $\square(E)$ and $\diamondsuit(E)$ both hold, then there exists a stationary $S\subseteq E$ such that $\mathrm{P}(\kappa,\kappa,{}^S\sqsubseteq,1,\{S\})$ holds.

***Corollary 4.6*** *Suppose that $2^{2^{\aleph_0}}=\aleph_2$, and that $S$ is a nonreflecting stationary subset of $E^{\aleph_2}_{\aleph_0}$. Then, there exists a normal prolific streamlined $\aleph_2$-Souslin tree $T$ such that $V(T)=S\cup E^{\aleph_2}_{\aleph_1}$.*

**Proof** By [BR19c, Lemma 3.2], the hypotheses implies that $\mathrm{P}(\aleph_2,\aleph_2,{}^S\sqsubseteq,1,\{S\})$ holds. Appealing to Theorem 4.4 with $(\kappa,\chi):=(\aleph_2,\aleph_1)$ provides us with a normal, prolific, streamlined $\aleph_2$-Souslin tree $T$ such that $V^-(T)\cap E^{\aleph_2}_{\aleph_0}=V(T)\cap E^{\aleph_2}_{\aleph_0}=S$. As $V^-(T)\cap E^{\aleph_2}_{\aleph_0}$ is a nonreflecting stationary set, Lemma 2.10(1) (using $(\varsigma,\chi,\kappa):=(2,\aleph_1,\aleph_2)$) implies that $V(T)\cap E^{\aleph_2}_{\aleph_1}=E^{\aleph_2}_{\aleph_1}$. ∎

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***Corollary 4.7*** *Suppose* CH *and* $\boxtimes_{\aleph_1}$ *both hold.*[16] *For every stationary* $S \subseteq E^{\aleph_2}_{\aleph_0}$, *there exists an* $\aleph_2$*-Souslin tree* $\mathbf{T}$ *such that* $V(\mathbf{T})$ *is a stationary subset of* $S$.

**Proof** $\boxtimes_{\aleph_1}$ implies $\square_{\aleph_1}$ which implies that for every stationary $S \subseteq E^{\aleph_2}_{\aleph_0}$ there exists a stationary $R \subseteq S$ that is nonreflecting. It thus follows from Corollary 4.6 that for every stationary $S \subseteq E^{\aleph_2}_{\aleph_0}$, there exist a stationary $R \subseteq S$ and an $\aleph_2$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = R \cup E^{\aleph_2}_{\aleph_1}$. In addition, $\boxtimes_{\aleph_1}$ yields a uniformly coherent $\aleph_2$-Souslin tree $\mathbf{S}$ (see [Vel86, Theorem 7] or [BR17a, Proposition 2.5 and Theorem 3.6]). By [RS23, Remark 2.20], then, $V(\mathbf{S}) = E^{\aleph_2}_{\aleph_0}$. Clearly, $\mathbf{T} + \mathbf{S}$ is an $\aleph_2$-Souslin tree, and, by Proposition 2.32(2), $V(\mathbf{T} + \mathbf{S}) = R$. ∎

***Theorem 4.8*** *Suppose that* $\kappa$ *is a strongly inaccessible cardinal, and that* $\mathrm{P}(\kappa, \kappa, {}^{S}\sqsubseteq, 1, \{S\})$ *holds for a given* $S \subseteq \operatorname{acc}(\kappa)$. *Then, there exists a normal, prolific, streamlined* $\kappa$*-Souslin tree* $T$ *such that* $V^-(T) = V(T) = S$.

**Proof** The proof is almost identical to that of Theorem 4.3, where the only change is that now, the definition of $T_\alpha$ for a limit $\alpha$ does not explicitly mention the $\mathbf{b}^{C,1}_x$'s. Instead, it is:

$$T_\alpha := \begin{cases} \{\mathbf{b}^{C,0}_x \mid C \in \mathcal{C}_\alpha, x \in T \restriction C\}, & \text{if } \alpha \in S; \\ \mathcal{B}(T \restriction \alpha), & \text{otherwise.} \end{cases}$$

The details are left to the reader. ∎

***Corollary 4.9*** *Suppose that* $\kappa$ *is a strongly inaccessible cardinal, and* $S$ *is a nonreflecting stationary subset of* $\operatorname{acc}(\kappa)$ *on which* $\diamondsuit$ *holds. Then, there exists a normal prolific streamlined* $\kappa$*-Souslin tree* $T$ *such that* $V^-(T) = V(T) = S$.

**Proof** By Theorem 4.8 together with [BR21, Theorem 4.26]. ∎

## 5 Realizing all points of some fixed cofinality

In this section, we deal with the problem of constructing a $\kappa$-Souslin tree $\mathbf{T}$ for which $V(\mathbf{T})$ is equal to the finite nonempty union of sets of the form $E^\kappa_\chi$. The proof approach is motivated by Proposition 2.30, hence, we shall be constructing a finite sequence of $\kappa$-Souslin trees whose product is still $\kappa$-Souslin, and each of these trees satisfies that its set of vanishing levels is equal to $E^\kappa_\chi$ for one of the cardinals $\chi$ of interest. The main result of this section is Theorem 5.10 below. A sample corollary of it reads as follows.

***Corollary 5.1*** *In* $\mathsf{L}$, *for every regular uncountable cardinal* $\kappa$ *that is not weakly compact, for every finite nonempty* $x \subseteq \operatorname{Reg}(\kappa)$ *with* $\max(x) \leq \operatorname{cf}(\sup(\operatorname{Reg}(\kappa)))$, *there exists a streamlined uniformly homogeneous* $\kappa$*-Souslin tree* $T$ *such that* $V^-(T) = \bigcup_{\chi \in x} E^\kappa_\chi$.

**Proof** Work in $\mathsf{L}$. Let $\kappa$ be regular uncountable cardinal that is not weakly compact, and let $\langle \chi_i \mid i \leq n\rangle$ be the increasing enumeration of a set $x$ as in the hypothesis. By GCH, $\lambda^{<\chi_n} < \kappa$ for every $\lambda < \kappa$. By [BR17a, Theorem 3.6] and [BR19a, Corollary 4.14], $\mathrm{P}(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq \chi_n}\})$ holds. So, by Theorem 5.10 below, using $(\nu, \chi, \chi') := (\aleph_0, \chi_0, \chi_n)$ and $S := {}^{<\kappa}1$, we may pick a streamlined, normal, 2-splitting, uniformly homogeneous,

[16] The definition of square with built-in diamond may be found at [BR17a, Definition 8.16].

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$\chi_0$-complete, $\chi_0$-coherent, $E^\kappa_{\geq\chi_0}$-regressive $\kappa$-Souslin tree $T^0$. Furthermore, $T^0$ is $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq\chi_n}\})$-respecting.

***Claim 5.1.1*** $V^-(T^0) = E^\kappa_{\chi_0}$. ∎

**Proof** Since $T^0$ is $\chi_0$-complete, $V^-(T^0) \cap E^\kappa_{<\chi_0} = \varnothing$, so that $\mathrm{Tr}(\kappa \setminus V^-(T^0))$ covers $E^\kappa_{\geq\chi_0}$. By GCH, $2^{<\chi_0} < 2^{\chi_0}$. Together with the fact that $T^0$ is $E^\kappa_{\chi_0}$-regressive, it follows from Lemma 2.10(2) that $E^\kappa_{\chi_0} \subseteq V^-(T^0)$. Finally, since $T^0$ is $\chi_0$-coherent and uniformly homogeneous, we get from Lemma 5.3 below that $V^-(T^0) \cap E^\kappa_{>\chi_0} = \varnothing$. ∎

If $n = 0$, then our proof is complete. Otherwise, one can continue by recursion, where the successive step is as follows: Suppose that $i < n$ is such that $\bigotimes_{j\leq i} T^j$ is a streamlined uniformly homogeneous normal $\kappa$-Souslin tree that is $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq\chi_n}\})$-respecting, and that $V(\bigotimes_{j\leq i} T^j) = \bigcup_{j\leq i} E^\kappa_{\chi_j}$. By Theorem 5.10 below, using $(\nu, \chi, \chi') := (\aleph_0, \chi_{i+1}, \chi_n)$ and $S := \bigotimes_{j\leq i} T^j$, we may pick a streamlined, normal, 2-splitting, uniformly homogeneous, $\chi_{i+1}$-complete, $\chi_{i+1}$-coherent, $E^\kappa_{\geq\chi_{i+1}}$-regressive $\kappa$-Souslin tree $T^{i+1}$. Furthermore, $S \otimes T^{i+1}$ is a normal $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq\chi_n}\})$-respecting $\kappa$-Souslin tree. By an analysis similar to that of Claim 5.1.1, $V^-(T^{i+1}) = E^\kappa_{\chi_{i+1}}$. Altogether, $\bigotimes_{j\leq i+1} T^j$ is a uniformly homogeneous normal $\kappa$-Souslin tree that is $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq\chi_n}\})$-respecting. In addition, by Proposition 2.30(2), $V(\bigotimes_{j\leq i+1} T^j) = \bigcup_{j\leq i+1} E^\kappa_{\chi_j}$.

We start by giving a definition.

***Definition 5.2*** A streamlined $\kappa$-tree $T$ is *$\chi$-coherent* iff for all $s, t \in T$, $\{\xi \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\xi) \neq t(\xi)\}$ has size $< \chi$.

***Lemma 5.3*** *Suppose that $\chi < \kappa$ is a cardinal, and that $T$ is a streamlined, $\chi$-coherent uniformly homogeneous $\kappa$-tree. Then, $V^-(T) \subseteq E^\kappa_{\leq\chi}$.*

**Proof** Let $\alpha \in E^\kappa_{>\chi}$. Suppose that $B \subseteq T$ is an $\alpha$-branch, and we shall show it is not vanishing.

For every $\beta < \alpha$, let $t_\beta$ denote the unique element of $T_\beta \cap B$. Fix a node $t \in T_\alpha$. For every $\beta \in E^\alpha_\chi$, by $\chi$-coherence, the following ordinal is smaller than $\beta$:

$$\varepsilon_\beta := \sup\{\xi < \beta \mid t_\beta(\xi) \neq t(\xi)\}.$$

As $\mathrm{cf}(\alpha) > \chi$, $E^\alpha_\chi$ is a stationary subset of $\alpha$, so we may fix a large enough $\varepsilon < \alpha$ for which $R := \{\beta \in E^\alpha_\chi \mid \varepsilon_\beta < \varepsilon\}$ is stationary. As $T$ is uniformly homogeneous, $t_\varepsilon * t$ is in $T_\alpha$. For every $\beta \in R$, $t_\beta = (t_\varepsilon * t) \restriction \beta$. But since $R$ is cofinal in $\alpha$, it is the case that $t_\varepsilon * t$ constitutes a limit for $B$. Therefore, $B$ is not vanishing. ∎

In the context of streamlined $\kappa$-trees, there is a neater way of presenting the operation of product (compare with Definition 2.28).

***Definition 5.4*** [BR19c, Section 4] For every function $x : \alpha \to {}^\tau H_\kappa$ and every $i < \tau$, we let $(x)_i : \alpha \to H_\kappa$ be $\langle x(\beta)(i) \mid \beta < \alpha\rangle$. Using this notation, for every sequence $\langle T^i \mid i < \tau\rangle$ of streamlined $\kappa$-trees, one may identify $\bigotimes_{i<\tau} T^i$ with the collection $T := \{x \in {}^{<\kappa}({}^\tau H_\kappa) \mid \forall i < \tau\,[(x)_i \in T^i]\}$, which is a streamlined $\kappa$-tree provided that $\lambda^\tau < \kappa$ for all $\lambda < \kappa$.

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***Remark 5.5*** The product of two uniformly homogeneous $\kappa$-trees is uniformly homogeneous.

Before we can state the main result of this section, we need one more definition.

***Definition 5.6*** [BR17b] A streamlined $\kappa$-tree $X$ is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\theta,\mathcal{S})$-*respecting* iff there exists a subset $\S\subseteq\kappa$ and a sequence of mappings $\langle d^C:(X\restriction C)\to{}^\alpha H_\kappa\cup\{\emptyset\}\mid \alpha<\kappa, C\in\mathcal{C}_\alpha\rangle$ such that:

(1) for all $\alpha\in\S$ and $C\in\mathcal{C}_\alpha$, $X_\alpha\subseteq\mathrm{Im}(d^C)$;
(2) $\vec{\mathcal{C}}=\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$ witnesses $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\theta,\{S\cap\S\mid S\in\mathcal{S}\})$;
(3) for all sets $D\sqsubseteq C$ from $\vec{\mathcal{C}}$ and $x\in X\restriction D$, $d^D(x)=d^C(x)\restriction\sup(D)$.

***Remark 5.7***

(1) If $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\theta,\mathcal{S})$ holds, then the normal streamlined $\kappa$-tree $X:={}^{<\kappa}1$ is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\theta,\mathcal{S})$-respecting;
(2) If $\kappa=\lambda^+$ for an infinite regular cardinal $\lambda$, and $\mathrm{P}^-_\lambda(\kappa,\mu,{}_\lambda\sqsubseteq,\theta,\{E^\kappa_\lambda\})$ holds, then every $\kappa$-tree is $\mathrm{P}^-_\lambda(\kappa,\mu,{}_\lambda\sqsubseteq,\theta,\{E^\kappa_\lambda\})$-respecting.

***Lemma 5.8*** *Suppose that:*

- *$X$ is a streamlined $\kappa$-tree that is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\mathcal{S})$-respecting, as witnessed by some $\vec{\mathcal{C}}$ and $\S$;*
- *$Y$ is a streamlined $\kappa$-tree that is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\{S\cap\S\mid S\in\mathcal{S}\})$-respecting, as witnessed by the same $\vec{\mathcal{C}}$.*

*Then, the product $X\otimes Y$ is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\mathcal{S})$-respecting.*

**Proof** In view of Definition 5.4, for every two functions $x,y$ from an ordinal $\alpha<\kappa$ to $H_\kappa$, we denote by $\ulcorner(x,y)\urcorner$ the unique function $p:\alpha\to{}^2H_\kappa$ such that $(p)_0=x$ and $(p)_1=y$. Note that $X\otimes Y=\bigcup_{\alpha<\kappa}\{\ulcorner(x,y)\urcorner\mid(x,y)\in X_\alpha\times Y_\alpha\}$.

Write $\vec{\mathcal{C}}$ as $\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$. Fix a sequence of mappings $\langle d^C:(X\restriction C)\to{}^\alpha H_\kappa\cup\{\emptyset\}\mid \alpha<\kappa, C\in\mathcal{C}_\alpha\rangle$ such that:

(1) for all $\alpha\in\S$ and $C\in\mathcal{C}_\alpha$, $X_\alpha\subseteq\mathrm{Im}(d^C)$;
(2) $\vec{\mathcal{C}}=\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$ witnesses $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\{S\cap\S\mid S\in\mathcal{S}\})$;
(3) for all sets $D\sqsubseteq C$ from $\vec{\mathcal{C}}$ and $x\in X\restriction D$, $d^D(x)=d^C(x)\restriction\sup(D)$.

Fix a stationary $\S'\subseteq\S$ and a sequence of mappings $\langle e^C:(Y\restriction C)\to{}^\alpha H_\kappa\cup\{\emptyset\}\mid \alpha<\kappa, C\in\mathcal{C}_\alpha\rangle$ such that:

(4) for all $\alpha\in\S'$ and $C\in\mathcal{C}_\alpha$, $Y_\alpha\subseteq\mathrm{Im}(e^C)$;
(5) $\vec{\mathcal{C}}=\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$ witnesses $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\{S\cap\S'\mid S\in\mathcal{S}\})$;
(6) for all sets $D\sqsubseteq C$ from $\vec{\mathcal{C}}$ and $y\in Y\restriction D$, $e^D(y)=e^C(y)\restriction\sup(D)$.

Let $\vec{B}=\langle B_{x,y}\mid(x,y)\in X\times Y\rangle$ be a partition of $\kappa$ into cofinal subsets of $\kappa$. Define a sequence of mappings $\langle b^C:(X\otimes Y)\restriction C\to{}^\alpha H_\kappa\cup\{\emptyset\}\mid\alpha<\kappa, C\in\mathcal{C}_\alpha\rangle$, as follows. Let $\alpha<\kappa$ and $C\in\mathcal{C}_\alpha$.

▸ For every $\beta\in C$, if there are $x\in X\restriction(C\cap\beta)$ and $y\in Y\restriction(C\cap\beta)$ such that $\beta\in B_{x,y}$, then since $\vec{B}$ is a sequence of pairwise disjoint sets, this pair $(x,y)$ is unique, and we let $b^C(p):=\ulcorner(d^C(x),e^C(y))\urcorner$ for every $p\in(X\otimes Y)_\beta$.

▸ For every $\beta\in C$ for which there is no such pair $(x,y)$, we let $b^C(p):=\emptyset$ for every $p\in(X\otimes Y)_\beta$.

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***Claim 5.8.1*** *Suppose $D \sqsubseteq C$ are sets from $\vec{\mathcal{C}}$. For every $p \in (X \otimes Y) \restriction D$, $b^D(p) = b^C(p) \restriction \sup(D)$.* ■

**Proof** Given $p \in (X \otimes Y) \restriction D$. Denote $\beta := \mathrm{dom}(p)$. Note that $D \cap \beta = C \cap \beta$. Now, there are two options:

▸ There are $x \in X \restriction (C \cap \beta)$ and $y \in Y \restriction (C \cap \beta)$ such that $\beta \in B_{x,y}$. Then, $b^D(p) = \ulcorner (d^D(x), e^D(y)) \urcorner$ and $b^C(p) = \ulcorner (d^C(x), e^C(y)) \urcorner$. Since $D \sqsubseteq C$, we know that $d^D(x) = d^C(x) \restriction \sup(D)$ and $e^D(y) = e^C(y) \restriction \sup(D)$. Therefore, $b^D(p) = d^C(p) \restriction \sup(D)$.

▸ There are no such $x$ and $y$. Then, $b^D(p) = \varnothing = d^C(p)$. ■

Consider the following set:

$$\S'' := \{\alpha \in \S' \mid \forall C \in \mathcal{C}_\alpha \forall x \in (X \restriction \alpha) \forall y \in (Y \restriction \alpha) \, [\sup(\mathrm{nacc}(C) \cap B_{x,y}) = \alpha]\}.$$

***Claim 5.8.2*** $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ *witnesses* $\mathrm{P}^-_\xi(\kappa, \mu, \mathcal{R}, \kappa, \{S \cap \S'' \mid S \in \mathcal{S}\})$.

**Proof** Let $\langle B_i \mid i < \kappa \rangle$ be a given sequence of cofinal subsets of $\kappa$. Let $\pi : \kappa \leftrightarrow \kappa \uplus (X \times Y)$ be a surjection. As $X$ and $Y$ are $\kappa$-trees, the set $D := \{\alpha < \kappa \mid \pi[\alpha] = \alpha \uplus ((X \restriction \alpha) \times (Y \restriction \alpha))\}$ is a club in $\kappa$. By Clause (5), then, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S \cap \S' \cap D$ such that for all $C \in \mathcal{C}_\alpha$ and $i < \alpha$, $\sup(\mathrm{nacc}(C) \cap B_{\pi(i)}) = \alpha$. In particular, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S \cap \S''$ such that for all $C \in \mathcal{C}_\alpha$ and $i < \alpha$, $\sup(\mathrm{nacc}(C) \cap B_i) = \alpha$. ■

***Claim 5.8.3*** *Let $\alpha \in \S''$ and $C \in \mathcal{C}_\alpha$. Then, $(X \otimes Y)_\alpha \subseteq \mathrm{Im}(b^C)$.*

**Proof** Let $(s, t) \in X_\alpha \times Y_\alpha$. As $\S'' \subseteq \S' \subseteq \S$, using Clauses (1) and (4), we may fix $x \in X \restriction C$ and $y \in Y \restriction C$ such that $d^C(x) = s$ and $e^C(y) = t$. As $\alpha \in \S''$, we may pick $\beta \in \mathrm{nacc}(C) \cap B_{x,y}$ above $\max\{\mathrm{dom}(x), \mathrm{dom}(y)\}$. Let $p$ be an arbitrary element of $(X \otimes Y)_\beta$. Then, $b^C(p) := \ulcorner (d^C(x), e^C(y)) \urcorner = \ulcorner (s, t) \urcorner$. ■

This completes the proof.

***Remark 5.9*** The preceding proof highlights a feature of respecting trees of independent interest, namely, for a streamlined $\mathrm{P}^-_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S})$-respecting $\kappa$-tree $X$, in case of $\theta = \kappa$, one may assume in Definition 5.6 that $d^C(x)$ depends only on $\mathrm{dom}(x)$ (and $C$), and Clause (1) may be strengthened to assert that for all $\alpha \in \S$, $C \in \mathcal{C}_\alpha$, and $x \in X_\alpha$, there are cofinally many $\beta \in \mathrm{nacc}(C)$ such that $d^C(x \restriction \beta) = x$.

***Theorem 5.10*** *Suppose that:*

- *$\varsigma < \kappa$ is a cardinal;*
- *$\nu \leq \chi \leq \chi' < \kappa$ are cardinals such that $\lambda^{<\chi} < \kappa$ for every $\lambda < \kappa$;*
- *$S$ is a $\mathrm{P}^-(\kappa, 2, {}_\nu{\sqsubseteq}, \kappa, \{E^\kappa_{\geq \chi'}\})$-respecting streamlined normal $\kappa$-tree with no $\kappa$-sized antichains;*
- *$\diamondsuit(\kappa)$ holds.*

*Then, there exists a streamlined, normal, $\varsigma$-splitting, prolific, uniformly homogeneous, $\chi$-complete, $\chi$-coherent, $E^\kappa_{\geq \chi}$-regressive $\kappa$-Souslin tree $T$ such that $S \otimes T$ is a normal $\mathrm{P}^-(\kappa, 2, {}_\nu{\sqsubseteq}, \kappa, \{E^\kappa_{\geq \chi'}\})$-respecting $\kappa$-Souslin tree.*

**Proof** Fix a stationary $\S \subseteq E^\kappa_{\geq \chi'}$ and a sequence $\langle d^\alpha : S \restriction C_\alpha \to {}^\alpha H_\kappa \cup \{\varnothing\} \mid \alpha < \kappa \rangle$ such that:

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(1) for every $\alpha \in \S$, $S_\alpha \subseteq \mathrm{Im}(d^\alpha)$;
(2) $\vec{C} := \langle C_\alpha \mid \alpha < \kappa \rangle$ witnesses $\mathrm{P}^-(\kappa, 2, {}_{\nu}{\sqsubseteq}, \kappa, \{\S\})$;
(3) for all $\alpha < \beta < \kappa$, if $C_\alpha \sqsubseteq C_\beta$, then $d^\alpha(x) = d^\beta(x) \restriction \alpha$ for every $x \in S \restriction C_\alpha$.

***Claim 5.10.1*** *We may assume that $C_{\alpha+1} = \{\alpha\}$ for every $\alpha < \kappa$ and that $\min(C_\alpha) = 0$ for every $\alpha \in \mathrm{acc}(\kappa)$.* ∎

**Proof** Consider the $C$-sequence $\vec{C^\bullet} = \langle C^\bullet_\alpha \mid \alpha < \kappa \rangle$ defined by letting $C^\bullet_0 := \emptyset$, $C^\bullet_{\alpha+1} := \{\alpha\}$ for every $\alpha < \kappa$, and

$$C^\bullet_\alpha := \{0\} \cup \{n + 1 \mid n \in C_\alpha \cap \omega\} \cup (C_\alpha \setminus \omega),$$

for every $\alpha \in \mathrm{acc}(\kappa)$,

It is clear that $\mathrm{acc}(C^\bullet_\alpha) = \mathrm{acc}(C_\alpha)$ for every $\alpha \in \mathrm{acc}(\kappa)$, and that for all nonzero $\beta < \alpha < \kappa$, $C^\bullet_\beta \sqsubseteq C^\bullet_\alpha$ iff ($\beta = 1$ and $\alpha \in \mathrm{acc}(\kappa)$) or ($\beta, \alpha \in \mathrm{acc}(\kappa)$ and $C_\beta \sqsubseteq C_\alpha$). Consequently, $\vec{C^\bullet}$ witnesses $\mathrm{P}^-(\kappa, 2, {}_{\nu}{\sqsubseteq}, \kappa, \{\S\})$. Next, for every $\alpha \in \mathrm{nacc}(\kappa)$, let $b^\alpha : S \restriction C^\bullet_\alpha \to \{\emptyset\}$ be a constant map. Then, for every $\alpha \in \mathrm{acc}(\kappa)$, define $b^\alpha : S \restriction C^\bullet_\alpha \to {}^\alpha H_\kappa \cup \{\emptyset\}$, as follows. Given $x \in S \restriction C^\bullet_\alpha$:

- If $x = \emptyset$, then let $b^\alpha(x) := \emptyset$;
- If $\mathrm{dom}(x) = n + 1$ for some $n < \omega$, then let $b^\alpha(x) := d^\alpha(x \restriction n)$;
- If $\mathrm{dom}(x) \geq \omega$, then let $b^\alpha(x) := d^\alpha(x)$.

As $S$ is normal, $\mathrm{Im}(b^\alpha) \supseteq \mathrm{Im}(d^\alpha)$ for every $\alpha \in \mathrm{acc}(\kappa)$. Finally, for all $\beta < \alpha < \kappa$, if $C^\bullet_\beta \sqsubseteq C^\bullet_\alpha$ and there exists $x \in S \restriction C^\bullet_\beta$ that is nonempty, then $C_\beta \sqsubseteq C_\alpha$ and $d^\beta(x) = d^\alpha(x) \restriction \beta$ for every $x \in S \restriction C_\beta$, from which it follows that $b^\beta(x) = b^\alpha(x) \restriction \beta$ for every $x \in S \restriction C^\bullet_\beta$. ∎

The upcoming construction follows the proof of [BR17a, Proposition 2.5]. Let $\langle R_i \mid i < \kappa \rangle$ and $\langle \Omega_\beta \mid \beta < \kappa \rangle$ together witness $\diamondsuit(H_\kappa)$. Let $\pi : \kappa \to \kappa$ be such that $\alpha \in R_{\pi(\alpha)}$ for all $\alpha < \kappa$. From $\diamondsuit(\kappa)$, we have $|H_\kappa| = \kappa$, thus let $\lhd$ be some well-ordering of $H_\kappa$ of order-type $\kappa$, and let $\phi : \kappa \leftrightarrow H_\kappa$ witness the isomorphism $(\kappa, \in) \cong (H_\kappa, \lhd)$. Put $\psi := \phi \circ \pi$.

We now recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa \rangle$ of levels whose union will ultimately be the desired tree $T$. Let $T_0 := \{\emptyset\}$, and for all $\alpha < \kappa$, let

$$T_{\alpha+1} := \{t^\frown \langle i \rangle \mid t \in T_\alpha, i < \max\{\varsigma, \omega, \alpha\}\}.$$

Next, suppose that $\alpha \in \mathrm{acc}(\kappa)$, and that $\langle T_\beta \mid \beta < \alpha \rangle$ has already been defined. We shall identify some $\mathbf{b}^\alpha \in \mathcal{B}(T \restriction \alpha)$, and then define the $\alpha^{\text{th}}$-level, as follows:

$$(\star) \qquad T_\alpha := \begin{cases} \mathcal{B}(T \restriction \alpha), & \text{if } \alpha \in E^\kappa_{<\chi}; \\ \{x * \mathbf{b}^\alpha \mid x \in T \restriction \alpha\}, & \text{if } \alpha \in E^\kappa_{\geq\chi}. \end{cases}$$

We shall obtain $\mathbf{b}^\alpha$ as a limit $\bigcup \mathrm{Im}(b^\alpha)$ of a sequence $b^\alpha \in \prod_{\beta \in C_\alpha} T_\beta$ that we define recursively, as follows. Let $b^\alpha(0) := \emptyset$. Next, suppose $\beta^- < \beta$ are two successive points of $C_\alpha$, and that $b^\alpha(\beta^-)$ has already been defined. There are two possible options:

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▸ If $\psi(\beta)$ happens to be a pair $(y,x)$ lying in $(S \restriction \beta^-) \times (T \restriction \beta^-)$, and the following set happens to be nonempty:

$$Q^{\alpha,\beta} := \{t \in T_\beta \mid \exists(\bar{s},\bar{t}) \in \Omega_\beta\,[\bar{s} \subseteq d^\alpha(y) \restriction \beta \;\&\; (\bar{t} \cup (x * b^\alpha(\beta^-))) \subseteq t]\},$$

then let $t$ denote its $\lhd$-least element, and put $b^\alpha(\beta) := b^\alpha(\beta^-) * t$.

▸ Otherwise, let $b^\alpha(\beta)$ be the $\lhd$-least element of $T_\beta$ that extends $b^\alpha(\beta^-)$.

As always, for all $\beta \in \operatorname{acc}(C_\alpha)$ such that $b^\alpha \restriction \beta$ has already been defined, we let $b^\alpha(\beta) := \bigcup \operatorname{Im}(b^\alpha \restriction \beta)$ and infer that it belongs to $T_\beta$. Indeed, either $\operatorname{cf}(\beta) < \chi$, and then $b^\alpha(\beta) \in \mathcal{B}(T \restriction \beta) = T_\beta$, or $\operatorname{cf}(\beta) \geq \chi \geq \nu$, and then $C_\beta = C_\alpha \cap \beta$ from which it follows that $b^\alpha(\beta) = \mathbf{b}^\beta \in T_\beta$. This completes the definition of $b^\alpha$, hence also that of $\mathbf{b}^\alpha$. Finally, let $T_\alpha$ be defined as promised in $(\star)$.

It is clear that $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a streamlined, normal, $\varsigma$-splitting, prolific, uniformly homogeneous, $\chi$-complete $\kappa$-tree.

***Claim 5.10.2*** *$T$ is $\chi$-coherent.*

**Proof** Suppose not, and let $\alpha$ be the least ordinal to accommodate $s,t \in T_\alpha$ such that $s$ differs from $t$ on a set of size $\geq \chi$. Clearly, $\alpha \in E^\kappa_{\geq\chi}$. So $s = x * \mathbf{b}^\alpha$ and $t = y * \mathbf{b}^\alpha$ for nodes $x,y \in T \restriction \alpha$, and hence $x$ and $y$ differ on a set of size $\geq \chi$, contradicting the minimality of $\alpha$. ∎

***Claim 5.10.3*** *$T$ is $E^\kappa_{\geq\chi}$-regressive.*

**Proof** To define $\rho : T \restriction E^\kappa_{\geq\chi} \to T$, let $\alpha \in E^\kappa_{\geq\chi}$. By the definition of $T_\alpha$, for every $t \in T_\alpha$, there exists some $x \in T \restriction \alpha$ such that $t = x * \mathbf{b}^\alpha$, so we let $\rho(t)$ be an element of $T \restriction \alpha$ such that $t = \rho(t) * \mathbf{b}^\alpha$. Now, if $s,t \in T_\alpha$ are such that $\rho(t) \subseteq s$ and $\rho(s) \subseteq t$, then $\rho(t) \subseteq \rho(s) * \mathbf{b}^\alpha$ and $\rho(s) \subseteq \rho(t) * \mathbf{b}^\alpha$. In particular, $\rho(s)$ is compatible with $\rho(t)$. Without loss of generality, $\rho(s) \subseteq \rho(t)$. Then, $t = \rho(s) * \mathbf{b}^\alpha = s$. ∎

***Claim 5.10.4*** *$T$ is $\mathrm{P}^-(\kappa, 2, {}_\nu{\sqsubseteq}, \kappa, \{\S\})$-respecting, as witnessed by $\vec{C}$.*

**Proof** Define $\langle e^\alpha : T \restriction C_\alpha \to T_\alpha \mid \alpha < \kappa\rangle$ via:

$$e^\alpha(x) := \begin{cases} x * \mathbf{b}^\alpha, & \text{if } x \neq \emptyset \text{ and } \alpha \in \operatorname{acc}(\kappa);\\ \emptyset, & \text{otherwise.}\end{cases}$$

Let $\alpha \in \S$ and we shall show that $T_\alpha \subseteq \operatorname{Im}(e^\alpha)$. To this end, let $y \in T_\alpha$. As $\S \subseteq E^\kappa_{\geq\chi'} \subseteq E^\kappa_{\geq\chi}$, we get from $(\star)$ the existence of some $x \in T \restriction \alpha$ such that $y = x * \mathbf{b}^\alpha$. By possibly extending $x$, we may assume that $x = y \restriction \beta$ for some nonzero $\beta \in C_\alpha$. Consequently, $e^\alpha(x) = y$.

By Claim 5.10.1, for all $\beta < \alpha < \kappa$ such that $C_\beta \sqsubseteq C_\alpha$, it is the case that $\beta \in \{0,1\} \cup \operatorname{acc}(\kappa)$ and $\alpha \in \operatorname{acc}(\kappa)$. If $\beta = 1$, then $e^\beta(x) = \emptyset = e^\alpha(x)$ for every $x \in T \restriction C_\beta$, and we are done. Otherwise, $\beta \in \operatorname{acc}(C_\alpha)$ hence $\mathbf{b}^\beta \subseteq \mathbf{b}^\alpha$ from which it follows that $e^\beta(x) = e^\alpha(x) \restriction \beta$ for every $x \in T \restriction C_\beta$. ∎

It thus follows from Lemma 5.8 that $S \otimes T$ is $\mathrm{P}^-(\kappa, 2, {}_\nu{\sqsubseteq}, \kappa, \{E^\kappa_{\geq\chi'}\})$-respecting. It is clear that $S \otimes T$ is normal, thus we are left with verifying that it is Souslin. To this end, let $A$ be a maximal antichain in $S \otimes T$. As both $S$ and $T$ are normal, it follows that for every $z \in T$, the following (upward-closed) set is cofinal in $S$:

$$D_z := \{s \in S \mid \exists(\bar{s},\bar{t}) \in A\,\exists t \in T \cap z^\uparrow\,[\operatorname{dom}(s) = \operatorname{dom}(t), \bar{s} \subseteq s, \bar{t} \subseteq t]\}.$$

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As an application of $\diamondsuit(H_\kappa)$, using the parameter $p := \{\phi, S, T, A, \langle D_z \mid z \in T\rangle\}$, we get that for every $i < \kappa$, the following set is cofinal (in fact, stationary) in $\kappa$:

$$B_i := \{\beta \in R_i \mid \exists \mathcal{M} \prec H_{\kappa^+} (p \in \mathcal{M}, \mathcal{M} \cap \kappa = \beta, \Omega_\beta = A \cap \mathcal{M})\}.$$

Note that $(S \upharpoonright \beta) \times (T \upharpoonright \beta) \subseteq \phi[\beta]$ for every $\beta \in \bigcup_{i<\kappa} B_i$. Now, as $\vec{C}$ witnesses $\mathrm{P}^-(\kappa, 2, {}_\nu\sqsubseteq, \kappa, \{\S\})$, we may fix some $\alpha \in \S$ such that, for all $i < \alpha$,

$$\sup(\mathrm{nacc}(C_\alpha) \cap B_i) = \alpha.$$

In particular, $(S \upharpoonright \alpha) \times (T \upharpoonright \alpha) \subseteq \phi[\alpha]$. As $\alpha \in \S$, we also know that $S_\alpha \subseteq \mathrm{Im}(d^\alpha)$ and that $\mathrm{cf}(\alpha) \geq \chi$.

***Claim 5.10.5*** *$A \subseteq (S \otimes T) \upharpoonright \alpha$. In particular, $|A| < \kappa$.*

**Proof** As $A$ is an antichain, it suffices to prove that every element of $(S \otimes T)_\alpha$ extends some element of $A$. To this end, fix $(s', t') \in (S \otimes T)_\alpha$. Since $S_\alpha \subseteq \mathrm{Im}(d^\alpha)$, we may fix a $y \in S \upharpoonright C_\alpha$ such that $d^\alpha(y) = s'$. Recalling $(\star)$, we may also fix some $x \in T \upharpoonright C_\alpha$ such that $t' = x * \mathbf{b}^\alpha$.

As the pair $(y, x)$ is an element of $(S \upharpoonright \alpha) \times (T \upharpoonright \alpha)$, we may find an $i < \alpha$ such that $\phi(i) = (y, x)$, and then find a $\beta \in \mathrm{nacc}(C_\alpha) \cap B_i$ such that $\beta^- := \sup(C_\alpha \cap \beta)$ is greater than $\max\{\mathrm{dom}(y), \mathrm{dom}(x)\}$. Note that $\psi(\beta) = \phi(\pi(\beta)) = \phi(i) = (y, x)$.

***Subclaim 5.10.5.1*** *$\Omega_\beta = A \cap ((S \otimes T) \upharpoonright \beta)$, and $Q^{\alpha,\beta} \neq \emptyset$.* ∎

**Proof** As $\beta \in B_i$, we may fix $\mathcal{M} \prec H_{\kappa^+}$ such that all of the following hold:

- $\{\phi, S, T, A, \langle D_z \mid z \in T\rangle\} \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa = \beta$;
- $\Omega_\beta = A \cap \mathcal{M}$.

By elementarity, $(S \otimes T) \cap \mathcal{M} = (S \otimes T) \upharpoonright \beta$, and $\Omega_\beta = A \cap \mathcal{M} = A \cap ((S \otimes T) \upharpoonright \beta)$. Then, $z := t' \upharpoonright \beta^-$ is in $\mathcal{M}$, and hence, so is $D_z$. Pick in $\mathcal{M}$ a maximal antichain $\bar{D}$ in $D_z$. Since $D_z$ is cofinal in $S$, $\bar{D}$ is a maximal antichain in $S$. Since $S$ has no $\kappa$-sized antichains, we may find a large enough $\gamma \in \mathcal{M} \cap \kappa$ such that $\bar{D} \subseteq S \upharpoonright \gamma$. It thus follows that $s' \upharpoonright \gamma$ extends an element of $\bar{D}$, but since $D_z$ is upward-closed, $s := s' \upharpoonright \gamma$ is in $D_z$. It follows that we may fix $(\bar{s}, \bar{t}) \in A$ and $t \in T_\gamma \cap z^\uparrow$ such that $\bar{s} \subseteq s$ and $\bar{t} \subseteq t$. As $\Omega_\beta = A \cap ((S \otimes T) \upharpoonright \beta)$, $(d^\alpha(y) \upharpoonright \beta) \upharpoonright \gamma = s$ and $x * b^\alpha(\beta^-) = z \subseteq t$, we infer that any element of $T_\beta$ extending $t$ is in $Q^{\alpha,\beta}$. ∎

It follows that $b^\alpha(\beta) = b^\alpha(\beta^-) * t$ for some $t \in Q^{\alpha,\beta}$. This means that we may pick $(\bar{s}, \bar{t}) \in \Omega_\beta \subseteq A$ such that $\bar{s} \subseteq s' \upharpoonright \beta$ and $\bar{t} \cup (x * b^\alpha(\beta^-)) \subseteq t$. Therefore, $\bar{t} \subseteq x * b^\alpha(\beta)$. Altogether, $(\bar{s}, \bar{t}) \in A$, $\bar{s} \subseteq s'$ and $\bar{t} \subseteq t'$.

This completes the proof.

We now arrive at the proof of Theorem A.

***Theorem 5.11*** *We have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3):*

(1) *there exists a streamlined $\kappa$-Souslin tree $T$ such that $V(T) = \emptyset$;*
(2) *there exists a normal and splitting $\kappa$-tree $\mathbf{T}$ such that $V(\mathbf{T})$ is nonstationary;*
(3) *$\kappa$ is not the successor of a cardinal of countable cofinality.*

*In addition, in $\mathsf{L}$, for $\kappa$ not weakly compact, (3) $\Longrightarrow$ (1).*

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**Proof** (1) $\Longrightarrow$ (2): If $\mathbf{T} = (T, <_T)$ is a $\kappa$-Souslin tree, then a standard argument (see [BR17b, Lemma 2.4]) shows that for some club $D \subseteq \kappa$, $\mathbf{T}' = (T \restriction D, <_T)$ is normal and splitting. Clearly, if $V(\mathbf{T}) = \emptyset$, then $V(\mathbf{T}') = \emptyset$, as well.

(2) $\Longrightarrow$ (3): Suppose that $\mathbf{T}$ is a normal and splitting $\kappa$-tree. If $\kappa$ is the successor of a cardinal of countable cofinality then by Corollary 2.12, $V(\mathbf{T})$ covers the stationary set $E^\kappa_\omega$.

Hereafter, work in $\mathsf{L}$, and suppose that $\kappa$ is a regular uncountable cardinal that is not weakly compact and not the successor of a cardinal of countable cofinality. Then, by Corollary 5.1 together with Proposition 2.6(2), there are streamlined $\kappa$-Souslin trees $T^0, T^1$ such that $V(T^0) = E^\kappa_\omega$ and $V(T^1) = E^\kappa_{\omega_1}$. The disjoint sum of the two $T := \sum\{T^0, T^1\}$ is clearly $\kappa$-Souslin. In addition, by Proposition 2.32(2), $V(T) = V(T^0) \cap V(T^1) = \emptyset$. ■

***Example 5.12*** A $\kappa$-tree $\mathbf{T} = (T, <_T)$ is *full* iff for every $\alpha \in \mathrm{acc}(\kappa)$, there is no more than one vanishing $\alpha$-branch in $\mathbf{T}$. Such a tree $\mathbf{T}$ must satisfy $V(\mathbf{T}) = \emptyset$, since for $\alpha \in V(\mathbf{T})$, it must be the case that $\mathbf{T}$ admits exactly one vanishing $\alpha$-branch and that the said branch contains all elements of $T \restriction \alpha$ which means that $T \restriction \alpha$ itself is the said vanishing $\alpha$-branch, so that $T_\alpha$ is empty. It thus follows from [RYY24, Theorem C and Proposition 2.6] that there consistently exists a family of $2^{\aleph_2}$-many $\aleph_2$-Souslin trees $\mathbf{T}$ with $V(\mathbf{T}) = \emptyset$ such that no two of them are club-isomorphic.

We conclude this section by pointing out that by using [BR17a, Theorem 3.6] and a proof similar to that of Theorem 5.11, we get more information on the model studied in Corollary 4.7.

***Corollary 5.13*** *Suppose that* CH *and* $\boxtimes_{\aleph_1}$ *both hold. Then, there are* $\aleph_2$*-Souslin trees* $\mathbf{T}^0, \mathbf{T}^1, \mathbf{T}^2, \mathbf{T}^3$ *such that:*

- $V(\mathbf{T}^0) = \emptyset$;
- $V(\mathbf{T}^1) = E^{\aleph_2}_{\aleph_0}$;
- $V(\mathbf{T}^2) = E^{\aleph_2}_{\aleph_1}$;
- $V(\mathbf{T}^3) = \mathrm{acc}(\aleph_2)$.

## 6 Souslin trees with an ascent path

The subject matter of this section is the following definition.

***Definition 6.1*** (Laver) Suppose that $\mathbf{T} = (T, <_T)$ is a $\kappa$-tree. A *$\mu$-ascent path* through $\mathbf{T}$ is a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \kappa\rangle$ such that:

- for every $\alpha < \kappa$, $f_\alpha : \mu \to T_\alpha$ is a function;
- for all $\alpha < \beta < \kappa$, there is an $i < \mu$ such that $f_\alpha(j) <_T f_\beta(j)$ whenever $i \le j < \mu$.

We will show that Souslin trees having a large set of vanishing levels are compatible with carrying an ascent path. For this, we shall make use of the following strengthening of $\mathrm{P}^-_\xi(\kappa, \mu^+, \sqsubseteq, \theta, \mathcal{S})$.

***Definition 6.2*** [BR21, Section 4.6] The principle $\mathrm{P}^-_\xi(\kappa, \mu^{\mathrm{ind}}, \sqsubseteq, \theta, \mathcal{S})$ asserts the existence of a $\xi$-bounded $\mathcal{C}$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ together with a sequence $\langle i(\alpha) \mid \alpha < \kappa\rangle$ of ordinals in $\mu$, such that:

- for every $\alpha < \kappa$, there exists a canonical enumeration $\langle C_{\alpha,i} \mid i(\alpha) \le i < \mu\rangle$ of $\mathcal{C}_\alpha$ satisfying that the sequence $\langle \mathrm{acc}(C_{\alpha,i}) \mid i(\alpha) \le i < \mu\rangle$ is $\subseteq$-increasing with $\bigcup_{i\in[i(\alpha),\mu)} \mathrm{acc}(C_{\alpha,i}) = \mathrm{acc}(\alpha)$;
- for all $\alpha < \kappa$, $i \in [i(\alpha), \mu)$ and $\bar\alpha \in \mathrm{acc}(C_{\alpha,i})$, it is the case that $i \ge i(\bar\alpha)$ and $C_{\bar\alpha,i} \sqsubseteq C_{\alpha,i}$;
- for every sequence $\langle B_\tau \mid \tau < \theta\rangle$ of cofinal subsets of $\kappa$, and every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that for all $C \in \mathcal{C}_\alpha$ and $\tau < \min\{\alpha, \theta\}$, $\sup(\mathrm{nacc}(C) \cap B_\tau) = \alpha$.

Conventions 3.4 and 3.5 apply to the preceding, as well.

***Lemma 6.3*** *Suppose that:*

(1) *$\mu < \kappa$ is an infinite cardinal;*
(2) *$K$ is a streamlined $\kappa$-tree;*
(3) *$\mathrm{P}^-(\kappa, \mu^{ind}, \sqsubseteq, 1)$ holds, and is witnessed by a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ such that $\bigcap \mathcal{C}_\alpha$ is cofinal in $\alpha$ for every $\alpha \in \mathrm{acc}(\kappa)$;*
(4) *$\diamondsuit(\kappa)$ holds.*

*Then, there exists a normal and splitting streamlined $\kappa$-Souslin tree $T$ with $V(T) \supseteq V^-(K)$ such that $T$ admits a $\mu$-ascent path.*

**Proof** As a preparatory step, we shall need the following simple claim.

***Claim 6.3.1*** *We may assume that $\mathcal{B}(K) \neq \emptyset$.* ∎

**Proof** For every $\eta \in K$, define a function $\eta' : \mathrm{dom}(\eta) \to H_\kappa$ via $\eta'(\alpha) := (\eta(\alpha), 0)$. Then, $K' := \{\eta' \mid \eta \in K\} \uplus {}^{<\kappa}1$ is a streamlined $\kappa$-tree with $V^-(K') = V^-(K)$ and, in addition, $\mathcal{B}(K') \neq \emptyset$. ∎

Write $\vec{\mathcal{C}}$ for $\langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$. In particular, $\vec{\mathcal{C}}$ is a $\mathrm{P}^-(\kappa, \kappa, \sqsubseteq, 1)$-sequence satisfying that, for all $\alpha \in \mathrm{acc}(\kappa)$ and $C, D \in \mathcal{C}_\alpha$, $\sup(C \cap D) = \alpha$. As always, we may also assume that $0 \in \bigcap_{0<\alpha<\kappa} \bigcap \mathcal{C}_\alpha$.

Using $\vec{\mathcal{C}}$ and $K$, construct the sequence of levels $\langle T_\alpha \mid \alpha < \kappa\rangle$ exactly as in the proof of Theorem 3.7, so that $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a normal and splitting streamlined $\kappa$-Souslin tree. From Claim 3.7.2, we infer that $V(T) \supseteq V^-(K)$.

In addition, the construction of Theorem 3.7 ensures that for every $\alpha \in \mathrm{acc}(\kappa)$, it is the case that

$$T_\alpha = \{\mathbf{b}_x^{C,\eta} \mid C \in \mathcal{C}_\alpha, \eta \in K_\alpha, x \in T \restriction C\}.$$

Fix $\zeta \in \mathcal{B}(K)$. Let $\langle i(\alpha) \mid \alpha < \kappa\rangle$ witness that $\vec{\mathcal{C}}$ is a $\mathrm{P}^-(\kappa, \mu^{\mathrm{ind}}, \sqsubseteq, 1)$-sequence. Similar to the proof of [BR21, Theorem 6.11], for every $\alpha \in \mathrm{acc}(\kappa)$, using the canonical enumeration $\langle C_{\alpha,i} \mid i(\alpha) \le i < \mu\rangle$ of $\mathcal{C}_\alpha$, we define a function $f_\alpha : \mu \to T_\alpha$ via

$$f_\alpha(j) := \mathbf{b}_\emptyset^{C_{\alpha,\max\{j,i(\alpha)\}},\zeta\restriction\alpha}.$$

***Claim 6.3.2*** *Let $\beta < \alpha$ be a pair of ordinals in $\mathrm{acc}(\kappa)$. Then, there exists an $i < \mu$ such that $f_\beta(j) \subseteq f_\alpha(j)$ whenever $i \le j < \mu$.*

**Proof** Note that by Claim 3.7.1, for all $C \in \mathcal{C}_\alpha$, $\eta \in K_\alpha$, and $x \in T \restriction (C \cap \beta)$, if $\beta \in \mathrm{acc}(C)$, then $\mathbf{b}_x^{C,\eta} \restriction \beta = \mathbf{b}_x^{C\cap\beta,\eta\restriction\beta}$.

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Now, by Definition 6.2, we may fix a large enough $i \in [i(\alpha), \mu)$ such that $\beta \in \mathrm{acc}(C_{\alpha,j})$ whenever $i \le j < \mu$. Let $j$ be such an ordinal. Then, $j \ge i(\beta)$ and $C_{\alpha,j} \cap \beta = C_{\beta,j}$, so that

$$f_\beta(j) = \mathbf{b}_\varnothing^{C_{\beta,j},\zeta\restriction\beta} = \mathbf{b}_\varnothing^{C_{\alpha,j},\zeta\restriction\alpha} \restriction \beta = f_\alpha(j) \restriction \beta,$$

as sought. ∎

It now easily follows that $T$ admits a $\mu$-ascent path.

***Corollary 6.4*** *Suppose that:*

- *$\lambda$ is an uncountable cardinal satisfying $\square_\lambda$ and $2^\lambda = \lambda^+$;*
- *$\mu < \lambda$ is an infinite regular cardinal satisfying $\lambda^\mu = \lambda$.*

*Then, there exists a streamlined $\lambda^+$-Souslin tree $T$ with $V(T) = \mathrm{acc}(\lambda^+)$ such that $T$ admits a $\mu$-ascent path.*

**Proof** By [LHL18, Theorem 3.4], in particular, $\square^{\mathrm{ind}}(\lambda^+, \mu)$ holds. Then, by [BR21, Theorem 4.44], $\mathrm{P}^-(\lambda^+, \mu^{\mathrm{ind}}, \sqsubseteq, 1)$ holds. Furthermore, its proof shows that starting with a $\square^{\mathrm{ind}}(\lambda^+, \mu)$-sequence $\vec{\mathcal{C}}$, there exists a triangular $\mathfrak{x} = \langle x_{\gamma,\beta} \mid \gamma < \beta < \kappa\rangle$ such that:

(i) for all $\gamma < \beta < \kappa$, $x_{\gamma,\beta}$ is a finite subset of $(\gamma, \beta]$ with $\beta \in x_{\beta,\gamma}$;
(ii) the corresponding postprocessing function $\Phi_\mathfrak{x}$ satisfies that $\langle \{\Phi_\mathfrak{x}(C) \mid C \in \mathcal{C}_\alpha\} \mid \alpha \in \mathrm{acc}(\kappa)\rangle$ witnesses $\mathrm{P}^-(\lambda^+, \mu^{\mathrm{ind}}, \sqsubseteq, 1)$.[17]

Recalling [BR21, Lemma 4.9], Clause (i) implies that $C \subseteq \Phi_\mathfrak{x}(C)$ for every $C \in \bigcup_{\alpha\in\mathrm{acc}(\kappa)} \mathcal{C}_\alpha$. Consequently, our witness to $\mathrm{P}^-(\lambda^+, \mu^{\mathrm{ind}}, \sqsubseteq, 1)$ satisfies Clause (3) of Lemma 6.3.

Meanwhile, by Shelah's theorem, $2^\lambda = \lambda^+$ implies $\diamondsuit(\lambda^+)$. In addition, it is a classical theorem of Jensen that $\square_\lambda$ gives a special $\lambda^+$-Aronszajn tree, so by Theorem 2.27, we may find a streamlined $\lambda^+$-tree $K$ such that $V(K) = \mathrm{acc}(\lambda^+)$. It now follows from Lemma 6.3 that there exists a normal and splitting streamlined $\lambda^+$-Souslin tree $T$ with $V(T) = \mathrm{acc}(\lambda^+)$ such that $T$ admits a $\mu$-ascent path. ∎

We now turn to combine the preceding construction with the study of large cardinals. The following cardinal characteristic $\chi(\kappa)$ provides a measure of how far $\kappa$ is from being weakly compact.

***Definition 6.5*** (The $C$-sequence number of $\kappa$, [LHR21]) If $\kappa$ is weakly compact, then let $\chi(\kappa) := 0$. Otherwise, let $\chi(\kappa)$ denote the least cardinal $\chi \le \kappa$ such that, for every $C$-sequence $\langle C_\beta \mid \beta < \kappa\rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta\in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

By [LHR21, Lemma 2.12(1)], if $\kappa$ is an inaccessible cardinal satisfying $\chi(\kappa) < \kappa$, then $\kappa$ is $\omega$-Mahlo. The following is an expanded form of Theorem E.

***Theorem 6.6*** *Assuming the consistency of a weakly compact cardinal, it is consistent that for some strongly inaccessible cardinal $\kappa$ satisfying $\chi(\kappa) = \omega$, the following two hold:*

[17] Strictly speaking, one needs to extend the definition to $\alpha$'s in $\mathrm{nacc}(\kappa)$, but this is trivial.

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- *Every $\kappa$-Aronszajn tree admits an $\omega$-ascent path;*
- *There is a streamlined $\kappa$-Souslin tree $T$ such that $V(T) = \mathrm{acc}(\kappa)$.*

**Proof** Suppose that $\kappa$ is a non-subtle weakly compact cardinal. By possibly using a preparatory forcing, we may assume that the non-subtle weak compactness of $\kappa$ is indestructible under forcing with $\mathrm{Add}(\kappa,1)$. Following the proof of [LHR21, Theorem 3.4], let $\mathbb{P}$ be the standard forcing to add a $\square^{\mathrm{ind}}(\kappa,\omega)$-sequence by closed initial segments, let $G$ be $\mathbb{P}$-generic, and let $\vec{\mathcal{C}} = \langle C_{\alpha,i} \mid \alpha < \kappa,\ i(\alpha) \leq i < \omega\rangle$ denote the generically-added $\square^{\mathrm{ind}}(\kappa,\omega)$-sequence. Work in $V[G]$. By Clauses (1), (2), and (4) of [LHR21, Theorem 3.4], $\kappa$ is strongly inaccessible, $\chi(\kappa) = \omega$, and every $\kappa$-Aronszajn tree admits an $\omega$-ascent path.

For every $\alpha \in \mathrm{acc}(\kappa)$, let

$$B_\alpha := \{\beta \in C_{\alpha,i(\alpha)} \mid \{\min(C_{\alpha,i(\alpha)}\setminus\beta+1) + l \mid l < \omega\} \subseteq C_{\alpha,i(\alpha)}\setminus\{\beta+1\}\}.$$

***Claim 6.6.1*** *For every cofinal $B \subseteq \kappa$, there exist $\alpha \in E^\kappa_\omega$ and $\varepsilon < \alpha$ such that $(B_\alpha\setminus\varepsilon) \subseteq B$, $i(\alpha) = 0$ and $\sup(\mathrm{nacc}(C_{\alpha,i}) \cap B_\alpha) = \alpha$ for every $i < \omega$.* ■

**Proof** We follow the proof of [LH17, Lemma 3.9]. Work in $V$. For every $\alpha \in \mathrm{acc}(\kappa)$, let $\dot{B}_\alpha$ be the canonical $\mathbb{P}$-name for $B_\alpha$. Next, let $\dot{B}$ be a $\mathbb{P}$-name for a cofinal subset of $\kappa$, and let $p_0$ be an arbitrary condition in $\mathbb{P}$. By possibly extending $p_0$, we may assume that $i(\gamma^{p_0})^{p_0} = 0$. We shall recursively define a decreasing sequence of conditions $\langle p_n \mid n < \omega\rangle$, and an increasing sequence of ordinals $\langle \beta_n \mid n < \omega\rangle$ such that for every $n < \omega$, all of the following hold:

(1) $p_{n+1} \leq p_n$;
(2) $i(\gamma^{p_{n+1}})^{p_{n+1}} = 0$;
(3) $p_{n+1} \Vdash$ "$\beta_n \in \dot{B}$ and $\dot{B}_{\gamma^{p_{n+1}}}\setminus(\gamma^{p_n}+1) = \{\beta_n\}$";
(4) For every $i \leq n$, $\beta_n \in \mathrm{nacc}(C^{p_{n+1}}_{\gamma^{p_{n+1}},i})$.
(5) For every $i < \omega$, $C^{p_{n+1}}_{\gamma^{p_{n+1}},i} \cap \gamma^{p_n} = C^{p_n}_{\gamma^{p_n},i}$.

Suppose $n < \omega$ is such that $\langle p_m \mid m \leq n\rangle$ and $\langle \beta_m \mid m < n\rangle$ have already been successfully defined. Find a $p^*_n \leq p_n$ and a $\beta_n > \gamma^{p_n}$ such that $p^*_n \Vdash$ "$\beta_n \in \dot{B}$". Without loss of generality, $\gamma^{p^*_n} > \beta_n$. Now, let $\gamma := \gamma^{p^*_n} + \omega$, so that

$$\gamma^{p_n} < \beta_n < \beta_n + 1 < \gamma^{p^*_n} < \gamma^{p^*_n} + \omega = \gamma.$$

Let $m < \omega$ be the least such that $m \geq \max\{n, i(\gamma^{p^*_n})^{p^*_n}\}$ and $\gamma^{p_n} \in \mathrm{acc}(C^{p^*_n}_{\gamma^{p^*_n},m})$. Then, let $p_{n+1}$ be the unique extension of $p^*_n$ with $\gamma^{p_{n+1}} = \gamma$ and $i(\gamma)^{p_{n+1}} = 0$ to satisfy the following for every $i < \omega$:

$$C^{p_{n+1}}_{\gamma^{p_{n+1}},i} = \begin{cases} C^{p_n}_{\gamma^{p_n},i} \cup \{\gamma^{p_n},\beta_n\} \cup \{\gamma^{p^*_n} + l \mid l < \omega\}, & \text{if } i \leq m;\\ C^{p^*_n}_{\gamma^{p^*_n},i} \cup \{\gamma^{p^*_n} + l \mid l < \omega\}, & \text{otherwise.}\end{cases}$$

Thus, we have maintained requirements (1)–(5).

Once completing the above recursion, we obtain a decreasing sequence of conditions $\langle p_n \mid n < \omega\rangle$. Let $\alpha := \sup\{\gamma^{p_n} \mid n < \omega\}$, and let $p$ be the unique lower bound of $\langle p_n \mid n < \omega\rangle$ to satisfy $\gamma^p = \alpha$, $i(\alpha)^p = 0$, and $C^p_{\alpha,i} = \bigcup_{n<\omega} C^{p_n}_{\gamma^{p_n},i}$ for every $i < \omega$.

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Then, $p$ is a legitimate condition satisfying $p \Vdash$ "$\dot{B}_\alpha \setminus (\gamma^{p_0} + 1) = \{\beta_n \mid n < \omega\} \subseteq \dot{B}$". In addition, for every $i < \omega$, $\{\beta_n \mid i \leq n < \omega\} \subseteq \mathrm{nacc}(C^p_{\alpha,i})$. So we are done. ∎

For each $\alpha < \kappa$, let $\mathcal{C}_\alpha := \{C_{\alpha,i} \mid i(\alpha) \leq i < \omega\}$. We claim that $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ is a $\mathrm{P}^-(\kappa, \omega^{\mathrm{ind}}, \sqsubseteq, 1)$-sequence satisfying that $\bigcap \mathcal{C}_\alpha$ is cofinal in $\alpha$ for every $\alpha \in \mathrm{acc}(\kappa)$. As we already know that $\vec{\mathcal{C}}$ is an $\square^{\mathrm{ind}}(\kappa, \omega)$-sequence, the first two bullets of Definition 6.2 are satisfied, and $\bigcap \mathcal{C}_\alpha = C_{\alpha, i(\alpha)}$ for every $\alpha \in \mathrm{acc}(\kappa)$. Thus, we are left with verifying the last bullet of Definition 6.2 with $\theta := 1$ and $\mathcal{S} := \{\kappa\}$. By the same argument from the proof of [BR21, Corollary 3.4], this boils down to showing that for every cofinal $B \subseteq \kappa$, there exists at least one $\alpha \in \mathrm{acc}(\kappa)$ such that $\sup(\mathrm{nacc}(C_{\alpha,i}) \cap B) = \alpha$ for every $i \in [i(\alpha), \omega)$. This is covered by Claim 6.6.1.

***Claim 6.6.2*** *$\diamondsuit(E^\kappa_\omega)$ holds.*

**Proof** This is a standard consequence of Claim 6.6.1 together with the fact that $\kappa^{<\kappa} = \kappa$, but we give the details. Let $\vec{X} = \langle X_\beta \mid \beta < \kappa \rangle$ be a repetitive enumeration of $[\kappa]^{<\kappa}$ such that each set appears cofinally often. Let us say that an ordinal $\alpha \in E^\kappa_\omega$ is *informative* if $\sup(B_\alpha) = \alpha$ and there are $\varepsilon < \kappa$ and a subset $A_\alpha \subseteq \alpha$ such that $A_\alpha \cap \gamma = X_\beta \cap \gamma$ for every pair $\gamma < \beta$ of ordinals from $B_\alpha \setminus \varepsilon$. Note that if $\alpha$ is informative, then the set $A_\alpha$ is uniquely determined. For a noninformative $\alpha \in E^\kappa_\omega$, we let $A_\alpha := \emptyset$.

To verify that $\langle A_\alpha \mid \alpha \in E^\kappa_\omega \rangle$ witnesses $\diamondsuit(E^\kappa_\omega)$, let $A$ be a subset of $\kappa$ and let $C$ be a club in $\kappa$, and we shall find an $\alpha \in C \cap E^\kappa_\omega$ such that $A \cap \alpha = A_\alpha$.

By the choice of $\vec{X}$, we may fix a strictly increasing function $f : \kappa \to \kappa$ satisfying that $A \cap \xi = X_{f(\xi)}$ for every $\xi < \kappa$. Consider the club $D := \{\delta \in C \mid f[\delta] \subseteq \delta\}$. Let $B$ be some cofinal subset of $\mathrm{Im}(f)$ sparse enough to satisfy that for every pair $\gamma < \beta$ of ordinals from $B$, there exists a $\delta \in D$ with $\gamma < \delta < \beta$. Using Claim 6.6.1, fix $\alpha \in E^\kappa_\omega$ and $\varepsilon < \alpha$ such that $(B_\alpha \setminus \varepsilon) \subseteq B$ and $\sup(B_\alpha) = \alpha$. Now, let $\gamma < \beta$ be a pair of ordinals in $B_\alpha \setminus \varepsilon$. As $\gamma, \beta \in B$, we may pick a $\delta \in D$ with $\gamma < \delta < \beta$. As $\beta \in B \subseteq \mathrm{Im}(f)$, we may also pick a $\xi < \kappa$ such that $\beta = f(\xi)$. Since $f[\delta] \subseteq \delta \subseteq \beta$, it must be the case that $\xi \geq \delta > \gamma$. So $A \cap \gamma = (A \cap \xi) \cap \gamma = X_\beta \cap \gamma$. Thus, we showed that $A \cap \gamma = X_\beta \cap \gamma$ for every pair $\gamma < \beta$ of ordinals in $B_\alpha \setminus \varepsilon$, and hence $\alpha$ is informative and $A_\alpha = A \cap \alpha$. In addition, for every pair $\gamma < \beta$ of ordinals in $B_\alpha \setminus \varepsilon$, there exists $\delta \in D$ with $\gamma < \delta < \beta$, and hence $\alpha \in \mathrm{acc}^+(D) \subseteq C$. ∎

Since $\kappa$ is a strongly inaccessible cardinal that is non-subtle, Corollary 2.22 implies that there exists a streamlined $\kappa$-tree $K$ such that $V^-(K)$ covers a club in $\kappa$. So by appealing to Lemma 6.3 and then to Lemma 2.15, we infer that there exists a streamlined $\kappa$-Souslin tree $T$ with $V(T) = \mathrm{acc}(\kappa)$.

By [RS23, Theorem 2.30], $\chi(\kappa) = 0$ refutes $\clubsuit_{\mathrm{AD}}(\mathrm{Reg}(\kappa))$. An easy variant of that proof yields that $\chi(\kappa) = 0$ furthermore refutes $\clubsuit_{\mathrm{AD}}(\mathrm{Reg}(\kappa) \cap D)$ for every club $D \subseteq \kappa$. It follows from the preceding theorem together with the proof of [RS23, Theorem 2.23] that $\chi(\kappa) = \omega$ is compatible with $\clubsuit_{\mathrm{AD}}(D)$ holding for some club $D \subseteq \kappa$. Whether this can be improved to $\chi(\kappa) = 1$ remains an open problem.

## 7 A new sufficient condition for a Dowker space

In this section, we shall present a new sufficient condition for the existence of a Dowker space of size $\kappa$, proving Theorem F. Our proof will go through the principle $\clubsuit_{\mathrm{AD}}$ to be

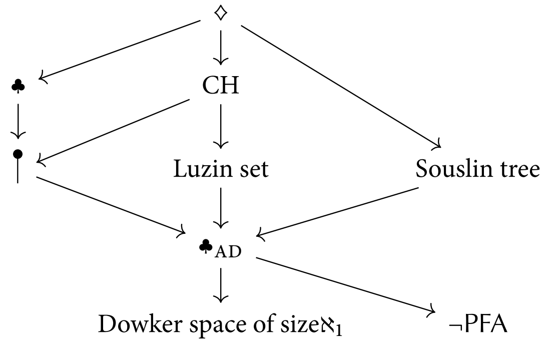


*Figure 2*: Diagram of implications, all at the level of $\aleph_1$.

defined momentarily. As mentioned in the article's Introduction, the existence of a $\kappa$-Souslin tree $\mathbf{T}$ for which $V(\mathbf{T})$ is stationary yields an instance of $\clubsuit_{\mathrm{AD}}$. Here, however, we shall obtain instances of $\clubsuit_{\mathrm{AD}}$ by pumping-up instances of the classical principle $\clubsuit$. For completeness, and upon the suggestion of the referee, we first include a diagram (see Figure 2) illustrating the results from [RS23, RST24].

The general case reads as follows.

***Definition 7.1*** [RS23] Let $\mathcal{S}$ be a collection of stationary subsets of a regular uncountable cardinal $\kappa$, and $\mu, \theta$ be nonzero cardinals below $\kappa$. The principle $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, \theta)$ asserts the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S} \rangle$ such that:

(1) For every $\alpha \in \mathrm{acc}(\kappa) \cap \bigcup \mathcal{S}$, $\mathcal{A}_\alpha$ is a pairwise disjoint family of $\mu$ many cofinal subsets of $\alpha$.
(2) For every $\mathcal{B} \subseteq [\kappa]^\kappa$ of size $\theta$, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that $\sup(A \cap B) = \alpha$ for all $A \in \mathcal{A}_\alpha$ and $B \in \mathcal{B}$[18] .
(3) For all $A \neq A'$ from $\bigcup_{S \in \mathcal{S}} \bigcup_{\alpha \in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

***Remark 7.2*** The variation $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, {<}\theta)$ asserts the existence of a sequence simultaneously witnessing $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, \vartheta)$ for all $\vartheta < \theta$.

By [RS23, Lemma 2.10], for a pair $\chi < \kappa$ of infinite regular cardinals, for a stationary subset $S$ of $E^\kappa_\chi$, Ostaszewski's principle $\clubsuit(S)$ implies $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \chi, {<}\omega)$ for some partition $\mathcal{S}$ of $S$ into $\kappa$ many stationary sets. The next lemma reduces the hypothesis "$S \subseteq E^\kappa_\chi$" down to "$S \cap \mathrm{Tr}(S) = \emptyset$".

***Lemma 7.3*** *Suppose:*

- *$\mu, \theta < \kappa = \kappa^{<\theta}$ are infinite cardinals;*
- *$S \subseteq E^\kappa_{\geq \max\{\mu,\theta\}}$ is stationary and* $\mathrm{Tr}(S) \cap S = \emptyset$*;*
- *$\clubsuit(S)$ holds.*

[18] Note that the existence of stationarily many such $\alpha \in S$ is no stronger than the existence of just one $\alpha \in S \cap \mathrm{acc}(\kappa)$. See [BR21, Corollary 3.4] for the prototype argument.

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*Then, $\clubsuit_{\mathrm{AD}}(\mathcal{S},\mu,{<}\theta)$ holds for some partition $\mathcal{S}$ of $S$ into $\kappa$ many stationary sets. More generally, for every $Z\subseteq\kappa$ such that $S\subseteq\operatorname{acc}^+(Z)$, there exists a matrix $\langle A_{\delta,i}\mid \delta\in S, i<\mu\rangle$ and a partition $\mathcal{S}$ of $S$ into $\kappa$ many pairwise disjoint stationary sets such that:*

(1) *For every $\delta\in S$, $\langle A_{\delta,i}\mid i<\mu\rangle$ is a sequence of pairwise disjoint subsets of $Z\cap\delta$, and* $\sup(A_{\delta,i})=\delta$.
(2) *For every $(\gamma,\delta)\in[S]^2$, for all $i,j<\mu$, $\sup(A_{\gamma,i}\cap A_{\delta,j})<\gamma$.*
(3) *For every $\vartheta<\theta$, every sequence $\langle B_\tau\mid\tau<\vartheta\rangle$ of cofinal subsets of $Z$ and every $S'\in\mathcal{S}$, there exists $\delta\in S'$ such that $\sup(A_{\delta,i}\cap B_\tau)=\delta$ for all $i<\mu$ and $\tau<\vartheta$.*

**Proof** By [BR21, Theorem 3.7], since $\clubsuit(S)$ holds, we may find a partition $\langle S_{\vartheta,\iota}\mid\vartheta<\theta,\iota<\kappa\rangle$ of $S$ into stationary sets such that $\clubsuit(S_{\vartheta,\iota})$ holds for all $\vartheta<\theta$ and $\iota<\kappa$. For all $\vartheta<\theta$ and $\iota<\kappa$, since $\clubsuit(S_{\vartheta,\iota})$ holds and $\kappa^\vartheta=\kappa$, by [BR21, Lemma 3.5], we may fix a matrix $\langle X^\tau_\delta\mid\delta\in S_{\vartheta,\iota},\tau<\vartheta\rangle$ such that, for every sequence $\langle X^\tau\mid\tau<\vartheta\rangle$ of cofinal subsets of $\kappa$, there are stationarily many $\delta\in S_{\vartheta,\iota}$, such that, for all $\tau<\vartheta$, $X^\tau_\delta\subseteq X^\tau\cap\delta$ and $\sup(X^\tau_\delta)=\delta$.

Now, let $Z\subseteq\kappa$ with $S\subseteq\operatorname{acc}^+(Z)$ be given. For all $\vartheta<\theta$, $\iota<\kappa$, $\delta\in S_{\vartheta,\iota}$ and $\tau<\vartheta$, we do the following:

- if $X^\tau_\delta\cap Z$ is a cofinal subset of $\delta$, then let $Y^\tau_\delta:=X^\tau_\delta\cap Z$. Otherwise, let $Y^\tau_\delta$ be an arbitrary cofinal subset of $Z\cap\delta$;
- since $\delta\in S\subseteq\kappa\setminus\operatorname{Tr}(S)$, we may fix a club $C_\delta\subseteq\delta$ disjoint from $S$, and then, by [BR21, Lemma 3.3], we may find a cofinal subset $Z^\tau_\delta$ of $Y^\tau_\delta$ such that in-between any two points of $Z^\tau_\delta$ there exists a point of $C_\delta$, so that $\operatorname{acc}^+(Z^\tau_\delta)\cap S=\emptyset$.

As $\operatorname{cf}(\delta)\geq\theta>\vartheta$ and by possibly thinning out, we may assume that $\langle Z^\tau_\delta\mid\tau<\vartheta\rangle$ consists of pairwise disjoint cofinal subsets of $Z\cap\delta$. As $\operatorname{cf}(\delta)\geq\mu$, for every $\tau<\vartheta$, we may fix a partition $\langle Z^{\tau,i}_\delta\mid i<\mu\rangle$ of $Z^\tau_\delta$ into cofinal subsets of $\delta$. For every $i<\mu$, let

$$A_{\delta,i}:=\bigcup_{\tau<\vartheta}Z^{\tau,i}_\delta.$$

For every $i<\mu$, since $\operatorname{acc}^+(Z^{\tau,i}_\delta)\cap S\subseteq\operatorname{acc}^+(Z^\tau_\delta)\cap S=\emptyset$, and since $\delta\in S\subseteq E^\kappa_{>\vartheta}$, we get that $\operatorname{acc}^+(A_{\delta,i})\cap S=\emptyset$. So $\langle A_{\delta,i}\mid i<\mu\rangle$ is a sequence of pairwise disjoint cofinal subsets of $\delta$, and for every $\gamma\in S\cap\delta$ and every cofinal subset $A\subseteq\gamma$, $\sup(A\cap A_{\delta,i})<\gamma$. Thus, we have already taken care of Clauses (1) and (2).

Next, consider $\mathcal{S}:=\{\bigcup_{\vartheta<\theta}S_{\vartheta,\iota}\mid\iota<\kappa\}$ which is a partition of $S$ into $\kappa$ many stationary sets. Now, given $\vartheta<\theta$, a sequence $\langle B_\tau\mid\tau<\vartheta\rangle$ of cofinal subsets of $Z$, and some $S'\in\mathcal{S}$, we may find $\iota<\kappa$ such that $S'\supseteq S_{\vartheta,\iota}$, and find $\delta\in S_{\vartheta,\iota}$ such that, for all $\tau<\vartheta$, $X^\tau_\delta\subseteq B_\tau\cap\delta$ and $\sup(X^\tau_\delta)=\delta$. In particular, for all $\tau<\vartheta$ and $i<\mu$, $Z^{\tau,i}_\delta\subseteq Z^\tau_\delta\subseteq Y^\tau_\delta=X^\tau_\delta\cap Z\subseteq B_\tau$. Therefore, for all $\tau<\vartheta$ and $i<\mu$, $\sup(A_{\delta,i}\cap B_\tau)=\delta$. ∎

***Corollary 7.4*** *Suppose that $\clubsuit(S)$ holds for some nonreflecting stationary subset $S$ of $\kappa$. Then, $\clubsuit_{\mathrm{AD}}(\mathcal{S},\omega,{<}\omega)$ holds for some partition $\mathcal{S}$ of $S$ into $\kappa$ many stationary sets.*

Using the preceding, we now obtain Theorem F which extends an old result of Good [Goo95].

***Corollary 7.5*** *If $\clubsuit(S)$ holds over a nonreflecting stationary $S\subseteq\kappa$, then there are $2^\kappa$ many pairwise nonhomeomorphic Dowker spaces of size $\kappa$.*

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**Proof** By [RST24, Theorem A.1], if $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,2)$ holds for a partition $\mathcal{S}$ of a nonreflecting stationary subset of $\kappa$ into $\kappa$ many stationary sets, then there are $2^\kappa$ many pairwise nonhomeomorphic Dowker spaces of size $\kappa$. ■

Our last corollary deals with the problem of getting $\clubsuit_{\mathrm{AD}}$ to hold over a club subset of a successor cardinal.

***Corollary 7.6*** *Suppose that $\kappa = \lambda^+$ for some infinite cardinal $\lambda$, and that $\clubsuit(E^\kappa_\theta)$ holds for every $\theta \in \mathrm{Reg}(\kappa)$. Then, there exists a partition $\mathcal{S}$ of some club $D \subseteq \mathrm{acc}(\kappa)$ into $\kappa$ many sets such that $\clubsuit_{\mathrm{AD}}(\mathcal{S},\omega,1)$ holds. Furthermore, there is a matrix $\langle A_{\delta,i} \mid \delta \in D, i < \mathrm{cf}(\delta)\rangle$ such that:*

(1) *For every $\delta \in D$, $\langle A_{\delta,i} \mid i < \mathrm{cf}(\delta)\rangle$ is sequence of pairwise disjoint cofinal subsets of $\delta$.*
(2) *For all $A \neq A'$ from $\{A_{\delta,i} \mid \delta \in D, i < \mathrm{cf}(\delta)\}$, $\sup(A \cap A') < \sup(A)$.*
(3) *For every cofinal $B \subseteq \kappa$, for every $S \in \mathcal{S}$, there are stationarily many $\delta \in S$ such that $\sup(A_{\delta,i} \cap B) = \delta$ for all $i < \mathrm{cf}(\delta)$.*

**Proof** Let $\langle Z_\mu \mid \mu \in \mathrm{Reg}(\kappa)\rangle$ be a partition of $\kappa$ into cofinal sets. Let $D := \bigcap_{\mu\in\mathrm{Reg}(\kappa)} \mathrm{acc}^+(Z_\mu)$. For every $\mu \in \mathrm{Reg}(\kappa)$, by appealing to Lemma 7.3 with the set $Z_\mu$ and the stationary set $E^\kappa_\mu \cap D$, we may fix a matrix $\langle A_{\delta,i} \mid \delta \in E^\kappa_\mu \cap D, i < \mu\rangle$ and a partition $\langle S_{\mu,\iota} \mid \iota < \kappa\rangle$ of $E^\kappa_\mu \cap D$ into $\kappa$ many pairwise disjoint stationary sets such that:

- For every $\delta \in E^\kappa_\mu \cap D$, $\langle A_{\delta,i} \mid i < \mu\rangle$ is a sequence of pairwise disjoint subsets of $Z_\mu \cap \delta$, and $\sup(A_{\delta,i}) = \delta$.
- For every $(\gamma,\delta) \in [E^\kappa_\mu \cap D]^2$, for all $i, j < \mu$, $\sup(A_{\gamma,i} \cap A_{\delta,j}) < \gamma$.
- For every cofinal $B \subseteq Z_\mu$, for every $\iota < \kappa$, there exists $\delta \in S_{\mu,\iota}$ such that $\sup(A_{\delta,i} \cap B) = \delta$ for every $i < \mu$.

Putting these matrices together, we get a matrix $\langle A_{\delta,i} \mid \delta \in D, i < \mathrm{cf}(\delta)\rangle$ satisfying Clause (1). In addition, since $Z_\mu \cap Z_{\mu'} = \emptyset$ for $\mu \neq \mu'$, Clause (2) is satisfied. Now, $\mathcal{S} := \{\bigcup_{\mu\in\mathrm{Reg}(\kappa)} S_{\mu,\iota} \mid \iota < \kappa\}$ is a partition of $D$ into $\kappa$ many stationary sets. By the pigeonhole principle, for every cofinal $B \subseteq \kappa$, there exists some $\mu \in \mathrm{Reg}(\kappa)$ such that $B \cap Z_\mu$ is cofinal in $\kappa$. So, for every $S \in \mathcal{S}$, there exists an $\iota < \kappa$ with $S_{\mu,\iota} \subseteq S$ and then there exists a $\delta \in S_{\mu,\iota}$ such that $\sup(A_{\delta,i} \cap B) = \delta$ for every $i < \mathrm{cf}(\delta)$. ■

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